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EXISTENCE OF PSEUDO-SYMMETRIC SOLUTIONS TO A P-LAPLACIAN FOUR-POINT BVPS INVOLVING DERIVATIVES ON TIME SCALES

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Abstract. We are concerned with a four-point boundary-value problem of the p-Laplacian dynamic equation on time scales where the nonlinear term contains the first-order derivatives of the dependent variable. By using Krasnosel'skii's fixed-point theorem, some new sufficient conditions are obtained for the existence of at least single or twin positive pseudo-symmetric solutions to this problem. We also establish the existence of at least triple or arbitrary odd positive pseudo-symmetric solutions to this problem by using the Avery-Peterson fixed-point theorem. As applications, two examples are given to illustrate and explain our main results.

1. Introduction

Recently, considerable attention has been attracted to the problem of three positive pseudo-symmetric solutions for the one–dimensional p-Laplacian dynamic equation, for example, works by Avery and Henderson [1, 2], and Ma and Ge [3]. They considered the p-Laplacian differential dynamic equation of the form

$$\begin{aligned}&\big(\varphi_p(u'(t))\big)' + h(t)f(t, u(t)) = 0 \quad \text{for} \quad t \in (0, 1),\\ &u(0) = 0 \ \ \text{and} \ \ u(\xi) = u(1),\end{aligned} \tag{1.1}$$

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where the p-Laplacian operator $\varphi_p(u) = |u|^{p-2}\,u$ for $p > 1$ with $(\varphi_p)^{-1} = \varphi_q$ and $1/p + 1/q = 1$. Here $\xi \in (0,\, 1)$ is a constant and $f(t,\, u)$: $(0,\, 1) \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. The investigation is focused on the existence of pseudo-symmetric solutions, that is, solutions that are symmetric over the interval $[\xi,\, 1]$. By assuming that the coefficient $h(t)$ is a nonnegative pseudo-symmetric function and $f(t,\, u(t))$ is nonnegative, the existence of three pseudo-symmetric positive solutions to the above problem (1.1) was proved [2] by the so-called five functionals fixed-point theorem in a cone [4]. Similar results were also presented in [1] for the discrete case. In addition, Ma and Ge [3] developed the existence of at least two positive pseudo-symmetric solutions to the boundary-value problem (1.1) by using a monotone iterative technique.

In [5], Ji and his co-workers studied the p-Laplacian differential dynamic equation

$$\begin{aligned} &\big(\varphi_p(u'(t))\big)' + h(t)f\,(t, u(t)) = 0 \text{ for } t \in (0,1),\\ &u(0) - \beta u'(\xi) = 0 \text{ and } u(\xi) - \delta u'(\eta) = u(1) + \delta u'(1 - \eta + \xi), \end{aligned} \tag{1.2}$$

where $\beta \geq 0$, $\delta \geq 0$, ξ and $\eta \in (0,\, 1)$ satisfy $\eta > \xi$. Equation (1.2) plays an important role in the study of the n-dimensional systems, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [6]. By means of a fixed-point theorem due to Avery and Peterson [7], they obtained sufficient conditions that guarantee the existence of at least triple positive pseudo-symmetric solutions to the above boundary-value problem (1.2).

In [8], Sun and Ge studied the p-Laplacian differential dynamic equation where f involves the first-order derivative

$$\begin{aligned} &\big(\varphi_p(u'(t))\big)' + h(t)f(t, u(t), u'(t)) = 0 \text{ for } t \in (0,1),\\ &u(0) = 0 \text{ and } u(\xi) = u(1), \end{aligned} \tag{1.3}$$

where $f(t, u, u')$: $(0,\, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. By applying the monotone iterative technique and constructing successive iterative schemes starting off with known functions, they explored the iteration and existence of positive pseudo-symmetric solutions for the three-point second-order p-Laplacian boundary-value problem (1.3) without requiring the existence of lower and upper solutions [8].

As for the p-Laplacian boundary-value problem on time scales $\mathbb{T}$ of the form

$$\begin{aligned} &\big(\phi_p(u^{\Delta}(t))\big)^{\nabla} + h(t)f(t, u(t)) = 0 \text{ for } t \in (0, T)_{\mathbb{T}},\\ &u(0) = 0 \text{ and } u(\xi) = u(T), \end{aligned} \tag{1.4}$$

where $\xi \in (0,T)_{\mathbb{T}}$ and $\mathbb{T}$ is symmetric in $[\xi,T]_{\mathbb{T}}$, by using the pseudo-symmetric technique and developing the five functionals fixed-point theorem in a cone, Su and his collaborators [9] proved the existence of at least three positive pseudo-symmetric solutions of the boundary-value problem (1.4). Su [10] also established some new sufficient conditions for the existence of at least single, twin, triple or arbitrary odd positive pseudo-symmetric solutions of this problem by applying a pseudo-symmetric technique and the fixed-point theorems in a cone.

So far, to the best of our knowledge, very little has been known about the existence of positive solutions for p-Laplacian dynamic equations involving derivatives on time scales. In [11, 12], Li and Su et al investigated the p-Laplacian multi-point boundary-value problem with dependence on the first-order derivative on time scales, and obtained a class of sufficient conditions for the existence of at least twin or arbitrary even positive solutions to the multi-point boundary-value problem. Naturally, it is quite necessary to consider the existence of positive pseudo-symmetric solutions for the p-Laplacian dynamic equations involving derivatives on time scales in all respects.

Motivated by all the works mentioned above, in the present paper we consider the p-Laplacian boundary-value problem on pseudo-symmetric time scales $\mathbb{T}$ with ξ on $[0,\, T]_{\mathbb{T}}$ of the form

$$\begin{gathered}\left(\varphi_p(u^{\Delta}(t))\right)^{\nabla} + h(t) f\left(t, u(t), u^{\Delta}(t)\right) = 0 \text{ for } t \in [0,T]_{\mathbb{T}}, \\ u(0) - \beta u^{\Delta}(\xi) = 0 \text{ and } u(\xi) - \delta u^{\Delta}(\eta) = u(T) + \delta u^{\Delta}(T - \eta + \xi),\end{gathered} \tag{1.5}$$

where $\beta \geq 0$, $\delta \geq 0$, ξ, η and $(\xi + T)/2 \in (0,T)_{\mathbb{T}}$ satisfy $\eta > \xi$ and $\eta \neq \frac{\xi+T}{2}$. The interesting and challenging point here is that the nonlinear term f involves the delta derivative.

The paper is organized as follows. At the end of Section 1, we introduce some basic definitions which may help the reader to better understand our main results. In Section 2 we present some technical lemmas which will be used in the proofs of our main results. In Section 3, we study the existence of single or twin positive pseudo-symmetric solutions to problem (1.5) by applying Krasnosel'skii's fixed-point theorem in a cone. In Section 4, we discuss the existence criteria for at least three and arbitrary odd positive pseudo-symmetric solutions to problem (1.5) under certain conditions by using the Avery-Peterson fixed-point theorem [7]. In Section 5, two examples illustrate our main results.

In order to present our main results in a straightforward way, here we introduce the following definitions [13, 14, 15, 16, 9]. For some well-known results of the p-Laplacian boundary-value problems on time scales we refer the readers to [16, 17, 18, 19, 20, 21] and the references therein.

Definition 1.1. *A time scale* $\mathbb{T}$ *is a nonempty closed subset of* $\mathbb{R}$*. For* $t \in \mathbb{T}$ *the forward and back jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ *are well defined, respectively, by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ *and* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$*. In this definition we take* $\inf \emptyset := \sup \mathbb{T}$ *and* $\sup \emptyset := \inf \mathbb{T}$*, where* $\emptyset$ *denotes the empty set. A point* $t \in \mathbb{T}$ *is called left-dense if* $\rho(t) = t$*, left-scattered if* $\rho(t) < t$*, right-dense if* $\sigma(t) = t$*, and right-scattered if* $\sigma(t) > t$*. If* $\mathbb{T}$ *has a right-scattered minimum* m*, we define* $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$*; otherwise, we set* $\mathbb{T}_\kappa = \mathbb{T}$*. If* $\mathbb{T}$ *has a left-scattered maximum* M*, we define* $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$*; otherwise, we set* $\mathbb{T}^\kappa = \mathbb{T}$*. The forward graininess is* $\mu(t) := \sigma(t) - t$*. Similarly, the backward graininess is* $\nu(t) := t - \rho(t)$*.*

Definition 1.2. *For* $x : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^\kappa$*, we define the delta derivative of* $x(t)$*,* $x^\Delta(t)$*, to be the number (when it exists) with the property that, for any* $\varepsilon > 0$*, there is a neighborhood* U *of* t *such that*

$$\left|[x(\sigma(t)) - x(s)] - x^\Delta(t)\,[\sigma(t) - s]\right| < \varepsilon\,|\sigma(t) - s|\,,$$

for all $s \in U$*. For* $x : \mathbb{T} \to R$ *and* $t \in \mathbb{T}_\kappa$*, we define the nabla derivative of* $x(t)$*,* $x^\nabla(t)$*, to be the number (when it exists) with the property that, for any* $\varepsilon > 0$*, there is a neighborhood* V *of* t *such that*

$$\left|[x(\rho(t)) - x(s)] - x^\nabla(t)\,[\rho(t) - s]\right| < \varepsilon\,|\rho(t) - s|\,,$$

for all $s \in V$*.*

If $\mathbb{T} = \mathbb{R}$, then $x^\Delta(t) = x^\nabla(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^\Delta(t) = x(t+1) - x(t)$ is the forward difference operator while $x^\nabla(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 1.3. *A function* $f : \mathbb{T} \to \mathbb{R}$ *is ld-continuous provided it is continuous at left-dense points in* $\mathbb{T}$ *and its right-sided limit exists (finite) at right-dense points in* $\mathbb{T}$*.*

Definition 1.4. *If* $F^\Delta(t) = f(t)$*, then we define the delta integral by*

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

If $F^\nabla(t) = f(t)$*, then we define the nabla integral by*

$$\int_a^t f(s)\nabla s = F(t) - F(a).$$

Now, we present some pseudo-symmetric definitions which can be found in [9].

