Differential and Integral Equations Volume 25, Numbers 11-12 (2012) , 1189–1202

NODAL DOMAINS FOR AN ELLIPTIC PROBLEM WITH THE SPECTRAL PARAMETER NEAR THE FOURTH EIGENVALUE

J. Fleckinger
Institut de Mathématique - CEREMATH-UT1
Université de Toulouse
31042 Toulouse Cedex, France

J.-P. Gossez
Département de Mathématique, C.P. 214
Université Libre de Bruxelles
1050 Bruxelles, Belgium

F. de Thélin
Institut de Mathématique, Université de Toulouse
31062 Toulouse Cédex, France

(Submitted by: Reza Aftabizadeh)

Abstract. We consider the Dirichlet problem $-\Delta u = \mu u + f$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^N$. Let $\hat{\lambda}$ be an eigenvalue with $\hat{\phi}$ an associated eigenfunction. We study the following question $(*)$: *Assuming $\int_\Omega f\hat{\phi} \neq 0$, has u the same number of nodal domains as $\hat{\phi}$ if μ is sufficiently close to $\hat{\lambda}$?* The answer to $(*)$ is known to be affirmative in various cases; see [1], [5], and [6]. Here we study a specific situation where, on the contrary, the answer to $(*)$ is not always affirmative: $\Omega =$ the unit disk in $\mathbb{R}^2$ and $\hat{\lambda} = \lambda_4 = \lambda_5$.

1. Introduction

Consider the Dirichlet problem

$$-\Delta u = \mu u + f \text{ in } \Omega\,,\ u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

with Ω a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $f \in L^q(\Omega)$, $q > N$. Let $\hat{\lambda}$ be an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, with $\hat{\phi}$ an associated eigenfunction. We are interested in the solution u of (1.1) when the spectral parameter μ stays near $\hat{\lambda}$, with $\mu \neq \hat{\lambda}$. More precisely we ask the following question:

Accepted for publication: March 2012.
AMS Subject Classifications: 35J25.

(∗) *Assuming* $\int_\Omega f\hat{\phi} \neq 0$*, has* u *the same number of nodal domains as* $\hat{\phi}$ *if* μ *is sufficiently close to* $\hat{\lambda}$*?*

When $\hat{\lambda}$ is the first eigenvalue, the answer to (∗) is affirmative. This is a consequence of the maximum and antimaximum principles as given in [1]. In two recent papers, we investigated question (∗) for $\hat{\lambda}$ a higher eigenvalue. In [5], it was shown that the answer to (∗) is affirmative if, among other hypotheses, the nodal domains of $\hat{\phi}$ are regular (in the sense that the interior ball condition holds at each of their boundary points). In [6], we investigated a few cases where this regularity hypothesis is not satisfied. We showed that the answer to (∗) is still affirmative when $\hat{\lambda}$ is the second eigenvalue and Ω is either a convex domain in $\mathbb{R}^2$ or a ball or an annulus in $\mathbb{R}^N$. In these examples, the lack of regularity of the nodal domains of $\hat{\phi}$ occurs at points located on $\partial\Omega$.

In this paper we will deal with a situation where this lack of regularity occurs also inside Ω. Specifically we will consider the case where Ω is the unit disk in $\mathbb{R}^2$ and $\hat{\lambda}$ is the fourth eigenvalue. The nodal domains of $\hat{\phi}$ are thus four quadrants, and their lack of regularity occurs not only at four points on $\partial\Omega$ but also at the center of the disk. We will see that the situation gets much more involved and that the answer to question (∗) is not anymore always affirmative. Some conditions involving the Fourier coefficients of f will play a role (see condition (2.1) in Theorem 2 and Hypothesis (H''_f) in Theorem 3 below).

The statements of our main results are given in Section 2 and their proofs are contained in Section 3. Some complementary results and remarks are given in Section 4.

2. STATEMENTS

From now on Ω is the open unit disk $D := D(0,1)$ in $\mathbb{R}^2$. We denote by J_n the Bessel function of order n and by $k_{n,p}$ its p-th positive zero. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta$ in $H_0^1(D)$, and let ϕ_i be the associated eigenfunctions (see e.g. [4], p. 304):

$$\lambda_1 = k_{0,1}^2,\ \phi_1(r,\theta) = J_0(k_{0,1}r),$$

$$\lambda_2 = k_{1,1}^2,\ \phi_2(r,\theta) = J_1(k_{1,1}r)\sin\theta,$$

$$\lambda_3 = k_{1,1}^2,\ \phi_3(r,\theta) = J_1(k_{1,1}r)\cos\theta,$$

$$\lambda_4 = k_{2,1}^2,\ \phi_4(r,\theta) = J_2(k_{2,1}r)\sin 2\theta,$$

$$\lambda_5 = k_{2,1}^2,\ \phi_5(r,\theta) = J_2(k_{2,1}r)\cos 2\theta,$$

$$\lambda_6 = k_{0,2}^2,\ \phi_6(r,\theta) = J_0(k_{0,2}r),$$
$$\lambda_7 = k_{3,1}^2,\ \phi_7(r,\theta) = J_3(k_{3,1}r)\sin 3\theta,$$
$$\lambda_8 = k_{3,1}^2,\ \phi_8(r,\theta) = J_3(k_{3,1}r)\cos 3\theta, \dots$$

In the present situation, question $(*)$ is thus solved affirmatively in [1] for μ near λ_1, in [5] for μ near λ_6 (and $\lambda_{15} = k_{0,3}^2, \dots$), and in [6] for μ near $\lambda_2 = \lambda_3$. Our purpose in this paper is to investigate question $(*)$ for μ near $\lambda_4 = \lambda_5$.

