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NONLOCAL INTERACTION EQUATIONS: STATIONARY STATES AND STABILITY ANALYSIS

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Abstract. In this paper, we are interested in the long-time behavior of solutions to a nonlocal interaction equation in dimension 1. We show that, up to an extraction, the solution converges to a steady-state. Then, we study the structure of stable steady-states.

1. Introduction

We are interested in the asymptotic behavior of a density $\rho(t,x)$ of particles or individuals at position $x\in\mathbb{R}^d$ $(d\geq 1)$ and at time $t\geq 0$, which evolves according to the nonlocal aggregation equation:

$$\partial_t\rho=\nabla_x\cdot(\rho\,\nabla_x[W*\rho+V]),\text{ for }(t,x)\in\mathbb{R}_+\times\mathbb{R}^d. \tag{1.1}$$

This equation can be seen as a many particles limit of discrete processes where particles (or individuals) can interact at a large distance, through an interaction potential W (see [27, 20]), and may be subjected to an external potential V. Such equations appear in various biological phenomena like swarming (see [7, 15]), distribution of actin-filament networks (see [16, 19]), as well as in physical problems, for example in the field of granular media (see [2, 32, 14]). For some interaction potentials, this equation can lead to surprisingly complicated patterns, such as solutions converging to singular steady states, as shown in [4, 29, 17, 18], or more recently in [23, 8, 1].

Many of the above models couple the long-range interaction between particles with a diffusive term. Nevertheless, in this paper we shall not consider a diffusion term, and focus our study on the effect of a long-range interaction.

Let us now describe typical interaction potentials W which appear in the models quoted above:

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- In [21, 29], interaction potentials are regular, repulsive at short range and attractive when particles are far apart, typically $W(x) = -x^2 + x^4$. In this case, the solution typically concentrates and tends to a finite number of Dirac masses, when time goes to infinity. These types of potentials have been studied in [12, 10, 14], but we don't know any general study of the case of regular interaction potentials so far.
- In chemotaxis models (see [28, 22, 5]), interaction potentials are singular at $x = 0$ and attractive, typically, in dimension 2, $W(x) := -\frac{1}{2\pi}\log|x|$. In this case, the solution usually (if there is no diffusion) blows-up in finite time. Potentials singular at $x = 0$ and attractive have been widely studied both with a diffusion term (see [6, 9]), or without diffusion (see [11, 24, 13, 4, 3]), for various types of attractive singularities.
- In swarming models (see [15, 26, 31]), interaction potentials are usually singular at $x = 0$ and repulsive; typical examples are the repulsive Morse potential $W(x) = -e^{-|x|}$, or the attractive-repulsive Morse potentials $W(x) = -C_a\, e^{-|x|/l_a} + C_r\, e^{-|x|/l_r}$ and $W(x) = -C_a\, e^{-|x|^2/l_a} + C_r\, e^{-|x|^2/l_r}$. Related interpolation potentials in physics are, for instance, the Lennard-Jones potential [30]. We don't know any qualitative study of such models.

We will show in this article that the asymptotic behavior of the solution of (1.1) highly depends on the type of singularity of W at the point $x = 0$.

In the present article, we shall focus on the one-dimensional case. We aim at understanding the dynamical behavior presented by a nonlocal interaction operator with even potential. This assumption is not crucial in this study, but is satisfied by the interaction kernels used in practice for this model.

Assumption 1:

$$\forall x \in \mathbb{R}, \quad W(x) = W(-x). \tag{1.2}$$

In this study, we shall focus on compactly supported densities; we shall thus only consider situations where a confinement exists, either from the external potential, or from the interaction potential itself. We shall assume the following.

Assumption 2: One of the following two conditions is satisfied:
There exists $C > 0$ such that

$$\|W'\|_{L^\infty([-2C,2C])} < \min\left(V'(C),\, -V'(-C)\right), \tag{1.3}$$

or

$$V = 0, \quad \exists C_1,\, C_2 > 0,\, \forall x \geq C_1 : \quad W'(x) \geq C_2\, x. \tag{1.4}$$

Assumption 3:

$$\rho^0 \in M^1(\mathbb{R}),\, \text{supp}\ \rho^0 \subset [-C, C], \tag{1.5}$$

where $C < \infty$. If $V \neq 0$, C must satisfy (1.3).

It has been proven in [11] that Assumptions 2 and 3 ensure that the support of $\rho(t,\cdot)$ is (uniformly with respect to time) bounded:

$$\exists C > 0,\, \forall t \geq 0, \quad \text{supp}\,\rho(t,\cdot) \subset [-C, C]. \tag{1.6}$$

Note that (1.1) formally conserves the total mass $\int \rho(t,x)\,dx$, which with no loss of generality we shall assume to be normalized $\int_{\mathbb{R}} \rho(x)\,dx = 1$. The quantity $\rho(t,\cdot)$ is then interpreted as a probability density. In particular, in the one-dimensional case, this enables a change of variables in which one introduces the pseudo-inverse of the distribution function $\int_{-\infty}^{x} d\rho$; i.e.,

$$u(t,z) = \inf\left\{ x \in \mathbb{R} : \int_{(-\infty,x]} \rho(t,y)\,dy > z \right\} \qquad z \in [0,1], \tag{1.7}$$

which transforms the evolution equations (1.1) for measure solutions $\rho(t,\cdot)$ into an integral equation for the nondecreasing pseudo-inverse $u(t,z)$ satisfying (see, e.g. [25, 5, 10])

$$\partial_t u(t,z) = \int W'\left(u(t,\xi) - u(t,z)\right)\, d\xi - V'(u(t,z)), \qquad \forall z \in [0,1]. \tag{1.8}$$

Since equation (1.8) is much more convenient than equation (1.1) for stability analysis, we shall often use it in this paper. In particular, atomic parts of measure solutions $\rho(x)$ correspond to constant parts of the pseudo-inverse $u(z)$. Notice also the useful change of variable

$$\int g(x)\rho(x)\,dx = \int_0^1 g(u(\xi))\,d\xi,$$

which holds for any $g \in L^1(\text{supp}\ \rho)$.

In the absence of a confining potential V (and if W is symmetric), the center of mass $\int_{\mathbb{R}} x\,\rho(t,x)\,dx$ is conserved by equation (1.1), or equivalently, $\int_0^1 u$ is preserved by (1.8):

$$\frac{d}{dt}\int_{\mathbb{R}} x\rho(t,x)\,dx = 0, \quad \frac{d}{dt}\int_0^1 u(t,z)\,dz = 0. \tag{1.9}$$

Note that equation (1.1) can be seen as a gradient-flow equation for the following energy (see [11]):

$$E(t) := \frac{1}{2}\int\int \rho(t,x)\rho(t,y)W(x-y)\,dx\,dy + \int_{\mathbb{R}} \rho(t,x)V(x)\,dx. \qquad (1.10)$$

In Section 2, we shall consider regular interaction potentials W. Proposition 1 shows that $\rho(t,\cdot)$ converges (in a sense to be made precise then) to a set of steady-states, as time goes to infinity. This result emphasizes the importance of steady-states, when one wishes to understand the long-time behavior of solutions to (1.1).

In Subsection 2.2, we show that stable steady-states of (1.1) are generically sums of Dirac masses. More precisely, we show in Proposition 2 that for analytic V, W, the steady-states of (1.1) are necessarily finite sums of Dirac masses. If V, W are only C^2, continuous steady-states may exist, but they cannot be linearly stable.

In Section 3, we consider interaction potentials having a singularity at $x = 0$.

In Subsection 3.1, we consider the steady-states of (1.1) for an interaction potential W having an attractive singularity at $x = 0$. Since (1.1) may develop blow-ups in L^∞ in finite time (see [4, 3]), we consider (following [11]) the extension (3.2) of (1.1) to measure-valued solutions. In Proposition 4, we show that a steady-state $\bar\rho$ of (3.2) such that supp $\bar\rho$ has an accumulation point (and a bit more, see (3.4)) is nonlinearly unstable.