Definition 1.5. *For any $\xi, T \in \mathbb{T}$ with $\xi < T$, a time scale $\mathbb{T}$ is said to be pseudo-symmetric about ξ on $[0,T]_{\mathbb{T}}$ if $\mathbb{T}$ is symmetric over the interval $[\xi,T]_{\mathbb{T}}$. That is, for any given $t \in [\xi,T]_{\mathbb{T}}$, we have $T - t + \xi \in [\xi,T]_{\mathbb{T}}$.*

Definition 1.6. *For $\xi \in (0,T)_{\mathbb{T}}$, a function $u : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is said to be pseudo-symmetric about ξ on $[0,T]_{\mathbb{T}}$ if u is symmetric over the interval $[\xi,T]_{\mathbb{T}}$, where $\mathbb{T}$ is symmetric in $[\xi,T]_{\mathbb{T}}$. That is, for any given $t \in [\xi,T]_{\mathbb{T}}$, we have $u(t) = u(T - (t - \xi))$.*

Remark 1.7. For $\xi \in (0,T)_{\mathbb{T}}$, if $u : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is pseudo-symmetric about ξ on the interval $[0,T]_{\mathbb{T}}$ and $u^{\Delta}(T)$ is well defined, we have $u^{\Delta}(t) = -u^{\Delta}(T + \xi - t)$ for $t \in [\xi,T]_{\mathbb{T}}$.

Definition 1.8. *We say that u is a pseudo-symmetric solution of the boundary-value problem (1.5) on the interval $[0,T]_{\mathbb{T}}$ provided u is a solution of the boundary-value problem (1.5) and is symmetric over the interval $[\xi,T]_{\mathbb{T}}$.*

Definition 1.9. *If $u^{\Delta\nabla}(t) \le 0$ on the interval $[0, T]_{\mathbb{T}}$, then we say that u is concave on $[0, T]_{\mathbb{T}}$.*

Throughout this paper, we study the p-Laplacian boundary-value problem (1.5) under the following two assumptions:

(H1) $f\left(t,u,u^{\Delta}\right) : [0,T]_{\mathbb{T}} \times [0,\infty) \times (-\infty,\infty) \to [0,\infty)$ is left-dense continuous and $f(t,0,0)$ does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$ and is expressed as, for $t \in [\xi,T]_{\mathbb{T}}$,

$$f\left(t,u,u^{\Delta}\right) = f(T-(t-\xi), u(T-(t-\xi)), u^{\Delta}(T-(t-\xi)))$$

(H2) $h(t) \in C_{ld}([0,T]_{\mathbb{T}}, [0,\infty))$ is symmetric on $[\xi,T]_{\mathbb{T}}$ and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $C_{ld}([0,T]_{\mathbb{T}}, [0,\infty))$ denotes the set of all left-dense continuous functions from $[0,T]_{\mathbb{T}}$ to $[0,\infty)$.

2. Preliminaries

In this section, in order to prove our main results smoothly, we present some technical lemmas. After introducing the definition of cone and the fixed-point theorems, we show that solving the boundary-value problem (1.5) is actually equivalent to finding the fixed point of a completely continuous operator.

Definition 2.1. [22] *Let E be a real Banach space. A nonempty, closed, convex set $P \in E$ is said to be a cone provided the following two conditions are satisfied:*

(*i*) *If $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;*
(*ii*) *If $x \in P$ and $-x \in P$, then $x = 0$.*

Lemma 2.2. [23, 22] *Let P be a cone in a Banach space E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|$, *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, *for all* $x \in P \cap \partial\Omega_2$*;*
(ii) *or* $\|Ax\| \geq \|x\|$, *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, *for all* $x \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define the set $P(\gamma, d) = \{x \in P : \gamma(x) < d\}$ for each $d > 0$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P, respectively. We define the following convex sets as $P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x),\ \gamma(x) \leq d\}$, $P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x),\ \theta(x) \leq c,\ \gamma(x) \leq d\}$, and a closed set $R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x),\ \gamma(x) \leq d\}$.

Lemma 2.3. [7] *Let P be a cone in a real Banach space E, and γ, θ, α and ψ be defined as above. Moreover, ψ satisfies $\psi(\lambda' x) \leq \lambda' \psi(x)$ for $0 \leq \lambda' \leq 1$ such that, for some positive numbers h and d, we have $\alpha(x) \leq \psi(x)$ and $\|x\| \leq h\gamma(x)$, for all $x \in \overline{P(\gamma, d)}$. Suppose that A: $\overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive real numbers a, b, c with $a < b$ such that:*

(i) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(A(x)) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;
(ii) $\alpha(A(x)) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(A(x)) > c$;
(iii) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(A(x)) < a$ *for all* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that $\gamma(x_i) \leq d$ for $i = 1, 2, 3$, $b < \alpha(x_1)$ and $a < \psi(x_2)$, $\alpha(x_2) < b$ with $\psi(x_3) < a$.

Now we show that solving the boundary-value problem (1.5) is equivalent to finding fixed points of a completely continuous operator.

Firstly, problem (1.5) implies that $u^{\Delta}(T)$ exists. Choose the Banach space as $E = C^1_{ld}([0, T]_{\mathbb{T}}, \mathbb{R})$, where $C^1_{ld}([0, T]_{\mathbb{T}}, \mathbb{R})$ denotes the set of functions

$u \in C_{ld}\left([0,T]_{\mathbb{T}}, \mathbb{R}\right)$ which are differentiable and $u^{\Delta} \in C_{ld}\left([0,T]_{\mathbb{T}}, \mathbb{R}\right)$. Define the norm $\|u\|$ as $\|u\| = \max\left\{\|u\|_1, \|u\|_2\right\}$, where $\|u\|_1 = \sup_{[0,T]_{\mathbb{T}}} |x(t)|$, and $\|u\|_2 = \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|$. Let the cone $P \subset E$ be defined by

$$P = \left\{ u \in E : \begin{array}{c} u(t) \geq 0, \quad u(0) - \beta u^{\Delta}(\xi) = 0, \\ u \text{ is concave on } [0,T]_{\mathbb{T}} \text{ and symmetric on } [\xi,T]_{\mathbb{T}}. \end{array} \right\}.$$

Secondly, if $t \in [0,\omega_1]_{\mathbb{T}}$, we integrate the first equality in (1.5) from s to ω_1, where $\omega_1 = (\xi+T)/2$, and obtain

$$\varphi_p(u^{\Delta}(s)) = \varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_s^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r, \tag{2.1}$$

which implies

$$\varphi_p\left(u^{\Delta}(\omega_1)\right) = \varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) - \int_{\rho(\omega_1)}^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \Delta r. \tag{2.2}$$

Integrating equality (2.1) from 0 to t, and using (1.5) and (2.2) we have

$$\begin{aligned} u(t) &= \beta\varphi_q\Big(\varphi_p(u^{\Delta}(\omega_1)) + \int_{\xi}^{\omega_1} h(r) f(r,u(r),u^{\Delta}(r))\Delta r\Big) \\ &\quad + \int_0^t \varphi_q\Big(\varphi_p(u^{\Delta}(\omega_1)) + \int_s^{\omega_1} h(r) f(r,u(r),u^{\Delta}(r))\nabla r\Big)\Delta s, \\ &= \beta\varphi_q\Big(u^{\Delta}(\rho(\omega_1)) + \int_{\xi}^{\rho(\omega_1)} h(r) f(r,u(r),u^{\Delta}(r))\Delta r\Big) \\ &\quad + \int_0^t \varphi_q\Big(\varphi_p(u^{\Delta}(\rho(\omega_1))) + \int_s^{\rho(\omega_1)} h(r) f(r,u(r),u^{\Delta}(r))\Delta r\Big)\nabla s. \end{aligned}$$

If $t \in [\omega_1, T]_{\mathbb{T}}$, we integrate the first equality in (1.5) from ω_1 to s, and obtain

$$\varphi_p\left(u^{\Delta}(s)\right) - \varphi_p(u^{\Delta}(\omega_1)) = -\int_{\omega_1}^{s} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r,$$

which can be reduced to

$$u(T) - u(t) = \int_t^T \varphi_q\Big(\varphi_p(u^{\Delta}(\omega_1)) - \int_{\omega_1}^{s} h(r) f(r,u(r),u^{\Delta}(r))\nabla r\Big)\Delta s.$$

In view of $u(\xi) - \delta u^{\Delta}(\eta) = u(T) + \delta u^{\Delta}(T+\xi-\eta)$ and (2.2), we get

$$\begin{aligned} u(t) &= u(\xi) - \delta u^{\Delta}(\eta) - \delta u^{\Delta}(T+\xi-\eta) \\ &\quad - \int_t^T \varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) - \int_{\omega_1}^{s} h(r) f\left(r,u(r),u^{\Delta}(r)\right)\nabla r\Big)\Delta s, \end{aligned}$$

$$
\begin{aligned}
&= \beta\varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_\xi^{\omega_1} h(r)f\left(r, u(r), u^\Delta(r)\right)\Delta r\Big)\\
&+ \int_0^\xi \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_s^{\omega_1} h(r)f\left(r, u(r), u^\Delta(r)\right)\nabla r\Big)\Delta s\\
&+ \int_t^T \varphi_q\Big(-\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{\omega_1}^s h(r)f\left(r, u(r), u^\Delta(r)\right)\nabla r\Big)\Delta s,\\
&= \beta\varphi_q\Big(\varphi_p\left(u^\Delta(\rho(\omega_1))\right) + \int_\xi^{\rho(\omega_1)} h(r)f\left(r, u(r), u^\Delta(r)\right)\Delta r\Big)\\
&+ \int_0^\xi \varphi_q\Big(\varphi_p(u^\Delta(\rho(\omega_1))) + \int_s^{\rho(\omega_1)} h(r)f(r, u(r), u^\Delta(r))\Delta r\Big)\nabla s\\
&+ \int_t^T \varphi_q\Big(-\varphi_p(u^\Delta(\omega_1)) + \int_{\omega_1}^s h(r)f(r, u(r), u^\Delta(r))\Delta r\Big)\nabla s.
\end{aligned}
$$

If $u \in P$, we can derive two lemmas immediately as follows.

