INFINITE SEMIPOSITONE PROBLEMS WITH ASYMPTOTICALLY LINEAR GROWTH FORCING TERMS

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Abstract. We study the existence of positive solutions to the singular problem

$$\begin{cases} -\Delta_p u = \lambda f(u) - \frac{1}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$, $0 < \alpha < 1$, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is asymptotically p-linear at ∞. We prove the existence of positive solutions for a certain range of λ using the method of sub-supersolutions. We also extend our study to classes of systems which have forcing terms satisfying a combined asymptotically p-linear condition at ∞ and to corresponding problems on exterior domains.

1. Introduction

Consider the boundary-value problem

$$\begin{cases} -\Delta_p u = g(\lambda, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g(\lambda, u) = \lambda f(u) - \frac{1}{u^\alpha}$, λ is a positive parameter,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

$p > 1$, Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$, $0 < \alpha < 1$, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is asymptotically

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p-linear at ∞. We prove the existence of a positive solution of (1.1) for a certain range of λ. We first state some hypotheses on f which we will use to state our main result.

(H_1) There exist $\sigma_1 > 0$, $k > 0$, and $s_0 > 0$ such that $f(s) \geq \sigma_1 s^{p-1} - k$ for every $0 \leq s \leq s_0$.

(H_2) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \sigma$ for some $\sigma > 0$.

Theorem 1.1. *Assume* (H_1)–(H_2). *Then there exist positive constants* $s_0^*(\sigma,\Omega)$, $J(\Omega)$, $\underline{\lambda}$, *and* $\hat{\lambda}(>\underline{\lambda})$ *such that if* $s_0 \geq s_0^*$ *and* $\frac{\sigma_1}{\sigma} \geq J$, (1.1) *has a positive solution for* $\lambda \in [\underline{\lambda}, \hat{\lambda}]$.

We also extend our results for (1.1) to the system

$$\begin{cases} -\Delta_p u = \lambda f_1(v) - \frac{1}{u^\alpha} & \text{in } \Omega, \\ -\Delta_p v = \lambda f_2(u) - \frac{1}{v^\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the nonlinearities $f_i's$, $i = 1, 2$ are continuous, nondecreasing, and satisfy

(H_3) There exist $\sigma_i > 0$, $k_i > 0$, and $s_i > 0$ such that $f_i(s) \geq \sigma_i s^{p-1} - k_i$ for every $0 \leq s \leq s_i$, $i = 1, 2$.

(H_4) $\lim_{s\to\infty} \frac{f_1([f_2(s)]^{p-1})}{s^{p-1}} = \sigma$ for some $\sigma > 0$.

(H_5) There exists $\tau \in \mathbb{R}$ such that for each $M > 0$, $f_1(Ms) \leq M^\tau f_1(s)$ for $s \gg 1$.

In particular we establish

Theorem 1.2. *Assume* (H_3)–(H_5). *Then there exist positive constants* $s_0^*(\sigma,\Omega)$, $J^*(\Omega)$, λ_*, *and* $\lambda_{**}(>\lambda_*)$ *such that if* $\min\{s_1, s_2\} \geq s_0^*$ *and* $\frac{\min(\sigma_1,\sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \geq J^*$, (1.2) *has a positive solution for* $\lambda \in [\lambda_*, \lambda_{**}]$.

Remark 1. Explicit expressions of s_0^*, J, J^*, $\underline{\lambda}$, $\hat{\lambda}(>\underline{\lambda})$, λ_*, and $\lambda_{**}(>\lambda_*)$ are given in the proofs of the respective theorems.

The case when $g(\lambda, 0) < 0$ (and finite) is referred to in the literature as a semipositone problem. Finding a positive solution for a semipositone problem is well known to be challenging (see [2] and [5]) . Here we consider the more challenging case when $\lim_{u\to 0+} g(\lambda, u) = -\infty$, which has received attention very recently and is referred to as an infinite semipositone problem. However, most of these studies have concentrated on the case when the nonlinear function satisfies a sublinear condition at ∞ (see [6], [7], and [9]). The only paper to our knowledge dealing with an infinite semipositone problem with an asymptotically linear nonlinearity is [8], where the author is restricted to the case $p = 2$. Also here the existence of a positive solution is focused near $\frac{\lambda_1}{\sigma}$, where λ_1 is the first eigenvalue of $-\Delta$. See also [1] and [11], where asymptotically linear nonlinearities have been discussed in the case of a nonsingular semipostione problem and an infinite positone problem. In this paper we focus on the existence of positive solutions in a region away from $\frac{\lambda_1}{\sigma}$ (see Figure 1).