Considering ϕ_4 as given above, the nodal lines are made of two diameters

$$\mathcal{N} := \overline{\{x \in D : \phi_4(x) = 0\}} = \{(r,\theta) : 0 \le r \le 1,\ \theta = j\pi/2,\ j = 0,1,2,3\},$$

and the nodal domains are the four quadrants

$$D_j := \{(r,\theta) : 0 < r < 1,\ j\pi/2 < \theta < (j+1)\pi/2\},\ j = 0,1,2,3.$$

The eigenfunction ϕ_4 is positive in D_0 and D_2; it is negative in D_1 and D_3. $\mathcal{N}$ intersects ∂D at exactly 4 points $P_j = (1, j\pi/2)$, $j = 0,1,2,3$. The lack of regularity of the nodal domains occurs at those 4 points P_j as well as at the center of the disk.

We assume that f in the right-hand side of (1.1) can be written as

$$f = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 + f^\perp,\ \alpha_4 \ne 0, \qquad (H_f)$$

with $f^\perp$ orthogonal to the eigenspaces associated with λ_1, $\lambda_2 = \lambda_3$ and $\lambda_4 = \lambda_5$, and also $f^\perp \in L^q(D)$ for some $q > 2$. Observe that (H_f) implies that f has no component with respect to ϕ_5.

The parameter μ is assumed to stay near $\lambda_4 = \lambda_5$, in the sense that $\lambda_3 < \mu < \lambda_4$ or $\lambda_5 < \mu < \lambda_6$. Let $u = u(f,\mu) \in H_0^1(\Omega)$ be the (unique) solution of (1.1), which belongs to $W^{2,q}(\Omega) \subset \mathcal{C}^1(\bar\Omega)$ (see e.g. [2], p. 198).

Theorem 1. *Assume (H_f). Then u has at least three nodal domains if μ is sufficiently close to λ_4.*

It turns out that in some cases, the solution u may have only three nodal domains. This occurs for instance for $f = \phi_1 + \phi_4$, or more generally as follows:

Theorem 2. *Assume (H_f) and that*

$$f \text{ can be written as a finite sum } f = \Sigma\alpha_i\phi_i. \qquad (H'_f)$$

Assume in addition

$$\Sigma_{j\in J}\frac{\alpha_j}{\lambda_j - \lambda_4} \ne 0, \qquad (2.1)$$

where in (2.1) j runs over the set J of indices of those ϕ_j which involve the Bessel function J_0 (i.e., $j = 1, 6, 15, \dots$). Then u has exactly three nodal domains if μ is sufficiently close to λ_4.

Roughly speaking, assuming $\alpha_4 > 0$ and $\mu < \lambda_4$ in Theorem 2, u will be negative in a small disk centered at 0, and the two negative nodal domains D_1 and D_3 of ϕ_4 will merge into a single negative nodal domain for u.

Our last general result leads to four nodal domains for u in the case where f has no component involving the Bessel functions J_0 and J_1. The following notation is used below:

$$D_{j,\sigma} := \{x \in D_j : \operatorname{dist}(x, \partial D_j) > \sigma\},$$

$$\tilde{D}_{j,\sigma} := \{x \in D : \operatorname{dist}(x, D_j) < \sigma\},$$

for $\sigma > 0$ and for $j = 0, 1, 2, 3$.

Theorem 3. *Assume (H_f) and (H'_f), as well as that*

$$f \text{ has no component involving the Bessel functions } J_0 \text{ and } J_1. \qquad (H''_f)$$

Assume also $\alpha_4 > 0$. Given $\sigma_0 > 0$, there exists $\delta = \delta(f, \sigma_0) > 0$ such that

(i) *if $\lambda_4 - \delta < \mu < \lambda_4$, then u has exactly four nodal domains $\mathcal{O}_j$ with the following two properties:*
 (i_1) *$D_{j,\sigma_0} \subset \mathcal{O}_j \subset \tilde{D}_{j,\sigma_0}$ for $j = 0, \dots, 3$.*
 (i_2) *$u(x)\phi_4(x) > 0$ for $x \in \mathcal{O}_j \cap D_j$, for $j = 0, \dots, 3$;*

(ii) *if $\lambda_4 < \mu < \lambda_4 + \delta$, then the same conclusion as in (i) above holds, with the only change being that now in (i_2), $u(x)\phi_4(x) < 0$ for $x \in \mathcal{O}_j \cap D_j$.*

The same conclusion also holds for $\alpha_4 < 0$, with just another change of sign in (i_2).

3. PROOFS

We start with the

Proof of Theorem 1. The solution $u = u(f, \mu)$ can be written as

$$u = \tilde{u} + \hat{u} + u^{\perp}, \tag{3.1}$$

where

$$\tilde{u} = \sum_{i=1}^{3} \frac{\alpha_i}{\lambda_i - \mu} \phi_i, \quad \hat{u} = \frac{\alpha_4}{\lambda_4 - \mu} \phi_4,$$

and $u^{\perp}$ is orthogonal to ϕ_1, ϕ_2, ϕ_3, ϕ_4, and ϕ_5 and solves

$$-\Delta u^{\perp} = \mu u^{\perp} + f^{\perp}, \text{ in } D,\ u^{\perp} = 0 \text{ on } \partial D.$$

Assuming that μ varies with, say, $\frac{\lambda_3+\lambda_4}{2} < \mu < \lambda_4$, one obtains by a standard argument (see [5], p. 821) that

$$\|\tilde{u} + u^{\perp}\|_{\mathcal{C}^1(\bar{D})} \le C_1, \tag{3.2}$$

with a constant $C_1 = C_1(f)$ independent of μ. This argument is based on the variational characterization of λ_6, the regularity theory for $-\Delta$, and the Sobolev imbedding theorem. The dominating term in (3.1) for μ near λ_4 is thus $\hat{u}$.