Lemma 2.4. *If $u \in P$, then the following statements are true:*

(i) $u(t) \geq \frac{u(w_1)}{\omega_1}\min\{t, T+\xi-t\} = \frac{\|u\|_1}{\omega_1}\min\{t, T+\xi-t\}$ *for* $t \in [0, T]_{\mathbb{T}}$*;*
(ii) $u(t) \geq \frac{\xi}{\omega_1}u(\omega_1) = \frac{\xi}{\omega_1}\|u\|_1$ *for* $t \in [\xi, \omega_1]_{\mathbb{T}}$;
(iii) $\max_{t\in[0,T]_{\mathbb{T}}} u(t) = u(\omega_1) = \|u\|_1$.

The proof is similar to that of Lemma 2.1 in reference [3], so we omit it.

Lemma 2.5. *If $u \in P$, then we have*

$$\|u\|_1 = \sup_{t\in[0,T]_{\mathbb{T}}} u(t) \leq (T+\beta)\sup_{t\in[0,T]_{\mathbb{T}}} |u^\Delta(t)| = (T+\beta)\|u\|_2.$$

Proof of Lemma 2.5. Since

$$u(t) = u(0) + \int_0^t u^\Delta(s)\Delta s,$$

it is easy to see that

$$
\begin{aligned}
\sup_{t\in[0,T]_{\mathbb{T}}} u(t) &= u(0) + \sup_{t\in[0,T]_{\mathbb{T}}}\int_0^t u^\Delta(s)\Delta s \leq \beta\left|u^\Delta(\xi)\right| + T\sup_{t\in[0,T]_{\mathbb{T}}}|u^\Delta(s)|\\
&\leq (\beta+T)\sup_{t\in[0,T]_{\mathbb{T}}}|u^\Delta(s)| = (T+\beta)\|u\|_2.
\end{aligned}
$$

□

To construct the completely continuous operator, we define the operator A: $P \to E$ by

$$(Au)(t) = \beta\varphi_q\Big(u^{\Delta}(\rho(\omega_1)) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u(r), u^{\Delta}(r)\right)\Delta r\Big) + \int_0^t \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + \int_s^{\rho(\omega_1)} h(r)f\left(r, u(r), u^{\Delta}(r)\right)\Delta r\Big)\nabla s$$

for $t \in [0, \omega_1]_{\mathbb{T}}$, or

$$(Au)(t) = \beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u(r), u^{\Delta}(r)\right)\Delta r\Big) + \int_0^{\xi} \varphi_q\Big(\varphi_p(u^{\Delta}(\rho(\omega_1))) + \int_s^{\rho(\omega_1)} h(r)f(r, u(r), u^{\Delta}(r))\Delta r\Big)\nabla s + \int_t^T \varphi_q\Big(-\varphi_p(u^{\Delta}(\omega_1)) + \int_{\omega_1}^s h(r)f(r, u(r), u^{\Delta}(r))\Delta r\Big)\nabla s$$

for $t \in [\omega_1, T]_{\mathbb{T}}$.

Lemma 2.6. *A: $P \to P$ is a completely continuous operator.*

Proof. It is easy to see that $(Au)(\xi) = (Au)(T)$, and $(Au)(0) - \beta(Au)^{\Delta}(\xi) = 0$. First, we show that $(Au)(t) \geq 0$ for $t \in [0, T]_{\mathbb{T}}$. If $\rho(\omega_1) = \omega_1$, by the pseudo-symmetry, we have $u^{\Delta}(\rho(\omega_1)) = 0$. If $\rho(\omega_1) < \omega_1$, then we have

$$u^{\Delta}(\rho(\omega_1)) = \tfrac{u(\omega_1) - u(\rho(\omega_1))}{\omega_1 - \rho(\omega_1)} > 0.$$

Combining these two cases, we have $u^{\Delta}(\rho(\omega_1)) \geq 0$. Hence, we find $(Au)(t) \geq 0$ for $t \in [0, \omega_1]_{\mathbb{T}}$.

If $\sigma(\omega_1) = \omega_1$, by the pseudo-symmetry, we have $u^{\Delta}(\omega_1) = 0$. If $\sigma(\omega_1) > \omega_1$, then we have

$$u^{\Delta}(\omega_1) = \tfrac{u(\sigma(\omega_1)) - u(\omega_1)}{\sigma(\omega_1) - \omega_1} < 0.$$

That is, in either case, $u^{\Delta}(\omega_1) \leq 0$. Thus, we find $(Au)(t) \geq 0$ for $t \in [\omega_1, T]_{\mathbb{T}}$.

Secondly, we prove that the operator A is pseudo-symmetric. Note that $\omega_1 = \frac{T+\xi}{2}$, so we have

$$\int_{s_1}^{T+\xi-\omega_1} h\left(r_1\right) f\left(r_1, u\left(r_1\right), u^{\Delta}\left(r_1\right)\right)\Delta r_1 = \int_{s_1}^{\omega_1} h\left(r_1\right) f\left(r_1, u\left(r_1\right), u^{\Delta}(r_1)\right)\Delta r_1.$$

Using the integral transform and substitutions $s = T - s_1 + \xi$ and $r = T - r_1 + \xi$ yields

$$\begin{aligned}&\int_{T-t+\xi}^{T} \varphi_q\Big(-\varphi_p(u^\Delta(\omega_1)) + \int_{\omega_1}^{s} h(r)f(r,u(r),u^\Delta(r))\Delta r\Big)\nabla s\\ &= -\int_{\xi}^{t} \varphi_q\Big(-\varphi_p(u^\Delta(\omega_1)) - \int_{s_1}^{T+\xi-\omega_1} h(r_1)f(r_1,u(r_1),u^\Delta(r_1))\Delta r_1\Big)\nabla s_1,\\ &= \int_{\xi}^{t} \varphi_q\Big(\varphi_p(u^\Delta(\omega_1)) + \int_{s_1}^{\omega_1} h\left(r_1\right) f\left(r_1, u\left(r_1\right), u^\Delta\left(r_1\right)\right)\Delta r_1\Big)\nabla s_1.\end{aligned}$$

In fact, for $t \in [\xi, \omega_1]_{\mathbb{T}}$, we have that $T - t + \xi \in [\omega_1, T]_{\mathbb{T}}$ and

$$\begin{aligned}&(Au)(T-t+\xi)\\ &= \beta\varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{\xi}^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\\ &+ \int_{0}^{\xi} \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{s}^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\nabla s\\ &+ \int_{T-t+\xi}^{T} \varphi_q\Big(-\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{\omega_1}^{s} h(r)f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\nabla s,\\ &= \beta\varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{\xi}^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\\ &+ \int_{0}^{\xi} \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{s}^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\nabla s\\ &+ \int_{\xi}^{t} \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_{s_1}^{\omega_1} h\left(r_1\right) f\left(r_1, u\left(r_1\right), u^\Delta\left(r_1\right)\right)\Delta r_1\Big)\nabla s_1\\ &= (Au)(t).\end{aligned}$$

For $t \in [\omega_1, T]_{\mathbb{T}}$, we note that $T - t + \xi \in [\xi, \omega_1]_{\mathbb{T}}$. Making use of the integral transform again gives

$$\begin{aligned}&\int_{\xi}^{T-t+\xi} \varphi_q\Big(\varphi_p(u^\Delta(\omega_1)) + \int_{s}^{\omega_1} h\left(r\right) f\left(r,u(r),u^\Delta(r)\right)\Delta r\Big)\nabla s\\ &= -\int_{t}^{T} \varphi_q\Big(-\varphi_p(u^\Delta(\omega_1)) - \int_{\omega_1}^{s_1} h\left(r_1\right) f\left(r_1,u(r_1),u^\Delta(r_1)\right)\Delta r_1\Big)\nabla s_1.\end{aligned}$$

Thus, we further deduce

$$
\begin{aligned}
&(Au)(T-t+\xi)=\beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_{\xi}^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\\
&+\int_0^{T-t+\xi}\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_s^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\nabla s\\
&=\beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_{\xi}^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\\
&+\int_0^{\xi}\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_s^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\nabla s\\
&+\int_{\xi}^{T-t+\xi}\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_s^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\nabla s\\
&=\beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_{\xi}^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\\
&+\int_0^{\xi}\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_s^{\omega_1}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\nabla s\\
&+\int_t^{T}\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right)+\int_{\omega_1}^{s_1}h\left(r_1\right)f\Big(r_1,u(r_1),u^{\Delta}(r_1)\Big)\Delta r_1\Big)\nabla s_1\\
&=(Au)(t).
\end{aligned}
$$

Consequently, A is pseudo-symmetric about ξ on $[0,T]_{\mathbb{T}}$.