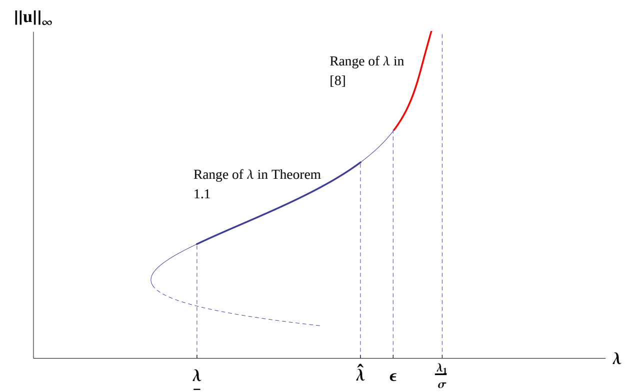

FIGURE 1.

We also establish an existence result for classes of systems which have nonlinearities satisfying a combined asymptotically p-linear condition at ∞.

Finally, in this paper we also study the existence of positive radial solutions to the problem

$$\begin{cases} -\Delta_p u = K(|x|)g(\lambda, u) & x \in \Omega, \\ u = 0 & \text{if } |x| = r_0, \\ u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

where λ and $g(\lambda, u)$ are as before, $r_0 > 0$, $\Omega = \{x \in \mathbb{R}^n : |x| > r_0\}$, and $n > p$ is an exterior domain. Here $K : [r_0, \infty) \to (0, \infty)$ belongs to a class of continuous functions such that $\lim_{r\to\infty} K(r) = 0$. By using transformations $r = |x|$ and $s = (\frac{r}{r_0})^{\frac{-n+p}{p-1}}$, we reduce (1.3) to the following two-point boundary-value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = h(s)g(\lambda, u), & 0 < s < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $h(s) = (\frac{p-1}{n-p})^p r_0^p s^{\frac{-p(n-1)}{n-p}} K(r_0 s^{\frac{-(p-1)}{n-p}})$. We will assume that $K(r) < \frac{1}{r^{n+\theta}}$ for $r \gg 1$ and for some $\theta > 0$. Then equivalently h satisfies

$$\text{there exist } \epsilon_1 > 0 \text{ and } d > 0 \text{ such that } h(s) \leq \frac{d}{s^\beta} \text{ for all } s \in (0, \epsilon_1], \quad (1.5)$$

where $\beta = \frac{n-p-\theta(p-1)}{n-p}$. Note that if $\theta \geq \frac{n-p}{p-1}$, $h(s)$ is nonsingular at 0 and then (1.4) can be studied similarly to (1.1). Hence we focus on the case when $\theta < \frac{n-p}{p-1}$, which forces h to be singular at 0. Thus we assume $K(r) < \frac{1}{r^{n+\theta}}$ for $r \gg 1$ and for $(\frac{n-p}{p-1})\alpha < \theta < \frac{n-p}{p-1}$. Since $\theta > (\frac{n-p}{p-1})\alpha$ we note $\beta < 1 - \alpha$. Also note that in this singular case $\hat{h} = \inf_{s\in(0,1)} h(s) > 0$. We establish the following result:

Theorem 1.3. *Assume* (H_1)–(H_2). *Then there exist positive constants* $s^*(\sigma, \Omega)$, $\bar{J}(\Omega)$, $\tilde{\underline{\lambda}}$, *and* $\hat{\lambda}(> \tilde{\underline{\lambda}})$ *such that if* $s_0 \geq s^*$ *and* $\frac{\sigma_1}{\sigma} \geq \bar{J}$, (1.3) *has a positive solution for* $\lambda \in [\tilde{\underline{\lambda}}, \hat{\lambda}]$.