The Hopf lemma (or the explicit form of ϕ_4) yields that for some $\varepsilon > 0$ and some $\eta > 0$,

$$\frac{\partial \phi_4}{\partial r} \le -\varepsilon \text{ (respectively } \ge \varepsilon)$$

on $B_j := \{(r,\theta) : r = 1;\, j\pi/2 + \eta \le \theta \le (j+1)\pi/2 - \eta\}$ for $j = 0, 2$ (respectively $j = 1, 3$). Combining with (3.2), one deduces the existence of $\delta = \delta(f) > 0$ such that if $\lambda_4 - \delta < \mu < \lambda_4$,

$$\frac{\partial u}{\partial r} \le -\varepsilon/2 \text{ (respectively } \ge \varepsilon/2)$$

on B_j for $j = 0, 2$ (respectively $j = 1, 3$). We have assumed here $\alpha_4 > 0$. It follows that $u > 0$ (respectively < 0) in D near B_j for $j = 0, 2$, (respectively $j = 1, 3$).

The conclusion now follows easily. Indeed, u clearly changes sign in D. If u has only 2 nodal domains, a positive one $\mathcal{O}_+$ and a negative one $\mathcal{O}_-$, then $\mathcal{O}_+$ (respectively $\mathcal{O}_-$) contains neighborhoods in D of B_0 and B_2 (respectively B_1 and B_3). But this is impossible by the connexity of $\mathcal{O}_+$ and $\mathcal{O}_-$.

A similar argument holds for $\mu > \lambda_4$ and for $\alpha_4 < 0$. □

Remark 4. The above argument clearly holds as soon as the geometry of the nodal domains of an eigenfunction $\hat{\phi}$ in a domain $\Omega \subset \mathbb{R}^2$ looks like that of ϕ_4 in the disk D.

In the proofs of Theorems 2 and 3, we will combine the study of the behavior of u near the origin with the study of the behavior of u away from the origin. We start with the latter and consider, for $0 < R < 1$, the annulus

$$A = A(R) := \{r, \theta\} : R < r < 1, 0 \le \theta \le 2\pi\},$$

as well as for $\sigma > 0$ and $j = 0, 1, 2, 3$, its subsets,

$$A_{j,\sigma} = A_{j,\sigma}(R) := \{x \in D_j \cap A(R) : \text{dist}(x, \mathcal{N}) > \sigma\},$$

$$\tilde{A}_{j,\sigma} = \tilde{A}_{j,\sigma}(R) := \{x \in A(R) : \mathrm{dist}(x, D_j) < \sigma\}.$$

Observe that for $\sigma > 0$ small, $A_{j,\sigma}$ touches part of the "interior" boundary $r = R$ and of the "exterior" boundary $r = 1$ of A.

Proposition 5. *Let $0 < R < 1$ and assume (H_f) with $f^\perp \in W^{1,q}(D)$ for some $q > 2$. Then u has exactly four nodal domains in $A(R)$ if μ is sufficiently close to λ_4. More precisely, given $\sigma_0 > 0$ and assuming $\alpha_4 > 0$, there exists $\delta = \delta(f, R, \sigma_0) > 0$ such that*
(i) if $\lambda_4 - \delta < \mu < \lambda_4$, then u has exactly four nodal domains $\mathcal{V}_j := \mathcal{V}_j(R)$ in $A(R)$ with the following properties:
(i_1) $A_{j,\sigma_0} \subset \mathcal{V}_j \subset \tilde{A}_{j,\sigma_0}$ for $j = 0, \ldots, 3$;
(i_2) $u(x)\phi_4(x) > 0$ for $x \in \mathcal{V}_j \cap D_j$, for $j = 0, \ldots, 3$;
(ii) if $\lambda_4 < \mu < \lambda_4 + \delta$, then the same conclusion as in (i) above holds, with the only change being that now in (i_2), $u(x)\phi_4(x) < 0$ for $x \in \mathcal{V}_j \cap D_j$.
The same conclusion also holds when $\alpha_4 < 0$ with just another change of sign in (i_2).

Proof of Proposition 5. Assume $\alpha_4 > 0$ and $\frac{\lambda_3+\lambda_4}{2} < \mu < \lambda_4$ (similar arguments in the other cases). We write $u = \tilde{u} + \hat{u} + u^\perp$ as in the proof of Theorem 1 above. By the added regularity of f, estimate (3.2) is now strengthened into

$$\|\tilde{u} + u^\perp\|_{\mathcal{C}^2(\bar{D})} \le C_1 \tag{3.3}$$

(see the argument on p. 395 in [6]).

The general idea of the proof is the following. We start with one nodal domain of ϕ_4 in $A(R)$, say $D_0(R) := D_0 \cap A(R)$, and try to control its deformation into a nodal domain of u when μ approaches λ_4.

In the first part of the proof, we look successively at u away from the boundary of $D_0(R)$, near the "exterior" boundary $r = 1$ of $D_0(R)$ and near the "interior" boundary $r = R$ of $D_0(R)$.

