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WEIGHT ESTIMATES FOR SOLUTIONS OF LINEAR SINGULAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND THE EVERITT-GIERTZ PROBLEM

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Abstract. We consider the equation

$$-y'(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R} \tag{0.1}$$

where $f\in L_p(\mathbb{R})$, $p\in[1,\infty]$ $(L_\infty(\mathbb{R}):=C(\mathbb{R}))$ and $0\le q\in L_1^{\text{loc}}(\mathbb{R})$. We assume that equation (0.1) is correctly solvable in $L_p(\mathbb{R})$. Let $y\in L_p(\mathbb{R})$ be a solution of (0.1). In the present paper we find minimal requirements for the weight function $r\in L_p^{\text{loc}}(\mathbb{R})$ under which the following estimate holds:

$$\|ry\|_p\le c(p)\|f\|_p,\quad \forall f\in L_p(\mathbb{R})$$

with an absolute constant $c(p)\in(0,\infty)$.

1. Introduction

In the present paper, we consider the equation

$$-y'(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R} \tag{1.1}$$

where $f\in L_p$, $p\in[1,\infty]$, $(L_p(\mathbb{R}):=L_p,\ p\in[1,\infty), L_\infty(\mathbb{R}):=C(\mathbb{R}))$ and

$$0\le q\in L_1^{\text{loc}}\quad (L_1^{\text{loc}}:=L_1^{\text{loc}}(\mathbb{R})). \tag{1.2}$$

By a solution of equation (1.1) we mean any absolutely continuous function y satisfying (1.1) almost everywhere in $\mathbb{R}$. In addition, we assume that equation (1.1) is correctly solvable in L_p. This means (see [3, Chapter III, Section 6, number 2]) that for a given $p\in[1,\infty]$ the following assertions hold:

I) For every function $f\in L_p$, there exists a unique solution of (1.1), $y\in L_p$.

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II) There exists an absolute positive constant $c(p)$ such that the solution of (1.1), $y \in L_p$, satisfies the inequality

$$\|y\|_p \le c(p)\|f\|_p, \qquad \forall f \in L_p \qquad (\|f\|_p := \|f\|_{L_p}). \tag{1.3}$$

See Section 2 below for precise requirements of q which guarantee I)–II). In the sequel, we denote by y a solution of (1.1) belonging to L_p. By c, $c(\cdot)$, $c_1, c_2, \dots$, we denote absolute positive constants which are not essential for exposition and may differ within a single chain of computations. In the present paper we study the problem of strengthening the estimate (1.3). More precisely, we find minimal requirements for a given weight function $r \in L_p^{\text{loc}}$ ($L_p^{\text{loc}} := L_p^{\text{loc}}(\mathbb{R})$), under which the following inequality holds:

$$\|ry\|_p \le c(p)\|f\|_p, \quad \forall f \in L_p. \tag{1.4}$$

For brevity, we call this "the (1.4) problem" or "the (1.4) question".

For differential equations (operators) of order higher than one, a similar question has been studied in many papers (see, e.g., the book [8] and comments and bibliography therein). We obtain necessary and sufficient (with an additional weak) conditions for the validity of estimates of type (1.4). A special feature of equation (1.1) is the following: one can first establish an unconditional criterion for solution of the (1.4) problem (this is the "easy" part of the paper that follows from a Hardy-type inequality and leads to conditions of fulfillment of (1.4) which is very difficult to check) and then use it for deducing convenient necessary and similar sufficient conditions for checking estimates (1.4) (this is the main and "harder" part of the paper that leads to easily verifiable conditions of fulfillment of (1.4)). Note that for $r \equiv q$ the (1.4) question is equivalent to the well-known Everitt-Giertz problem on separability in L_p of the differential operator $L = -\frac{d}{dx} + q(x)$ (see [4, 5] and also [8] and Section 3 below).

Therefore, our criterion for validity of (1.4) for $r \equiv q$ becomes a criterion for separability of the operator L in L_p. This is the first instance of a complete solution of the Everitt-Giertz problem.

We now describe the structure of the paper. For the reader's convenience, the paper is divided into Sections 2 through 5. In Section 2 we present preliminaries, Section 3 contains the list of all our results, in Section 4 all proofs are presented, and, finally, Section 5 is devoted to examples.

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2. Preliminaries

Theorem 2.1. [12, Chapter 2, Section 7] *Let $p \in (1,\infty)$, let μ, θ be continuous positive functions defined on $\mathbb{R}$, and let K be an integral operator:*

$$(Kf)(t) = \mu(t)\int_t^\infty \theta(\xi)f(\xi)d\xi, \quad t \in \mathbb{R}. \tag{2.1}$$

Then the operator $K : L_p \to L_p$ is bounded if and only if $H_p < \infty$.

Here $H_p = \sup_{x\in\mathbb{R}} H_p(x)$,

$$H_p(x) = \Big[\int_{-\infty}^x \mu(t)^p dt\Big]^{1/p}\Big[\int_x^\infty \theta(t)^{p'}dt\Big]^{1/p'}, \quad p' = \frac{p}{p-1} \tag{2.2}$$

and

$$H_p \le \|K\|_{p\to p} \le (p)^{1/p}(p')^{1/p'}H_p. \tag{2.3}$$

Remark 2.2. Theorem 2.1 follows from an inequality of Hardy type (see [9]). In particular, see [2] for such a proof. In [12] one can find an original direct proof of this theorem (under weaker requirements on μ and θ).

Theorem 2.3. [6, Chapter V, Section 2, numbers 4–5] *Let K denote the integral operator* (2.1). *Then*

$$\|K\|_{1\to 1} = \sup_{x\in\mathbb{R}} \theta(x)\int_{-\infty}^x \mu(t)dt, \tag{2.4}$$

$$\|K\|_{C(\mathbb{R})\to C(\mathbb{R})} = \sup_{x\in\mathbb{R}} \mu(x)\int_x^\infty \theta(t)dt. \tag{2.5}$$

Theorem 2.4. [7] *Let $p \in [1,\infty]$. Then equation* (1.1) *is correctly solvable in L_p if and only if*

$$\exists a \in (0,\infty): \quad q_0(a) \overset{def}{=} \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)dt > 0. \tag{2.6}$$

Corollary 2.5. [7], [1] *If equation* (1.1) *is correctly solvable in L_p, $p \in [1,\infty]$, then its solution $y \in L_p$ is of the form*

$$y(x) \overset{def}{=} (Gf)(x) = \int_x^\infty \exp\Big(-\int_x^t q(\xi)d\xi\Big)f(t)dt, \quad x \in \mathbb{R}. \tag{2.7}$$

Assume that conditions (1.2) and (2.6) are satisfied. For a given $x \in \mathbb{R}$ define a function $d(x)$:

$$d(x) = \inf_{d\ge 0}\Big\{d : \int_{x-d}^{x+d} q(t)dt = 2\Big\}. \tag{2.8}$$

In other words, $d(x)$ is the smallest positive root of the equation in $d \geq 0$:

$$\int_{x-d}^{x+d} q(t)dt = 2, \quad x \in \mathbb{R}. \tag{2.9}$$

Note that the function $q^*(x) = 1/d(x)$, $x \in \mathbb{R}$, can be interpreted as the Steklov average with step $d(x)$ of the function $q(t)$ at the point $t = x$ (see [13]). Indeed, from (2.9) it follows that

$$q^*(x) = \frac{1}{d(x)} = \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)dt, \quad x \in \mathbb{R}.$$

The function d was introduced in [1]. Various functions similar to (2.8) were first introduced by M. Otelbaev (see [8]).

Lemma 2.6. [2] *The function $d(x)$ is positive and continuous for $x \in \mathbb{R}$.*

Lemma 2.7. [2] *Let $x \in \mathbb{R}$. The inequality $\eta \geq d(x)$ $(0 \leq \eta \leq d(x))$ holds if and only if*

$$F(\eta) \overset{def}{=} \int_{x-\eta}^{x+\eta} q(t)dt \geq 2 \quad (F(\eta) \leq 2). \tag{2.10}$$

Lemma 2.8. [1] *Condition* (2.6) *holds if and only if for every $x \in \mathbb{R}$ equation* (2.9) *has at least one solution and $d_0 < \infty$. Here*

$$d_0 = \sup_{x\in\mathbb{R}} d(x). \tag{2.11}$$

Lemma 2.9. [2] *Let $x \in \mathbb{R}$, $\varepsilon \in [0,1]$, $t \in [x - \varepsilon d(x),\, x + \varepsilon d(x)$. Then*

$$(1-\varepsilon)d(x) \leq d(t) \leq (1+\varepsilon)d(x). \tag{2.12}$$

Definition 2.10. [2] *Let the following be given: $x \in \mathbb{R}$, a positive continuous function $\varkappa$ defined on $\mathbb{R}$, a sequence $\{x_n\}_{n\in\mathbb{N}'}$, $N' = \{\pm1, \pm2, \dots\}$. Consider the segments*

$$\Delta_n = [\Delta_n^-, \Delta_n^+], \quad \Delta_n^\pm = x_n \pm \varkappa(x_n), \quad n \in \mathbb{N}'.$$

We say that the sequence of segments $\{\Delta_n\}_{n=1}^\infty$ $\left(\{\Delta_n\}_{n=-\infty}^{-1}\right)$ forms an $\mathbb{R}(x,\varkappa)$-covering of $[x,\infty)$ (respectively $(-\infty,x]$) if the following conditions hold:

$$1) \qquad \Delta_n^+ = \Delta_{n+1}^- \quad \text{for} \quad n \geq 1 \quad (\Delta_{n-1}^+ = \Delta_n^- \quad \text{for} \quad n \leq -1) \tag{2.13}$$

$$2) \qquad \Delta_1^- = \Delta_{-1}^+ = x, \quad \bigcup_{n\geq1} \Delta_n = [x,\infty) \quad \Big(\bigcup_{n\leq-1} \Delta_n = (-\infty,x]\Big). \tag{2.14}$$

Lemma 2.11. [2] *Suppose that a positive continuous function* $\varkappa$ *defined on* $\mathbb{R}$ *satisfies the condition*

$$\lim_{t\to\infty}(t-\varkappa(t))=\infty \quad \Big(\lim_{t\to-\infty}(t+\varkappa(t))=-\infty\Big). \tag{2.15}$$

Then for every $x\in\mathbb{R}$ *there exists an* $\mathbb{R}(x,\varkappa)$*-covering of* $[x,\infty)$ *(an* $\mathbb{R}(x,\varkappa)$*-covering of* $(-\infty,x]$*).*

Remark 2.12. Assertions similar to Lemmas 2.9 and 2.11 were first made by Otelbaev (see [8]).

3. Statements of results

In the sequel (without special mention), we assume that for a given $p\in[1,\infty]$ the function r belongs to L_p^{loc} and the function q satisfies conditions (1.2) and (2.6). Under these assumptions we mainly consider the problem of boundedness of the operator $rG: L_p\to L_p$ for a fixed $p\in[1,\infty]$. (Clearly, by Theorem 2.4 and Corollary 2.5 this operator is bounded if and only if inequality (1.4) holds for the solutions $y\in L_p$ of equation (1.1).)

Theorem 3.1. *Let* $p\in[1,\infty]$, *and let* $p'=p(p-1)^{-1}$ *for* $p\in(1,\infty)$. *The operator* $rG: L_p\to L_p$ *is bounded if and only if* $\mathcal{A}_p(r,q)<\infty$. *Here*

$$\mathcal{A}_p(r,q)=\begin{cases}\sup\limits_{x\in\mathbb{R}} I_1(x), & \text{if} \quad p=1\\ \sup\limits_{x\in\mathbb{R}}(I_p(x))^{1/p}(J_{p'}(x))^{1/p'}, & \text{if} \quad p\in(1,\infty)\\ \sup\limits_{x\in\mathbb{R}} r(x)J_1(x), & \text{if} \quad p=\infty,\end{cases} \tag{3.1}$$

$$I_s(x)=\int_{-\infty}^x |r(t)|^s\exp\Big(-s\int_t^x q(\xi)d\xi\Big)dt, \quad x\in\mathbb{R}, \quad s\ge 1, \tag{3.2}$$

$$J_s(x)=\int_x^\infty \exp\Big(-s\int_x^t q(\xi)d\xi\Big)dt, \quad x\in\mathbb{R}, \quad s\ge 1. \tag{3.3}$$

Definition 3.2. *Let* $p\in[1,\infty]$, *and suppose that equation* (1.1) *is correctly solvable in* L_p. *Denote by* $\mathcal{D}_p$ *the set consisting of the solutions* $y\in L_p$ *of equation* (1.1) *in which* f *runs over the whole space* L_p *and* $\mathcal{L}_p$ *is the differential operator given by the expression* $\ell=-\frac{d}{dx}+q(x)$ *defined on* $\mathcal{D}_p$. *We say that the operator* $\mathcal{L}_p$ *is separable in* L_p *if*

$$\|y'\|_p+\|qy\|_p\le c(p)\|\mathcal{L}_p y\|_p, \qquad \forall y\in\mathcal{D}_p. \tag{3.4}$$

Note that the problem on separability of differential operators was first studied in [4, 5]. In [1] it was shown that the operator $\mathcal{L}_1$ is separable. For $p \in (1, \infty]$ the operator $\mathcal{L}_p$ is not always separable. A criterion for its separability is given in the following statement which follows from Theorem 3.1.

Corollary 3.3. *For $p \in (1, \infty]$ the operator $\mathcal{L}_p$ is separable if and only if $\mathcal{A}_p(q, q) < \infty$.*

Remark 3.4. See Section 5 for an example of a nonseparable operator $\mathcal{L}_p$, $p \in (1, \infty]$.

We give an example of an application of Corollary 3.3. Towards this end, we study below the class of equations (1.1) with coefficients q which are weakly equivalent to nondecreasing functions in $\mathbb{R}$. We need the following additional notions.