In Subsection 3.2, we consider the steady-states of (1.1) for an interaction potential W having a repulsive singularity at $x = 0$. In Proposition 5, we provide an existence proof for (1.1) with a regular initial condition (until now, no existence result had been written down for such interaction potentials). In particular, Proposition 5 provides a uniform bound on the solution in $L^\infty(\mathbb{R})$. The situation is therefore completely different from the two other cases: no blow-up can occur.

2. Regular interaction potentials

In this first section, we make the following regularity assumptions on V and W.

Assumption 4:

$$V \in C^2(\mathbb{R}),\ W \in C^2(\mathbb{R}), \qquad (2.1)$$

$$W \in W^{2,\infty}(\mathbb{R}). \qquad (2.2)$$

We shall use in the following the measure space

$$\mathcal{P}_\infty(\mathbb{R}) := \{\rho \in M^1(\mathbb{R}) : \text{ supp } \rho \text{ is bounded}\},$$

together with the Wasserstein distance

$$W_\infty(\rho_1, \rho_2) := \|u_1 - u_2\|_\infty, \tag{2.3}$$

where u_1, u_2 are the pseudo-inverses of ρ_1, ρ_2.

Under Assumptions 1 to 4, it has been proven in [10] that a unique solution $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ to (1.1) exists. The support of $\rho(t,\cdot)$ is uniformly bounded with respect to time thanks to [11]

2.1. **Asymptotic behavior of the solution.** In this subsection, we show that we cannot expect the solution to converge to anything other than a set of steady-states, using an energy dissipation argument. In particular, no periodic limit cycles exist. We define a steady-state of (1.1) as a probability measure $\bar\rho \in \mathcal{P}_\infty(\mathbb{R})$ such that the velocity field it generates is equal to 0 on the support of $\bar\rho$; that is,

$$\nabla_x[W * \bar\rho + V] = 0 \quad \text{on supp } \bar\rho.$$

Proposition 1. *Let* ρ^0, V, W *satisfy Assumptions* 1-4. *Let* $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ *be the unique solution of* (1.1) *given by* [10]. *Then,*

(1)
$$\int \rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)^2 dx \to 0 \text{ as } t \to \infty.$$

(2) *For any sequence* $t_k \to \infty$*, there exists a subsequence, still denoted* (t_k)*, such that*

$$W_1\left(\rho(t_k,\cdot), \bar\rho\right) \to 0 \quad \text{as } k \to \infty, \tag{2.4}$$

where W_1 *denotes the* $1-$*Wasserstein distance, and* $\bar\rho$ *is a steady-state of* (1.1).

Remark 1. The limit $\bar\rho$ of $\rho(t_k,\cdot)$ in (2.4) is not necessarily unique; it may depend both on the sequence (t_k) and the extracted sequence.

Proof of Proposition 1.

Step 1. Proof of 1. We first show that the energy (1.10) is nonincreasing in time, using integrations by parts:

$$\frac{dE}{dt}(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x\Big(\rho(t,x)\Big(\int W'(x-z)\rho(t,z)\,dz + V'(x)\Big)(t,x)\Big) \times \rho(t,y)W(x-y)\,dx\,dy$$

$$+ \int_{\mathbb{R}} \partial_x \Big(\rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\Big) V(x)\,dx$$
$$= -\int \rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)^2 dx \leq 0. \tag{2.5}$$

Next, we have the following estimate on the regularity of the energy dissipation:

$$\begin{aligned}
\frac{d^2E}{dt^2} =& -\int \partial_x \Big(\rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\Big) \\
&\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)^2 dx \\
&- 2\int \rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\int W'(x-y) \\
&\partial_y\Big(\rho(t,y)\Big(\int W'(y-z)\rho(t,z)\,dz + V'(y)\Big)\Big)\,dy\,dx \\
=&2\int \rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)^2 \\
&\partial_x\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\,dx \\
&+ 2\int \rho(t,x)\Big(\int W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\int \partial_y(W'(x-y)) \\
&\Big(\rho(t,y)\Big(\int W'(y-z)\rho(t,z)\,dz + V'(y)\Big)\Big)\,dy\,dx.
\end{aligned}$$

Since V, $W \in C^2(\mathbb{R})$, we can estimate $\frac{d^2E}{dt^2}$ as follows:

$$\begin{aligned}
\left|\frac{d^2E}{dt^2}\right| \leq& 2\left(\|V\|_{W^{2,\infty}(-C,C)} + \|W\|_{W^{2,\infty}(-2C,2C)}\right) \\
&\times \left(\|W\|_{W^{2,\infty}(-2C,2C)} + \|V\|_{W^{2,\infty}(-C,C)}\right)^2 \leq C,
\end{aligned} \tag{2.6}$$

where $C < +\infty$ is a constant.

Finally, notice that the energy is bounded from below:

$$E \geq -\left(\tfrac{1}{2}\|W\|_{L^\infty(-2C,2C)} + \|V\|_{L^\infty(-C,C)}\right). \tag{2.7}$$

To prove that $\frac{dE}{dt}(t) \to 0$, we use an interpolation between $E(t) \to \bar{E}$ and the bounded $\frac{d^2}{dt^2}E(t)$.

Let $\varepsilon > 0$. Since the energy E is nonincreasing ((2.5)) and bounded from below ((2.7)), E has a limit $\bar{E}$ when $t \to \infty$. Let $t > 0$ and $\tau \in (0, \frac{t}{2}]$. Then

$$\begin{aligned}\Big|\frac{dE}{dt}(t)\Big| &= \Big|\frac{1}{\tau}\int_{t-\tau}^{t}\Big[\frac{dE}{dt}(s) + \int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\Big]\,ds\Big| \\ &= \Big|\frac{1}{\tau}\big[E(t) - E(t-\tau)\big] + \frac{1}{\tau}\int_{t-\tau}^{t}\int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\,ds\Big| \\ &\le \frac{2}{\tau}\|E - \bar{E}\|_{L^\infty([\frac{t}{2},\infty))} + \frac{\tau}{2}\Big\|\frac{d^2E}{dt^2}\Big\|_{L^\infty([0,\infty))}.\end{aligned}$$

For $t > 0$ large enough, $\tau := \frac{\|E-\bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}}{\big\|\frac{d^2E}{dt^2}\big\|^{\frac{1}{2}}_{L^\infty([0,\infty))}} < \frac{t}{2}$, and then

$$\Big|\frac{dE}{dt}(t)\Big| \le \frac{5}{2}\|E - \bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}\Big\|\frac{d^2E}{dt^2}\Big\|^{\frac{1}{2}}_{L^\infty([0,\infty))},$$

which implies $\frac{dE}{dt}(t) \to 0$ as $t \to \infty$.