The eigenfunction ϕ_4 is positive on $D_0(R)$ and so, for some $\varepsilon > 0$ and $\nu > 0$, $\phi_4 \ge \varepsilon$ on

$$B := \{x \in D_0(R) : \mathrm{dist}(x, \partial D_0(R)) \ge \nu\}.$$

As in the proof of Theorem 1 above, one also has, for some $\varepsilon' > 0$, $\nu' > 0$, and $\eta' > 0$, $\frac{\partial \phi_4}{\partial r}(r, \theta) \le -\varepsilon'$ on

$$B' := \{(r, \theta) : 1 - \nu' \le r \le 1,\ \eta' \le \theta \le \frac{\pi}{2} - \eta'\}.$$

One also clearly has, for some $\varepsilon'' > 0$, $\eta'' > 0$, and $\nu'' > 0$, $\phi_4 \ge \varepsilon''$ on

$$B'' := \{(r, \theta) : R \le r \le R - \nu'',\ \eta'' \le \theta \le \frac{\pi}{2} - \eta''\}.$$

Given any $\sigma > 0$, one can in addition always arrange things so that

$$B \cup B' \cup B'' \supset A_{0,\sigma}.$$

Consequently, using (3.3) (a $\mathcal{C}^1$ estimate suffices here) one deduces the existence of $\delta_1 = \delta_1(f, R, \sigma) > 0$ such that, if $\lambda_4 - \delta_1 < \mu < \lambda_4$, then $u \geq \varepsilon/2$ on B, $\frac{\partial u}{\partial r} \leq -\varepsilon'/2$ on B', and $u \geq \varepsilon''/2$ on B''.

At this stage we have reached the property (P): for any $\sigma > 0$, there exists $\delta_1 = \delta_1(f, R, \sigma) > 0$ such that, if $\lambda_4 - \delta_1 < \mu < \lambda_4$, then u is > 0 in $A_{0,\sigma}$. A similar statement of course holds for $A_{j,\sigma}$, $j = 1, 2, 3$, taking δ_1 smaller if necessary.

We will now investigate the behavior of u near the singular points $(1, j\frac{\pi}{2})$, $j = 0, 1, 2, 3$, as well as near the nodal lines $\{(r, \theta) : R \leq r \leq 1,\ \theta = j\frac{\pi}{2}\}$. The argument below is adapted from [5] and [6].

We start looking at u near the point $(1, 0)$. Using the Serrin corner point lemma [8] as given on page 214 in [3] (or using the explicit form of ϕ_4), one has

$$\frac{\partial^2 \phi_4}{\partial r \partial \theta}(1, 0) > 0.$$

One can then argue as in Lemma 3.3 from [6], applying the mean-value theorem to $\frac{\partial \phi_4}{\partial \theta}$ between (r, θ) and $(1, \theta)$, to get that for some $\eta > 0$, $\nu > 0$, and $C > 0$,

$$\frac{\partial \phi_4}{\partial \theta}(r, \theta) \geq C(1 - r) \text{ on } E_0, \tag{3.4}$$

where $E_0 := \{(r, \theta) : 1 - \nu < r < 1, -\eta < \theta < \eta\}$. Using the $\mathcal{C}^2$ estimate (3.3) and applying the mean-value theorem to $\frac{\partial(\tilde{u} + u^{\perp})}{\partial \theta}$ between (r, θ) and $(1, \theta)$, one can now argue as in Lemma 3.4 from [6] to deduce from (3.4) the existence of $\delta_2 = \delta_2(f) > 0$ such that if $\lambda_4 - \delta_2 < \mu < \lambda_4$, then

$$\frac{\partial u}{\partial \theta}(r, \theta) > 0 \text{ on } E_0. \tag{3.5}$$

Writing

$$E_0^- = \{(r, \theta) \in E_0 : 1 - \nu < r < 1,\ -\eta \leq \theta \leq -\eta/2\},$$

and

$$E_0^+ = \{(r, \theta) \in E_0 : 1 - \nu < r < 1,\ \eta/2 \leq \theta \leq \eta\},$$

and replacing the given $\sigma_0 > 0$ by a smaller $\sigma_1 > 0$ if necessary, one can always arrange things so that

$$E_0^+ \subset A_{0,\sigma_1} \text{ and } E_0^- \subset A_{3,\sigma_1}. \tag{3.6}$$

It is worth observing here that E_0 depends only on ϕ_4 and so σ_1 depends only on ϕ_4 and the given σ_0. Conclusions similar to (3.5) and (3.6) of course hold on sets E_j, E_j^+, and E_j^- near the points $(1, j\frac{\pi}{2})$, $j = 1, 2, 3$, taking δ_2 smaller if necessary.