Definition 3.5. *We say that the finite positive functions φ and ψ defined in $\mathbb{R}$ are weakly equivalent if there is an absolute constant $c \geq 1$ such that, for all $x \in \mathbb{R}$, the following inequalities hold:*

$$c^{-1}\varphi(x) \leq \psi(x) \leq c\varphi(x). \tag{3.5}$$

Theorem 3.6. *Let the function q, defined in $\mathbb{R}$, be finite and positive. Then it is weakly equivalent to some nondecreasing finite positive function if and only if there is an absolute constant $\gamma \geq 1$ such that, for all $x \in \mathbb{R}$, the following estimate holds:*

$$\sup_{t \leq x} q(t) \leq \gamma q(x). \tag{3.6}$$

Theorem 3.7. *Consider equation* (1.1)*. Suppose that condition* (1.2) *holds and, in addition, the function q is finite, positive and satisfies condition* (3.6)*. Then* (1.1) *is correctly solvable in L_p, $p \in [1, \infty]$ if and only if the following condition holds:*

$$q_0 > 0, \qquad q_0 = \inf_{x \in \mathbb{R}} q(x). \tag{3.7}$$

In the latter case, the operator $\mathcal{L}_p$ is separable in L_p for all $p \in [1, \infty]$.

Corollary 3.8. *Consider equation* (1.1)*. Let the condition* (1.2) *hold, and in addition assume the function q is decreasing. Then the equation* (1.1) *is correctly solvable in $\mathcal{L}_p$ if and only if $q_0 > 0$ (see* (3.7)*). When $q_0 > 0$, the operator $\mathcal{L}_p$ is separable in $\mathcal{L}_p$ for all $p \in [1, \infty]$.*

Remark 3.9. During our discussion of an earlier version of Corollary 3.8, Prof. I.R. Liflyand suggested the idea to study the properties of equation

(1.1) with the condition that the function q is weakly equivalent to some nondecreasing finite positive function.

We now make some comments on Theorem 3.1. Checking the condition $\mathcal{A}(r,q) < \infty$ reduces to the problem on sharp by order two-sided estimates for the improper integrals $I_s(x)$ and $J_s(x)$ for $x \in \mathbb{R}$. The complexity of the latter problem depends on individual properties of the functions r and q, and its solution cannot be standardized. Thus, Theorem 3.1 is not convenient enough for practical applications. However, it can be used for proving other more efficient criteria for boundedness of the operator $rG : L_p \to L_p$, $p \in [1,\infty]$. For example, Theorem 3.1 was used to obtain the following assertions (and Theorems 3.12 and 3.14 below).

Theorem 3.10. *Let $r \in L_1^{\text{loc}}$. Then the operator $rG : L_1 \to L_1$ is bounded if and only if $m(r,q) < \infty$. Here*

$$m(r,q) = \sup_{x\in\mathbb{R}} \int_{x-d(x)}^{x+d(x)} |r(t)|dt. \tag{3.8}$$

Consider the main features of this criterion. First, Theorem 3.1 (for $p = 1$) is equivalent to Theorem 3.10. However, Theorem 3.10 is essentially different than Theorem 3.1. The main difference is as follows: checking the condition $m(r,q) < \infty$ can be made standard under some additional (and not too rigid) requirements on r and q. A detailed exposition of algorithms for such a check is contained in Section 5. Here we only mention the main reason that Theorem 3.10 is practically more efficient. Since equation (1.1) is correctly solvable, we have $d_0 < \infty$ (see Lemma 2.8), and regardless of $x \in \mathbb{R}$ the integral in (3.8) can be continued to a segment of finite length $\ell \le 2d_0$. In addition, for $d_0 < \infty$ the function d usually admits sharp by order two-sided, pointwise estimates (see Theorem 3.18 below). These estimates reduce the checking of $m(r,q) < \infty$ to a local problem on estimating an integral of the form

$$\int_{x-\varphi(x)}^{x+\varphi(x)} |r(t)|dt$$

with the already known absolutely bounded function φ. At this point the potential advantages of Theorem 3.10 (compared with Theorem 3.1 for $p = 1$) become obvious. Thus, for straightforward analysis of the (1.4) problem, Theorem 3.10 is substantially more convenient than Theorem 3.1 (for $p = 1$).

Therefore, now our main goal is to find analogues of Theorem 3.10 in the cases $p \in (1,\infty)$ and $p = \infty$.

We start with the following definition.

Definition 3.11. *Let q be a function such that for some $a \ge 1$, $b > 0$ and $x_0 \gg 1$, for all $|x| \ge x_0$, we have the inequalities*

$$a^{-1}d(x) \le d(t) \le ad(x) \quad for \quad |t-x| \le bd(x). \tag{3.9}$$

Then we say that the function q belongs to the class $\mathcal{K}(\gamma)$ (and write $q \in \mathcal{K}(\gamma)$) where $\gamma = a\exp(-b/a^2)$.

Theorem 3.12. *Let $p \in (1,\infty)$, $r \in L_p^{\mathrm{loc}}$. Then for the operator $rG : L_p \to L_p$ to be bounded, it is necessary and under the condition $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$ also sufficient that $h_p(r,q) < \infty$. Here*

$$h_p(r,q) = \sup_{x\in\mathbb{R}} d(x)^{1/p'}\Big[\int_{x-d(x)/2}^{x+d(x)/2} |r(t)|^p dt\Big]^{1/p}, \quad p' = \frac{p}{p-1}. \tag{3.10}$$

Corollary 3.13. *Let $p \in (1,\infty)$. For the operator $\mathcal{L}_p$ to be separable in L_p, it is necessary and under the condition $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$ also sufficient that $h_p(q,q) < \infty$.*

Note that the case $p = 1$ is excluded from the statement of Theorem 3.12 because Theorem 3.10 above gives (in contrast to Theorem 3.12) an unconditional criterion for boundedness of the operator $rG : L_1 \to L_1$. Thus Theorem 3.12 for $p = 1$ becomes redundant.

Theorem 3.14. *Let $r \in C^{\mathrm{loc}}(\mathbb{R})$. Then for the operator $rG : C(\mathbb{R}) \to C(\mathbb{R})$ to be bounded, it is necessary and under the condition $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$ also sufficient that $h_\infty(r,q) < \infty$. Here*

$$h_\infty(r,q) = \sup_{x\in\mathbb{R}} r(x)d(x). \tag{3.11}$$

Corollary 3.15. *For the operator $\mathcal{L}$ to be separable in $C(\mathbb{R})$, it is necessary and under the condition $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$, also sufficient that $h_\infty(q,q) < \infty$.*

In connection with the above statements, let us show that the requirement $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$ is not too rigid. This is a consequence of the following theorem.

Theorem 3.16. *Denote by $L_{1,\mathrm{loc}}^+$ the set of functions q satisfying the two conditions* (1.2) *and* (2.6)*. Then for every $\gamma > 1$ we have*

$$\mathcal{K}(\gamma) = L_{1,\mathrm{loc}}^+. \tag{3.12}$$

Remark 3.17. A general scheme, consisting of analogues of Definition 3.11 and Theorems 3.12 and 3.16, was first proposed by Otelbaev (see [8, 11, 10]).

To conclude, we present several convenient assertions for checking the condition $h_p(r,q) < \infty$, $p \in [1,\infty]$.

Theorem 3.18. *Suppose the function q can be decomposed into a sum*

$$q(x) = q_1(x) + q_2(x), \quad x \in \mathbb{R} \tag{3.13}$$

where the function $q_1(x)$ is positive for $x \in \mathbb{R}$ and absolutely continuous together with its derivative, and $q_2 \in L_1^{\mathrm{loc}}$. Let $\varkappa_1(x) \to 0$, $\varkappa_2(x) \to 0$ as $|x| \to \infty$ where

$$\varkappa_1(x) = \frac{1}{q_1(x)^2} \sup_{|\xi| \le 2/q_1(x)} \Big| \int_{x-\xi}^{x+\xi} q_1''(s)ds \Big|, \quad x \in \mathbb{R}, \tag{3.14}$$

$$\varkappa_2(x) = \sup_{|\xi| \le 2/q_1(x)} \Big| \int_{x-\xi}^{x+\xi} q_2(s)ds \Big|, \quad x \in \mathbb{R}. \tag{3.15}$$

Then the following relations hold:

$$d(x)q_1(x) = 1 + \varepsilon(x), \quad |\varepsilon(x)| \le \varkappa_1(x) + \varkappa_2(x), \quad |x| \gg 1, \tag{3.16}$$

$$c^{-1} \le d(x)q_1(x) \le c, \quad x \in \mathbb{R}. \tag{3.17}$$

Theorem 3.19. *Let the hypotheses of Theorem 3.18 hold. Suppose in addition that*

$$\lim_{|x|\to\infty} \frac{q_1'(x)}{q_1^2(x)} = 0, \tag{3.18}$$

$$\lim_{|x|\to\infty} |x| q_1(x) = \infty. \tag{3.19}$$

Then $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$.

Corollary 3.20. *Let q_1 satisfy the hypotheses of Theorems 3.18 and 3.19. Then we have the inequalities*

$$c^{-1} q_1(x) \le q_1(t) \le c q_1(x) \quad \text{for} \quad |t - x| \le 2/q_1(x), \quad x \in \mathbb{R}. \tag{3.20}$$

Corollary 3.21. *Let the hypotheses of Theorems 3.18 and 3.19 hold. Suppose in addition that $r(x) > 0$ for $x \in \mathbb{R}$ and*

$$\sup_{x\in\mathbb{R}} \frac{|r'(x)|}{r(x)q_1(x)} < \infty. \tag{3.21}$$

Then the operator $rG : L_p \to L_p$, $p \in [1,\infty)$ is bounded if and only if

$$\sup_{x\in\mathbb{R}} \frac{r(x)}{q_1(x)} < \infty. \tag{3.22}$$

Finally, we present some assertions which are specially adapted for the study of equation (1.1) with coefficient q oscillating at infinity. (We say that a non-negative, continuous function $q(x) \in \mathbb{R}$ oscillates at infinity if on every finite interval it has at most finitely many zeros, and the set of all zeros of q is not bounded in $\mathbb{R}$.)

Lemma 3.22. *Let q be a non-negative function of the form* (3.13) *where $0 < q_1 \in L_1^{\text{loc}}$, $q_2 \in L_1^{\text{loc}}$, and assume the following conditions hold:*

1) *There exists $a_0 \in (0, \infty)$ such that*

$$\inf_{x\in\mathbb{R}} \int_{x-a_0}^{x+a_0} q_1(\xi)d\xi = \varepsilon > 0. \tag{3.23}$$

2) *$\tau(a) \to 0$ as $a \to \infty$ where*

$$\tau(a) = \sup_{x\in\mathbb{R}} \Big| \int_{x-a}^{x+a} q_2(\xi)d\xi \Big| \cdot \Big(\int_{x-a}^{x+a} q_1(\xi)d\xi \Big)^{-1}, \quad a \ge a_0. \tag{3.24}$$

Then there exists $a_1 \ge a_0$ such that $q_0(a_1) > 0$ (see (2.6)*).*

Theorem 3.23. *Suppose that the following conditions hold:*

1) *The function q has zeros at the points $\{x_k\}_{k=1}^{\infty}$, and $|x_k| \to \infty$ as $k \to \infty$.*
2) *The function q is absolutely continuous together with its derivatives $q^{(i)}$, $i = 1, 2, 3$, and $q''(x_k) > 0$ for $k \gg 1$.*

Denote

$$\mathcal{A}_k = \Big[0, \frac{2}{3\sqrt[3]{q''(x_k)}}\Big], \quad \sigma_k = \sup_{t\in\mathcal{A}_k} \Big| \int_{x_k-t}^{x_k+t} q^{(4)}(\xi)d\xi \Big|, \tag{3.25}$$

$$\delta_k = \frac{\sigma_k}{(q''(x_k))^{4/3}}, \quad k \gg 1. \tag{3.26}$$

Then, if $\delta_k \to 0$ as $k \to \infty$, we have the relation

$$d(x_k) = \sqrt[3]{\tfrac{6}{q''(x_k)}}(1 + \nu_k), \quad |\nu_k| \le c\delta_k, \quad k \gg 1. \tag{3.27}$$

See Section 5 for applications of Theorems 3.18 and 3.19 and their corollaries.

4. PROOFS

Proof of Theorem 3.1. From (2.7), we deduce the following relations:

$$r(Gf)(x) = r(x) \int_x^{\infty} \exp\Big(- \int_x^t q(\xi)d\xi\Big) f(t)dt = \mu(x) \int_x^{\infty} \theta(t)f(t)dt, \quad x \in \mathbb{R}; \tag{4.1}$$

$$\mu(x) = r(x) \exp\Big(\int_0^x q(\xi)d\xi\Big), \quad \theta(x) = \exp\Big(-\int_0^x q(\xi)d\xi\Big), \quad x \in \mathbb{R}. \tag{4.2}$$

The statement of the theorem now follows from (4.1) and (4.2) and Theorems 2.1 and 2.3. □

Proof of Corollary 3.3. It is easy to see that the operator $\mathcal{L}_p$, $p \in [1, \infty]$ is separable if and only if the operator $q\mathcal{L}_p^{-1} : L_p \to L_p$ is bounded. Since $\mathcal{L}_p^{-1} = G$ (see (2.7)), it remains to apply Theorem 3.1 with $r \equiv q$. □

Proof of Theorem 3.6. *Necessity.* Suppose that there exists a nondecreasing finite positive function φ for $x \in \mathbb{R}$ and an absolute constant $c \geq 1$ such that the following inequalities hold:

$$c^{-1}\varphi(x) \leq q(x) \leq c\varphi(x), \quad \forall x \in \mathbb{R}. \tag{4.3}$$

Assume that (3.6) does not hold. Then for every $n \geq 1$ there exist α_n and x_n such that

$$\alpha_n < x_n \quad \text{and} \quad q(\alpha_n) \geq nq(x_n). \tag{4.4}$$

From (4.3), we get the estimates

$$c^{-1}\varphi(\alpha_n) \leq q(\alpha_n) \leq c\varphi(\alpha_n), \quad n = 1, 2, \ldots \tag{4.5}$$

$$c^{-1}\varphi(x_n) \leq q(x_n) \leq c\varphi(x_n), \quad n = 1, 2, \ldots \tag{4.6}$$

Now from (4.3), (4.4), (4.5), (4.6) and properties of φ we obtain

$$c^{-1}n\varphi(\alpha_n) \leq c^{-1}n\varphi(x_n) \leq nq(x_n) \leq q(\alpha_n) \leq c\varphi(\alpha_n).$$

Hence,

$$c^{-1}n\varphi(\alpha_n) \leq c\varphi(\alpha_n), \quad n = 1, 2, \ldots \;\Rightarrow\; n \leq c^2 < \infty,$$

which is a contradiction.