Step 2. Proof of 2. The pseudo-inverse $u(t,\cdot)$ of $\rho(t,\cdot)$ is an increasing function, and is uniformly bounded thanks to (1.6). The sequence $u(t_k,\cdot)$ is then a uniformly bounded sequence of $BV([0,1])$. There exists then a subsequence, still denoted $u(t_k,\cdot)$, that converges in L^1 to a limit denoted by $\bar{u}$: $\|u(t_k,\cdot) - \bar{u}\|_{L^1} \to 0$. Our aim is to prove that $\bar{u}$ is a steady-state of (1.8). In order to prove that, we shall use the estimate obtained above, and the fact that $\frac{dE}{dt}(t_k) \to 0$. Let us write this estimate in the pseudo-inverse setting:

$$\begin{aligned}\frac{dE}{dt}(t_k) &= -\int \rho(t_k,x)\Big(\int W'(x-y)\rho(t_k,y)\,dy + V'(x)\Big)^2 dx \\ &= -\int_0^1 \Big(\int W'(u(t_k,z) - u(t_k,\xi))\,d\xi + V'(u(t_k,z))\Big)^2 dz.\end{aligned}$$

We define

$$\bar{F} := -\int_0^1 \Big(\int W'(\bar{u}(z) - \bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\Big)^2 dz.$$

Then

$$\begin{aligned}\bar{F} - \frac{dE}{dt} = \int_0^1 &\Big(\int W'(u(z) - u(\xi))\,d\xi + V'(u(z))\Big)^2 \\ &- \Big(\int W'(\bar{u}(z) - \bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\Big)^2 dz\end{aligned}$$

$$= \int_0^1 \Big(\int W'(u(z) - u(\xi))\, d\xi + V'(u(z)) + \int W'(\bar u(z) - \bar u(\xi))\, d\xi + V'(\bar u(z)) \Big)$$
$$\times \Big(\int W'(u(z) - u(\xi))\, d\xi - \int W'(\bar u(z) - \bar u(\xi))\, d\xi + V'(u(z)) - V'(\bar u(z)) \Big)\, dz$$
$$\le C \Big\| \int W'(u(z) - u(\xi))\, d\xi - \int W'(\bar u(z) - \bar u(\xi))\, d\xi \Big\|_{L^1} + C\|u - \bar u\|_{L^1}$$
$$\le C\|W'\|_{L^\infty(-2C,2C)}\, \|u - \bar u\|_{L^1} + C\|u - \bar u\|_{L^1}.$$

Finally,

$$\bar F \le \frac{dE}{dt}(t_k) + C\|u(t_k,\cdot) - \bar u\|_{L^1} \to 0 \quad \text{as } k \to \infty.$$

Then, $\bar F = 0$; that is,

$$\text{supp } \bar\rho \subset \Big\{ x \in \mathbb{R} : \int W'(x-y)\bar\rho(y)\, dy + V'(x) = 0 \Big\},$$

and $\bar\rho$ is a steady-state of (1.1). □

2.2. **Study of the steady states.** In the previous subsection, we showed that, for any regular potential W satisfying Assumption 4, the sequence $\rho(t_k,\cdot)$ converges, up to an extraction, to a steady solution of (1.1). In this subsection, we shall try to characterize the steady-states of (1.1).

In the case of an analytical interaction potentials W and analytical external field V, we show in the following proposition that steady-states are necessarily finite sums of Dirac masses.

Proposition 2. *Assume W and V are analytical. Then, every steady state $\bar\rho \in M^1(\mathbb{R})$ of (1.1) with bounded support is a finite sum of Dirac masses:*

$$\bar\rho = \sum_{i=1}^N \bar\rho_i \delta_{\bar u_i},$$

with $\bar\rho_1, \dots, \bar\rho_N > 0$, $\bar u_1, \dots, \bar u_N \in \mathbb{R}$.

Proof. Let us consider a steady solution $\bar\rho$ of (1.1). For $x \in \text{supp } \bar\rho$,

$$0 = \nabla\Big[\int W(x-y)\, d\bar\rho(y) - V(x) \Big] = -(W' * \bar\rho + V')(x).$$

Since W and V are analytic, so is $W' * \bar\rho + V'$, and if supp $\bar\rho$ has an accumulation point, then

$$\forall x \in \mathbb{R}, \quad (W' * \bar\rho)(x) + V'(x) = 0,$$

which is not possible since V, W satisfy (1.3) or (1.4). Then, supp $\bar{\rho}$ cannot have any accumulation point, and is thus a finite set of points. □

For less regular potentials, for instance when W is only C^2, the same result cannot be expected to hold anymore, as the following example shows.

Example 1. Consider the interaction potential $W(x) := (\mathrm{dist}(x, [-1, 1]))^3$, where $\mathrm{dist}(x, y) := |x - y|$, and $V = 0$. W is C^2 (one could even consider a smoothed (C^∞) version of the potential), but (1.1) admits the $L^1(\mathbb{R})$ steady state: $\bar{\rho} = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$.

Nevertheless, the following proposition shows that steady states which are linearly stable (in a sense made clear in the following proposition) have to be sums of Dirac masses.

Proposition 3. *Let V, W satisfy Assumptions* 1–4. *Let $\bar{\rho} \in M^1(\mathbb{R})$ be a compactly supported steady state of* (1.1)*, and $\bar{u}$ be its pseudo-inverse. If $\bar{\rho}$ is such that* $\mathrm{supp}(\bar{\rho})$ *has an accumulation point x_0, then the pseudo-inverse equation* (1.8) *linearized around $\bar{u}$ in L^1 has no spectral gap.*

Remark 2. Since the perturbations u^ε of $\bar{u}$ used in the proof of Proposition 3 satisfy $\int_0^1 u^\varepsilon = \int_0^1 \bar{u}$, Proposition 3 remains true if we only consider perturbations preserving the center of mass $\int x\bar{\rho}(x)\, dx$ of $\bar{\rho}$ (this is important since (1.1) is invariant with respect to translations along x).

Remark 3. For a stability analysis of steady-states $\bar{\rho}$ that are sums of Dirac masses, see [17, 18]. In [17], a necessary and sufficient condition for local stability of such steady-states with respect to perturbations ρ of $\bar{\rho}$ such that $W_\infty(\bar{\rho}, \rho)$ is small (where W_∞ denotes the $\infty-$Wasserstein distance) is discussed.

Recently, several papers have shown that, in several dimensions, interaction potentials that are singular repulsive locally and attractive at long range can lead to very complicated patterns, see [8, 23, 1].

The idea of the proof is to construct a measure ρ^ε arbitrarily close to $\bar{\rho}$ by collapsing the mass of $\bar{\rho}$ around x_0 into a single Dirac mass (a similar construction will be employed in the proof of Proposition 4). If we denote by L the linearization of (1.8) around $\bar{u}$, then we show that $\|L(v^\varepsilon)\|_{L^1} = o_\varepsilon(1)\|v^\varepsilon\|_{L^1}$, which implies that L cannot have any spectral gap.

Proof of Proposition 3. We begin by linearizing (1.8) around $\bar{u}$, with $u = \bar{u} + \delta v$, $\delta > 0$:

$$\partial_t u(t, z) = \int_0^1 W'(u(t, \xi) - u(t, z))\, d\xi - V'(u(t, z))$$

$$= \int_0^1 W'(\bar u(t,\xi) - \bar u(t,z))\,d\xi - V'(\bar u(t,z))$$
$$+ \delta\Big(\int_0^1 W''(\bar u(\xi) - \bar u(z))(v(t,\xi) - v(t,z))\,d\xi - V''(\bar u(z))v(t,z)\Big) + o(\delta)$$
$$= \delta\Big(\int_0^1 W''(\bar u(\xi) - \bar u(z))v(t,\xi)\,d\xi - \int_0^1 W''(\bar u(\xi) - \bar u(z))\,d\xi v(t,z)$$
$$- V''(\bar u(z))v(t,z)\Big) + o(\delta),$$

so that the linearization of (1.8) around $\bar u$ yields the linear operator $L : L^1([0,1]) \to L^1([0,1])$:

$$L(v)(z) = \int_0^1 W''(\bar u(\xi) - \bar u(z))v(\xi)\,d\xi - \Big[\int_0^1 W''(\bar u(\xi) - \bar u(z))\,d\xi + V''(\bar u(z))\Big]v(z). \tag{2.8}$$

We now shall show that, if supp $\bar\rho$ has an accumulation point x_0, then we can build a sequence (v^ε) of perturbations of u such that

$$\frac{\|L(v^\varepsilon)\|_{L^1}}{\|v^\varepsilon\|_{L^1}} \to 0,$$

which shows that the linear operator L does not have any spectral gap. Since we are dealing with pseudo-inverses, we must however restrict to perturbations v such that, for some $\alpha > 0$, $u = \bar u + \alpha v$ is nondecreasing.