We now look at u near the nodal line $\{(r,\theta) : R \leq r \leq 1,\ \theta = 0\}$. Here we have by the Hopf lemma (or the explicit form of ϕ_4) that for some $\varepsilon > 0$, $\eta > 0$, and $\nu' > 0$ (with ν' smaller than ν in E_0 above),

$$\frac{\partial \phi_4}{\partial \theta}(r,\theta) \geq \varepsilon > 0 \tag{3.7}$$

on

$$F_0 := \{(r,\theta) : R < r < 1 - \nu',\ -\eta < \theta < \eta\}.$$

Using (3.3) (a $\mathcal{C}^1$ estimate suffices here), one can then deduce from (3.7) the existence of $\delta_3 = \delta_3(f, R) > 0$ such that if $\lambda_4 - \delta_3 < \mu < \lambda_4$, then

$$\frac{\partial u}{\partial \theta}(r,\theta) > 0 \quad \text{on} \quad F_0. \tag{3.8}$$

Writing

$$F_0^- = \{(r,\theta) \in F_0 : R < r < 1 - \nu',\ -\eta < \theta < -\eta/2\},$$

and

$$F_0^+ = \{(r,\theta) \in F_0 : R < r < 1 - \nu',\ \eta/2 < \theta < \eta\},$$

and replacing $\sigma_1 > 0$ by a smaller $\sigma_2 > 0$ if necessary, one can always arrange things so that

$$F_0^+ \subset A_{0,\sigma_2} \text{ and } F_0^- \subset A_{3,\sigma_2}. \tag{3.9}$$

Again here σ_2 depends only on ϕ_4 and the given σ_0. Conclusions similar to (3.8) and (3.9) of course also hold on sets F_j, F_j^+, and F_j^- near the lines $\{(r,\theta) : R \leq r \leq 1,\ \theta = j\frac{\pi}{2}\}$, $j = 1, 2, 3$, taking δ_3 smaller if necessary.

We now put everything together. Using property (P) above with $\sigma = \sigma_2$, we take

$$\delta := \min\{\delta_1(f, R, \sigma_2),\ \delta_2(f),\ \delta_3(f, R)\} > 0,$$

and define for $\lambda_4 - \delta < \mu < \lambda_4$,

$$\begin{aligned}\mathcal{V}_0 := {} & A_{0,\sigma_2} \cup \{x \in E_0 : u(x) > 0\} \cup \{x \in F_0 : u(x) > 0\} \cup \\ & \{x \in E_1 : u(x) > 0\} \cup \{x \in F_1 : u(x) > 0\}.\end{aligned}$$

This set can be seen as a perturbation of the nodal domain $D_0 \cap A(R)$ of ϕ_4 in $A(R)$.

We claim that $\mathcal{V}_0$ is connected. Indeed, A_{0,σ_2} is clearly connected, and we will see that all the points in $\mathcal{V}_0$ which are not in A_{0,σ_2} can be joined within $\mathcal{V}_0$ to A_{0,σ_2}. Given $x = (\overline{r}, \overline{\theta}) \in E_0$, with $u(x) > 0$, we move from

x along the circle $r = \overline{r}$ with θ increasing until we reach E_0^+. Along that curve u increases by (3.5) and consequently remains > 0. That curve is thus included in $\mathcal{V}_0$ and connects x with E_0^+ which, by (3.6), is included in $A_{0,\sigma_1} \subset A_{0,\sigma_2}$ (since $\sigma_2 < \sigma_1$). A similar argument holds for a point $x \in F_0$ with $u(x) > 0$, using (3.8) and (3.9). The same arguments are also valid for E_1 and F_1. A similar "positive" set $\mathcal{V}_2 \supset A_{2,\sigma_2}$ as well as similar "negative" sets $\mathcal{V}_1 \supset A_{1,\sigma_2}$ and $\mathcal{V}_3 \supset A_{3,\sigma_2}$ can of course be constructed.

By construction, $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 = \{x \in A(R) : u(x) \neq 0\}$ and $\partial\mathcal{V}_j \subset \{x \in A(R) : u(x) = 0\} \cup \partial A(R)$. This implies that the $\mathcal{V}_j$'s are the nodal domains of u. Finally, properties (i_1) and (i_2) are clear from the construction (and the observation that if (i_1) holds for σ_2, then it also holds for $\sigma_0 > \sigma_2$). This concludes the proof of Proposition 5. $\square$

We are now ready to start the

Proof of Theorem 2. We assume $\alpha_4 > 0$ and $\frac{\lambda_3+\lambda_4}{2} < \mu < \lambda_4$. The solution u can be written as

$$u \,=\, \Sigma_{j\in J}\frac{\alpha_j}{\lambda_j-\mu}J_0(k_{0,j}r) + \frac{\alpha_4}{\lambda_4-\mu}J_2(k_{2,1}r)\sin(2\theta) + \Sigma_{i\in I}\frac{\alpha_i}{\lambda_i-\mu}\phi_i, \quad (3.10)$$

where the first sum collects all the terms in the Fourier series of u involving the Bessel function J_0 (i.e., $j = 1, 6, 15, \ldots$) and where the last sum is orthogonal to ϕ_4 and ϕ_5 and to all the ϕ_j with $j \in J$ (i.e., $I = \mathbb{N}_0 \backslash (J \cup \{4, 5\})$).

Denoting by $g(r, \mu)$ the first sum in (3.10) and by $e(r, \theta, \mu)$ the last sum in (3.10), one can rewrite u as

$$u \,=\, (g(r,\mu) + e(r,\theta,\mu))\,(1 - h(r,\theta,\mu)\sin 2\theta)\,, \qquad (3.11)$$

where

$$h(r,\theta,\mu) := \Big(\frac{\alpha_4}{\lambda_4-\mu}\Big)\Big(\frac{J_2(k_{2,1}r)}{-g(r,\mu)-e(r,\theta,\mu)}\Big).$$

Note that, by (H'_f), the sums in the definition of $g(r, \mu)$ and $e(r, \theta, \mu)$ are finite. By Hypothesis (2.1), $g(0, \lambda_4) \neq 0$, and we will assume below that $g(0, \lambda_4) < 0$ (with a similar argument in the other case). Since $e(0, \theta, \lambda_4) = 0$, one deduces by continuity the existence of $\varepsilon = \varepsilon(f) > 0$, $c = c(f) > 0$, $R_1 = R_1(f) > 0$, and $\delta_1 = \delta_1(f) > 0$ such that

$$c \,\geq\, -g(r,\mu) - e(r,\theta,\mu) \,\geq\, -g(r,\mu)/2 \,\geq\, \varepsilon, \qquad (3.12)$$

for $0 \leq r \leq R_1$, $\lambda_4 - \delta_1 \leq \mu < \lambda_4$, and any θ. In particular, formula (3.11) makes sense for these values of r, θ, and μ; moreover, the functions J_2, g, e, and h are $\mathcal{C}^\infty$ functions there.