Thirdly, we show that, for $t\in[0,\omega_1]_{\mathbb{T}}$,

$$
(Au)^{\nabla}(t)=\varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right)+\int_t^{\rho(\omega_1)}h(r)f\left(r,u(r),u^{\Delta}(r)\right)\Delta s\Big)\geq 0
$$

is continuous and nonincreasing in $[0,\omega_1]_{\mathbb{T}}$, where $\varphi_q(x)$ is a monotone increasing and continuously differentiable function. By using the inequality

$$
\Big(\int_t^{\omega_1}h(s)f(s,u(s),u^{\Delta}(s))\Delta s\Big)^{\Delta}=-h(t)f\left(t,u(t),u^{\Delta}(t)\right)\leq 0,\ t\in[0,\omega_1]_{\mathbb{T}},
$$

it is easy to find that $(Au)^{\nabla\Delta}(t)\leq 0$ for $t\in[0,\omega_1]_{\mathbb{T}}$. Using a similar argument, we can derive that $(Au)^{\nabla\Delta}(t)\leq 0$ for $t\in[\omega_1,T]_{\mathbb{T}}$. So, we obtain that $A:P\to P$.

Finally, we prove $A:P\to P$ is completely continuous.

Step 1. We show that A maps a bounded set into a bounded set. Assume that $c>0$ is a constant and $u\in\overline{P}_c=\{u\in P:\|u\|=\max\{\|u\|_1,\|u\|_2\}\leq c\}$.

Note that continuity of f guarantees that there is a $C > 0$ such that

$$f(t, u, u^{\Delta}) \leq \varphi_p(C) \quad \text{for } u \in \overline{P}_c.$$

If $t \in [0, \omega_1]_{\mathbb{T}}$, it follows from Lemma 2.5 that

$$\begin{aligned}
&\|Au\| \leq \max\{T+\beta, 1\} \left|(Au)^{\Delta}(0)\right| \\
&= \max\{T+\beta, 1\} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_0^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r\Big) \\
&\leq \max\{T+\beta, 1\} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + \int_0^{\rho(\omega_1)} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r\Big) \\
&\leq \max\{T+\beta, 1\} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r) \nabla r\Big).
\end{aligned}$$

Consequently, $A\overline{P}_c$ is uniformly bounded. When $t_1, t_2 \in [0, \omega_1]_{\mathbb{T}}$, we have

$$\begin{aligned}
&|(Au)(t_1) - (Au)(t_2)| \\
&= \Big| \int_{t_1}^{t_2} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + \int_s^{\rho(\omega_1)} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r\Big) \Delta s \Big| \\
&\leq \Big| \int_{t_1}^{t_2} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r) \nabla r\Big) \Delta s \Big| \\
&\leq |t_2 - t_1| \, \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r) \nabla r\Big).
\end{aligned}$$

When $t_1, t_2 \in [\omega_1, T]_{\mathbb{T}}$, using a closely similar technique we can obtain an analogous inequality. Hence, it follows from the Arzela-Ascoli theorem on time scales that $A\overline{P}_c$ is relatively compact.

Step 2. We claim that A: $\overline{P}_c \to P$ is continuous. Assume that $\{u_n\}_{n=1}^{\infty} \subset \overline{P}_c$ and $\|u_n - u_0\| \to 0$ for $t \in [0, T]_{\mathbb{T}}$, which implies $|u_n - u_0| \to 0$ and $|u_n^{\Delta} - u_0^{\Delta}| \to 0$ for $t \in [0, T]_{\mathbb{T}}$. Thus $\{(Au_n)(t)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0, T]_{\mathbb{T}}$. The Arzela-Ascoli theorem on time scales implies that there exists a uniformly convergent subsequence in $\{(Au_n)(t)\}_{n=1}^{\infty}$. Letting $\left\{\left(Au_{n(m)}\right)(t)\right\}_{m=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0, T]_{\mathbb{T}}$, we have

$$\begin{aligned}
0 \leq (Au_n)(t) &\leq \beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r) \nabla r\Big) \\
&+ \int_0^{\omega_1} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r) \Delta r\Big) \nabla s \quad \text{for } t \in [0, \omega_1]_{\mathbb{T}},
\end{aligned}$$

and

$$0 \le (Au_n)(t) \le (T+\beta)\varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right) + C\int_0^{\rho(\omega_1)} h(r)\nabla r\Big)$$
$$+ \int_{\omega_1}^{T} \varphi_q\Big(-\varphi_p\left(u^{\Delta}(\omega_1)\right) + C\int_{\omega_1}^{T} h(r)\Delta r\Big)\nabla s \quad \text{for } t \in [\omega_1, T]_{\mathbb{T}}\,.$$

We can observe that

$$(Au_n)(t) = \beta\varphi_q\Big(u_n^{\Delta}(\rho(\omega_1)) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u_n(r), u_n^{\Delta}(r)\right)\Delta r\Big)$$
$$+ \int_0^{t} \varphi_q\Big(\varphi_p\left(u_n^{\Delta}(\rho(\omega_1))\right) + \int_{s}^{\rho(\omega_1)} h(r)f\left(r, u_n(r), u_n^{\Delta}(r)\right)\Delta r\Big)\nabla s$$

for $t \in [0, \omega_1]_{\mathbb{T}}$, or

$$(Au_n)(t) = \beta\varphi_q\Big(\varphi_p\left(u_n^{\Delta}(\rho(\omega_1))\right) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u_n(r), u_n^{\Delta}(r)\right)\Delta r\Big)$$
$$+ \int_0^{\xi} \varphi_q\Big(\varphi_p(u_n^{\Delta}(\rho(\omega_1))) + \int_{s}^{\rho(\omega_1)} h(r)f(r, u_n(r), u_n^{\Delta}(r))\Delta r\Big)\nabla s$$
$$+ \int_t^{T} \varphi_q\Big(-\varphi_p(u_n^{\Delta}(\omega_1)) + \int_{\omega_1}^{s} h(r)f(r, u_n(r), u_n^{\Delta}(r))\Delta r\Big)\nabla s$$

for $t \in [\omega_1, T]_{\mathbb{T}}$. By the Lebesgue dominated convergence theorem on time scales, we insert $u_{n(m)}$ into the above equality and let m go to infinity, then we have

$$v(t) = \beta\varphi_q\Big(u_0^{\Delta}(\rho(\omega_1)) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u_0(r), u_0^{\Delta}(r)\right)\Delta r\Big)$$
$$+ \int_0^{t} \varphi_q\Big(\varphi_p\left(u_0^{\Delta}(\rho(\omega_1))\right) + \int_{s}^{\rho(\omega_1)} h(r)f\left(r, u_0(r), u_0^{\Delta}(r)\right)\Delta r\Big)\nabla s$$

for $t \in [0, \omega_1]_{\mathbb{T}}$ or

$$v(t) = \beta\varphi_q\Big(\varphi_p\left(u_0^{\Delta}(\rho(\omega_1))\right) + \int_{\xi}^{\rho(\omega_1)} h(r)f\left(r, u_0(r), u_0^{\Delta}(r)\right)\Delta r\Big)$$
$$+ \int_0^{\xi} \varphi_q\Big(\varphi_p(u_0^{\Delta}(\rho(\omega_1))) + \int_{s}^{\rho(\omega_1)} h(r)f(r, u_0(r), u_0^{\Delta}(r))\Delta r\Big)\nabla s$$

$$+\int_t^T \varphi_q\Big(-\varphi_p(u_0^\Delta(\omega_1))+\int_{\omega_1}^s h(r)f(r,u_0(r),u_0^\Delta(r))\Delta r\Big)\nabla s$$

for $t\in[\omega_1,T]_{\mathbb{T}}$. From the definition of A, we know that $v(t)=Au_0(t)$ on $[0,T]_{\mathbb{T}}$. This shows that each subsequence of $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $(Au_0)(t)$. Thus the sequence $\{(Au_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $(Au_0)(t)$, which implies that A is continuous at $u_0\in\overline{P}_c$. In other words, A is continuous on $\overline{P}_c$ since u_0 is arbitrary. Consequently, A is completely continuous. □

Therefore, all solutions of the boundary-value problem (1.5) are fixed points of the completely continuous operator A.

3. Single or twin solutions

In this section, we study the existence of single or twin positive pseudo-symmetric solutions to problem (1.5) by means of Krasnosel'skii's fixed-point theorem in a cone.