We assume without any loss of generality that x_0 is an accumulation point of $\mathrm{supp}(\bar\rho) \cap [x_0, \infty)$. Then, for any $\varepsilon > 0$,

$$\int_{(x_0,x_0+\varepsilon)} d\bar\rho > 0. \tag{2.9}$$

For a given $\varepsilon > 0$, we define $z_0 := \inf\{z \in (0,1) : \bar u(z) > x_0\}$, $z_1^\varepsilon := \sup\{z \in (0,1) : \bar u(z) < x_0 + \varepsilon\}$, and $Z^\varepsilon := [z_0, z_1^\varepsilon]$.

We define the following perturbation u^ε of $\bar u$:

$$u^\varepsilon(z) := \left|\begin{array}{l} \bar u(z) \text{ on } (Z^\varepsilon)^c, \\ \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon} \bar u(y)\,dy \text{ on } Z^\varepsilon, \end{array}\right.$$

and we write $v^\varepsilon := u^\varepsilon - \bar u$. The function u^ε is then the pseudo-inverse of the measure

$$\rho^\varepsilon = \bar\rho|_{[x_0,x_0+\varepsilon]^c} + \Big(\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx\Big)\delta_{\tilde x},$$

where

$$\tilde{x} = \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon} \bar{u}(y)\,dy = \int_{[x_0,x_0+\varepsilon]} x\bar{\rho}(x)\,\frac{dx}{\int_{[x_0,x_0+\varepsilon]}\bar{\rho}(x)\,dx}.$$

- We estimate $\int_0^1 W''(\bar{u}(\xi)-\bar{u}(z))v^\varepsilon(\xi)\,d\xi$:

$$\begin{aligned}&\int_0^1 W''(\bar{u}(\xi)-\bar{u}(z))v^\varepsilon(\xi)\,d\xi\\ &= \int_0^1 W''(\bar{u}(\xi)-x_0)v^\varepsilon(\xi)\,d\xi + \int_0^1 o_\varepsilon(1)v^\varepsilon(\xi)\,d\xi = o_\varepsilon(1)\|v^\varepsilon\|_{L^1}.\end{aligned}\tag{2.10}$$

- We estimate $\left[\int_0^1 W''(\bar{u}(\xi)-\bar{u}(z))\,d\xi + V''(\bar{u}(z))\right]v^\varepsilon(z)$:
  Since $\bar{u}$ is a steady state of (1.8),

$$\forall x\in \operatorname{supp}\bar{\rho},\quad (W'*\bar{\rho})(x)+V'(x)=0.$$

  Thanks to Assumption 4, $W'*\bar{\rho}+V'\in C^1(\mathbb{R})$ is differentiable at $x=x_0$. Since x_0 is an accumulation point of $\operatorname{supp}\bar{\rho}$, there exists a sequence $(x^k)_k\in(\operatorname{supp}\bar{\rho})^{\mathbb{N}}$ such that $x^k\to x_0$. Then

$$\begin{aligned}&(W''*\bar{\rho})(x_0)+V''(x_0)\\ &= \lim_{k\to\infty}\frac{((W'*\bar{\rho})(x_0)+V'(x_0)) - \big((W'*\bar{\rho})(x^k)+V'(x^k)\big)}{x_0-x_k} = \lim_{k\to\infty} 0 = 0.\end{aligned}$$

  Since $W''*\rho+V''$ is continuous, and thanks to the definition of z_0, z_1^ε, for any $z\in\operatorname{supp}(v)\subset[z_0,z_1^\varepsilon]$,

$$\left[(W''*_x\bar{\rho})(\bar{u}(z))+V''(\bar{u}(z))\right]v^\varepsilon(z) = \big(0+o_{\bar{u}(z)-x_0}(1)\big)\,v^\varepsilon(z) = o_\varepsilon(1)v^\varepsilon(z).\tag{2.11}$$

Finally, using (2.11) and (2.10) in (2.8), we get $\|L(v^\varepsilon)\|_{L^1}=o_\varepsilon(1)\|v^\varepsilon\|_{L^1}$, which proves the proposition. □

3. Singular interaction potentials

In this section, we shall consider interaction potentials having a singularity at $x=0$:

- Interaction potentials having an attractive singularity at $x=0$, satisfying Assumption 5 (see Subsection 3.1),
- Interaction potentials having a repulsive singularity at $x=0$, satisfying Assumption 6 (see Subsection 3.2).

(1.6) shows that the support of $\rho(t,\cdot)$ is uniformly bounded with respect to time; we shall therefore only consider compactly supported solutions. We shall show that those two cases have a very different dynamics: If Assumption 5 is satisfied, every steady-state apart from sums of Dirac masses are nonlinearly unstable, whereas if Assumption 6 is satisfied, the solution (of the time-dependent equation) is uniformly bounded in $L^\infty(\mathbb{R})$.

3.1. **Interaction potentials having an attractive singularity at $x = 0$.** We shall consider in this section potentials having an attractive singularity at $x = 0$; that is, interaction potentials W such that $W'(0) > 0$.

Assumption 5. $V \in C^2(\mathbb{R})$, $W \in C^0(\mathbb{R})$, and there exists $W'(0^+) > 0$ such that

$$x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x| \in C^2(\mathbb{R}). \tag{3.1}$$

It is well known that, in this case, classical solutions of (1.1) may blow up in finite time (see [4, 3]). Following [11], we extend (1.1) to measure-valued solutions with the following equation:

$$\partial_t \rho(t,x) = \partial_x \Big[\rho(t,x)\Big(\int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x)\Big)\Big], \tag{3.2}$$

where we write, with a slight abuse of notation, $\rho(t,y)dy$ instead of $d\rho(t,\cdot)(y)$. If Assumptions 1 to 3, and 5 are satisfied, then it has been proven in [11] that a unique solution $\rho \in \mathrm{AC}_{loc}([0,\infty),\mathcal{P}_2(\mathbb{R}))$ to (3.2) exists. Note that the energy (1.10) is also a Lyapounov functional for (3.2).

One can check that the pseudo-inverse $u(t,z)$ of the solution $\rho(t,x)$ to (3.2) satisfies

$$\partial_t u(t,z) = \int_{\{\xi\in[0,1];\, u(t,\xi)\neq u(t,z)\}} W'(u(t,\xi)-u(t,z))\,d\xi - V'(u(t,z)). \tag{3.3}$$

For regular potentials, we showed that if a (compactly supported) steady-state $\bar\rho \in M^1(\mathbb{R})$ of (1.1) is such that supp $\bar\rho$ has an accumulation point, then $\bar\rho$ cannot be linearly stable (in a sense defined in Proposition 3). In the case of interaction potentials having an attractive singularity at $x = 0$, we shall show that if a (compactly supported) steady-state $\bar\rho \in M^1(\mathbb{R})$ of (3.2) is such that supp $\bar\rho$ has an accumulation point (and a bit more, see (3.4)), then $\bar\rho$ is actually nonlinearly unstable in the sense that there exists arbitrarily close measures of strictly smaller energy, as we show in the proposition below.

Proposition 4. *Let V, W satisfy Assumptions* 1 *and* 5. *Let $\bar\rho$ be a compactly supported steady-state of* (3.2). *If supp $\bar\rho$ has an accumulation point x_0 such* *that*

$$\exists C>0,\, \exists \eta>0,\, \forall \gamma\in(0,\eta), \quad \frac{1}{\gamma}\int_{x_0}^{x_0+\gamma}\bar\rho(y)\,dy\geq C \tag{3.4}$$

(*or the same estimate with* $-\eta<\varepsilon<0$), *then it is locally unstable: For any* $\varepsilon>0$, *there exists* $\rho^\varepsilon\in M^1(\mathbb{R})$, *such that* $W_1(\rho^\varepsilon,\bar\rho)\leq\varepsilon$ *and*

$$E(\rho^\varepsilon)<E(\bar\rho), \tag{3.5}$$

where E *is the energy defined by* (1.10).

Remark 4. As in the case of regular potentials, there may exist L^1 steady-states of (3.2): For example, if $V(x):=\frac{-x^2}{2}$, $W(x):=|x|$,

$$\bar\rho:=\tfrac{1}{2}\mathbb{I}_{[-1,1]} \tag{3.6}$$

is a steady-state of (3.2). Proposition 4 shows that such steady-states are unstable.