Proof of Theorem 3.6. *Sufficiency.* Suppose that (3.6) holds. Then for all $x \in \mathbb{R}$ we have

$$\frac{1}{\gamma}\sup_{t\leq x} q(t) \leq q(x) \leq \sup_{t\leq x} q(t) \leq \gamma \sup_{t\leq x} q(t). \tag{4.7}$$

Let

$$\varphi(x) = \sup_{t\leq x} q(t), \quad x \in \mathbb{R}. \tag{4.8}$$

Clearly, the function φ is not decreasing and is finite and positive in $\mathbb{R}$. Together with (4.7), this proves the assertion of the theorem. □

Proof of Theorem 3.7. *Necessity.* Conditions (1.2) and (3.6) imply the relations (4.7)–(4.8). Since equation (1.1) is correctly solvable in L_p, $p \in [1,\infty]$, by Theorem 2.4 there is $a>0$ such that $q_0(a)>0$ (see (2.6)). Hence for such an a and every $x\in\mathbb{R}$, we get from (4.7)–(4.8)

$$\begin{aligned} &q_0(a) \le \int_{x-a}^{x+a} q(t)dt \le \gamma \int_{x-a}^{x+a}\varphi(t)dt \le 2a\gamma\varphi(x+a) \quad \Rightarrow \\ &\varphi(x+a) \ge (2a\gamma)^{-1} q_0(a), \quad \forall x\in\mathbb{R} \quad \Rightarrow \\ &\qquad\qquad \varphi_0>0, \qquad \varphi_0 = \inf_{x\in\mathbb{R}} \varphi(x). \end{aligned}$$

Using (4.7)–(4.8) once again, we now obtain

$$q(x) \ge \gamma^{-1}\varphi(x) \ge \gamma^{-1}\varphi_0 \quad \forall x\in\mathbb{R} \ \Rightarrow\ \inf_{x\in\mathbb{R}} q(x) = q_0 > 0.$$

Proof of Theorem 3.7. *Sufficiency.* If $q_0>0$, then $q_0(a)>0$ for every $a>0$ (see (2.6) and (3.7)), and therefore equation (1.1) is correctly solvable in L_p, $p\in[1,\infty]$ by Theorem 2.4 It remains to prove the separability of the operator $\mathcal{L}_p$, $p\in[1,\infty]$. Towards this end, we use (4.7)–(4.8) in order to estimate $A_p(q,q)$:

$$\begin{aligned} A_p(q,q) &= \sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} q(t)^p \exp\Big(-p\int_t^x q(\xi)d\xi\Big)dt\Big]^{1/p} \\ &\qquad \times \Big[\int_x^{\infty} \exp\Big(-p'\int_x^t q(\xi)d\xi\Big)dt\Big]^{1/p'} \\ &\le \sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} \gamma^p\varphi(t)^p \exp\Big(-\frac{p}{\gamma}\int_t^x \varphi(\xi)d\xi\Big)dt\Big]^{1/p} \\ &\qquad \times \Big[\int_x^{\infty} \exp\Big(-\frac{p'}{\gamma}\int_x^t \varphi(\xi)d\xi\Big)dt\Big]^{1/p'} \\ &\le c\sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} \varphi(t) \exp\Big(-\frac{p}{\gamma}\int_t^x \varphi(\xi)d\xi\Big)dt\Big]^{1/p}\varphi(x)^{1/p'} \\ &\qquad \times \Big[\int_x^{\infty} \exp\Big(-\frac{p'}{\gamma}\int_x^t \varphi(\xi)d\xi\Big)dt\Big]^{1/p'} \\ &\le c(p)\sup_{x\in\mathbb{R}} \Big[\int_x^{\infty} \varphi(t)\exp\Big(-\frac{p'}{\gamma}\int_t^x \varphi(\xi)d\xi\Big)dt\Big]^{1/p'} = c(p)<\infty. \end{aligned}$$

It remains to refer to Corollary 3.3. □

Proof of Corollary 3.8. This assertion follows from Theorem 3.7 because the requirements on q imply that condition (3.6) holds automatically with $\gamma = 1 : \sup_{t\le x} q(t) = q(x)$. □

Proof of Theorem 3.10. *Necessity.* If the operator $rG : L_1 \to L_1$ is bounded, we have $\mathcal{A}_1(r,q) < \infty$ (see (3.1)). Then for every $x \in \mathbb{R}$, in view of (2.9), we have

$$\begin{aligned}\mathcal{A}_1(r,q) &\ge \int_{-\infty}^{x+d(x)} |r(t)| \exp\Big(-\int_t^{x+d(x)} q(\xi)d\xi\Big)dt\\ &\ge \int_{x-d(x)}^{x+d(x)} |r(t)| \exp\Big(-\int_t^{x+d(x)} q(\xi)d\xi\Big)dt\\ &\ge \exp\Big(-\int_{x-d(x)}^{x+d(x)} q(\xi)d\xi\Big)\int_{x-d(x)}^{x+d(x)} |r(t)|dt = e^{-2}\int_{x-d(x)}^{x+d(x)}|r(t)|dt\\ &\Rightarrow \quad m(r,q) \le e^2\mathcal{A}_1(r,q) < \infty.\end{aligned}$$

Proof of Theorem 3.10. *Sufficiency.* Let $\{\Delta_n\}_{n=-\infty}^{-1}$ stand for the segments of an $\mathbb{R}(x)$-covering of $(-\infty, x]$ (see Lemmas 2.6, 2.8 and 2.11). Then for $t \in \Delta_n$ the following estimate holds:

$$\int_t^x q(\xi)d\xi \ge 2(|n|-1). \tag{4.9}$$

Indeed, for $n = -1$ inequality (4.9) is obvious, and for $n \le -2$ we get

$$\int_t^x q(\xi)d\xi = \sum_{k=n+1}^{-1}\int_{\Delta_k} q(\xi)d\xi + \int_t^{\Delta_n^+} q(\xi)d\xi \ge \sum_{k=n+1}^{-1} 2 = 2(|n|-1).$$

From (3.1)–(3.2), (2.13)–(2.14),(2.9) and (4.9), it now follows that

$$\begin{aligned}\mathcal{A}_1(r,q) &= \sup_{x\in\mathbb{R}}\int_{-\infty}^x |r(t)|\exp\Big(-\int_t^x q(\xi)d\xi\Big)dt\\ &= \sup_{x\in\mathbb{R}}\sum_{n=-\infty}^{-1}\int_{\Delta_n}|r(t)|\exp\Big(-\int_t^x q(\xi)d\xi\Big)dt\\ &\le \sup_{x\in\mathbb{R}}\sum_{n=-\infty}^{-1}\Big(\int_{\Delta_n}|r(t)|dt\Big)\exp(-2(|n|-1))\\ &\le m(r,q)\sum_{k=1}^{\infty}\exp(-2(k-1)) = cm(r,q) < \infty.\end{aligned}$$

It remains to apply Theorem 3.1. □

Proof of Theorem 3.12. *Necessity.* Suppose that for some $p \in (1,\infty)$ the operator $rG : L_p \to L_p$ is bounded. Then $\mathcal{A}_p(r,q) < \infty$ by Theorem 3.1. Denote $\omega(x) = [\omega^-(x), \omega^+(x)]$, $\omega^\pm(x) = x \pm \frac{d(x)}{2}$, $x \in \mathbb{R}$. In the following relations we use Lemmas 2.8 and 2.9:

$$\begin{aligned}\mathcal{A}_p(r,q) &\geq \Big[\int_{-\infty}^{\omega^+(x)} |r(t)|^p \exp\Big(-p\int_t^{\omega^+(x)} q(\xi)d\xi\Big)dt\Big]^{1/p}\\ &\quad\times\Big[\int_{\omega^+(x)}^{\infty} \exp\Big(-p'\int_{\omega^+(x)}^{t} q(\xi)d\xi\Big)\Big]^{1/p'}\\ &\geq \Big[\int_{\omega(x)} |r(t)|^p \exp\Big(-p\int_{\omega(x)} q(\xi)d\xi\Big)dt\Big]^{1/p}\\ &\quad\times\Big[\int_{\omega^+(x)}^{x+d(x)} \exp\Big(-p'\int_{\omega(x)} q(\xi)d\xi\Big)\Big]^{1/p'}\\ &\geq \exp\Big(-2\int_{x-d(x)}^{x+d(x)} q(\xi)d\xi\Big)\frac{1}{2^{1/p'}}h_p(x) = \frac{e^{-4}}{2^{1/p'}}h_p(x)\\ &\Rightarrow \quad h_p(r,q) \leq 2e^4\mathcal{A}_p(r,q) < \infty.\end{aligned}$$

Proof of Theorem 3.12. *Sufficiency.* We need the following lemmas.

Lemma 4.1. *Let $a \geq 1$, $b > 0$, $\gamma = a\exp(-b/a^2)$. Then $b \geq 1$ provided $\gamma \leq e^{-1}$.*

Proof. Assume the contrary: $b<1$. Then $e \leq a^{-1}\exp(b/a^2) < a^{-1}\exp(1/a^2) \leq e$, which is a contradiction. □

Remark 4.2. Until the end of the proof of Theorem 3.12, the main requirement that $q \in \mathcal{K}(\gamma)$, $\gamma = a\exp\left(-\frac{b}{a^2}\right) < e^{-1}$ is assumed to be satisfied and is not mentioned in the statements.

Lemma 4.3. *There exists an absolute constant $c_1 \in (0,\infty)$ such that, for all $x \in \mathbb{R}$, the following inequalities hold:*

$$\frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} - \int_x^t q(\xi)d\xi \leq c_1, \quad \forall t \geq x, \tag{4.10}$$

$$\frac{1}{a}\int_t^x \frac{d\xi}{d(\xi)} - \int_t^x q(\xi)d\xi \leq c_1, \quad \forall t \leq x. \tag{4.11}$$

Proof of Lemma 4.3. We need the following two auxiliary assertions.

Lemma 4.4. *Denote* $\Delta(s) = [s - d(s), s + d(s)]$, $s \in \mathbb{R}$. *Let* $t_1, t_2 \in \Delta(s)$ *and* $\Delta(s) \cap [-x_0, x_0) = \emptyset$ *(see Definition* 3.11*). Then the following inequality holds:*

$$\frac{1}{a}\Big|\int_{t_1}^{t_2} \frac{d\xi}{d(\xi)}\Big| - \Big|\int_{t_1}^{t_2} q(\xi)d\xi\Big| \leq 2. \tag{4.12}$$

Proof. In the next chain of calculations, we use (3.9) and Lemma 4.1.

$$\frac{1}{a}\Big|\int_{t_1}^{t_2} \frac{d\xi}{d(\xi)}\Big| - 2 \leq \frac{1}{a}\int_{\Delta(s)} \frac{d\xi}{d(\xi)} - 2 = \frac{1}{a}\int_{\Delta(s)} \frac{d(s)}{d(\xi)}\frac{d\xi}{d(s)} - 2$$
$$\leq \int_{\Delta(s)} \frac{d\xi}{d(s)} - 2 = 0 \leq \Big|\int_{t_1}^{t_2} q(\xi)d\xi\Big|. \qquad \square$$

Let $\tilde{x}_0 = x_0 + 2d_0 + 1$ (see (2.11) and Definition 3.11).

Lemma 4.5. *Let* $t \geq x$, $x \in \mathbb{R}$ *and*

$$[x, t] \cap [-\tilde{x}_0, x_0) = \emptyset. \tag{4.13}$$

Then for $c_1 = 2$ *inequality* (4.10) *holds.*

Proof. By the hypothesis of the lemma, there are two possibilities:
1) $x < -\tilde{x}_0$ and 2) $x \geq x_0$.

Let $\{\Delta_n\}_{n=1}^{\infty}$ be the segments of an $\mathbb{R}(x, d)$-covering of $[x, \infty)$, and $t \in \Delta_n$, $n \geq 1$. Then

$$\Big(\bigcup_{k=1}^{n} \Delta_k\Big) \cap [-x_0, x_0] = \emptyset. \tag{4.14}$$

Indeed, in the case 2) equality (4.14) is obvious.

Consider 1). Let $t \in \Delta_n = [\Delta_n^-, \Delta_n^+]$ and $t < -\tilde{x}_0$. Then

$$\Delta_n^+ = \Delta_n^- + 2d(x_n) < -\tilde{x}_0 + 2d_0 = -x_0 - 1 < -x_0 \;\Rightarrow\; (4.14)$$

Let us now go over to the conclusion of the lemma. If $t \in \Delta_1$, then (4.10) follows from (4.12). Let $t \in \Delta_n$, $n \geq 2$. In the following relations, we use (2.9), Definitions 2.10 and 3.11, and Lemmas 4.1 and 4.4:

$$\int_x^t q(\xi)d\xi = \sum_{k=1}^{n-1}\int_{\Delta_k} q(\xi)d\xi + \int_{\Delta_n^-}^t q(\xi)d\xi \geq \sum_{k=1}^{n-1} 2 + \frac{1}{a}\int_{\Delta_n^-}^t \frac{d\xi}{d(\xi)} - 2$$
$$= \sum_{k=1}^{n-1}\int_{\Delta_k} \frac{d(\xi)}{d(x_k)}\frac{d\xi}{d(\xi)} + \frac{1}{a}\int_{\Delta_n^-}^t \frac{d\xi}{d(\xi)} - 2$$

$$\geq \sum_{k=1}^{n-1} \frac{1}{a}\int_{\Delta_k} \frac{d\xi}{d(\xi)} + \frac{1}{a}\int_{\Delta_n^-}^{t} \frac{d\xi}{d(\xi)} - 2 = \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} - 2. \qquad \square$$

Proof of Lemma 4.3. Note that, according to Lemma 4.5, it remains to prove (4.10) in the cases σ) and τ):

σ) $x \in [-\tilde{x}_0, x_0)$, $t \geq x$ $\quad\tau$) $x \leq -\tilde{x}_0$, $t \geq -\tilde{x}_0$.