Theorem 3.1. *Suppose that the following two conditions hold:*

(i) *there exists a constant* $p^*>0$ *such that*

$$f\left(t,u,u^\Delta\right)\le\varphi_p(p^*\Lambda_1)\quad \text{for } \left(t,u,u^\Delta\right)\in[0,\omega_1]_{\mathbb{T}}\times[0,p^*]\times[-p^*,p^*],$$

where

$$\Lambda_1=\Big(\max\{T+\beta,1\}\,\varphi_q\Big(\int_0^{\omega_1}h(r)\nabla r\Big)\Big)^{-1};$$

(ii) *there exists a constant* $q^*>0$ *such that*

$$f\left(t,u,u^\Delta\right)\ge\varphi_p(q^*\Lambda_2)$$

for $\left(t,u,u^\Delta\right)\in[\xi,\rho(\omega_1)]_{\mathbb{T}}\times[0,q^*]\times[-q^*,q^*]$, *where*

$$\Lambda_2=\Big(\varphi_q\Big(\int_\xi^{\rho(\omega_1)}h(r)\nabla r\Big)\Big)^{-1}$$

and $p^*\ne q^*$. *Then problem* (1.5) *has at least one positive pseudo-symmetric solution* u *such that* $\|u\|$ *lies between* p^* *and* q^*.

Proof. Without loss of generality, we may assume that $p^*<q^*$. Let

$$\Omega_{p^*}=\{u\in E:\|u\|<p^*\}.$$

For any $u\in P\cap\partial\Omega_{p^*}$, since

$$|u^\Delta(t)|\le\|u\|<p^*\text{ and }u(t)\le p^*\text{ for }t\in[0,T]_{\mathbb{T}},$$

by virtue of Lemma 2.5 and condition (i), we have

$$\begin{aligned}\|Au\| &\le \max\{T+\beta,1\}|(Au)^{\Delta}(0)|\\ &\le \max\{T+\beta,1\}\varphi_q\Big(\int_0^{\omega_1} h(r)f\left(r,u(r),u^{\Delta}(r)\right)\nabla r\Big)\\ &\le p^*\Lambda_1 \max\{T+\beta,1\}\,\varphi_q\Big(\int_0^{\omega_1} h(r)\nabla r\Big), = p^*,\end{aligned}$$

which implies that

$$\|Au\| \le \|u\| \text{ for } u\in P\cap\partial\Omega_{p^*}. \tag{3.1}$$

Then, we set $\Omega_{q^*}=\{u\in E:\|u\|<q^*\}$. For $u\in P\cap\partial\Omega_{q^*}$, since $u(t)\le q^*$, and $|u^{\Delta}(t)|\le\|u\|\le q^*$, using condition (ii) we have

$$\begin{aligned}\|Au\| \ge &|(Au)^{\Delta}(0)| \ge \varphi_q\Big(\varphi_p\left(u^{\Delta}(\rho(\omega_1))\right)+\int_{\xi}^{\rho(\omega_1)} h(r)f\left(r,u(r),u^{\Delta}(r)\right)\nabla r\Big)\\ &\ge q^*\Lambda_2\varphi_q\Big(\int_{\xi}^{\rho(\omega_1)} h(r)\nabla r\Big)=q^*.\end{aligned}$$

If we take $\Omega_{q^*}=\{u\in E:\|u\|<q^*\}$, then we have

$$\|Au\|\ge\|u\|,\quad u\in P\cap\partial\Omega_{q^*}. \tag{3.2}$$

In view of $p^*<q^*$, and inequalities (3.1) and (3.2), it follows from Lemma 2.2 that problem (1.5) has a positive pseudo-symmetric solution u in $P\cap(\overline{\Omega}_{q^*}\backslash\Omega_{p^*})$. □

For $u\in P$, we denote

$$\begin{aligned} f^0 &= \lim_{(u_1,u_2)\to(0,0)}\ \sup_{t\in[0,T]_{\mathbb{T}}}\frac{f(t,u_1,u_2)}{\varphi_p(|u_2|)},\\ f_\infty &= \lim_{u_1+|u_2|\to\infty}\ \inf_{t\in[0,T]_{\mathbb{T}}}\frac{f(t,u_1,u_2)}{\varphi_p(u_1+|u_2|)},\\ f_0 &= \lim_{(u_1,u_2)\to(0,0)}\ \inf_{t\in[0,T]_{\mathbb{T}}}\frac{f(t,u_1,u_2)}{\varphi_p(u_1+|u_2|)},\\ f^\infty &= \lim_{(u_1,u_2)\to(\infty,\infty)}\ \sup_{t\in[0,T]_{\mathbb{T}}}\frac{f(t,u_1,u_2)}{\varphi_p(|u_2|)}.\end{aligned}$$

Now we present our first result on the p-Laplacian boundary-value problem (1.5).

Theorem 3.2. *Suppose that* $f^0 \in [0, \varphi_p(\Lambda_1))$ *and* $f_\infty \in (\varphi_p(\Lambda_2), \infty) \cup \{\infty\}$. *Then problem* (1.5) *has at least one positive pseudo-symmetric solution.*

Proof. Since $f^0 < \varphi_p(\Lambda_1)$, there exists a sufficiently small $p^* > 0$ such that

$$\begin{aligned} f\left(t, u, u^\Delta\right) &\leq \varphi_p(\Lambda_1)\,\varphi_p\left(|u^\Delta|\right) \\ &\leq \varphi_p(\Lambda_1 p^*) \text{ for } \left(t, u, u^\Delta\right) \in [0, T]_{\mathbb{T}} \times [0, p^*] \times [-p^*, p^*]. \end{aligned}$$

This implies that condition (i) in Theorem 3.1 holds. Thus if we take $\Omega_{p^*} = \{u \in E : \|u\| < p^*\}$, then inequality (3.1) holds.

According to $f_\infty > \varphi_p(\Lambda_2)$, there exists an $H > 4p^*$ such that

$$f\left(t, u, u^\Delta\right) \geq \varphi_p(\Lambda_2)\varphi_p(u + |u^\Delta|) \geq \varphi_p(\Lambda_2)\varphi_p(\|u\|), \tag{3.3}$$

where $t \in [0, T]_{\mathbb{T}}$ and $u + \left|u^\Delta\right| \geq H$. Set $\Omega_H = \left\{u \in E : u + \left|u^\Delta\right| < H\right\}$. It is easy to see that $\overline{\Omega}_{p^*} \subset \Omega_H$. For $u \in P \cap \partial\Omega_H$, we have $u + \left|u^\Delta\right| = H$. Hence, using inequality (3.3) we have

$$\begin{aligned} \|Au\| \geq |(Au)^\Delta(0)| &\geq \varphi_q\Big(\varphi_p(u^\Delta(\omega_1)) + \int_0^{\omega_1} h(r) f(r, u(r), u^\Delta(r))\nabla r\Big) \\ &\geq \varphi_q\Big(\varphi_p \int_\xi^{\rho(\omega_1)} h(r) f\left(r, u(r), u^\Delta(r)\right)\nabla r\Big) \\ &\geq \Lambda_2 \|u\|\, \varphi_q\Big(\varphi_p \int_\xi^{\rho(\omega_1)} h(r)\nabla r\Big) = \|u\|. \end{aligned}$$

It follows from Lemma 2.2 that problem (1.5) has a positive pseudo-symmetric solution u in $P \cap (\overline{\Omega}_H \backslash \Omega_{p^*})$. Consequently, the proof is completed. □

Theorem 3.3. *Suppose that* $f_0 \in (\varphi_p(\Lambda_2), \infty) \cup \{\infty\}$ *and* $f^\infty \in [0, \varphi_p(\Lambda_1))$. *Then problem* (1.5) *has at least one positive pseudo-symmetric solution.*

Proof. Since $f_0 > \varphi_p(\Lambda_2)$, there exists a sufficiently small $q^* > 0$ such that

$$f\left(t, u, u^\Delta\right) \geq \varphi_p(\Lambda_2)\,\varphi_p(\max\{\|u\|_1, \|u\|_2\})$$

for $\left(t, u, u^\Delta\right) \in [0, T]_{\mathbb{T}} \times [0, q^*] \times [-q^*, q^*]$. If $\left(t, u, u^\Delta\right) \in [\xi, \rho(\omega_1)]_{\mathbb{T}} \times [0, q^*] \times [-q^*, q^*]$, we have

$$f\left(t, u, u^\Delta\right) \geq \varphi_p(\Lambda_2)\,\varphi_p(\max\{\|u\|_1, \|u\|_2\}) = \varphi_p(\Lambda_2 q^*),$$

which indicates that condition (ii) in Theorem 3.1 holds. That is, if we take $\Omega_{q^*} = \{u \in E : \|u\| < q^*\}$, then inequality (3.2) holds.

Note that $f^{\infty} < \varphi_p(\Lambda_1)$. We let $\varepsilon_1 = \varphi_p(\Lambda_1) - f^{\infty}$, then there exists a p_1 $(> q^*)$ such that

$$\frac{f\left(t,u,u^{\Delta}\right)}{\varphi_p(|u^{\Delta}|)} \leq \varepsilon_1 + f^{\infty} = \varphi_p(\Lambda_1), \tag{3.4}$$

where $\left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times [p_1,\infty) \times (-\infty,-p_1] \cup [p_1,+\infty)$.