Equation (3.3) is not linearizable around steady-states (in L^1) in general. As a consequence, in order to define the nonlinear instability of steady-states like (3.6), we use the energy E (which is a Lyapounov functional of (1.1)), see (3.5).

Sketch of the proof of Proposition 4. The main idea is to construct a measure ρ^ε arbitrarily close to $\bar\rho$ by collapsing the mass of $\bar\rho$ around x_0 into a single Dirac mass (see (3.9)):

$$\rho^\varepsilon=\bar\rho|_{[x_0,x_0+\varepsilon]^c}+\Big(\int_{[x_0,x_0+\varepsilon]}\bar\rho(x)\,dx\Big)\delta_{\bar x^\varepsilon}.$$

Then, in Step 2, we estimate the difference of energy between $\bar\rho$ and ρ^ε to get (see(3.12))

$$\begin{aligned} E(\rho^\varepsilon)-E(\bar\rho) &= -\frac{W'(0^+)+O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz \\ &\quad +\tfrac{1}{2}\left(-\omega^\varepsilon+o_\varepsilon(1)\right)\|v^\varepsilon\|_{L^2}^2. \end{aligned}$$

In steps 3 and 4, we estimate respectively $\|v^\varepsilon\|^2_{L^2}$ and ω^ε, to show that the first term on the right-hand side of (3.7), which is negative, is dominant, and thus, $E(\rho^\varepsilon)-E(\bar\rho)<0$.

Proof of Proposition 4.

Step 1. We define a sequence of measures (ρ^ε) approaching $\bar\rho$. We assume with no loss of generality that x_0 is an accumulation point of supp $\bar\rho\cap[x_0,\infty)$ such that (3.4) is satisfied. We define for $\varepsilon>0$ such that $x_0+\varepsilon\in$ supp $\bar\rho$:

$z_0 := \inf\{z \in (0,1) : \bar{u}(z) \geq x_0\}$, $z_1^\varepsilon := \sup\{z \in (0,1) : \bar{u}(z) \leq x_0 + \varepsilon\}$, and $Z^\varepsilon := [z_0, z_1^\varepsilon]$. Since x_0, $x_0 + \varepsilon \in$ supp $\bar{\rho}$ and $\bar{\rho}$ is a steady-state of (3.2),

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 - y)\bar{\rho}(y)\,dy + V'(x_0) = -\int_{y \in (x_0, x_0+\varepsilon]} W'(x_0 - y)\bar{\rho}(y)\,dy,$$

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy + V'(x_0 + \varepsilon)$$
$$= -\int_{y \in [x_0, x_0+\varepsilon)} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy.$$

If $\varepsilon > 0$ is small enough, then, $\text{sign}(W'(x)) = \text{sign}(x)$ for $x \in [-\varepsilon, \varepsilon]$. Then

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 - y)\bar{\rho}(y)\,dy + V'(x_0) > 0$$
$$> \int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy + V'(x_0 + \varepsilon).$$

On $[x_0, x_0 + \varepsilon]$,

$$\begin{aligned} F(x) &= \int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x - y)\bar{\rho}(y)\,dy + V'(x) \\ &= W'(0^+)\int_{(-\infty, x_0)} \bar{\rho}(y)\,dy - W'(0^+)\int_{(x_0, +\infty)} \bar{\rho}(y)\,dy \\ &\quad + \int_{\{y \notin [x_0, x_0+\varepsilon]\}} \tilde{W}'(x - y)\bar{\rho}(y)\,dy + V'(x), \end{aligned}$$

where $\tilde{W}$ is defined in (3.1), and F is then continuous on $[x_0, x_0 + \varepsilon]$. There exists then $\bar{x}^\varepsilon \in [x_0, x_0 + \varepsilon]$ such that

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(\bar{x}^\varepsilon - y)\bar{\rho}(y)\,dy + V'(\bar{x}^\varepsilon) = 0. \tag{3.7}$$

We define the following perturbation u^ε of $\bar{u}$:

$$u^\varepsilon(z) := \left| \begin{array}{l} \bar{u}(z) \text{ on } (Z^\varepsilon)^c, \\ \bar{x}^\varepsilon \text{ on } Z^\varepsilon, \end{array} \right. \quad , \quad v^\varepsilon := u^\varepsilon - \bar{u}. \tag{3.8}$$

u^ε is then the pseudo-inverse of the measure

$$\rho^\varepsilon = \bar{\rho}|_{[x_0, x_0+\varepsilon]^c} + \Big(\int_{[x_0, x_0+\varepsilon]} \bar{\rho}(x)\,dx\Big)\delta_{\bar{x}^\varepsilon}. \tag{3.9}$$

Notice that $W_1(\rho^\varepsilon, \bar{\rho}) \leq \varepsilon$.

Step 2. We estimate $E(\rho^\varepsilon) - E(\bar\rho)$. We use the symmetry of W and the fact that $u^\varepsilon = \bar u$ on $(Z^\varepsilon)^c$ to compute

$$\begin{aligned}E(\rho^\varepsilon) - E(\bar\rho) &= \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(u^\varepsilon(\xi) - u^\varepsilon(z))\,d\xi\,dz \\ &\quad - \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(\bar u(\xi) - \bar u(z))\,d\xi\,dz + \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi))\,d\xi\,dz \\ &\quad - \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(\bar u(z) - \bar u(\xi))\,d\xi\,dz + \int_{Z^\varepsilon} V(u^\varepsilon(z))\,dz - \int_{Z^\varepsilon} V(\bar u(z))\,dz.\end{aligned}$$

Since u^ε is constant on Z^ε (see (3.8)), the first term can be computed. We estimate the second term using the expansion $W(x) = W(0) + W'(0)|x| + \tilde W'(0)x + O(x^2)$ (thanks to Assumption 5), where the notation $O(x^2)$ stands for a term such that $\frac{1}{x^2}O(x^2)$ is bounded when x is in a neighborhood of 0. Notice that $\tilde W'(0) = 0$ thanks to Assumption 1. We use Taylor expansions on the fourth and sixth terms to get

$$\begin{aligned}&E(\rho^\varepsilon) - E(\bar\rho) \\ &= \frac{W(0)}{2}\left(|Z^\varepsilon|^2 - |Z^\varepsilon|^2\right) - \frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ &+ \int_{Z^\varepsilon}\Big\{\Big[\int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi))\,d\xi\Big] + V(u^\varepsilon(z))\Big\}\,dz \\ &- \int_{Z^\varepsilon}\Big\{\Big[\int_{(Z^\varepsilon)^c} W(\bar x^\varepsilon - \bar u(\xi))\,d\xi\Big] + V(\bar x^\varepsilon)\Big\}\,dz \\ &+ \int_{Z^\varepsilon}\Big\{\Big[\int_{(Z^\varepsilon)^c} W'(\bar x^\varepsilon - \bar u(\xi))\,d\xi\Big] + V'(\bar x^\varepsilon)\Big\}(\bar x^\varepsilon - \bar u(z))\,dz \\ &- \frac{1}{2}\int_{Z^\varepsilon}\Big\{\Big[\int_{(Z^\varepsilon)^c} W''(\theta_1(\xi,z) - \bar u(\xi)))\,d\xi\Big] + V''(\theta_2(z))\Big\}(\bar x^\varepsilon - \bar u(z))^2\,dz,\end{aligned}$$

where $\theta_1(\xi,z)$, $\theta_2(z) \in [(\bar u(z), \bar x^\varepsilon)]$. Since $u^\varepsilon(z) = \bar x^\varepsilon$ on Z^ε, the third and fourth line cancel. The fifth line is equal to 0 thanks to the definition of $\bar x^\varepsilon$ (see (3.7)). Then,

$$\begin{aligned}E(\rho^\varepsilon) - E(\bar\rho) &= -\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ &\quad - \frac{1}{2}\int_{Z^\varepsilon}\Big\{\Big[\int_{(Z^\varepsilon)^c} W''(\theta_1(\xi,z) - \bar u(\xi))\,d\xi\Big] + V''(\theta_2(z))\Big\}(\bar x^\varepsilon - \bar u(z))^2\,dz.\end{aligned}$$