First consider σ). Let us introduce a domain $\mathcal{D}_1$ and a function f_1 :

$$\mathcal{D}_1 = \{(x,t) : x \in [-\tilde{x}_0, x_0],\, x \leq t \leq x_0\},$$

$$f_1(x,t) = \int_x^t q(\xi)d\xi - \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)},\ (x,t) \in \mathcal{D}_1.$$

From (1.2) and Lemma 2.6 it follows that f_1 is a continuous function, and, as the domain $\mathcal{D}_1$ is closed, we have $M_1 < \infty$, where $M_1 = \max_{(x,t)\in\mathcal{D}_1} |f_1(x,t)|$. Further we distinguish between the following cases:

$$\sigma_1)\quad x \in [\tilde{x}_0, x_0], \quad t \in [x, x_0]; \qquad \sigma_2) \quad x \in [-\tilde{x}_0, x_0], \quad t \geq x_0.$$

In the case σ_1), we have

$$\int_x^t q(\xi)d\xi = \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} + f_1(x,t) \geq \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} - |f_1(x,t)|$$
$$\geq \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} - M_1 \Rightarrow (4.10).$$

In the case σ_2), from the considerations in the case σ_1) and Lemma 4.5, we have

$$\int_x^t q(\xi)d\xi = \int_x^{x_0} q(\xi)d\xi + \int_{x_0}^t q(\xi)d\xi \geq \left[\frac{1}{a}\int_x^{x_0} \frac{d\xi}{d(\xi)} - M_1\right] + \left[\frac{1}{a}\int_{x_0}^t \frac{d\xi}{d(\xi)} - 2\right]$$
$$= \frac{1}{a}\int_x^t \frac{d\xi}{d(\xi)} - (M_1 + 2) \ \Rightarrow\ (4.10).$$

In both the cases σ_1) and τ), inequality (4.10) follows from the considerations in the case σ) and Lemma 4.5. $\square$

Lemma 4.6. *The following assertions hold:*

A) *Let $x \in \mathbb{R}$, let $\{\Delta_n\}_{n=1}^\infty$ be an $\mathbb{R}(x, bd)$-covering of $[x, \infty)$, let $t \geq x$ and $t \in \Delta_n$, $n \geq 1$. Then there exists an absolute constant $c_2 \in (0, \infty)$ such that the following inequality holds:*

$$\int_x^t \frac{d\xi}{d(\xi)} \geq \frac{2b}{a}n - c_2. \tag{4.15}$$

B) *Let* $x \in \mathbb{R}$, *let* $\{\Delta_n\}_{n=-\infty}^{-1}$ *be an* $\mathbb{R}(x, bd)$*-covering of* $(-\infty, x]$, *let* $t \le x$ *and* $t \in \Delta_n$, $n \le -1$. *Then there is an absolute constant* $c_2 \in (0, \infty)$ *such that the following inequality holds:*

$$\int_t^x \frac{d\xi}{d(\xi)} \ge \frac{2b}{a}|n| - c_2. \tag{4.16}$$

Proof. Assertions A) and B) are proved in the same way, and therefore below we only consider A). In the course of the proof of (4.15), we consider separate mutually exclusive cases corresponding to different positions of t on the semi-axis $[x, \infty)$, $x \in \mathbb{R}$.

Lemma 4.7. *Let* $x \in \mathbb{R}$, $t \ge x$ *and* (*see* (2.11) *and* (3.9))

$$[x, t] \cap (-\hat{x}_0, x_0) = \emptyset, \qquad \hat{x}_0 = x_0 + 2bd_0 + 1. \tag{4.17}$$

Then inequality (4.15) *holds with* $c_2 = \frac{2b}{a}$.

Proof. We consider two cases:

1) $x \le t \le -\hat{x}_0$ and 2) $t \ge x \ge x_0$.

These cases are similar, and therefore below we only consider 1). Clearly, the following possibilities exist:

1a) $t \in \Delta_1$ and 1b) $t \in \Delta_n$, $n \ge 2$.

In the case 1a), we have

$$\int_x^t \frac{d\xi}{d(\xi)} \ge 0 \ge \frac{2b}{a}n - \frac{2b}{a}\bigg|_{n=1} = \frac{2b}{a}1 - c_2.$$

The proof of (4.15) in the case 1b) is based on Definitions 2.10 and 3.11:

$$\int_x^t \frac{d\xi}{d(\xi)} = \sum_{k=1}^{n-1} \int_{\Delta_k} \frac{d\xi}{d(\xi)} + \int_{\Delta_n^-}^t \frac{d\xi}{d(\xi)} \ge \sum_{k=1}^{n=1} \int_{\Delta_k} \frac{d(x_k)}{d(\xi)} \frac{d\xi}{d(x_k)}$$
$$\ge \sum_{k=1}^{n-1} \frac{1}{a} \int_{\Delta_k} \frac{d\xi}{d(x_k)} = \sum_{k=1}^{n-1} \frac{2b}{a} = \frac{2b}{a}n - \frac{2b}{a} \ \Rightarrow \ (4.15). \qquad \square$$

Corollary 4.8. *Suppose that the hypotheses of Lemma* 4.6 (*Part A*) *are satisfied, equality* (4.17) *holds, and* $t \in \Delta_n$, $n \ge 1$. *Then for every* $s = \overline{1, n}$, *the following inequality holds:*

$$\int_{\Delta_s^-}^t \frac{d\xi}{d(\xi)} \ge \frac{2b}{a}(n - s + 1) - \frac{2b}{a}. \tag{4.18}$$

Proof. The estimate (4.18) is checked similarly to (4.15) in the Case 1b) of Lemma 4.7. $\square$

Note that, by Lemma 4.7, it only remains to check (4.15) in the following cases:

α) $x \in [-\hat{x}_0, x_0]$, $t \geq x$; β) $x \leq -\hat{x}_0$, $t \geq -\hat{x}_0$.

Below we consider in α) two separate subcases α_1) and α_2):

α_1) $x \in [-\hat{x}_0, x_0]$, $t \in [x, x_0]$;

α_2) $x \in [-\hat{x}_0, x_0]$, $t \geq x_0$.

Let $x \leq \hat{x}_0$ (see (4.17)), let $\{\Delta_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $[x, \infty)$, and let $N(x)$ be the number of segments Δ_n, $n \geq 1$ such that

$$\Delta_n \cap [-\hat{x}_0, \hat{x}_0] \neq \emptyset$$

(see (2.9) and Lemma 2.6).

Lemma 4.9. *We have*

$$N_0 < \infty, \qquad N_0 = \sup_{x \leq \hat{x}_0} N(x). \tag{4.19}$$

Proof. Denote by $\mu(a, b)$ the length $(b - a)$ of the segment (a, b) and $\hat{d}_0 := \min_{x \in [-\hat{x}_0, \hat{x}_0]} d(x)$. Then since $\mu(-\hat{x}_0, \hat{x}_0) = 2 + 2x_0 + 4bd_0$ and $\mu(\Delta_n) = 2bd(x_n) \leq 2bd_0$ (see (2.11) and Definition 2.10), there exist points $x \leq \hat{x}_0$ such that $N(x) \geq 3$. Denote the set of such points by S. Let $x \in S$, and let $\{\Delta_n\}_{n=n_1}^{n_2}$ be the subset of segments of $\{\Delta_n\}_{n=1}^{\infty}$, an $\mathbb{R}(x, bd)$-covering of $[x, \infty)$ such that $\Delta_n \cap [-\hat{x}_0, \hat{x}_0] \neq \emptyset$. Clearly, $\bigcup_{n=n_1+1}^{n_2-1} \Delta_n \subseteq [-\hat{x}_0, \hat{x}_0]$. According to Definition 2.10, this implies

$$\begin{aligned}
\mu(-\hat{x}_0, \hat{x}_0) = 2 + 2x_0 + 4bd_0 &\geq \sum_{k=n_1+1}^{n_2-1} \mu(\Delta_k) = \sum_{k=n_1+1}^{n_2-1} 2bd(x_k) \\
&\geq \sum_{k=n_1+1}^{n_2-1} 2b\hat{d}_0 = 2b\hat{d}_0(n_2 - n_1 - 1) = 2b\hat{d}_0(N(x) - 2) \\
&\Rightarrow \ N(x) \leq 2 + \left[\frac{1 + x_0 + 2bd_0}{b\hat{d}_0}\right] := N_0,
\end{aligned}$$

and the lemma is proved. □

In the case α_1), let us introduce a domain $\mathcal{D}_2$ and a function f_2 :

$$\mathcal{D}_2 = \{(x, t, s) : x \in [-\hat{x}_0, \hat{x}_0],\ x \leq t \leq \hat{x}_0,\ 0 \leq s \leq N_0\}$$

$$f_2(x, t, s) = \int_x^t \frac{d\xi}{d(\xi)} - \frac{2b}{a} s, \quad (x, t, s) \in \mathcal{D}_2.$$

From Lemma 2.6, it follows that $M_2 < \infty$ where $M_2 = \max_{\mathcal{D}_2} |f_2(x,t,s)|$. Let us now consider the case $\alpha_1)$: $x \in [-\hat{x}_0, x_0]$, $t \in [x, x_0]$. Let $\{\Delta_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $[x,\infty)$ and $t \in \Delta_{n_0}$. Then we have

$$\Delta_{n_0}^+ = x_{n_0} + bd(x_{n_0}) = \Delta_{n_0}^- + 2bd(x_{n_0}) \le x_0 + 2bd_0 < \hat{x}_0.$$

Hence $n_0 \le N_0$ (see (4.19)), and we thus obtain

$$\begin{aligned}\int_x^t \frac{d\xi}{d(\xi)} &= \frac{2b}{a}n_0 + \Big[\int_x^t \frac{d\xi}{d(\xi)} - \frac{2b}{a}n_0\Big] \ge \frac{2b}{a}n_0 - |f_2(x,t,n_0)| \\ &\ge \frac{2b}{a}n_0 - M_2 \ \Rightarrow \ (4.15).\end{aligned}$$

Let us now consider the case $\alpha_2)$: $x \in [-\hat{x}_0, x_0]$, $t \ge x_0$. Let $\{\Delta_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $[x,\infty)$, $x_0 \in \Delta_s$, $t \in \Delta_n$. Then we repeat the argument of case $\alpha_1)$ and obtain $s \le N_0$. Therefore, below we use the results of the case $\alpha_1)$, Corollary 4.8 and the equality $\Delta_s^+ = \Delta_{s+1}^-$ (see Definition 2.10):

$$\begin{aligned}\int_x^t \frac{d\xi}{d(\xi)} &= \int_x^{\Delta_s^+} \frac{d\xi}{d(\xi)} + \int_{\Delta_{s+1}^-}^t \frac{d\xi}{d(\xi)} \ge \Big(\frac{2b}{a}s - M_2\Big) + \frac{2b}{a}(n-(s+1)+1) - \frac{2b}{a} \\ &= \frac{2b}{a}n - \Big(M_2 + \frac{2b}{a}\Big) \ \Rightarrow \ (4.15).\end{aligned}$$

Let us now go to the case $\beta)$: $x \le -\hat{x}_0$, $t \ge -\hat{x}_0$. This case is easily reduced to a combination of the situations considered above, in particular, in the following obvious equalities:

$$\int_x^t \frac{d\xi}{d(\xi)} = \int_x^{-\hat{x}_0} \frac{d\xi}{d(\xi)} + \int_{-\hat{x}_0}^t \frac{d\xi}{d(\xi)}, \quad x \le -\hat{x}_0, \quad t \le x_0,$$
$$\int_x^t \frac{d\xi}{d(\xi)} = \int_x^{-\hat{x}_0} \frac{d\xi}{d(\xi)} + \int_{-\hat{x}_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)}, \quad x \le -\hat{x}_0, \quad t \ge x_0.$$

Their summands can be estimated as above, and we thus arrive at inequality (4.15). □

Lemma 4.10. *Let $x \in \mathbb{R}$, let $\{\Delta_n\}_{n=-\infty}^{-1}$ and $\{\Delta_n\}_{n=1}^{\infty}$ be $\mathbb{R}(x, bd)$-coverings of $(-\infty, x]$ and $[x, \infty)$, respectively, and let $\{x_n\}_{n\in\mathbb{N}'}$, $N' = \{\pm1, \pm2, \dots\}$ be the centers of the segments Δ_n. Then there exists an absolute constant $c_3 \ge 1$ such that the following inequalities hold:*

$$\frac{1}{c_3 a^{2|n|}} \le \frac{d(x_n)}{d(x)} \le c_3 a^{2|n|}, \quad n \in \mathbb{N}'. \tag{4.20}$$

Proof. We check inequalities (4.20) for $n \ge 1$. For $n \le -1$ the arguments are similar.

Below we use the following lemma.

Lemma 4.11. *Suppose the following condition holds (see Definition 3.11):*

$$[x, \Delta_n^+] \cap (-x_0, x_0) = \emptyset. \tag{4.21}$$

Then equalities (4.20) *hold with* $c_3 = 1$.

Proof. For $n = 1$ the assertion of the lemma follows from Definition 3.11 and the construction of the segment Δ_1 :

$$x = \Delta_1^- \ \Rightarrow\ \frac{1}{a^2} \le \frac{1}{a} \le \frac{d(\Delta_1^-)}{d(x_1)} = \frac{d(x)}{d(x_1)} \le a \le a^2.$$

Let now $n \ge 2$, $m = 1, 2, \dots, n-1$. Then, from Definitions 2.10 and 3.11, it follows that

$$\left.\begin{aligned} &\frac{1}{a} < \frac{d(x_{m+1})}{d(\Delta_{m+1}^-)} \le a \\ &\frac{1}{a} < \frac{d(\Delta_m^+)}{d(x_m)} \le a \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\frac{1}{a^2} < \frac{d(x_{m+1})}{d(x_m)} \le a^2 \\ &m = 1, 2, \dots, n-1 \end{aligned}\right\}. \tag{4.22}$$

Multiplying the inequalities (4.22), we obtain

$$\frac{1}{a^{2n-2}} \le \frac{d(x_n)}{d(x_1)} \le a^{2n-2}, \quad n = 1, 2, \dots .$$

These estimates together with (3.9):

$$\frac{1}{a} \le \frac{d(x_1)}{d(x)} \le a$$

imply the relations

$$\frac{1}{a^{2n}} \le \frac{1}{a^{2n-1}} \le \frac{d(x_n)}{d(x)} \le a^{2n-1} \le a^{2n} \ \Rightarrow\ (4.20). \qquad \square$$

Clearly, for $x \ge x_0$, the inequalities (4.20) hold by Lemma 4.11 (with $c_3 = 1$). Therefore it remains to consider the cases μ) and ν):

μ) $x \in [-\hat{x}_0, \hat{x}_0]$; ν) $x \le -\hat{x}_0$, $\hat{x}_0 = x_0 + 2bd_0 + 1$

(see (2.11) and (3.9)). Denote $\alpha^2 = \max\left\{\frac{d_0}{\hat{d}_0}, a\right\}$ (see (3.9) and the notation of Lemma 4.9).