Since $f \in C_{ld}([0,T]_{\mathbb{T}} \times [0,\infty) \times (-\infty,\infty),[0,\infty))$, there exists a constant $C_4 > 0$ such that

$$f\left(t,u,u^{\Delta}\right) \leq \varphi_p(C_4) \text{ for } \left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times [0,p_1] \times [-p_1,p_1].$$

By virtue of inequality (3.4), when $\left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times [0,\infty) \times (-\infty,\infty)$ we have

$$f\left(t,u,u^{\Delta}\right) \leq \max\left\{\varphi_p(C_4), \varphi_p(\Lambda_1)\varphi_p\left(|u^{\Delta}|\right)\right\}.$$

Let $p_2^* > \max\{C_4/\Lambda_1, 2q^*\}$, and $\Omega_{p_2^*} = \{u \in E : \|u\| < p_2^*\}$. If $u \in P \cap \partial\Omega_{p_2^*}$, we have $\|u\| = p_2^*$ and

$$\begin{aligned}
\|Au\| &\leq \max\{T+\beta,1\}\varphi_q\Big(\int_0^{\omega_1} h(r)f\left(r,u(r),u^{\Delta}(r)\right)\nabla r\Big)\\
&\leq \max\{T+\beta,1\}\varphi_q\Big(\int_0^{\omega_1} h(r)\max\{\varphi_p(C_4),\varphi_p(\Lambda_1)\varphi_p(|u^{\Delta}(r)|)\}\nabla r\Big)\\
&\leq \max\{T+\beta,1\}\varphi_q\Big(\int_0^{\omega_1} h(r)\max\{\varphi_p(C_4),\varphi_p(\Lambda_1)\varphi_p(\|u\|)\}\nabla r\Big)\\
&\leq \max\{T+\beta,1\}\Lambda_1\|u\|\varphi_q\Big(\int_0^{\omega_1} h(s)\Delta s\Big) \leq \|u\|.
\end{aligned}$$

By virtue of Lemma 2.2, we can conclude that problem (1.5) has at least one positive pseudo-symmetric solution. Hence, the proof is completed. □

Next, we present two theorems on the existence of at least two distinct positive pseudo-symmetric solutions to problem (1.5).

Theorem 3.4. *Suppose that $f_0 = \infty$ and $f_\infty = \infty$, and condition* (i) *of* Theorem 3.1 *hold; then problem* (1.5) *has at least two distinct positive pseudo-symmetric solutions u_1, $u_2 \in P$.*

Proof. Since $f_0 = \infty$, there exists an H_1 such that $0 < H_1 < p^*$ and

$$f\left(t,u,u^{\Delta}\right) \geq \varphi_p(m)\varphi_p(u+|u^{\Delta}|) \geq \varphi_p(m)\varphi_p(\|u\|), \tag{3.5}$$

where $\left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times (0,H_1] \times [-H_1,H_1]$ and m is taken to satisfy

$$m\varphi_q\Big(\int_{\xi}^{\rho(\omega_1)} h(r)\nabla r\Big) \geq 1. \tag{3.6}$$

If $u \in P$ with $\|u\| = H_1$, then we get $|u^\Delta(t)| \leq \|u\| = H_1$ for $t \in [0,T]_{\mathbb{T}}$, and $u(t) \leq H_1$ for $t \in [0,T]_{\mathbb{T}}$. From (3.5) and (3.6), we have

$$\begin{aligned}\|Au\| &\geq |(Au)^\Delta(0)| \geq \varphi_q\Big(\varphi_p(u^\Delta(\rho(\omega_1))) + \int_\xi^{\rho(\omega_1)} h(r)f(r,u(r),u^\Delta(r))\nabla r\Big)\\ &\geq m\|u\|\varphi_q\Big(\int_\xi^{\rho(\omega_1)} h(r)\nabla r\Big) = \|u\|\,.\end{aligned}$$

Letting $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$, we deduce $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{H_1}$. By condition (i) of Theorem 3.1, if we take $\Omega_{p^*} = \{u \in E : \|u\| < p^*\}$, then inequality (3.1) holds. It follows from Lemma 2.2 that problem (1.5) has a positive pseudo-symmetric solution u_1 in $P \cap (\overline{\Omega}_{p^*} \backslash \Omega_{H_1})$.

On the other hand, since $f_\infty = \infty$, there exists an $H_2 > 4p^*$ such that

$$f\left(t,u,u^\Delta\right) \geq \varphi_p(k)\varphi_p(u + |u^\Delta|) \geq \varphi_p(k)\varphi_p\left(\|u\|\right), \tag{3.7}$$

where $t \in [0,T]_{\mathbb{T}}$ and $u + \left|u^\Delta\right| \geq H_2$. In addition, k satisfies

$$k\varphi_q\Big(\int_\xi^{\rho(\omega_1)} h(r)\nabla r\Big) \geq 1.$$

Let $\Omega_{H_2} = \left\{u \in \Omega_H : u + |u^\Delta| < H_2\right\}$. It is easy to see that $\overline{\Omega}_{p^*} \subset \Omega_{H_2}$. For $u \in P \cap \partial\Omega_{H_2}$, we have $u + |u^\Delta| = H_2$. Using inequality (3.7), we get

$$\begin{aligned}\|Au\| &\geq |(Au)^\Delta(0)| \geq \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_0^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\nabla r\Big)\\ &\geq \varphi_q\Big(\varphi_p\left(u^\Delta(\omega_1)\right) + \int_\xi^{\omega_1} h(r)f\left(r,u(r),u^\Delta(r)\right)\nabla r\Big)\\ &\geq \varphi_q\Big(\varphi_p\left(u^\Delta(\rho(\omega_1))\right) + \int_\xi^{\rho(\omega_1)} h(r)f\left(r,u(r),u^\Delta(r)\right)\nabla r\Big)\\ &\geq \varphi_q\Big(\int_\xi^{\rho(\omega_1)} h(r)f\left(r,u(r),u^\Delta(r)\right)\nabla r\Big)\\ &\geq k\|u\|\varphi_q\Big(\int_\xi^{\rho(\omega_1)} h(r)\nabla r\Big) \geq \|u\|\,.\end{aligned}$$

Thus, by virtue of condition (i) of Lemma 2.2, problem (1.5) has at least a single positive pseudo-symmetric solution u_2 in $P \cap (\overline{\Omega}_{H_2} \backslash \Omega_{p^*})$ with $p^* \leq \|u_2\|$ and $u_2 + |u_2^\Delta| \leq H_2$. It is apparent that u_1 and u_2 are distinct. Consequently, we have completed the proof of Theorem 3.4. $\square$

Theorem 3.5. *Suppose that* $f^0 = 0$, $f^\infty = 0$, *and condition* (ii) *of* Theorem 3.1 *hold; then problem* (1.5) *has at least two distinct positive pseudo-symmetric solutions* u_1, $u_2 \in P$.

The proof of this theorem can be carried out by using a closely similar argument to that of Theorem (3.4), so we omit it.

4. Multiple solutions

In the preceding section, some results are established on the existence of at least one or two positive pseudo-symmetric solutions to problem (1.5). In this section, we will further discuss the existence criteria for at least three and arbitrary odd positive pseudo-symmetric solutions to problem (1.5) by using the Avery-Peterson fixed-point theorem [7].

Letting $r \in (\eta, \omega_1)_{\mathbb{T}}$, for notational convenience, we denote

$$lM = (\beta + \omega_1)\varphi_q\Big(\int_0^{\omega_1} h(r)\nabla r\Big), \quad N = \xi\varphi_q\Big(\int_\xi^{\rho(\omega_1)} h(r)\nabla r\Big),$$

and

$$W = \max\{T + \beta,\, 1\}\,\varphi_q\Big(\int_0^{\omega_1} h(r)\nabla r\Big).$$

Define the nonnegative continuous convex functionals θ and γ, the nonnegative continuous concave functional α, and the nonnegative continuous functional ψ on P respectively by $\gamma(u) = \sup_{t\in[0,\omega_1]_{\mathbb{T}}} |u^\Delta(t)|$, $\psi(u) = \theta(u) = \max_{t\in[0,\omega_1]_{\mathbb{T}}} u(t) = u(\omega_1)$, and $\alpha(u) = \min_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u(t) = u(\xi)$.

Now we present our first result on the existence of three positive pseudo-symmetric solutions of problem (1.5).

Theorem 4.1. *Suppose that there exist constants* a^*, b^* *and* d^* *such that* $0 < a^* < b^* < \xi d^*$. *In addition,* f *satisfies the following conditions:*

- (i) $f(t, u, u^\Delta) \le \varphi_p(\frac{d^*}{W})$ *when* $(t, u, u^\Delta) \in [0, \omega_1]_{\mathbb{T}} \times [0, (T + \beta)d^*] \times [-d^*, d^*]$;
- (ii) $f(t, u, u^\Delta) > \varphi_p(\frac{b^*}{N})$ *when* $(t, u, u^\Delta) \in [\xi, \rho(\omega_1)]_{\mathbb{T}} \times [b^*, (T + \beta)d^*] \times [-d^*, d^*]$;
- (iii) $f(t, u, u^\Delta) < \varphi_p(\frac{a^*}{M})$ *when* $(t, u, u^\Delta) \in [0, \omega_1]_{\mathbb{T}} \times [0, a^*] \times [-d^*, d^*]$.