Since $\bar\rho$ is compactly supported, W'', V'' are continuous, and $\theta_1(\xi,z)$, $\theta_2(z)\in[(\bar u(z),\bar x^\varepsilon)]$, we have uniform estimates

$$\sup_{\{\xi\in(Z^\varepsilon)^c,\,z\in Z^\varepsilon\}}\left|W''(\theta_1(\xi,z)-\bar u(\xi))-W''(\bar x^\varepsilon-\bar u(\xi))\right|=o_\varepsilon(1), \tag{3.10}$$
$$\sup_{\{z\in Z^\varepsilon\}}\left|V''(\theta_2(z))-V''(\bar x^\varepsilon)\right|=o_\varepsilon(1),$$

where the notation $o_\varepsilon(1)$ stands for a a term such that $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. Then, if we define

$$\omega^\varepsilon:=\int_{(Z^\varepsilon)^c}W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi+V''(\bar x^\varepsilon), \tag{3.11}$$

we get

$$\begin{aligned}E(\rho^\varepsilon)-E(\bar\rho) \;&=\; -\frac{W'(0^+)+O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz\\ &\quad+\frac12\left(-\omega^\varepsilon+o_\varepsilon(1)\right)\|v^\varepsilon\|_{L^2}^2.\end{aligned} \tag{3.12}$$

In order to prove the proposition, we shall show that the first term of (3.12) is strictly negative and dominates the second term (which is strictly positive). Then, $E(\rho^\varepsilon)-E(\bar\rho)<0$ if $\varepsilon>0$ is small enough. However, the two terms of (3.12) are of the same order in ε; we shall thus need to estimate precisely the second term.

Step 3. We estimate $\|v^\varepsilon\|_{L^2}^2$. Since $\bar u$ is a steady-state, for any $z\in Z^\varepsilon$,

$$\begin{aligned}0&=\int_{\{\xi;\,\bar u(\xi)\neq\bar u(z)\}}W'(\bar u(\xi)-\bar u(z))\,d\xi-V'(\bar u(z))\\ &=\Big[\int_{(Z^\varepsilon)^c}W'(\bar u(\xi)-\bar u(z))\,d\xi-V'(\bar u(z))\Big]+\int_{\{\xi\in Z^\varepsilon;\,\bar u(\xi)\neq\bar u(z)\}}W'(\bar u(\xi)-\bar u(z))\,d\xi.\end{aligned}$$

We estimate the first term through Taylor expansions of $x\mapsto W'(\bar u(\xi)-x)$, $x\mapsto V'(x)$ around $\bar x^\varepsilon$ (the rest of the term is estimated as in (3.10)), and the second term using $W'(x)=W'(0^+)\mathrm{sign}(x)+\tilde W'(x)=W'(0^+)\mathrm{sign}(x)+\tilde W''(\theta)x$ and $\mathrm{sign}(0)=0$ to get

$$\begin{aligned}0\;&=\;\Big[\int_{(Z^\varepsilon)^c}W'(\bar u(\xi)-\bar x^\varepsilon)\,d\xi-V'(\bar x^\varepsilon)\Big]\\ &\quad+\Big[\int_{(Z^\varepsilon)^c}W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi+V''(\bar x^\varepsilon)\Big](\bar x^\varepsilon-\bar u(z))+o_\varepsilon(1)(\bar x^\varepsilon-\bar u(z))\end{aligned}$$

$$
\begin{aligned}
& +W'(0^+)\int_{Z^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi+\int_{Z^\varepsilon}W''(\theta)(\bar u(\xi)-\bar u(z))\,d\xi\\
= \;& 0+\omega^\varepsilon\, v^\varepsilon(z)+W'(0^+)\int_{Z^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\\
& +O(1)\int_{Z^\varepsilon}|\bar u(\xi)-\bar u(z)|\;d\xi+o_\varepsilon(1)v^\varepsilon(z),
\end{aligned}
$$

thanks to the definition of $\bar x^\varepsilon$. We can then estimate v^ε (see (3.8); we also recall the definition of ω^ε (3.11)) as follows:

$$
\begin{aligned}
\|v^\varepsilon\|^2_{L^2} = & \int_{Z^\varepsilon} v^\varepsilon(z)^2 dz = \int_{Z^\varepsilon}\Big[\frac{W'(0^+)}{-\omega^\varepsilon}\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\Big]v^\varepsilon(z)\,dz\\
& +\frac{1}{-\omega^\varepsilon}O(1)\|v^\varepsilon\|_\infty\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz+\frac{o_\varepsilon(1)}{\omega^\varepsilon}\|v^\varepsilon\|^2_{L^2}. \quad (3.13)
\end{aligned}
$$

Let $z\in[0,1]$, and $\zeta:=\inf\{\xi\in[z_0,z_1^\varepsilon]:\ \bar u(\xi)=\bar u(z)\}$, $\zeta':=\sup\{\xi\in[z_0,z_1^\varepsilon]:\ \bar u(\xi)=\bar u(z)\}$. Then

$$
\begin{aligned}
& \int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi=\int_{[z_0,z_1^\varepsilon]\setminus(\zeta,\zeta')}\operatorname{sign}(\xi-z)\,d\xi+\int_\zeta^{\zeta'}0\,d\xi\\
& =\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\xi-z)\,d\xi-\int_\zeta^{\zeta'}\operatorname{sign}(\xi-z)\,d\xi\\
& =[(z_1^\varepsilon-z)-(z-z_0)]-[(\zeta'-z)-(z-\zeta)]=-2\big[z-\tfrac{z_0+z_1^\varepsilon}{2}\big]+2\big[z-\tfrac{\zeta+\zeta'}{2}\big].
\end{aligned}
$$

Then, since $\bar u$ is constant on (ζ,ζ'), so is $z\mapsto v^\varepsilon(z)=\bar x^\varepsilon-\bar u(z)=v^\varepsilon(\frac{\zeta+\zeta'}{2})$, and

$$
\begin{aligned}
& \int_\zeta^{\zeta'}\Big[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\Big]v^\varepsilon(z)\,dz\\
& \quad =-2\int_\zeta^{\zeta'}\Big[z-\frac{z_0+z_1^\varepsilon}{2}\Big]v^\varepsilon(z)\,dz+2v^\varepsilon\Big(\frac{\zeta+\zeta'}{2}\Big)\int_\zeta^{\zeta'}\Big[z-\frac{\zeta+\zeta'}{2}\Big]\,dz\\
& \quad =-2\int_\zeta^{\zeta'}\Big[z-\frac{z_0+z_1^\varepsilon}{2}\Big]v^\varepsilon(z)\,dz. \qquad (3.14)
\end{aligned}
$$

We consider

$$
\begin{aligned}
\Omega:=\big\{&(\zeta,\zeta')\subset Z^\varepsilon:\bar u \text{ is constant on } (\zeta,\zeta'),\\
&(\zeta,\zeta')\text{ being the maximal interval such that this is true}\big\}.
\end{aligned}
$$