Note that the following obvious inequalities hold:

$$\frac{1}{\alpha^2} \le \frac{\hat{d}_0}{d_0} \le \frac{d(t)}{d(x)} \le \frac{d_0}{\hat{d}_0} \le \alpha^2 \quad \text{for} \quad t, x \in [-\hat{x}_0, \hat{x}_0]. \tag{4.23}$$

Let $\{\Delta_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $[x, \infty)$.

Consider μ). In this case we consider two separate subcases:

μ_1) $\hat{x}_0 \in \Delta_1$; μ_2) $\hat{x}_0 \in \Delta_s$, $s \ge 2$.

Case μ_1). If $\hat{x}_0 \in \Delta_1$, then

$$x_0 + 2bd_0 + 1 = \hat{x}_0 \le \Delta_1^+ = x + 2bd(x_1) \le x + 2bd_0 \ \Rightarrow \ x \ge x_0 + 1.$$

Therefore, by Lemma 4.11, we have inequalities (4.20) (with $c_3 = 1$).

Case μ_2). If $\hat{x}_0 \in \Delta_s$, $s \ge 2$, then

$$x_0 + 2bd_0 + 1 = \hat{x}_0 \le \Delta_s^- + 2bd(x_s) \le \Delta_s^- + 2bd_0 \ \Rightarrow \ \Delta_s^- \ge x_0 + 1.$$

We now use Definition 2.10. As $\Delta_{s-1}^+ = \Delta_s^- \le \hat{x}_0$, we have the inequalities

$$-\hat{x}_0 \le x < x_1 < x_2 < \cdots < x_{s-1} \le \hat{x}_0, \tag{4.24}$$

and therefore by (4.23) we obtain

$$\frac{1}{\alpha^2} \le \frac{d(x_m)}{d(x)} \le \alpha^2, \quad m = 1, 2, \dots s - 1. \tag{4.25}$$

Note that since we have both $\Delta_s^- \in [-\hat{x}_0, \hat{x}_0]$ and $\Delta_s^- \ge x_0 + 1$, from (4.23) and (3.9) it follows that

$$\frac{1}{\alpha^2} \le \frac{d(\Delta_s^-)}{d(x)} \le \alpha^2 \quad \text{and} \quad \frac{1}{a} \le \frac{d(x_s)}{d(\Delta_s^-)} \le a \quad \Rightarrow$$
$$\frac{1}{\alpha^2 a^2} \le \frac{1}{\alpha^2 a} \le \frac{d(x_s)}{d(x)} \le \alpha^2 a \le \alpha^2 a^2. \tag{4.26}$$

Furthermore, using the inequality $\Delta_s^- \ge x_0 + 1$ and (4.22) we get

$$\frac{1}{a^2} \le \frac{d(x_{m+1})}{d(x_m)} \le a^2, \qquad m = s, s+1, \dots, n-1, \dots. \tag{4.27}$$

Multiplying the estimates (4.27), we obtain

$$\frac{1}{a^{2(n-s)}} \le \frac{d(x_n)}{d(x_s)} \le a^{2(n-s)}, \qquad n \ge s. \tag{4.28}$$

Finally, from (4.26) and (4.28) it follows that

$$\frac{1}{\alpha^2 a^{2n-2s+2}} \le \frac{d(x_n)}{d(x)} \le \alpha^2 a^{2n-2s+2}, \qquad n \ge s. \tag{4.29}$$

It is now clear that the inequalities (4.20) with $c_3 = \alpha^2$ appear as a result of an obvious replacement of (4.25) and (4.29) with rougher estimates.

Let us now consider the case $\nu)$ $x \le -\hat{x}_0$.
Let $\{\Delta_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $[x, \infty)$. Below we consider two separate subcases:

$\nu_1)$ $(-\hat{x}_0) \in \Delta_1$; $\nu_2)$ $(-\hat{x}_0) \in \Delta_s$, $s \ge 2$.

Case $\nu_1)$ Let $(-\hat{x}_0) \in \Delta_1$ $\Rightarrow$ (see (2.11)):

$$-x_0 - 2bd_0 - 1 = -\hat{x}_0 \ge \Delta_1^- \ge \Delta_1^+ - 2bd_0 \;\Rightarrow\; \Delta_1^+ \le -x_0 - 1.$$

Hence, by Definition 2.10 and (3.9), we have

$$\frac{1}{a} \le \frac{d(x_1)}{d(x)} \le a, \qquad \frac{1}{a} \le \frac{d(\Delta_1^+)}{d(x_1)} \le a. \tag{4.30}$$

Since $\Delta_2^- = \Delta_1^+ \ge -\hat{x}_0$ from the case $\mu)$ studied above, we obtain

$$\frac{1}{c_3}\frac{1}{a^{2(n-1)}} \le \frac{d(x_n)}{d(\Delta_2^-)} \le c_3 a^{2(n-1)}, \qquad n = 2, 3, \dots . \tag{4.31}$$

Further, since (4.30) implies the inequalities

$$\frac{1}{a^2} \le \frac{d(\Delta_1^+)}{d(x)} \le a^2 \tag{4.32}$$

by multiplying (4.31) and (4.32), we obtain

$$\frac{1}{c_3}\frac{1}{a^{2n}} \le \frac{d(x_n)}{d(x)} \le c_3 a^{2n}, \qquad n \ge 2. \tag{4.33}$$

The estimates (4.20) follow from (4.30) and (4.33).

Case $\nu_2)$ Let $(-\hat{x}_0) \in \Delta_s$, $s \ge 2$. Then, as in the case $\nu_1)$, we obtain $\Delta_s^+ \le -x_0 - 1$, and therefore by Lemma 4.11 we get

$$\frac{1}{a^{2n}} \le \frac{d(x_n)}{d(x)} \le a^{2n}, \qquad n = 1, 2, \dots, s. \tag{4.34}$$

On the other hand, $\Delta_s^+ \in [-\hat{x}_0, \infty)$. Hence from the case $\nu_1)$ considered above it follows that

$$\frac{1}{c_3 a^{2(n-s)}} \le \frac{d(x_n)}{d(\Delta_{s+1}^-)} \le c_3 a^{2(n-s)}, \qquad n = s+1, s+2, \dots . \tag{4.35}$$

From (4.34) (for $n = s$) and (3.9) (recall that $\Delta_s^+ \le -x_0 - 1$), it follows that

$$\left.\begin{aligned} \frac{1}{a^{2s}} \le \frac{d(x_s)}{d(x)} \le a^{2s} \\ \frac{1}{a} < \frac{d(\Delta_s^+)}{d(x_s)} \le a \end{aligned}\right\} = \frac{1}{a^{2s+1}} \le \frac{d(\Delta_s^+)}{d(x_s)} \le a^{2s+1}. \tag{4.36}$$

We now multiply (4.36) and (4.35) and obtain (recall that $\Delta_s^+ = \Delta_{s+1}^-$)

$$\frac{1}{(c_3a)}\,\frac{1}{a^{2n}} \le \frac{d(x_n)}{d(x)} \le (c_3a)a^{2n}, \qquad n = s+1, s+2, \dots . \tag{4.37}$$

Let $c_3 := c_3 a$. Then we make estimates (4.34) rougher and obtain

$$\frac{1}{c_3 a^{2n}} \le \frac{1}{a^{2n}} \le \frac{d(x_n)}{d(x)} \le a^{2n} \le c_3 a^{2n}, \qquad n = 1, 2, \dots, s. \tag{4.38}$$

From (4.37) and (4.38) we obtain (4.20). □

Let $\theta \in L_p^{\mathrm{loc}}$, $p \in [1, \infty)$. Denote

$$\mathcal{A}_p(x) = \int_x^\infty |\theta(t)|^p \exp\Big(-p\int_x^t q(\xi)d\xi\Big)dt, \quad x \in \mathbb{R}, \tag{4.39}$$

$$\mathcal{A}_p^*(x) = \int_x^\infty |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt, \quad x \in \mathbb{R}, \tag{4.40}$$

$$\mathcal{B}_p(x) = \int_{-\infty}^x |\theta(t)|^p \exp\Big(-p\int_t^x q(\xi)d\xi\Big)dt, \quad x \in \mathbb{R}, \tag{4.41}$$

$$\mathcal{B}_p^*(x) = \int_{-\infty}^x |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_t^x \frac{d\xi}{d(\xi)}\Big)dt, \quad x \in \mathbb{R}. \tag{4.42}$$

Remark 4.12. Below we estimate the integrals (4.39)–(4.42). Note that such inequalities for the first two and the last two integrals are checked in the same way, and therefore below we only estimate the integrals $\mathcal{A}_p$ and $\mathcal{A}_p^*$.

Lemma 4.13. *Let $p \in [1, \infty)$, and suppose that the integral $\mathcal{A}_p^*(x)$ ($\mathcal{B}_p^*(x)$) converges. Then so does the integral $\mathcal{A}_p(x)$ ($\mathcal{B}_p(x)$). Moreover, there exists a constant $c = c(\theta, q)$ such that for all $x \in \mathbb{R}$ we have the inequality*

$$\mathcal{A}_p(x) \le c\mathcal{A}_p^*(x), \qquad (\mathcal{B}_p(x) \le c\mathcal{B}_p^*(x)), \quad x \in \mathbb{R}. \tag{4.43}$$

Proof. Below we use Lemma 4.3:

$$\mathcal{A}_p(x) = \int_x^\infty |\theta(t)|^p \exp\Big(-p\int_x^t q(\xi)d\xi\Big)dt$$

$$= \int_x^\infty |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)\cdot \exp\Big(\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)} - \int_x^t q(\xi)d\xi\Big)dt$$
$$\le e^{c_1}\cdot \int_x^\infty |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt = c\mathcal{A}_p^*(x). \qquad \square$$

Lemma 4.14. *Let* $p \in [1,\infty)$, $p' = p(p-1)^{-1}$, *and let*

$$\tilde h_p(\theta,q) = \sup_{x\in\mathbb{R}} d(x)^{1/p'}\Big[\int_{x-bd(x)}^{x+bd(x)} |\theta(t)|^p dt\Big]^{1/p}.$$

Then (*see* (3.10))

$$c^{-1}h_p(\theta,q) \le \tilde h_p(\theta,q) \le ch_p(\theta,q). \tag{4.44}$$

Proof. Since the function $d(x)$ is positive and continuous for $x \in \mathbb{R}$ (see Lemma 2.6), the inequalities (3.9) can be extended to the segment $[-x_0, x_0]$ (replacing, if necessary, the constant a with a larger constant $\alpha \in [1,\infty)$). Thus, by Lemma 2.6, from (3.9) we deduce

$$\alpha^{-1}d(x) \le d(t) \le \alpha d(x), \quad |t-x| \le bd(x), \quad x \in \mathbb{R}. \tag{4.45}$$

Let $\omega(x) = [\omega^-(x), \omega^+(x)]$, $\omega^\pm(x) = x \pm bd(x)$, $x \in \mathbb{R}$, and $\tilde N(x)$ denotes the number of the segments from a $\{\Delta_n\}_{n=1}^\infty - \mathbb{R}\big(x - bd(x), \frac{d}{2}\big)$-covering of $[x - bd(x), \infty)$ such that $\Delta_n \cap \omega(x) \ne \emptyset$. Denote by $\{x_n\}_{n=1}^\infty$ the centers of the segments $\{\Delta_n\}_{n=1}^\infty$. We show that $\tilde N_0 < \infty$ where $\tilde N_0 = \sup_{x\in\mathbb{R}} \tilde N(x)$. If $\tilde N(x) \equiv 1$, $x \in \mathbb{R}$, then $\tilde N_0 = 1$. Let $\tilde N(x) \ge 2$ for some $x \in \mathbb{R}$. Then, from (2.13)–(2.14), it follows that

$$\bigcup_{n=1}^{\tilde N(x)-1} \Delta_n \subseteq \omega(x), \quad x \in \mathbb{R} \quad \Rightarrow$$

$$\begin{aligned} 2bd(x) &\ge \sum_{n=1}^{\tilde N(x)-1} d(x_n) = \sum_{n=1}^{\tilde N(x)-1} \frac{d(x_n)}{d(x)}\cdot d(x) \\ &\ge \sum_{n=1}^{\tilde N(x)-1} \frac{d(x)}{\alpha} = \frac{\tilde N(x)-1}{\alpha} d(x) \ge \frac{\tilde N(x)}{2\alpha} d(x) \\ &\Rightarrow \quad \tilde N(x) \le 4\alpha b \quad \Rightarrow \quad \tilde N_0 \le 4\alpha b, \end{aligned}$$

as required. Let us now go to (4.44). Since $b \ge 1$ (see Lemma 4.1), the lower estimate in (4.44) is obvious. To prove the upper estimate in (4.44), we use the notation introduced in the course of the proof of the inequality $\tilde N_0 <$

∞. With this notation, let us consider the cases $\tilde N(x)=1$ and $\tilde N(x)\ge 2$ separately. If $\tilde N(x)=1$, then $\omega(x)\subseteq\Delta_1$ and $d(x)\le 2d(x_1)$ (see Lemma 2.9 for $\varepsilon=1/2$). Hence,

$$d(x)^{1/p'}\Big(\int_{\omega(x)}|\theta(t)|^p dt\Big)^{1/p}\le 2^{1/p'}d(x_1)^{1/p'}\Big(\int_{\Delta_1}|\theta(t)|^p dt\Big)^{1/p}\le 2^{1/p'}h_p(\theta,q).$$

Let $N(x)\ge 2$. Then from (4.45), (2.13)–(2.14) and Lemma 2.9 (for $\varepsilon=1/2$), it follows that

$$\left.\begin{array}{l} \dfrac{1}{\alpha}\le\dfrac{d(x_n)}{d(x)}\le\alpha,\qquad n=\overline{1,\tilde N(x)-1}\\ \dfrac{1}{\alpha}\le\dfrac{d(\omega^+(x))}{d(x)}\le\alpha\\ \dfrac{1}{2}\le\dfrac{d(x_{N(x)})}{d(\omega^+(x))}\le 2 \end{array}\right\}$$
$$\Rightarrow\quad \left.\frac{1}{2\alpha}\le\frac{d(x_n)}{d(x)}\le 2\alpha,\ n=\overline{1,\tilde N(x)}\ \right\}. \tag{4.46}$$