Then problem (1.5) *has at least three positive pseudo-symmetric solutions* u_1, u_2 *and* u_3 *such that*

$$\sup_{t\in[0,\omega_1]_{\mathbb{T}}} |u_i^\Delta(t)| \le d^*,\ i = 1, 2, 3, \quad b^* < \min_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_1(t), \quad a^* < \max_{t\in[0,\omega_1]_{\mathbb{T}}} u_2(t),$$

$$\min_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_2(t) < b^* \textit{ with } \max_{t\in[0,\omega_1]_{\mathbb{T}}} u_3(t) < a^*.$$

Proof. By the definition of the completely continuous operator A and its properties, it suffices to show that all conditions of Lemma 2.3 hold with respect to A. It is easy to verify that

$$\psi(\lambda' u) = \lambda' u(\omega_1) = \lambda'\psi(u) \text{ for } 0 < \lambda' < 1, \qquad \alpha(u) \le \psi(u) \text{ for all } u \in P$$

and

$$\|u\| \le \max\{T+\beta, 1\} \sup_{t\in[0,T]_{\mathbb{T}}} |u^{\Delta}(t)| = \max\{T+\beta,1\}\,\gamma(u) \text{ for all } u \in P.$$

Step 1. We show that A: $\overline{P(\gamma, d^*)} \to \overline{P(\gamma, d^*)}$. For any $u \in \overline{P(\gamma, d^*)}$, we have $u(t) \le (T+\beta)\gamma(u) \le (T+\beta)d^*$ and $|u^{\Delta}(t)| \le \gamma(u) \le d^*$for $t \in [0,T]_{\mathbb{T}}$. Hence, assumption (i) implies that

$$\begin{aligned}\|Au\| &\le \max\{T+\beta,1\}\left|(Au)^{\Delta}(0)\right| \\ &\le \max\{T+\beta,1\}\,\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_0^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right)\Big) \\ &\le \frac{d^*}{W}\max\{T+\beta,1\}\,\varphi_q\Big(\int_0^{\omega_1} h(r)\nabla r\Big) = d^*.\end{aligned}$$

Based on the above analysis, next we wish to show that all conditions (i)-(iii) of Lemma 2.3 hold for A.

Step 2. We verify that condition (i) of Lemma 2.3 holds. Letting $u(t) \equiv \frac{tb^*}{\xi} + b^*$, $t \in [0,T]_{\mathbb{T}}$, one can see that $\alpha(u) = u(\xi) = 2b^* > b^*$, and $\theta(u) = u(\omega_1) = \frac{\omega_1 b^*}{\xi} + b^* \le \frac{\omega_1 b^*}{\xi} + b^*$. Note that $\gamma(u) = \frac{b^*}{\xi} < d^*$, thus, we conclude

$$\left\{u \in P(\gamma,\theta,\alpha,b^*,\frac{\omega_1 b^*}{\xi} + b^*, d^*) : \alpha(x) > b^*\right\} \ne \emptyset.$$

For any $u \in P(\gamma,\theta,\alpha,b^*,\frac{\omega_1 b^*}{\xi} + b^*, d^*)$, we deduce that

$$|u^{\Delta}(t)| \le \gamma(u) = d^* \text{ for all } t \in [\xi, \rho(\omega_1)]_{\mathbb{T}},$$

$$b^* \le \alpha(u) \le u(t) \le (T+\beta)\gamma(u) = (T+\beta)d^* \text{ for all } t \in [\xi,\rho(\omega_1)]_{\mathbb{T}},$$

and

$$\begin{aligned}\alpha(Au) = (Au)(\xi) &= \beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_\xi^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right)\Delta r\Big) \\ &\quad + \int_0^{\xi} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_s^{\omega_1} h(r) f\left(r,u(r),u^{\Delta}(r)\right)\Delta r\Big)\nabla s\end{aligned}$$

$$\geq \int_0^{\xi} \varphi_q\Big(\int_{\xi}^{\rho(\omega_1)} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \nabla r\Big) \Delta s$$
$$> \frac{b^*}{N} \xi \varphi_q\Big(\int_{\xi}^{\rho(\omega_1)} h(r) \nabla r\Big) = b^*.$$

Step 3. We prove that condition (ii) of Lemma 2.3 holds. In fact, we find $\alpha(Au) = Au(\xi)$, and $\theta(Au) = \max_{t\in[0,\omega_1]_{\mathbb{T}}} A(u) = Au(\omega_1)$. For any $u \in P(\gamma, \alpha, b^*, d^*)$ with $\theta(Au) > \frac{\omega_1 b^*}{\xi} + b^*$, Lemma 2.4 implies that

$$\alpha(Au) = Au(\xi) \geq \frac{\xi}{\omega_1}\theta(Au) > b^* + \frac{\omega_1 b^*}{\xi} > b^*.$$

Step 4. We are left to check condition (iii) of Lemma 2.3. Since $\psi(0) = 0 < a^*$, we have $0 \notin R(\gamma, \psi, a^*, d^*)$. If u satisfies $u \in R(\gamma, \psi, a^*, d^*)$, with $\psi(u) = \max_{t\in[0,\omega_1]_{\mathbb{T}}} u(t) = u(\omega_1) = a^*$, then we have $0 \leq u(t) \leq a^*$ for all $t \in [0, \omega_1]_{\mathbb{T}}$, and $\max_{t\in[0,T]_{\mathbb{T}}} |u^{\Delta}(t)| = \gamma(u) \leq d^*$. Furthermore, we have

$$\begin{aligned}
\psi(Au) = (Au)(\omega_1) &= \beta\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_{\xi}^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \Delta r\Big) \\
&+ \int_0^{\omega_1} \varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_s^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \Delta r\Big) \nabla s \\
&\leq (\beta + \omega_1)\varphi_q\Big(\varphi_p\left(u^{\Delta}(\omega_1)\right) + \int_0^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \Delta r\Big) \\
&\leq (\beta + \omega_1)\varphi_q\Big(\int_0^{\omega_1} h(r) f\left(r, u(r), u^{\Delta}(r)\right) \Delta r\Big) \\
&\leq (\beta + \omega_1)\frac{a^*}{M}\varphi_q\Big(\int_0^{\omega_1} h(r) \Delta r\Big) = a^*.
\end{aligned}$$

Consequently, all three conditions of Lemma 2.3 are satisfied. By virtue of Lemma 2.3, we have completed the proof of Theorem 4.1. □

Here we remark that condition (i) in Theorem 4.1 can actually be replaced by

$$\text{(i}'\text{)} \qquad \lim_{(u,u^{\Delta})\to(\infty,\infty)} \sup_{t\in[0,T]_{\mathbb{T}}} \frac{f\left(t, u, u^{\Delta}\right)}{\varphi_p\left(\min\{u, |u^{\Delta}|\}\right)} \leq \varphi_p\left(\tfrac{1}{W}\right).$$

Thus we obtain a corollary as follows.

Corollary 4.2. *If condition* (i) *of Theorem* 4.1 *is replaced by the above condition* (i'), *then the conclusion of Theorem* 4.1 *holds too.*

Proof. From Theorem 4.1, we only need to prove that the above condition (i') implies that condition (i) of Theorem 4.1 holds. That is, if (i') holds, then there exists a number $d^* \geq \frac{1}{\xi} b^*$ such that

$$f\left(t,u,u^{\Delta}\right) \leq \varphi_p\left(\frac{d^*}{W}\right) \text{ for } \left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times [0,(T+\beta)d^*] \times [-d^*,d^*].$$

Suppose on the contrary that for any $d^* \geq \frac{1}{\xi} b^*$ there exists $(u_c, u_c^{\Delta}) \in [0,(T+\beta)d^*] \times [-d^*,d^*]$ such that

$$f\left(t,u_c,u_c^{\Delta}\right) > \varphi_p\left(\frac{d^*}{W}\right) \text{ for } t \in [0,T]_{\mathbb{T}}.$$

Hence, one can choose $c_n^* > \frac{1}{\xi} b^* (n = 1, 2, \ldots)$ with $c_n^* \to \infty$, such that there exists $(u_n, u_n^{\Delta}) \in [0,(T+\beta)c_n^*] \times [-c_n^*, c_n^*]$ which satisfies

$$f\left(t,u_n,u_n^{\Delta}\right) > \varphi_p\left(\frac{c_n^*}{W}\right) \text{ for } t \in [0,T]_{\mathbb{T}}, \tag{4.1}$$

and

$$\lim_{n\to\infty} f(t,u_n,u_n^{\Delta}) = \infty \text{ for } t \in [0,T]_{\mathbb{T}}. \tag{4.2}$$

Since condition (i') holds, for $\left(t,u,u^{\Delta}\right) \in [0,T]_{\mathbb{T}} \times [(T+\beta)\tau,\infty) \times (-\infty,\tau] \cup [\tau,\infty)$, there exists a number $\tau > 0$ such that

$$f\left(t,u,u^{\Delta}\right) \leq \varphi_p\left(\frac{\min\{u,|u^{\Delta}|\}}{W}\right). \tag{4.3}$$

Thus we have $|u_n^{\Delta}(t)| \leq \tau$ for $t \in [0,T]_{\mathbb{T}}$, which implies $u_n(t) \leq (T+\beta)\tau$ for $t \in [0,T]_{\mathbb{T}}$. Otherwise, if we have $|u_n^{\Delta}(t)| > \tau$ and $u_n(t) > (T+\beta)\tau$ for $t \in [0,T]_{\mathbb{T}}$, it follows from (4.3) that

$$f(t,u_n,u_n^{\Delta}) \leq \varphi_p\left(\frac{u_n}{W}\right) \leq \varphi_p\left(\frac{c_n^*}{W}\right) \text{ for } t \in [0,T]_{\mathbb{T}}.$$

This yields a contradiction with inequality (4.1).