Since each element of Ω contains a rational number, Ω is at most countable, and then, thanks to (3.14),

$$\begin{aligned}
&\int_{Z^\varepsilon}\Big[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\Big]v^\varepsilon(z)\,dz\\
&=\int_{Z^\varepsilon\setminus\left(\cup_{(\zeta,\zeta')\in\Omega}(\zeta,\zeta')\right)}\Big[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\Big]v^\varepsilon(z)\,dz\\
&\quad+\sum_{(\zeta,\zeta')\in\Omega}\int_\zeta^{\zeta'}\Big[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\Big]v^\varepsilon(z)\,dz\\
&=\int_{Z^\varepsilon\setminus\left(\cup_{(\zeta,\zeta')\in\Omega}(\zeta,\zeta')\right)}\Big[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\xi-z)\,d\xi\Big]v^\varepsilon(z)\,dz\\
&\quad+\sum_{(\zeta,\zeta')\in\Omega}-2\int_\zeta^{\zeta'}\Big[z-\frac{z_0+z_1^\varepsilon}{2}\Big]v^\varepsilon(z)\,dz\\
&=-2\int_{Z^\varepsilon}\Big[z-\frac{z_0+z_1^\varepsilon}{2}\Big]v^\varepsilon(z)\,dz. \qquad (3.15)
\end{aligned}$$

Thanks to (3.15), (3.13) becomes

$$\begin{aligned}
\Big(1-\frac{o_\varepsilon(1)}{\omega^\varepsilon}\Big)\|v^\varepsilon\|_{L^2}^2 &= -2\frac{W'(0^+)}{-\omega^\varepsilon}\int_{Z^\varepsilon}\Big(z-\frac{z_0+z_1^\varepsilon}{2}\Big)v^\varepsilon(z)\,dz\\
&\quad+\frac{1}{-\omega^\varepsilon}O(\varepsilon)\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\;d\xi\,dz. \qquad (3.16)
\end{aligned}$$

We notice that

$$\begin{aligned}
&\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz = 2\int\int_{(Z^\varepsilon)^2,\,\xi\ge z}[\bar u(\xi)-\bar u(z)]\;d\xi\,dz\\
&=2\int_{Z^\varepsilon}[(z-z_0)\bar u(z)-(z_1^\varepsilon-z)\bar u(z)]\;dz = 4\int_{Z^\varepsilon}\Big(z-\frac{z_0+z_1^\varepsilon}{2}\Big)\bar u(z)\,dz,
\end{aligned}$$

and since $\int_{Z^\varepsilon}(z-\frac{z_0+z_1^\varepsilon}{2})\,dz=0$, we have

$$\begin{aligned}
\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz &= 4\int_{Z^\varepsilon}\Big(z-\frac{z_0+z_1^\varepsilon}{2}\Big)\left(\bar u(z)-\bar x^\varepsilon\right)\;dz\\
&=-4\int_{Z^\varepsilon}\Big(z-\frac{z_0+z_1^\varepsilon}{2}\Big)v^\varepsilon(z)\,dz. \qquad (3.17)
\end{aligned}$$

Finally, thanks to (3.17), (3.16) becomes

$$\|v^\varepsilon\|_{L^2}^2 = \frac{W'(0^+) + O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)} \int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)| \, d\xi \, dz. \tag{3.18}$$

Step 4. We estimate ω^ε. See (3.11) for the definition of ω^ε. In this step, we denote by $\bar{\rho}((a,b))$, with a, $b \in \mathbb{R} \cup \{-\infty, +\infty\}$, the $\bar{\rho}$-measure of the open interval (a,b). Since x_0, $x_0 + \varepsilon \in \operatorname{supp} \bar{\rho} = \overline{\bar{u}([0,1])}$ and $\bar{u}$ is a steady-state of (3.3),

$$\begin{aligned}
0 &= \Big(\int_{\{\xi \in [0,1];\, \bar{u}(\xi) \neq x_0 + \varepsilon\}} W'(\bar{u}(\xi) - (x_0 + \varepsilon)) \, d\xi - V'(x_0 + \varepsilon) \Big) \\
&\quad - \Big(\int_{\{\xi \in [0,1];\, \bar{u}(\xi) \neq x_0\}} W'(\bar{u}(\xi) - x_0) \, d\xi - V'(x_0) \Big) \\
&= \Big(\int_0^1 \Big(W'(0^+) \operatorname{sign}(\bar{u}(\xi) - (x_0 + \varepsilon)) + \tilde{W}'(\bar{u}(\xi) - (x_0 + \varepsilon)) \Big) d\xi - V'(x_0 + \varepsilon) \Big) \\
&\quad - \Big(\int_0^1 \Big(W'(0^+) \operatorname{sign}(\bar{u}(\xi) - x_0) + \tilde{W}'(\bar{u}(\xi) - x_0) \Big) \, d\xi - V'(x_0) \Big) \\
&= \Big(W'(0^+) \Big(\bar{\rho}\big((x_0 + \varepsilon, +\infty)\big) - \bar{\rho}\big((-\infty, x_0 + \varepsilon)\big) \Big) \\
&\qquad + \int_0^1 \tilde{W}'(\bar{u}(\xi) - (x_0 + \varepsilon)) \, d\xi - V'(x_0 + \varepsilon) \Big) \\
&\quad - \Big(W'(0^+) \Big(\bar{\rho}\big((x_0, +\infty)\big) - v\bar{\rho}\big((-\infty, x_0)\big) \Big) + \int_0^1 W'(\bar{u}(\xi) - x_0) d\xi - V'(x_0) \Big) \\
&= -W'(0^+) \left[\bar{\rho}(\{x_0, x_0 + \varepsilon\}) + 2\bar{\rho}\big((x_0, x_0 + \varepsilon)\big) \right] \\
&\quad - \Big[\int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon) \, d\xi + V''(\bar{x}^\varepsilon) \Big] \varepsilon + o(\varepsilon),
\end{aligned}$$

where we applied a Taylor expansion to the regular terms $x \mapsto \tilde{W}'(\bar{u}(\xi) - x)$ and $x \mapsto V'(x)$ at the point $x = \bar{x}^\varepsilon$ (the rest fot the term is estimated as in (3.10)). We notice that

$$\int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon) \, d\xi + V''(\bar{x}^\varepsilon) = \omega^\varepsilon + \int_{Z^\varepsilon} \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon) \, d\xi = \omega^\varepsilon + O(|Z^\varepsilon|),$$

and then,

$$-\varepsilon(\omega^\varepsilon + O(|Z^\varepsilon|)) = W'(0^+) \left[\bar{\rho}(\{x_0, x_0 + \varepsilon\}) + 2\bar{\rho}\big((x_0, x_0 + \varepsilon)\big) \right] + o(\varepsilon). \tag{3.19}$$

Since $|\omega^\varepsilon| \leq \|W''\|_{L^\infty(\text{supp }\bar\rho-\text{supp }\bar\rho)} + \|V''\|_{L^\infty(\text{supp }\bar\rho)}$, we have in particular that $|Z^\varepsilon|$ is of order ε:

$$|Z^\varepsilon| = \bar\rho([x_0, x_0+\varepsilon]) = O(\varepsilon), \tag{3.20}$$

and then, using again (3.19), we get that, for ε small enough,

$$\begin{aligned} -\omega^\varepsilon &= \frac{W'(0^+)}{\varepsilon}\left[\bar\rho(\{x_0, x_0+\varepsilon\}) + 2\bar\rho\big((x_0, x_0+\varepsilon)\big)\right] + o_\varepsilon(1) \\ &\geq W'(0^+)\,\frac{1}{\varepsilon}\bar\rho\big([x_0, x_0+\varepsilon]\big) + o_\varepsilon(1). \end{aligned}$$

We assumed (see (3.4)) that

$$\frac{1}{\varepsilon}\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx > C > 0$$

for ε small enough. Then, for $\varepsilon > 0$ small enough,

$$-\omega^\varepsilon \geq C > 0. \tag{3.21}$$

Step 5. We conclude. Thanks to (3.18), (3.12) becomes

$$\begin{aligned} E(\rho^\varepsilon) - E(\bar\rho) = &-\frac{W'(0^+)+O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ &+ \frac{1}{2}\left(-\omega^\varepsilon + o_\varepsilon(1)\right)\frac{W'(0^+)+O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ = &-\left[\frac{W'(0^+)}{4} + o_\varepsilon(1)\right]\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz, \end{aligned}$$

thanks to (3.21). Finally, we assumed that x_0 is an accumulation point of $\text{supp }\rho^0 \cap [x_0, \infty)$; ε can thus be chosen small enough for $o_\varepsilon(1) \leq \frac{W'(0^+)}{8}$ to hold, and then

$$E(\rho^\varepsilon) - E(\bar\rho) \leq -\frac{W'(0^+)}{8}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz. \tag{3.22}$$

Since x_0 is an accumulation point of $\text{supp }\bar\rho \cap [x_0, x_0+\varepsilon] = \bar u(Z^\varepsilon)$, $\bar u$ cannot be constant on Z^ε, and then

$$E(\rho^\varepsilon) - E(\bar\rho) < 0. \tag{3.23}$$

3.2. **Potentials having a repulsive singularity at $x = 0$.** In this section, we shall consider potentials having a repulsive singularity at $x = 0$; that is, interaction potentials W such that $W'(0) < 0$.