We next extend our results for (1.3) to the following system:

$$\begin{cases} -\Delta u = K_1(|x|)\big(\lambda f_1(v) - \frac{1}{u^\alpha}\big), & x \in \Omega, \\ -\Delta v = K_2(|x|)\big(\lambda f_2(u) - \frac{1}{v^\alpha}\big), & x \in \Omega, \\ u, v = 0 & \text{if } |x| = r_0, \\ u, v \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.6}$$

where the f_i's are continuous, nondecreasing functions satisfying (H_3)–(H_5) and the $K_i : [r_0, \infty) \to (0, \infty)$ are continuous functions which satisfy for $i = 1, 2$,

$$K_i(r) < \frac{1}{r^{n+\theta}} \text{ for } r \gg 1 \text{ and for } \theta \text{ such that } \left(\frac{n-p}{p-1}\right)\alpha < \theta < \frac{n-p}{p-1}.$$

By using the same transformations as above we reduce (1.6) to

$$\begin{cases} -u'' = h_1(t)\big(\lambda f_1(v) - \frac{1}{u^\alpha}\big), & 0 < t < 1, \\ -v'' = h_2(t)\big(\lambda f_2(u) - \frac{1}{v^\alpha}\big), & 0 < t < 1, \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.7}$$

where $h_i(s) = (\frac{p-1}{n-p})^p r_0^p s^{\frac{-p(n-1)}{n-p}} K_i(r_0 s^{\frac{-(p-1)}{n-p}})$, $i = 1, 2$. Again we note here that the h_i's satisfy

$$\text{there exist } \epsilon > 0 \text{ and } d > 0 \text{ such that } h_i(s) \leq \frac{d}{s^\beta} \text{ for all } s \in (0, \epsilon],\ i = 1, 2,$$

where $\beta = \frac{n-p-\theta(p-1)}{n-p}$. Then we establish

Theorem 1.4. *Assume* (H_3)–(H_5). *Then there exist positive constants* $s^*(\sigma, \Omega)$, $\bar{J}^*(\Omega)$, $\tilde{\lambda_*}$, *and* $\lambda_{**}(> \tilde{\lambda_*})$ *such that if* $\min\{s_1, s_2\} \geq s^*$ *and* $\frac{\min(\sigma_1,\sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \geq \bar{J}^*$, (1.6) *has a positive solution for* $\lambda \in [\tilde{\lambda_*}, \lambda_{**}]$.

Remark 2. Explicit expressions of s^*, $\bar{J}$, $\bar{J}^*$, $\underline{\tilde{\lambda}}$, $\hat{\lambda}(> \underline{\tilde{\lambda}})$, $\tilde{\lambda_*}$, and $\lambda_{**}(> \tilde{\lambda_*})$ are given in the proofs of the respective theorems.

We prove our existence results by the method of sub-supersolutions. By a subsolution of (1.1) we mean a function $\psi \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ that satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \leq \int_\Omega g(\lambda, \psi)w, & \text{for every } w \in W, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and by a supersolution we mean a function $Z \in W^{1,p}(\Omega) \bigcap C(\bar{\Omega})$ that satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla Z|^{p-2}\nabla Z \cdot \nabla w \geq \int_\Omega g(\lambda, Z)w, & \text{for every } w \in W, \\ Z > 0 & \text{in } \Omega, \\ Z = 0 & \text{on } \partial\Omega, \end{cases}$$

where $W = \{\xi \in C_0^\infty(\Omega) : \xi \geq 0 \text{ in } \Omega\}$. The following lemma was established by Cui in [3].

Lemma 1.5. (See [3].) *Let* ψ *be a subsolution of* (1.1) *and* Z *be a supersolution of* (1.1) *such that* $\psi \leq Z$ *in* Ω. *Then* (1.1) *has a solution* u *such that* $\psi \leq u \leq Z$ *in* Ω.

Remark 3. This method of sub-supersolutions naturally extends to cooperative systems.

Construction of a subsolution is quite challenging in the semipositone case. Here our test functions for a positive subsolution must come from positive functions ψ such that $-\Delta_p\psi < 0$ near the boundary and $-\Delta_p\psi > 0$ in a large part of the interior (see Figure 2). Infinite semipositone problems are more challenging because in this case a subsolution must also satisfy $\lim_{x\to\partial\Omega} -\Delta_p\psi = -\infty$ since $\lim_{s\to 0^+} g(\lambda, s) = -\infty$.

We prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 and Theorem 1.4 in Section 4, and give examples of nonlinear functions satisfying our hypotheses in Section 5.

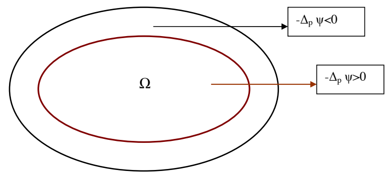

FIGURE 2.