Letting $W^* = \max_{(t,u,u^{\Delta}) \in [0,T]_{\mathbb{T}^\kappa} \times [0,(T+\beta)\tau] \times [-\tau,\tau]} f\left(t,u,u^{\Delta}\right)$, we deduce

$$f(t,u_n,u_n^{\Delta}) \leq W^* (n = 1, 2, \ldots),$$

which contradicts inequality (4.2). Consequently, combining the above analysis we have completed the proof. □

Theorem 4.3. *Suppose that there exist constants a_i^*, b_i^* and d_i^* such that*

$$0 < a_1^* < b_1^* < \xi d_2^* < a_2^* < b_2^* < \xi d_3^* < \ldots < a_n^* < b_n^* < \xi d_{n+1}^* \text{ and } d_i^* \leq b_i^*,$$

where $n \in \mathbb{N}$. Suppose that f satisfies the following conditions:

(i) $f\left(t,u,u^{\Delta}\right) \leq \varphi_p\left(\frac{d_i^*}{W}\right)$ *for* $\left(t,u,u^{\Delta}\right) \in [0,\omega_1]_{\mathbb{T}} \times [0,(T+\beta)d_i^*] \times [-d_i^*,d_i^*]$;

(ii) $f\left(t,u,u^{\Delta}\right) > \varphi_p\left(\frac{b_i^*}{N}\right)$ *for* $\left(t,u,u^{\Delta}\right) \in [\xi,\rho(\omega_1)]_{\mathbb{T}} \times [b_i^*,(T+\beta)d_i^*] \times [-d_i^*,d_i^*]$;

(iii) $f\left(t,u,u^{\Delta}\right) < \varphi_p\left(\frac{a^*}{M}\right)$ *for* $\left(t,u,u^{\Delta}\right) \in [0,\omega_1]_{\mathbb{T}} \times [0,a_i^*] \times [-d_i^*,d_i^*]$.

Then problem (1.5) *has at least* $2n-1$ *positive pseudo-symmetric solutions.*

Proof. This theorem can be proved directly by using the principle of mathematical induction. When $n=1$, it follows from condition (i) that $A : \overline{P}_{a_1^*} \to P_{a_1^*} \subset \overline{P}_{a_1^*}$. By the Schauder fixed-point theorem, we know that A has at least one fixed point $u_1 \in \overline{P}_{a_1^*}$, which indicates that $\|u_1\| \le a_1^*$. When $n=2$, it is clear that Theorem 4.1 holds (with $a^* = a_1^*$, $b^* = b_1^*$, $d^* = d_2^*$). Then there exist at least three positive pseudo-symmetric solutions u_1, u_2 and u_3 satisfying $\gamma(u_1) \le d_1^*$, $\gamma(u_2) \le d_1^*$, $\gamma(u_3) \le d_1^*$, $b_1^* < \min_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_1(t)$, $a_1^* < \max_{t\in[0,\omega_1]_{\mathbb{T}}} u_2(t)$, $\min_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_2(t) < b_1^*$ with $\max_{t\in[0,\omega_1]_{\mathbb{T}}} u_3(t) < a_1^*$. Assume that the statement holds if $n=k$; that is, problem (1.5) has at least $2n-1$ positive pseudo-symmetric solutions u_i, $i = 1, 2, \cdots, 2k-1$. From the above analysis, we have

$$\|u_i\| \le d_k^*, \quad i = 1,\, 2,\, \cdots,\, 2k-1. \tag{4.4}$$

When $n = k+1$, we observe that Theorem 4.1 holds (with $a^* = a_k^*$, $b^* = b_k^*$, $d^* = d_{k+1}^*$). Then there exist at least three positive pseudo-symmetric solutions u_{k1}, u_{k2} and u_{k3} satisfying

$$\begin{aligned} &\gamma(u_{k1}) \le d_{k+1}^*, \quad \gamma(u_{k2}) \le d_{k+1}^*, \quad \gamma(u_{k3}) \le d_{k+1}^*, \\ &b_k^* < \min\nolimits_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_{k1}(t), \quad a_k^* < \max\nolimits_{t\in[0,\omega_1]_{\mathbb{T}}} u_{k2}(t), \\ &\min\nolimits_{t\in[\xi,\rho(\omega_1)]_{\mathbb{T}}} u_{k2}(t) < b_k^* \text{ with } \max\nolimits_{t\in[0,\omega_1]_{\mathbb{T}}} u_{k3}(t) < a_k^*. \end{aligned} \tag{4.5}$$

Taking into account $d_k^* \le b_k^*$, as well as inequalities (4.4) and (4.5), we have $u_i \ne u_{k1} \ne u_{k2}$, $i = 1, 2, \ldots, 2k-1$. Consequently, problem (1.5) has at least $2k+1$ positive pseudo-symmetric solutions when $n = k+1$. Therefore, the proof has been completed. □

5. Examples

In this section, we present two examples to illustrate our main results.

Example 5.1. Let

$$\mathbb{T} = \{0.1^n\} \cup \{0.12^n\} \cup \{0, 0.2, 0.25, 0.3, 0.34, 0.86, 0.9, 0.95, 1\} \cup [0.35, 0.85].$$

Consider the following boundary-value problem:

$$(\varphi_p(u^{\nabla}(t)))^{\Delta} + t(t+1+|u^{\Delta}(t)|^{p-2}) = 0, \quad t \in (0,1)_{\mathbb{T}}, \tag{5.1}$$

$$u(0)-3u^{\Delta}(0.2)=0 \text{ and } u(0.2)-4u^{\Delta}(0.35)=u(1)-4u^{\Delta}(0.85).$$

It is obvious that T=1. Note that

$$f_0=\lim_{(u,u^{\Delta})\to(0,0)}\inf_{t\in[0,1]_{\mathbb{T}}}\frac{t+1+|u^{\Delta}(t)|^{p-2}}{(|u^{\Delta}(t)|+u(t))^{p-1}}=\infty \text{ for } t\in[0,1]_{\mathbb{T}},$$

and

$$f^{\infty}=\lim_{(u,u^{\Delta})\to(\infty,\infty)}\sup_{t\in[0,1]_{\mathbb{T}}}\frac{t+1+|u^{\Delta}(t)|^{p-2}}{|u^{\Delta}(t)|^{p-1}}=0 \text{ for } t\in[0,1]_{\mathbb{T}}.$$

Hence, Theorem 3.3 implies that the boundary-value problem (5.1) has at least one positive-symmetric solution.

Example 5.2. Let

$\mathbb{T}=\{0.1^n\}\cup\{0.12^n\}\cup\{0,0.2,\ 0.25,\ 0.3,\ 0.9,\ 0.95,1\}\cup[0.35,0.85]$ and T=1.

Consider the following boundary-value problem with $p=3$:

$$(\varphi_p(u^{\nabla}(t)))^{\Delta}+h(t)f(t,u(t),u^{\Delta}(t))=0,\ t\in(0,1)_{\mathbb{T}}, \tag{5.2}$$
$$u(0)-0.01u^{\Delta}(0.2)=0 \text{ and } u(0.2)-4u^{\Delta}(0.35)=u(1)-4u^{\Delta}(0.85),$$

where $h(t)=t+\sigma(t)$, and for $t\in[0,1]_{\mathbb{T}}$ and $u^{\Delta}\in[-5.5,5.5]$. Let

$$f\left(t,u,u^{\Delta}\right)=\begin{cases} t+4+\left(\frac{u^{\Delta}}{6.05}\right)^2, & u\in[0,\ 0.9],\\ t+750u-671+\left(\frac{u^{\Delta}}{6.05}\right)^2, & u\in[0.9,\ 1],\\ t+79+\left(\frac{u^{\Delta}}{6.05}\right)^2, & u\in[1,\ 5.56].\end{cases}$$

Note that $\xi=0.2$, $\eta=0.35$ and $\omega_1=\rho(\omega_1)=0.6$; then a direct calculation shows that

$$M=(\beta+\omega_1)\varphi_q\Big(\int_0^{\omega_1}h(r)dr\Big)=0.3606,\ N\approx0.1131 \text{ and } W=0.66.$$

If we take $a'=0.9$, $b'=1$ and $d'=5.5$, then the relation $a'<b'<0.2d'$ holds. Furthermore, we have

$$f\left(t,u,u^{\Delta}\right)<82<82.645\approx\varphi_p\left(\tfrac{d'}{W}\right)$$
$$\text{for } \left(t,u,u^{\Delta}\right)\in[0,0.6]_{\mathbb{T}}\times[0,5.56]\times[-5.5,5.5],$$
$$f\left(t,u,u^{\Delta}\right)>79>78.176\approx\varphi_p\left(\tfrac{b'}{N}\right)$$
$$\text{for } \left(t,u,u^{\Delta}\right)\in[0.2,0.6]_{\mathbb{T}}\times[1,5.56]\times[-5.5,5.5],$$
$$f\left(t,u,u^{\Delta}\right)<6.2<6.25=\varphi_p\left(\tfrac{a'}{M}\right)$$

$$\text{for } \left(t,u,u^{\Delta}\right) \in [0,0.6]_{\mathbb{T}} \times [0,0.9] \times [-5.5,5.5].$$

From Theorem 4.1, we see that the boundary-value problem (5.2) has at least *three* positive pseudo-symmetric solutions u_1, u_2 and u_3 such that

$$\begin{aligned}
&\max\nolimits_{t\in[0,0.6]_{\mathbb{T}}} |u^{\Delta}(t)| \leq 5.5 \text{ for i=}1,2,3,\\
&1 < \min\nolimits_{t\in[0.2,0.6]_{\mathbb{T}}} u_1(t), \quad 0.9 < \max\nolimits_{t\in[0,0.6]_{\mathbb{T}}} u_2(t),\\
&\min\nolimits_{t\in[0.2,0.6]_{\mathbb{T}}} u_2(t) < 1 \text{ with } \max\nolimits_{t\in[0,0.6]_{\mathbb{T}}} u_3(t) < 0.9.
\end{aligned}$$

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