Assumption 6. $V \in C^2(\mathbb{R})$, $W \in C^0(\mathbb{R})$, and there exists $W'(0^+) < 0$ such that

$$(x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x|) \in C^2(\mathbb{R}).$$

For such potentials, we do not know any existence theory; we thus prove in Proposition 5 that if Assumptions 1, 2, 3 and 6 are satisfied, and if $\rho^0 \in W^{2,\infty}(\mathbb{R})$, then there exists a unique solution $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap \mathrm{Lip}_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$.

Proposition 5. *Let ρ^0, V, W satisfy Assumptions 1, 2, 3, and 6. Assume moreover that $\rho^0 \in W^{2,\infty}(\mathbb{R})$. Then there exists a unique solution*

$$\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$$

to (1.1).

If $\rho^0 \in W^{N,\infty}(\mathbb{R})$ and $V \in W^{N+2,\infty}(\mathbb{R})$ (for $N \in \mathbb{N}$), then

$$\rho \in Lip_{loc}(\mathbb{R}_+, W^{N,\infty}(\mathbb{R})).$$

Remark 5. The uniform bound $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ ensures that the solution does not converge to any singular measure. The behavior of the solution in this case is then very different from the two other cases (Assumptions 4 or 5) studied in this paper, where the solution generically converges to a sum of Dirac masses. For a short investigation on the transition from the situation of regular kernels to the situation where W has a singularity at $x = 0$ and is locally repulsive, see [17].

Proof of Proposition 5. Step 1. We show some a priori estimates on ρ, using maximum principle arguments: We consider first $x \in \mathbb{R}$ such that $\rho(t,x) = \|\rho(t,\cdot)\|_\infty$. Then $\partial_x \rho(t,x) = 0$, and

$$\begin{aligned} \partial_t \rho(t,x) &= \partial_x \rho(t,x)(W' * \rho)(t,x) + \rho(t,x)\left((\tilde{W}'' * \rho)(t,x) + V''(x)\right) \\ &\quad -2W'(0^+)\rho(t,x)^2 \\ &= \left((\tilde{W}'' * \rho)(t,x) + V''(x) - 2W'(0^+)\rho(t,x)\right)\rho(t,x) \\ &\leq \left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty - 2W'(0^+)\|\rho(t,\cdot)\|_\infty\right)\|\rho(t,\cdot)\|_\infty. \end{aligned}$$

Then,

$$\|\rho(t,\cdot)\|_\infty \leq \max\left(\|\rho^0\|_\infty, \frac{1}{2|W'(0^+)|}\left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty\right)\right). \qquad (3.24)$$

Let now $N \in \mathbb{N}$ and $x \in \mathbb{R}$ be such that $|\partial_x^N \rho(t,x)| = \|\partial_x^N \rho(t,\cdot)\|_\infty$. With no loss of generality, $\partial_x^N \rho(t,x) \geq 0$, then

$$\begin{aligned}
&\partial_t \partial_x^N \rho(t,x) = \partial_x^{N+1} \left(\rho(W' * \rho + V')\right)(t,x) \\
&= \sum_{n=0}^{N+1} \begin{pmatrix} N \\ n \end{pmatrix} \partial_x^n \rho(t,x) \partial_x^{N+1-n} (W' * \rho + V')(t,x) \\
&= \sum_{n=1}^{N} \begin{pmatrix} N \\ n \end{pmatrix} \partial_x^n \rho(t,x) (\tilde{W}'' * \partial_x^{N-n}\rho - 2W'(0^+)\partial_x^{N-n}\rho + \partial_x^{N+2-n} V)(t,x) \\
&+ \partial_x(\partial_x^N \rho)(t,x)(W' * \rho + V')(t,x) \\
&+ \rho(t,x)\Big[- 2W'(0^+)\partial_x^N \rho(t,x) + \tilde{W}'' * \partial_x^N \rho + \partial_x^{N+2} V\Big] \\
&\leq \sum_{n=1}^{N} \begin{pmatrix} N \\ n \end{pmatrix} \Big[\Big(\|\tilde{W}''\|_{L^1([-2C,2C])} + 2W'(0^+)\Big) \|\partial_x^n \rho(t,\cdot)\|_\infty \|\partial_x^{N-n}\rho(t,\cdot)\|_\infty \\
&\quad + \|\rho(t,\cdot)\|_{W^{N,\infty}} \|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
&+ 0 + \|\rho(t,\cdot)\|_\infty \Big[\|\tilde{W}''\|_{L^1([-2C,2C])} \|\rho(t,\cdot)\|_{W^{N,\infty}} + \|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
&\leq C\left(1 + \|\rho(t,\cdot)\|_{W^{N-1,\infty}}\right) \|\rho(t,\cdot)\|_{W^{N,\infty}},
\end{aligned}$$

where we used the assumption on x to get $\partial_x(\partial_x^N \rho)(t,x) = 0$, the assumption $\partial_x^N \rho(t,x) \geq 0$ to get $\rho(t,x)\big[- 2W'(0^+)\partial_x^N \rho(t,x)\big] \leq 0$, and (1.6) to get that supp $\rho(t,\cdot) \subset [-C,C]$ (uniformly in time).

Since this inequality holds for any $N \geq 1$, and $\|\rho(t,\cdot)\|_{L^\infty} < Cst$ by (3.24), an induction argument shows that, if $\rho^0 \in W^{N,\infty}$, there exists $C = C(N, \|\rho^0\|_{W^{N,\infty}})$ such that

$$\|\rho(t,\cdot)\|_{W^{N,\infty}} \leq \|\rho^0\|_{W^{N,\infty}} e^{Ct}. \tag{3.25}$$

Step 2. We build the solution using the above a priori estimates: In order to prove the existence of a solution $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap \mathrm{Lip}_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$ to (1.1), we use the inductive scheme $\rho_0(t,x) := \rho^0(x)$, and

$$\left\{ \begin{array}{l} \rho_{n+1}(0,\cdot) = \rho^0, \\ \partial_t \rho_{n+1}(t,x) = \partial_x \left(\rho_{n+1} W' * \rho_n + V'\right). \end{array} \right.$$

Thanks to estimates similar to the a priori estimates done in the first part of this proof, one gets the following (uniform in n) estimates:

$$\|\rho_{n+1}(t,\cdot)\|_\infty \leq \|\rho^0\|_\infty e^{C\,t},$$

and there exist $C, T > 0$ such that, for all $t \leq T$,

$$\|\partial_x \rho_{n+1}(t,\cdot)\|_\infty \leq C\|\partial_x \rho^0\|_\infty, \quad \|\partial_t \rho_{n+1}(t,\cdot)\|_\infty \leq C\left(\|\partial_x \rho^0\|_\infty + \|\rho^0\|_\infty\right).$$

Those estimates show that (ρ_n) converges in $L^\infty([0,T]\times\mathbb{R})$ up to an extraction. A further study of $(\rho_{n+1}-\rho_n)$ shows that the whole sequence converges to the unique strong solution ρ of (1.1). Finally, estimate (3.25) shows the propagation of regularity announced in Proposition 5. □

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