The following relations are based on (2.13)–(2.14), (4.46) and the inequality

$$\begin{aligned}\tilde h_p(\theta,q)&=\sup_{x\in\mathbb{R}}d(x)^{1/p'}\Big[\int_{\omega(x)}|\theta(t)|^p dt\Big]^{1/p}\\ &\le\sup_{x\in\mathbb{R}}d(x)^{1/p'}\Big[\sum_{n=1}^{\tilde N(x)}\int_{\Delta_n}|\theta(t)|^p dt\Big]^{1/p}\\ &=\sup_{x\in\mathbb{R}}\Big[\sum_{n=1}^{\tilde N(x)}\Big(\frac{d(x)}{d(x_n)}\Big)^{p/p'}\Big(d(x_n)^{p/p'}\int_{\Delta_n}|\theta(t)|^p dt\Big)\Big]^{1/p}\\ &\le c\sup_{x\in\mathbb{R}}\Big[\sum_{n=1}^{\tilde N(x)}h_p(\theta,q)^p\Big]^{1/p}=c\sup_{x\in\mathbb{R}}\tilde N(x)^{1/p}h_p(\theta,q)\\ &\le c\tilde N_0^{1/p}h_p(\theta,q)=ch_p(\theta,q).\end{aligned}$$
□

Lemma 4.15. *Let $q\in K(\gamma)$, $\gamma\le e^{-1}$, $p\in[1,\infty)$, and let $h_p(\theta,q)<\infty$. Then the integrals $\mathcal{A}_p^*(x)$ and $\mathcal{B}_p^*(x)$, $x\in\mathbb{R}$, converge (see (4.40), (4.42)), and there exists a constant $c=c(\theta,q)$ such that the following inequalities hold:*

$$\mathcal{A}_p^*(x)\le c\frac{h^p(\theta,q)}{d(x)^{p-1}},\quad \mathcal{B}_p^*(x)\le c\frac{h^p(\theta,q)}{d(x)^{p-1}},\quad x\in\mathbb{R}. \tag{4.47}$$

Moreover, there exists a constant $c = c(q)$ such that the integral $J_p(x)$ (see (3.3)) admits the estimates

$$c^{-1}d(x) \le J_p(x) \le cd(x), \quad x \in \mathbb{R}. \tag{4.48}$$

Proof. Let $x \in \mathbb{R}$, let $\{\omega_n\}_{n=1}^{\infty}$ be an $\mathbb{R}(x, bd)$-covering of $\mathbb{R}$, and let $\Delta(t) = [t - \frac{d(t)}{2}, t + \frac{d(t)}{2}]$ for $t \in \mathbb{R}$. In the following relations we use Definition 2.10, Lemma 4.6, 4.14, 4.10 and the condition $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$:

$$\begin{aligned}
\mathcal{A}_p^*(x) &= \int_x^{\infty} |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt \\
&= \sum_{n=1}^{\infty} \int_{\omega_n} |\theta(t)|^p \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt \\
&\le \sum_{n=1}^{\infty} \Big(\int_{\omega_n} |\theta(t)|^p dt\Big) \cdot e^{\frac{p}{a}c_2} \cdot e^{-\frac{2bp}{a^2}n} \\
&= c\sum_{n=1}^{\infty} \Big[d(x_n)^{\frac{p}{p'}} \cdot \int_{\omega_n} |\theta(t)|^p dt\Big] \frac{1}{d(x_n)^{p-1}} \cdot e^{-\frac{2bp}{a^2}n} \\
&\le c\frac{\tilde{h}^p(\theta, q)}{d(x)^{p-1}} \sum_{n=1}^{\infty} \Big(\frac{d(x)}{d(x_n)}\Big)^{p-1} \cdot e^{-\frac{2bp}{a^2}n} \le c\frac{h^p(\theta, q)}{d(x)^{p-1}} \sum_{n=1}^{\infty} \frac{a^{2n(p-1)}}{e^{\frac{2pb}{a^2}n}} \\
&\le c\frac{h^p(\theta, q)}{d(x)^{p-1}} \sum_{n=1}^{\infty} \gamma^{2np} = c\frac{h^p(\theta, q)}{d(x)^{p-1}} \Rightarrow (4.47).
\end{aligned}$$

Note that $J_p \equiv \mathcal{A}_p$ for $\theta \equiv 1$ (see (3.3) and (4.39). Therefore, using (4.43) and the same arguments as in the proof of (4.47), we obtain the upper estimate from (4.48):

$$\begin{aligned}
J_p(x) &= \int_x^{\infty} \exp\Big(-p\int_x^t q(\xi)d\xi\Big)dt \\
&= \int_x^{\infty} \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big) \cdot \exp\Big(\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)} - \int_x^t q(\xi)d\xi\Big)dt \\
&\le e^{c_1} \cdot \int_x^{\infty} \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt = c\sum_{n=1}^{\infty}\int_{\omega_n} \exp\Big(-\frac{p}{a}\int_x^t \frac{d\xi}{d(\xi)}\Big)dt \\
&\le c\sum_{n=1}^{\infty} 2bd(x_n) \cdot e^{\frac{p}{a}c_2} \cdot e^{-\frac{2bp}{a^2}n} = cd(x)\sum_{n=1}^{\infty}\Big(\frac{d(x_n)}{d(x)}\Big)e^{-\frac{2bp}{a^2}n}
\end{aligned}$$

$$\le cd(x)\sum_{n=1}^{\infty} a^{2np}\cdot e^{-\frac{2pb}{a^2}n} = cd(x)\sum_{n=1}^{\infty}\gamma^{2np} = cd(x) \ \Rightarrow \ (4.48).$$

It remains to check the lower estimate for $J_p(x)$. In the following relations $x\in\mathbb{R}$, and we only use (2.6), (2.9) and (1.2):

$$J_p(x) = \int_x^{\infty}\exp\Big(-p\int_x^t q(\xi)d\xi\Big)dt \ge \int_x^{x+d(x)}\exp\Big(-p\int_x^t q(\xi)d\xi\Big)dt$$
$$\ge \exp\Big(-p\int_{x-d(x)}^{x+d(x)} q(\xi)d\xi\Big)d(x) = e^{-2p}d(x) \ \Rightarrow (4.48). \qquad \square$$

The proof of Theorem 3.12 follows from Theorem 3.1 and Lemmas 4.13 and 4.15:

$$\mathcal{A}_p(r,q) = \sup_{x\in\mathbb{R}}(I_p(x))^{1/p}(J_{p'}(x))^{1/p'} = \sup_{x\in\mathbb{R}}[B_p(r,x)]^{1/p}\cdot(J_{p'}(x))^{1/p'}$$
$$\le c\sup_{x\in\mathbb{R}}[B_p^*(r,x)]^{1/p}\cdot d(x)^{1/p'}$$
$$\le c\sup_{x\in\mathbb{R}}\Big[\frac{h_p^p(r,q)}{d(x)^{p-1}}\Big]^{1/p}\cdot d(x)^{1/p'} = ch_p(r,q). \qquad \square$$

Proof of Corollary 3.13. This assertion follows from Corollary 3.3 and Theorem 3.12. $\square$

Proof of Theorem 3.14. *Necessity.* Note that in the proof of the lower estimate in (4.48), we did not use the condition $q\in K(\gamma)$, $\gamma\le e^{-1}$. Hence (see (3.1), (3.3) and (2.8)):

$$\mathcal{A}_\infty(r,q) = \sup_{x\in\mathbb{R}} r(x)J_1(x) \ge c^{-1}\sup_{x\in\mathbb{R}} r(x)d(x) = c^{-1}h_\infty(r,q),$$

and the statement follows from Theorem 3.1.

Proof of Theorem 3.14. *Sufficiency.* If $q\in\mathcal{K}(\gamma)$, $\gamma\le e^{-1}$, then from (2.8) we get

$$\mathcal{A}_\infty(r,q) = \sup_{x\in\mathbb{R}} r(x)J_1(x) \le c\sup_{x\in\mathbb{R}} r(x)d(x) = ch_\infty(r,q)$$

and it remains to refer to Theorem 3.1. $\square$

Proof of Corollary 3.15. This assertion follows from Corollary 3.3 and Theorem 3.14. $\square$

Proof of Theorem 3.16. From (2.12) it follows that for $x \in \mathbb{R}$ and $\varepsilon \in [0,1]$, the following inequalities hold:

$$1-\varepsilon \le \frac{d(t)}{d(x)} \le 1+\varepsilon \le \frac{1}{1-\varepsilon} \qquad \text{for} \qquad |t-x| \le \varepsilon d(x).$$

Hence, according to (3.9), we conclude that $q \in \mathcal{K}(\gamma)$ with $\gamma = a\exp(-b/a^2)$ where $a = (1-\varepsilon)^{-1}$ and $b = \varepsilon$; i.e.,

$$\gamma \overset{\text{def}}{=} \gamma(\varepsilon) = (1-\varepsilon)^{-1}\exp(-\varepsilon(1-\varepsilon)^2), \quad \varepsilon \in [0,1].$$

It is easy to see that $\gamma(0)=1$, $\gamma(1)=\infty$ and $\gamma'(\varepsilon)>0$ for $\varepsilon \in (0,1)$. Hence for any $\gamma_0 > 1$ there exists $\varepsilon_0 \in (0,1)$ such that $\gamma_0 = \gamma(\varepsilon_0)$, and therefore $q \in \mathcal{K}(\gamma_0)$. □

Proof of Theorem 3.18. It is easy to see that the following identity holds:

$$\int_{x-\eta}^{x+\eta} q_1(t)dt = 2q_1(x)\eta + \int_0^\eta \int_0^t \int_{x-\xi}^{x+\xi} q_1''(s)dsd\xi dt, \quad x \in \mathbb{R}, \quad \eta \ge 0. \quad (4.49)$$

Let $\eta(x) = (1+\varkappa_1(x)+\varkappa_2(x))/q_1(x)$. Then $\varkappa_1(x)+\varkappa_2(x) \le 1/3$ for all $|x| \gg 1$, and by (4.49), (3.14) and (3.15), we obtain

$$\begin{aligned}
\int_{x-\eta(x)}^{x+\eta(x)} q(t)dt &= \int_{x-\eta(x)}^{x+\eta(x)} q_1(t)dt + \int_{x-\eta(x)}^{x+\eta(x)} q_2(t)dt \\
&\ge 2q_1(x)\eta(x) - \left|\int_0^{\eta(x)}\int_0^t\int_{x-\xi}^{x+\xi} q_1''(s)dsd\xi dt\right| - \varkappa_2(x) \\
&\ge 2(1+\varkappa_1(x)+\varkappa_2(x)) - \frac{\varkappa_1(x)}{2}(1+\varkappa_1(x)+\varkappa_2(x))^2 - \varkappa_2(x) \\
&\ge 2+\varkappa_1(x)+\varkappa_2(x) \ge 2 \quad \Rightarrow \quad \eta(x) \ge d(x) \quad (\text{see } (2.10)).
\end{aligned}$$

Similarly, let $\eta(x) = (1-\varkappa_1(x)-\varkappa_2(x))/q_1(x)$, $|x| \gg 1$. Then

$$\begin{aligned}
\int_{x-\eta(x)}^{x+\eta(x)} q(t)dt &= \int_{x-\eta(x)}^{x+\eta(x)} q_1(t)dt + \int_{x-\eta(x)}^{x+\eta(x)} q_2(t)dt \\
&\le 2q_1(x)\eta(x) + \left|\int_0^{\eta(x)}\int_0^t\int_{x-\xi}^{x+\xi} q_1''(s)dsd\xi dt\right| + \varkappa_2(x) \\
&\le 2(1-\varkappa_1(x)-\varkappa_2(x)) + \varkappa_1(x)\frac{(1-\varkappa_1(x)-\varkappa_2(x))^2}{2} + \varkappa_2(x) \\
&\le 2-\varkappa_1(x)-\varkappa_2(x) \le 2 \quad \Rightarrow \quad \eta(x) \le d(x) \quad (\text{see } (2.10)).
\end{aligned}$$

Thus, (3.16) is proved. The estimates (3.17) follow from (3.16) and the fact that the function d is continuous and positive (see Lemma 2.6). □

Proof of Theorem 3.19. We need the following assertion.

Lemma 4.16. *Let c be the constant from* (3.17), *and let b be a given positive number. Then, under the hypotheses of Theorems* 3.18 *and* 3.19, *there exists $x_0 = x_0(bc)$ such that for all $|x| \ge x_0$ the following inequalities hold:*

$$\frac{1}{2} \le \frac{q_1(s)}{q_1(x)} \le 2 \qquad \text{for} \qquad |s-x| \le \frac{bc}{q_1(x)}. \tag{4.50}$$

Proof. By (3.19), there exists $x_1 = x_1(bc)$ such that

$$|x|q_1(x) \ge 4bc \qquad \text{for} \qquad |x| \ge x_2. \tag{4.51}$$

Let $\varepsilon = (4bc)^{-1}$. By (3.18), there exists $x_2 = x_2(\varepsilon)$ such that

$$|q_1'(x)| \le \varepsilon q_1^2(x) \qquad \text{for} \qquad |x| \ge x_2. \tag{4.52}$$

Let $x_0 = \max\{x_1, x_2\}$. Then we have

$$\Big[x - \frac{bc}{q_1(x)}, x + \frac{bc}{q_1(x)}\Big] \cap [-x_0, x_0] = \emptyset \qquad \text{for} \qquad |x| \ge 2x_0. \tag{4.53}$$

Indeed, if $x \ge 2x_0$ ($x \le -2x_0$), then we have the relations (see (4.51))

$$x - \frac{bc}{q_1(x)} = x\Big[1 - \frac{bc}{xq_1(x)}\Big] \ge x(1 - \tfrac14) \ge \frac{x}{2} \ge x_0$$

$$\Big(x + \frac{bc}{q_1(x)} = x\Big[1 + \frac{bc}{xq_1(x)}\Big] \le x\left(1 - \tfrac14\right) \ge \frac{x}{2} \le -x_0\Big).$$

Let $t \in [0, \frac{bc}{q_1(x)}]$ and $|x| \ge 2x_0$. Then (4.53) implies

$$\left.\begin{aligned} \int_{x-t}^{x} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \le \varepsilon t \le \frac{\varepsilon bc}{q_1(x)} = \frac{1}{4q_1(x)} \\ \int_{x}^{x+t} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \le \varepsilon t \le \frac{\varepsilon bc}{q_1(x)} = \frac{1}{4q_1(x)} \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned}
\frac{1}{q_1(x)} - \frac{1}{q_1(x+t)} &= \int_x^{x+t} \frac{q_1'(\xi)}{q_1^2(\xi)} d\xi \le \int_x^{x+t} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \le \frac{1}{4q_1(x)}, \\
\frac{1}{q_1(x)} - \frac{1}{q_1(x+t)} &= \int_x^{x+t} \frac{q_1'(\xi) d\xi}{q_1^2(\xi)} \ge -\int_x^{x+t} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \ge -\frac{1}{4q_1(x)}, \\
\frac{1}{q_1(x-t)} - \frac{1}{q_1(x)} &= \int_{x-t}^{x} \frac{q_1'(\xi)}{q_1^2(\xi)} d\xi \le \int_{x-t}^{x} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \le \frac{1}{4q_1(x)}, \\
\frac{1}{q_1(x-t)} - \frac{1}{q_1(x)} &= \int_{x-t}^{x} \frac{q_1'(\xi)}{q_1^2(\xi)} d\xi \ge -\int_{x-t}^{x} \frac{|q_1'(\xi)|}{q_1^2(\xi)} d\xi \ge -\frac{1}{4q_1(x)}.
\end{aligned}$$

These relations immediately imply the inequalities

$$\frac{1}{2}\le\frac{4}{5}\le\frac{q_1(x\pm t)}{q_1(x)}\le\frac{4}{3}\le 2\quad\text{for}\quad t\in\Big[0,\frac{bc}{q_1(x)}\Big],\quad |x|\ge 2x_0 \tag{4.54}$$

which are equivalent to (4.50). □

We now go to the proof of Theorem 3.19.