We first look at

$$t_{\mu,\theta}(r) := \frac{J_2(k_{2,1}r)}{-g(r,\mu) - e(r,\theta,\mu)}$$

as well as at

$$t_\mu(r) := \frac{2J_2(k_{2,1}r)}{-g(r,\mu)}.$$

Since the Bessel function of order n is

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{q=0}^{\infty} \frac{(-1)^q (\frac{x}{2})^{2q}}{\Gamma(n+q+1)} \tag{3.13}$$

(see e.g. [4] and [7]), $t_{\mu,\theta}(r)$ can be written as

$$t_{\mu,\theta}(r) = a_0(r^2 + d_1 r^3 + \cdots + d_j r^{j+2} + \cdots),$$

with $d_j = d_j(\mu,\theta)$ and $a_0 = a_0(\mu) = \frac{-\lambda_4}{8g(0,\mu)}$. Moreover, by (3.12), $a_0 \geq \bar{a}_0 > 0$ and $|d_j| \leq \bar{d}_j$, where $\bar{a}_0$ and $\bar{d}_j$ are independent of θ and μ. Similar relations hold for t_μ. The existence of constants $R_2 = R_2(f)$ and $\delta_2 = \delta_2(f)$ independent of θ and μ, $0 < R_2 \leq R_1$, $0 < \delta_2 \leq \delta_1$ follows, such that for each μ with $\lambda_4 - \delta_2 \leq \mu < \lambda_4$ and each θ, the functions $t_{\mu,\theta}$ and t_μ are strictly increasing on $[0, R_2]$.

We now apply Proposition 5 to the annulus $A(R_2/2)$ with $\sigma_0 > 0$ sufficiently small so that $\overline{A_{0,\sigma_0}}$ contains, say, $\{(r,\theta) : r = R_2/2,\ \pi/8 \leq \theta \leq 3\pi/8\}$. This yields the existence of $\delta_3(f, R_2/2, \sigma_0) > 0$ such that u has exactly four nodal domains $\mathcal{V}_j$ for $j = 0,1,2,3$ in $A(R_2/2)$ if $\lambda_4 - \delta_3 < \mu < \lambda_4$, with in addition the properties stated in Proposition 5.

We fix μ with $\lambda_4 - \delta < \mu < \lambda_4$, where $\delta := \min\{\delta_2(f), \delta_3(f, R_2/2, \sigma_0)\}$ and investigate the sign of $u = u(f,\mu)$ in $D(0,R_2)$. Since $h \geq 0$ by (3.12), formula (3.11) implies $u < 0$ for $0 \leq r \leq R_2$ and $\pi/2 \leq \theta \leq \pi$ or $3\pi/2 \leq \theta \leq 2\pi$.

We now look at $t_{\mu,\theta}$ and t_μ. By the inequality involving $-g(r,\mu)/2$ in (3.12), one has $t_{\mu,\theta} \leq t_\mu$. Since $t_\mu(0) = 0$, there exists $r_0(\mu) \in (0, R_2)$ such that $t_\mu(r_0) < \frac{\lambda_4 - \mu}{\alpha_4}$. This implies $t_{\mu,\theta}(r) < \frac{\lambda_4-\mu}{\alpha_4}$ for all $r \in [0, r_0]$ and all θ. It follows that $h < 1$ in $D(0, r_0)$, and consequently $u < 0$ in $D(0, r_0)$.

We go on investigating the sign of u in $D(0,R_2)$ and first concentrate on the first quadrant. Let us fix θ with $0 < \theta < \pi/2$ and look at the strictly increasing function $t_{\mu,\theta}$ on $[0,R_2]$. Either $t_{\mu,\theta}(R_2)\sin 2\theta < \frac{\lambda_4-\mu}{\alpha_4}$ and then $u(r,\theta)$ remains < 0 on $[0,R_2]$, or $t_{\mu,\theta}(R_2)\sin 2\theta \geq \frac{\lambda_4-\mu}{\alpha_4}$ and there exists a unique $r(\theta,\mu) \in [0,R_2]$ such that $t_{\mu,\theta}(r(\theta,\mu))\sin 2\theta = \frac{\lambda_4-\mu}{\alpha_4}$. In this latter case $u(r,\theta)$ is < 0 (respectively $= 0$, > 0) for $r < r(\theta,\mu)$ (respectively $r = r(\theta,\mu)$, $r > r(\theta,\mu)$).

We get in this way, when θ varies, a curve in the first quadrant of $D(0, R_2)$ on which u vanishes, u is < 0 on the "interior" side of this curve, and u is > 0 on the "exterior" side of this curve. Note that by the choice of σ_0 this "exterior" side has a nonempty intersection with $\mathcal{V}_0$. Observe that obviously $r(\theta, \mu) > r_0(\mu)$ defined above. A similar curve can of course be constructed in the third quadrant, with also there $u < 0$ (respectively > 0) on the "interior" (respectively "exterior") side of this curve.