Let $|x|\ge 2x_0$ and $s\in[x-bd(x),x+bd(x)]$. Then, from (3.17), it follows that

$$[x-bd(x),x+bd(x)]\subseteq\Big[x-\frac{bc}{q_1(x)},x+\frac{bc}{q_1(x)}\Big],$$

and therefore by (4.50) and (3.17) for such s and x, we have

$$\frac{d(s)}{d(x)}=d(s)q_1(s)\cdot\frac{q_1(x)}{q_1(s)}\cdot\frac{1}{d(x)q_1(x)}\le 2c^2,$$
$$\frac{d(s)}{d(x)}=d(s)q_1(s)\cdot\frac{q_1(x)}{q_1(s)}\cdot\frac{1}{d(x)q_1(x)}\ge\frac{1}{2c^2}.$$

Thus for $|x|\ge 2x_0=2x_0(bc)$, we have the inequalities

$$\frac{1}{2c^2}\le\frac{d(s)}{d(x)}\le 2c^2\quad\text{for}\quad |s-x|\le bd(x). \tag{4.55}$$

According to (4.55), we have $q\in\mathcal{K}(\gamma)$, $\gamma=2c^2\exp\left(-\frac{b}{4c^4}\right)$. Here c is an absolute constant. Choose $b\gg 1$ (and thus automatically increase $x_0(bc)$, see the proof of Lemma 4.16). We obtain $\gamma=2c^2\exp(-b/4c^4)\le e^{-1}$, as required. □

Proof of Corollary 3.20. Let c be the constant from (3.17), and let $b>0$. Set $bc=2$. Then inequalities (4.54) are equivalent to the estimates (3.20). □

Proof of Corollary 3.21. From (3.22) we get

$$-cq_1(\xi)\le\frac{r'(\xi)}{r(\xi)}\le cq_1(\xi),\quad \xi\in\mathbb{R}.$$

Let $t\in[0,\frac{2}{q_1(x)}]$, $x\in\mathbb{R}$. Then (3.20) implies the relations

$$ln\frac{r(x+t)}{r(x)}\le c\int_x^{x+t}q_1(\xi)d\xi\le c\int_x^{x+2/q_1(x)}\frac{q_1(\xi)}{q_1(x)}\cdot q_1(x)d\xi\le c,$$
$$ln\frac{r(x+t)}{r(x)}\ge -c\int_x^{x+t}q_1(\xi)d\xi\ge -c\int_x^{x+2/q_1(x)}\frac{q_1(\xi)}{q_1(x)}\cdot q_1(x)d\xi\ge -c.$$

Hence,

$$c^{-1} \le \frac{r(s)}{r(x)} \le c \quad \text{for} \quad s \in [x, x + \tfrac{2}{q_1(x)}], \tag{4.56}$$

and similarly,

$$c^{-1} \le \frac{r(s)}{r(x)} \le c \quad \text{for} \quad s \in [x - \tfrac{2}{q_1(x)}, x]. \tag{4.57}$$

Note that, for $|x| \gg 1$ from (3.16), it follows that $[x - d(x), x + d(x)] \subseteq [x - \frac{2}{q_1(x)}, x + \frac{2}{q_1(x)}]$, and therefore by (4.56)–(4.57) we have

$$c^{-1} \le \frac{r(t)}{r(x)} \le c \quad \text{for} \quad |t - x| \le d(x), \quad |x| \gg 1. \tag{4.58}$$

Since $r(x)$ is a positive continuous function for $x \in \mathbb{R}$, the inequalities (4.58) remain true (possibly with a bigger constant c) for all $x \in \mathbb{R}$:

$$c^{-1} \le \frac{r(t)}{r(x)} \le c \quad \text{for} \quad |t - x| \le d(x), \quad x \in \mathbb{R}. \tag{4.59}$$

By Theorem 3.19, we have $q \in \mathbb{K}(\gamma)$, $\gamma \le e^{-1}$. Therefore, to prove the assertion, it remains to show that $h_p(r, q) < \infty$ if and only if condition (3.22) holds (see Theorem 3.12). From (4.59) and (3.17), it follows that

$$\begin{aligned} h_p(r,q) &= \sup_{x\in\mathbb{R}} d(x)^{1/p'} \Big(\int_{|t-x|\le \frac{1}{2}d(x)} r(t)^p dt \Big)^{1/p} \\ &= \sup_{x\in\mathbb{R}} d(x)^{1/p'} \Big[\int_{|t-x|\le \frac{1}{2}d(x)} r(x)^p \cdot \Big(\frac{r(t)}{r(x)}\Big)^p dt \Big]^{1/p} \\ &\lesseqgtr c^{\mp 1} \sup_{x\in\mathbb{R}} r(x) d(x) \lesseqgtr c^{\mp 1} \sup_{x\in\mathbb{R}} \frac{r(x)}{q_1(x)}. \end{aligned}$$

□

Proof of Lemma 3.22. Let $\tau(a_1) \le \frac{1}{2}$ for $a_1 \gg a_0$. Then

$$\begin{aligned} q_0(a_1) &= \int_{x-a_1}^{x+a_1} q(t)dt = \int_{x-a_1}^{x+a_1} q_1(t)dt + \int_{x-a_1}^{x+a_1} q_2(t)dt \\ &= \int_{x-a_1}^{x+a_1} q_q(t)dt \Big[1 + \Big(\int_{x-a_1}^{x+a_1} q_2(t)dt\Big)\Big(\int_{x-a_1}^{x+a_1} q_1(t)dt\Big)^{-1}\Big] \\ &\ge \varepsilon(1 - \tau(a_1)) \ge \frac{\varepsilon}{2} > 0. \end{aligned}$$

□

Proof of Theorem 3.23. In the following transformations of integrals, we only use the following standard changes of variables:

$$\begin{aligned}\int_{x-d}^{x+d} q(\xi)d\xi &= \int_{x-d}^{x} q(t)dt + \int_{x}^{x+d} q(t)dt = \int_0^d [q(x+\xi_1)+q(x-\xi_1)]d\xi_1 \\ &= 2q(x)d + \int_0^d [q(x+\xi_1)-2q(x)+q(x-\xi_1)]d\xi_1 \\ &= 2q(x)d + \int_0^d \int_0^{\xi_1} \int_{x-\xi_2}^{x+\xi_2} q''(\xi_3)d\xi_3 d\xi_2 d\xi_1. \end{aligned} \tag{4.60}$$

Iterating (4.60) once again, we obtain

$$\int_{x-d}^{x+d} q(\xi)d\xi = 2q(x)d + 2q''(x)\frac{d^3}{6} + \int_0^d \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \int_{x-\xi_4}^{x+\xi_4} q^{(4)}(\xi_5)d\xi_5 \dots d\xi_1. \tag{4.61}$$

Hence the function $F(\eta)$ (see (2.10)) is of the form

$$F(\eta) = 2q(x)\eta + 2q''(x)\frac{\eta^3}{6} + \int_0^\eta \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \int_{x-\xi_4}^{x+\xi_4} q^{(4)}(\xi_5)d\xi_5 \dots d\xi_1. \tag{4.62}$$

Let now

$$\eta_k = \eta(x_k) = \sqrt[3]{\tfrac{6}{q''(x_k)}}(1-\varepsilon_k)^{1/3}, \quad \varepsilon_k = \tfrac{11\delta_k}{48}, \quad k \gg 1. \tag{4.63}$$

Then, from (4.62) and (4.63), we get

$$\begin{aligned} F(\eta_k) &= 2 - 2\varepsilon_k + \int_0^{\eta_k} \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \int_{x-\xi_4}^{x+\xi_4} q^{(4)}(\xi_5)d\xi_5 \dots d\xi_1 \\ &\le 2 - 2\varepsilon_k + \sigma_k \frac{\eta_k^4}{24} = 2 - 2\varepsilon_k + \frac{6^{4/3}}{24}\frac{\sigma_k}{q''(x_k)^{4/3}}(1-\varepsilon_k)^{4/3} \\ &\le 2 - 2\varepsilon_k + \frac{6^{4/3}}{24}\delta_k \le 2. \end{aligned}$$

Hence, $d(x_k) \ge \eta_k$ by Lemma 2.7. Similarly, set

$$\eta_k = \eta(x_k) = \sqrt[3]{\tfrac{6}{q''(x_k)}}(1+\varepsilon_k)^{1/3}, \quad \varepsilon_k = \tfrac{7}{12}\delta_k, \quad k \gg 1. \tag{4.64}$$

Then from (4.62) and (4.64), we get

$$F(\eta_k) = 2 + 2\varepsilon_k + \int_0^{\eta_k} \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \int_{x-\xi_4}^{x+\xi_4} q^{(4)}(\xi_5)d\xi_5 \dots d\xi_1$$

$$\geq 2 + 2\varepsilon_k - \frac{6^{4/3}}{24}\frac{\sigma_k}{q''(x_k)^{4/3}}(1+\varepsilon_k)^{4/3} \geq 2 + 2\varepsilon_k - \frac{12^{4/3}}{24}\delta_k \geq 2.$$

Hence $d(x_k) \leq \eta_k$ by Lemma 2.7. The statement of the lemma now follows from the proven estimates for $d(x_k)$, $x \gg 1$. □

5. Examples

Example 1. Let us show that for $p \in (1,\infty)$ there exist operators $\mathcal{L}_p$ not separable in L_p. Let $\alpha > 0$ and $\beta \in (\frac{\alpha}{p}, \alpha)$, $p \in (1,\infty)$, $\omega_n = \big[n - n^{-\alpha}, n + n^{-\alpha}\big]$, $n = 1,2,3,\ldots$. Set

$$q(x) = \begin{cases} 1, & \text{if} \quad x \notin \omega_n, \quad n = 1,2,\ldots \\ n^\beta, & \text{if} \quad x \in \omega_n, \quad n = 1,2,\ldots \end{cases} \tag{5.1}$$

Since $q(x) \geq 1$ for $x \in \mathbb{R}$, we have

$$q_0(1) = \inf_{x\in\mathbb{R}} \int_{x-1}^{x+1} q(t)dt \geq \inf_{x\in\mathbb{R}} \int_{x-1}^{x+1} 1dt = 2 > 0,$$

and by *Theorem* 2.4 we conclude that equation (1.1) is correctly solvable in L_p. Furthermore, it is easy to see that

$$d(n) = 1 + \frac{1}{n^\alpha} - \frac{1}{n^{\alpha-\beta}}, \qquad n \geq 2. \tag{5.2}$$

Indeed, denote

$$\omega_n = [n - n^{-\alpha}, n + n^{-\alpha}], \quad \tilde{\omega}_n = [n - n^{-\alpha} - 1 + n^{\beta-\alpha}, n - n^{-\alpha}],$$
$$\hat{\omega}_n = [n + n^{-\alpha}, n + n^{-\alpha} + 1 - n^{\beta-\alpha}].$$

Then in view of the obvious equality $[n - d(n), n + d(n)] = \omega_n \cup \tilde{\omega}_n \cup \hat{\omega}_n$, $n \geq 2$, we obtain from (5.1)

$$\int_{n-d(n)}^{n+d(n)} q(t)dt = \int_{\omega_n} n^\beta dt + \int_{\tilde{\omega}_n} 1dt + \int_{\tilde{\omega}_n} 1dt$$
$$= \frac{2}{n^{\alpha-\beta}} + \Big(1 - \frac{1}{n^{\alpha-\beta}}\Big) + \Big(1 - \frac{1}{n^{\alpha-\beta}}\Big) = 2 \quad \Rightarrow \quad (5.2).$$

From (5.2) it follows that $d(n) \geq \frac{1}{2}$ for $n \geq n_0 \gg 1$, and therefore the following relations hold:

$$h_p(q,q) = \sup_{x\in\mathbb{R}} d(x)^{1/p'} \Big[\int_{|t-x|\leq 2^{-1}d(x)} q(t)^p dt\Big]^{1/p}$$

$$\geq \sup_{n\geq n_0} d(n)^{1/p}\Big[\int_{|t-n|\leq\frac{1}{2}d(n)} q(t)^p dt\Big]^{1/p}$$
$$\geq \sup_{n\geq n_0} \frac{1}{2^{1/p'}}\Big[\int_{\omega_n} q(t)^p dt\Big]^{1/p} = \sup_{n\geq n_0} \frac{1}{2^{1/p'}} n^{\beta-\frac{\alpha}{p}} = \infty.$$

Taking into account Corollary 3.13, this implies that the operator is not separable in L_p.

Remark 5.1. One can check in a similar way that the operator $\mathcal{L}_\infty$ is not separable in $C(\mathbb{R})$ (see Corollary 3.15).

Example 2. Consider the equations

$$-y'(x) + e^{|x|}y(x) = f(x), \quad x \in \mathbb{R}, \tag{5.3}$$

$$-z'(x) + (e^{|x|} + e^{|x|}\cos e^{\alpha|x|})z(x) = f(x), \quad x \in \mathbb{R}, \tag{5.4}$$

where $\alpha \in (0,\infty)$. Let us show that they are both correctly solvable in L_p, $p \in [1,\infty]$, and for the solutions $y \in L_p$ of equation (5.3) the following inequality holds:

$$\|e^{|x|}y\|_p \leq c(p)\|f\|_p, \quad \forall f \in L_p. \tag{5.5}$$

Our main problem is to find all $\alpha \in (0,\infty)$ such that for the solutions $z \in L_p$ of the "perturbed equation" (5.4), we have the inequality

$$\|e^{|x|}z\|_p \leq c(p)\|f\|_p, \quad \forall f \in L_p, \tag{5.6}$$

similar to inequality (5.5) for the solutions $y \in L_p$ of the original ("non-perturbed") equation (5.3). In particular, using the results of Section 3, we show that the inequality (5.6) holds if and only if $\alpha \geq 1$. For the reader's convenience, we divide the proof of this fact into several steps. Note that we present all the calculations, estimates, etc. only for $x \geq 0$ or $x \gg 1$ because all the functions under consideration are even and the cases $x \leq 0$ and $x \ll -1$ are similar. Finally, we adopt the convention that for positive continuous functions φ and ψ the symbol $\varphi(x) \asymp \psi(x)$, $x \in (a,b)$, means that the following inequalities hold:

$$c^{-1}\varphi(x) \leq \psi(x) \leq c\varphi(x), \quad x \in (a,b).$$

I. Correct solvability. Since $e^{|x|} \geq 1$, $x \in \mathbb{R}$, equation (5.3) is correctly solvable in L_p, $p \in [1,\infty]$ (see Example 1). Let us show that equation (5.4) is also correctly solvable in L_p, $p \in [1,\infty]$ for all $\alpha \in (0,\infty)$. Below we use the structure of the proof of Lemma 3.22.

The following estimates are obtained by using the second average theorem (see [14, Section 12.3]) (for $a \gg 1$ and $x \ge a$) :

$$\begin{aligned} f(x) &\stackrel{\text{def}}{=} \int_{x-a}^{x+a} (e^t + e^t \cos e^{\alpha t}) dt \\ &= (e^{x+a} - e^{x-a}) + \int_{x-a}^{x+a} \frac{e^{(1-\alpha)t}}{\alpha} [\alpha e^{\alpha t} \cos e^{\alpha t}] dt \\ &\ge \frac{e^{x+a}}{2} - c \max\{e^{(1-\alpha)(x-a)}, e^{(1-\alpha)(x+a)}\} \\ &= \frac{e^{x+a}}{2} \{1 - c \max\{e^{-\alpha(x-a)-2a}, e^{-\alpha(x+a)}\}\} \\ &\ge \tfrac{1}{2}\{1 - c\max\{e^{-2a}, e^{-2\alpha a}\}\} \ge \tfrac{1}{4}. \end{aligned}$$

In addition, it is clear that $\inf_{x\in[0,a]} f(x) > 0$. Therefore, we finally conclude that $\inf_{x\in\mathbb{R}} f(x) > 0$, as required (see Theorem 2.4).

II. Proof of inequality (5.5)**.** Let us show that one can apply Theorems 3.18 and 3.19 to the function $q(x) = e^{|x|}$, $x \in \mathbb{R}$. Let $q_1(x), q_2(x) \equiv 0$, $x \in \mathbb{R}$. Then for $x \gg 1$ we have (3.14):

$$\varkappa_1(x) = \frac{1}{e^{2x}} \sup_{|\xi| \le 2e^{-x}} \left| \int_{x-\xi}^{x+\xi} e^s ds \right| \le \frac{c}{e^{2x}} \sup_{|\xi|\le 2e^{-x}} e^x |\xi| \le \frac{c}{e^{2x}}.$$

Hence, $\varkappa_1(x) \to 0$ as $|x| \to \infty$, $\varkappa_2(x) \equiv 0$, $x \in \mathbb{R}$ and therefore (see (3.17)) we have $d(x) \asymp e^{-|x|}$, $x \in \mathbb{R}$. In addition, since

$$\lim_{x\to\infty} \frac{q_1'(x)}{q_1^2(x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0, \qquad \lim_{|x|\to\infty} |x| e^{|x|} = \infty, \tag{5.7}$$

we have (see Theorem 3.19) $q \in \mathcal{K}(\gamma)$, $\gamma \le e^{-1}$. Let $r(x) = e^{|x|}$, $x \in \mathbb{R}$. Then

$$\frac{|r'(x)|}{r(x) q_1(x)} = \frac{1}{e^x} \le 1, \qquad \frac{r(x)}{q_1(x)} = 1, \qquad x \ge 0,$$

and therefore inequality (5.5) holds because of Corollary 3.21.

III. Proof of inequality (5.6) **for** $\alpha > 1$**.** Let us prove that for $\alpha > 1$ the function q

$$q(x) = q_1(x) + q_2(x), \quad q_1(x) = e^{|x|}, \quad q_2(x) = e^{|x|} \cos e^{\alpha |x|}, \quad x \in \mathbb{R} \tag{5.8}$$

satisfies the hypotheses of Theorems 3.18 and 3.19. As shown above (see II), $\varkappa_1(x) \to 0$ as $|x| \to \infty$ for $q_1(x) = e^{|x|}$. To estimate $\varkappa_2(x)$, $x \gg 1$ (see

(3.15)) in the case (5.8), we use the second average theorem (see [14, Section 12.3]) and the notation

$$\omega(x) = [x-2e^{-x}, x+2e^{-x}], \quad x\ge 0:$$

$$\varkappa_2(x) = \sup_{|\xi|\le 2e^{-x}}\Big|\int_{x-\xi}^{x+\xi} e^{\xi}\cos^{\alpha\xi} d\xi\Big| = \sup_{|\xi|\le 2e^{-x}}\Big|\int_{x-\xi}^{x+\xi}\frac{e^{(1-\alpha)\xi}}{\alpha}[\alpha e^{\alpha\xi}\cos e^{\alpha\xi}]d\xi\Big|$$
$$\le ce^{(1-\alpha)x}\sup_{\alpha,\beta\in\omega(x)}\Big|\int_{\alpha}^{\beta}\alpha e^{\alpha\xi}\cos e^{\alpha\xi}d\xi\Big|\le ce^{(1-\alpha)x}.$$

Thus $\varkappa_1(x)\to 0$, $\varkappa_2(x)\to 0$ as $|x|\to\infty$, and, in addition, according to (5.7), the hypotheses of Theorems 3.18 and 3.19 are satisfied. Having this at our disposal, we finish the proof of inequality (5.6) in the same way as for inequality (5.5).

IV. Proof of inequality (5.6) **for** $\alpha = 1$**.** For $\alpha=1$, the function q in the case (5.8) does not satisfy the hypotheses of Theorems 3.18 and 3.23. Therefore we establish all the needed estimates of the function d using Lemma 2.7. First we check that for $x\ge 0$ and $d\in[0,1]$ we have the inequalities

$$2de^x \le e^{x+d} - x^{x-d} \le 8\cdot 3^{-1}de^x. \tag{5.9}$$

Indeed, the lower estimate in (5.9) immediately follows from the expansion of the exponent

$$e^{x+d}-x^{x-d} = 2e^x\Big[d+\frac{d^3}{3!}+\frac{d^5}{5!}+\dots\Big].$$

To prove the upper estimate in (5.9), we use the additional relations

$$e^{x+d}-e^{x-d}\le 2e^x d\Big[1+\frac{d^2}{2^2}+\frac{d^4}{2^4}+\dots\frac{d^{2n}}{2^{2n}}+\dots\Big] = \frac{2e^x d}{1-4^{-1}d^2}\le\frac{8}{3}de^x.$$

Let now $\eta(x) = 2e^{-x}$, $x\ge 0$. Then (5.9) implies

$$\int_{x-\eta(x)}^{x+\eta(x)}(e^t+e^t\cos e^t)dt \ge e^{x+\eta}-e^{x-\eta}-\Big|\sin e^t\Big|_{x-\eta}^{x+\eta}\Big|\ge 4-2=2,$$

and therefore $d(x)\le 2e^{-x}$, $x\ge 0$ because of Lemma 2.7. Let now $\eta(x) = (4e^x)^{-1}$, $x\ge 0$. Then by (5.9) we have

$$\int_{x-\eta}^{x+\eta}(e^t+e^t\cos e^t)dt = e^{x+\eta}-e^{x-\eta}+2\sin\Big(\frac{e^{x+\eta}-e^{x-\eta}}{2}\Big)\cos\Big(\frac{e^{x+\eta}-e^{x-\eta}}{2}\Big)$$
$$\le\frac{8}{3}\eta e^x+2\Big|\sin\Big(\frac{e^{x+\eta}-e^{x-\eta}}{2}\Big)\Big|\le\frac{2}{3}+\big(e^{x+\eta}-e^{x-\eta}\big)\le\frac{2}{3}+\frac{8e^x\eta}{3}=\frac{4}{3}\le 2,$$

and therefore $d(x) \ge (4e^x)^{-1}$, $x \ge 0$. Thus

$$\frac{1}{4}\,\frac{1}{e^{|x|}} \le d(x) \le \frac{2}{e^{|x|}}, \qquad x \in \mathbb{R}. \tag{5.10}$$

Let now $x \in \mathbb{R}$, $b \ge 1$ and $t \in [x - bd(x), x + bd(x)]$. From (5.10), it follows that

$$[x - bd(x), x + bd(x)] \subseteq [x - 2be^{-|x|}, x + 2be^{-|x|}]. \tag{5.11}$$

From (5.10) and (5.11), we now obtain

$$\frac{d(t)}{d(x)} = (d(t)e^{|t|})e^{|x|-|t|}\frac{1}{(d(x)e^{|x|})} \le 8e^{|t-x|} = 8e^{be^{-|x|}},$$
$$\frac{d(t)}{d(x)} = (d(t)e^{|t|})e^{|x|-|t|}\frac{1}{(d(x)e^{|x|})} \ge \frac{1}{8e^{|t-x|}} \ge \frac{1}{8e^{be^{-|x|}}}.$$

Let $b \gg 1$, $x_0 = lnb$. Then the obtained estimates imply

$$24^{-1} \le \frac{d(t)}{d(x)} \le 24 \qquad \text{for} \qquad |t - x| \le bd(x), \qquad |x| \ge x_0.$$

Hence (see (3.9)) $q \in \mathcal{K}(\gamma)$, $\gamma = 24\exp(-b/24^2)$. Finally, choose $b \gg 1$ in order to satisfy the inequality $\gamma \le e^{-1}$. We can now apply Theorem 3.12. In the following, estimates we use the inequalities (5.10):

$$h_p(e^{|x|}, q) = \sup_{x\in\mathbb{R}} d(x)^{1/p'}\Big[\int_{|t-x|\le 2^{-1}d(x)} e^{p|t|}dt\Big]^{1/p} \le c\sup_{x\in\mathbb{R}} d(x)e^{|x|} = c < \infty.$$

Hence the inequality (5.6) holds because of Theorem 3.12.

V. Case of $\alpha \in (0, 1)$. Below we mainly use Theorems 3.23 and 3.12. It is easy to see that in the case (5.8) the following relations hold:

$$|q^{(i)}(x)| \le ce^{(1+\alpha i)|x|}, \quad i = \overline{0, 4}, \tag{5.12}$$

$$q(x_k) = 0 \quad \text{for} \quad x_k = \frac{1}{\alpha}ln[(2k + 1)\pi], \quad k = 0, 1, 2, \dots , \tag{5.13}$$

$$q''(x_k) = \alpha^2[(2k + 1)\pi]^{\frac{1+2\alpha}{\alpha}}, \qquad k = 0, 1, 2, 3, \dots . \tag{5.14}$$

According to (3.25) and (3.26) and (5.12)–(5.14), we have, for $k \gg 1$,

$$A_k = \Big[0, \tfrac{2}{3}\sqrt[3]{\tfrac{1}{q''(x_k)}}\,\Big] = \Big[0, \tfrac{2}{3}\alpha^{-2/3}[(2k + 1)\pi]^{-\frac{1+2\alpha}{3\alpha}}\Big],$$

$$\sigma_k = \sup_{t\in\mathcal{A}_k}\Big|\int_{x_k-t}^{x_k+t} q^{(4)}(\xi)d\xi\Big| \le ce^{(1+4\alpha)x_k}(2k + 1)^{-\frac{1+2\alpha}{3\alpha}} = c(2k + 1)^{\frac{2+10\alpha}{3\alpha}},$$

$$\delta_k = \frac{\sigma_k}{q''(x_k)^{4/3}} \le c\frac{(2k + 1)^{\frac{2+10\alpha}{3\alpha}}}{(2k + 1)^{\frac{4+8\alpha}{3\alpha}}} = c(2k + 1)^{-\frac{2(1-\alpha)}{3\alpha}}.$$

Thus $\delta_k \to 0$ as $k \to \infty$. Hence by Theorem 3.23 we get (see (3.27))

$$d(x_k) \asymp \sqrt[3]{\frac{6}{q''(x_k)}} \asymp \frac{1}{(2k+1)^{\frac{1+2\alpha}{3\alpha}}}, \qquad k \gg 1. \tag{5.15}$$

From (5.15) for $k \gg 1$, it now follows that

$$d(x_k)^{1/p'}\Big(\int_{|t-x_k|\le 2^{-1}d(x_k)} e^{p|t|}dt\Big)^{1/p} \asymp d(x_k)e^{x_k} \asymp \frac{(2k+1)^{1/\alpha}}{(2k+1)^{\frac{1+2\alpha}{3\alpha}}}$$
$$= (2k+1)^{\frac{2(1-\alpha)}{3\alpha}} \to \infty \text{ as } k \to \infty.$$

Thus $h_p(e^{|t|}, q) = \infty$, and by Theorem 3.12 inequality (5.6) does not hold for $p \in (1, \infty)$.

Remark 5.2. The cases $p = 1$ and $p = \infty$ are treated in a similar way.

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