The union of the two "interior" sides constructed above with $D(0, r_0) \cup \mathcal{V}_1 \cup \mathcal{V}_3$ provides a single negative nodal domain for u. The union of the "exterior" side in the first quadrant with $\mathcal{V}_0$ provides a positive nodal domain for u, and the union of the "exterior" side in the third quadrant with $\mathcal{V}_2$ provides another positive nodal domain for u. This completes the proof of Theorem 2. □

We now turn to the

Proof of Theorem 3. By (3.13) there exist positive constants $R_0 < 1/2$, K_1, and K_2 with the following property:

$$K_1 r^2 \leq J_2(k_{2,1} r) \leq K_2 r^2$$

for $0 < r < 2R_0$. We thus have

$$\frac{\partial \phi_4}{\partial \theta}(r, \theta) = 2J_2(k_{2,1} r) \cos(2\theta) \geq K_1 r^2$$

for $0 < r < 2R_0$ and $-\pi/6 < \theta < \pi/6$. On the other hand, by (H_f') and (H_f''), $f^\perp$ is a finite sum $\Sigma_{j>5} \alpha_j \phi_j$, where the ϕ_j involve only J_n for $n \geq 2$. Hence we have

$$\Big|\frac{\partial u^\perp}{\partial \theta}(r, \theta)\Big| \leq \Sigma_{j>5} \frac{|\alpha_j|}{|\lambda_j - \mu|} \Big|\frac{\partial \phi_j}{\partial \theta}(r, \theta)\Big| \leq K_3 r^2,$$

for $0 < r < 2R_0$ and $-\pi/6 < \theta < \pi/6$, where K_3 is independent of μ, r, and θ. Therefore, for μ sufficiently close to λ_4, we have

$$\Big|\frac{\partial u}{\partial \theta}(r, \theta)\Big| \geq \Big(\frac{|\alpha_4|}{\lambda_4 - \mu} K_1 - K_3\Big) r^2 \geq K_4 r^2 > 0$$

for $0 < r < 2R_0$ and $-\pi/6 < \theta < \pi/6$. We also have

$$\phi_4(r, \theta) = J_2(k_{2,1} r) \sin(2\theta) \geq \frac{\sqrt{2}}{2} K_1 r^2$$

and $|u^\perp(r,\theta)| \le K_3 r^2$ for $0 < r < 2R_0$ and $\pi/8 < \theta < 3\pi/8$. Therefore, for μ sufficiently close to λ_4,

$$u(r,\theta) = \frac{\alpha_4}{\lambda_4 - \mu}\phi_4(r,\theta) + u^\perp(r,\theta) \ge (K_1 \frac{\sqrt{2}}{2}\frac{\alpha_4}{\lambda_4 - \mu} - K_3)r^2 > 0$$

for $0 < r < 2R_0$ and $\pi/8 < \theta < 3\pi/8$. Similarly for μ sufficiently close to λ_4, we have $u(r,\theta) < 0$ for $0 < r < 2R_0$ and $-3\pi/8 < \theta < -\pi/8$. Therefore, for each given $r \in (0, 2R_0)$, u vanishes once and only once for θ in the interval $(-\pi/4, \pi/4)$. Analoguous considerations show that u vanishes only once in $(\pi/4, 3\pi/4)$, only once in $(3\pi/4, 5\pi/4)$, and only once in $(5\pi/4, 7\pi/4)$. Applying Proposition 1 on the annulus $A(R_0)$ and connecting the sets where $u > 0$ and $u < 0$, we deduce Theorem 3. □

Remark 6. Theorems 1, 2, and 3 remain valid if ϕ_4 is replaced by any eigenfunction $\hat{\phi}$ associated to $\lambda_4 = \lambda_5$. It is however unclear whether Proposition 7 from the next section remains valid for instance when $f = \phi_2 + \phi_5$.

4. Complementary results

From the proof of Theorem 3 above, one sees that the nodal lines of the solution u intersect at the origin, as those of ϕ_4. In contrast, in the following proposition where u still has four nodal domains, the nodal lines of u do not intersect at the origin.

Proposition 7. *For $f = \phi_2 + \phi_4$ and μ sufficiently close to λ_4, u has exactly four nodal domains.*

Proof of Proposition 7. Choose $\lambda_2 < \mu < \lambda_4$; the case $\lambda_5 < \mu < \lambda_6$ can be treated similarly. The solution u of (1.1) can be written as

$$u = \frac{1}{\lambda_2 - \mu} J_1(k_{1,1}r)\sin\theta(1 - h\cos\theta), \tag{4.1}$$

where

$$h = h(r,\mu) := 2\left(\frac{\mu - \lambda_2}{\lambda_4 - \mu}\right)\left(\frac{J_2(k_{2,1}r)}{J_1(k_{1,1}r)}\right).$$

By (3.13), there exists $R_0 < 1/2$ such that for $0 < r < 2R_0$, the function $t_1 : r \to \frac{J_2(k_{2,1}r)}{J_1(k_{1,1}r)}$ is strictly increasing. Hence, for μ sufficiently close to λ_4, there exists a unique $r_0 = r_0(\mu)$ such that $1 + 2\left(\frac{\lambda_2-\mu}{\lambda_4-\mu}\right)t_1(r_0) = 0$; we compute $t_1(r_0) = \frac{\lambda_4-\mu}{2(\mu-\lambda_2)}$.

We distinguish two cases: (i) $0 < r \le r_0$ and (ii) $r_0 < r \le 2R_0$.

In case (i),

$$\left|2\Big(\frac{\mu-\lambda_2}{\lambda_4-\mu}\Big)\,\frac{J_2(k_{2,1}r)}{J_1(k_{1,1}r)}\cos\theta\right|, \leq 2\Big(\frac{\mu-\lambda_2}{\lambda_4-\mu}\Big)t_1(r_0)=1$$

and the equality holds if and only if $r = r_0$ and $\theta = n\pi$ for $n = 0, 1$. Therefore, by (4.1), we have

$u < 0$ in $B_-(0, r_0(\mu)) := \{(r,\theta) : 0 < r < r_0;\ 0 < \theta < \pi\}$,

$u > 0$ in $B_+(0, r_0(\mu)) := \{(r,\theta) : 0 < r < r_0;\ \pi < \theta < 2\pi\}$.

Note that in the disk $D(0, r_0(\mu))$, u has only two nodal domains.

In case (ii), $t_1(r) > t_1(r_0)$, and so $h > 1$. It follows that $u < 0$ for $\frac{\pi}{2} < \theta < \pi$ and $u > 0$ for $\pi < \theta < \frac{3\pi}{2}$ since $\cos\theta < 0$. We now compute the sign of u for $\cos\theta > 0$. There exists a unique $\theta_r \in (0, \frac{\pi}{2})$ such that $\cos\theta_r = \frac{1}{h}$. So $1 - h\cos\theta < 0$ if and only if $\cos\theta > \cos\theta_r$, which happens for $-\theta_r < \theta < \theta_r$. On the other hand, $1 - h\cos\theta > 0$ if and only if $\theta_r < \theta < 2\pi - \theta_r$. Hence u is > 0 on the two sets

$$\mathcal{E}_0 := \{(r,\theta) : 0 < \theta < \theta_r;\ r_0 < r < 2R_0\},$$

$$\mathcal{E}_2 := \{(r,\theta) : \pi < \theta < 2\pi - \theta_r;\ r_0 < r < 2R_0\},$$

and u is < 0 on the two sets

$$\mathcal{E}_1 := \{(r,\theta) : \theta_r < \theta < \pi;\ r_0 < r < 2R_0\},$$

$$\mathcal{E}_3 := \{(r,\theta) : 2\pi - \theta_r < \theta < 2\pi;\ r_0 < r < 2R_0\}.$$

Finally we apply Proposition 1 on the annulus $A(R_0)$. We obtain in this way $u > 0$ in $\mathcal{O}_0 := \mathcal{V}_0 \cup \mathcal{E}_0$ and in $\mathcal{O}_2 := \mathcal{V}_2 \cup \mathcal{E}_2 \cup B_+(0, r_0(\mu))$, and $u < 0$ in $\mathcal{O}_1 := \mathcal{V}_1 \cup \mathcal{E}_1 \cup B_-(0; r_0(\mu))$ and in $\mathcal{O}_3 := \mathcal{V}_3 \cup \mathcal{E}_3$. Since there are continuous paths joining $\mathcal{V}_j$ and $\mathcal{E}_j$ in $\mathcal{O}_j$, the sets $\mathcal{O}_j$ for $j = 0, 1, 2, 3$ are connected and are the four nodal domains of u. □

Remark 8. The conclusion of Proposition 7 remains valid if (H_f) and (H'_f) hold and f can be written as

$$f \;=\; \Sigma_{j\in J}\alpha_l\phi_j + \alpha_4\phi_4 + \Sigma_{i\in I}\alpha_i\phi_i,$$

where the first sum collects all the terms involving J_1 and where the last sum does not contain any term involving J_0 or J_1, with moreover

$$\Sigma_{j\in J}\frac{\alpha_j k_{1,j}}{\lambda_j - \lambda_4} \neq 0.$$

Remark 9. Note that for $f = \phi_1 + \phi_4$, we get three nodal domains (see Theorem 2), while for $f = \phi_2 + \phi_4$, we get four nodal domains (see Proposition 7).

Remark 10. For μ near λ_7, the situation is rather similar to that in the present paper. The solution u will have six nodal domains, as ϕ_7, under suitable assumptions on f of the type of those in Theorem 3. On the contrary, u will have only four nodal domains if, for example, $f = \phi_1 + \phi_7$.

REFERENCES

[1] D. Arcoya and J. Gamez, *Bifurcation theory and related problems: antimaximum principle and resonance,* Comm. Part. Diff. Equat., 26 (2001), 1879–1911.
[2] H. Brezis, "Analyse Fonctionnelle," Masson, Paris, 1983.
[3] B. Gidas, W. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle,* Comm. Math. Phys., 68 (1979), 209–243.
[4] R. Courant and D. Hilbert, "Methods of Mathematical Physics," vol. 1, Interscience, New York, 1966.
[5] J. Fleckinger, J.P. Gossez, and F. de Thélin, *Maximum and antimaximum principles: Beyond the first eigenvalue,* Diff. Int. Equations, 22 N. 9–10 (2009), 815–828.
[6] J. Fleckinger, J.P. Gossez, and F. de Thélin, *Maximum and antimaximum principles near the second eigenvalue,* Diff. Int. Equations, 24 N. 3–4 (2011), 389–400.
[7] L. Schwartz, "Méthodes Mathématiques pour la Physique," Hermann, Paris, 1955.
[8] J. Serrin, *A symmetry problem in potential theory,* Arch. Rat. Mech., 43 (1971), 304–318.