2. PROOF OF THEOREM 1.1

We first construct a supersolution for (1.1). Let $Z = M_\lambda e_p$, where $M_\lambda \gg 1$ and e_p is the unique positive solution of

$$\begin{cases} -\Delta_p e_p = 1 & \text{in } \Omega, \\ e_p = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\tilde{f}(s) = \max_{t\in[0,s]} f(t)$. Then $f(s) \leq \tilde{f}(s)$, $\tilde{f}(s)$ is increasing, and $\lim\limits_{u\to\infty} \frac{\tilde{f}(u)}{u^{p-1}} = \sigma$. Hence, we can choose $M_\lambda \gg 1$ such that

$$2\sigma \geq \frac{\tilde{f}(M_\lambda \|e_p\|_\infty)}{(M_\lambda \|e_p\|_\infty)^{p-1}}.$$

Now let $\hat{\lambda} = \frac{1}{2\sigma\|e_p\|_\infty^{p-1}}$. For $\lambda \leq \hat{\lambda}$,

$$-\Delta_p Z = M_\lambda^{p-1} \geq \frac{\tilde{f}(M_\lambda\|e_p\|_\infty)}{2\sigma\|e_p\|_\infty^{p-1}} \geq \lambda \tilde{f}(M_\lambda e_p) \geq \lambda f(M_\lambda e_p) \geq \lambda f(Z) - \frac{1}{Z^\alpha}.$$

Hence, Z is a supersolution of (1.1) if $\lambda \leq \hat{\lambda}$. Next we construct a subsolution. Consider the boundary-value problem

$$\begin{cases} -\Delta_p z - \mu |z|^{p-2} z = -1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

By the anti-maximum principle established in [10], there exists a constant $\xi = \xi(\Omega) > 0$ such that if $\mu \in (\mu_1, \mu_1 + \xi)$, where μ_1 is the principal eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, then the solution z of (2.1) is positive in Ω and $\frac{\partial z}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal vector. Now fix $\mu \in (\mu_1, \mu_1 + \xi)$ and let z_μ denote the solution of (2.1). Since $z_\mu > 0$ in Ω and $\frac{\partial z_\mu}{\partial \nu} < 0$ on $\partial\Omega$ there exist $m > 0$, $A > 0$, and $\delta > 0$ such that

$|\nabla z_\mu| \geq m$ in Ω_δ and $z_\mu \geq A$ in $\Omega - \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$. Define $\psi = k_0 z_\mu^{\frac{p}{p-1+\alpha}}$, where $k_0 > 0$ is such that

$$\frac{1}{k_0^{p-1+\alpha}}\Big(1+\frac{kk_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big) \leq \min\Big\{\frac{p^{p-1}(1-\alpha)(p-1)m^p}{(p-1+\alpha)^p}, \Big(\frac{p}{p-1+\alpha}\Big)^{p-1} A\Big\}. \tag{2.2}$$

Then

$$\begin{aligned}
&\nabla\psi = k_0\Big(\frac{p}{p-1+\alpha}\Big) z_\mu^{\frac{1-\alpha}{p-1+\alpha}} \nabla z_\mu,\\
&-\Delta_p\psi = -\mathrm{div}(|\nabla\psi|^{p-2}\nabla\psi)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} \mathrm{div}\Big(z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} |\nabla z_\mu|^{p-2}\nabla z_\mu\Big)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{(\nabla z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}) \cdot |\nabla z_\mu|^{p-2}\nabla z_\mu + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \Delta_p z_\mu\Big\}\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{\frac{(1-\alpha)(p-1)}{p-1+\alpha} z_\mu^{\frac{-\alpha p}{p-1+\alpha}} |\nabla z_\mu|^p\\
&\qquad + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} (1-\mu z_\mu^{p-1})\Big\}\\
&= k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} \mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\\
&\qquad - \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p}.
\end{aligned} \tag{2.3}$$

Now we let $s_0^*(\sigma, \Omega) = k_0\|z_\mu^{\frac{p}{p-1+\alpha}}\|_\infty$. If we can prove

$$-\Delta_p\psi \leq \lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - \lambda k - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}, \tag{2.4}$$

then (H_1) implies $-\Delta_p\psi \leq \lambda f(\psi) - \frac{1}{\psi^\alpha}$, and ψ will be a subsolution of (1.1). We will now prove (2.4) by comparing terms in (2.3) and (2.4). Let $\underline{\lambda} = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\sigma_1}$. For $\lambda \geq \underline{\lambda}$,

$$k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} \mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \leq \lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}}. \tag{2.5}$$

Also since $\lambda \le \hat{\lambda} = \frac{1}{2\sigma\|e_p\|_\infty^{p-1}}$,

$$\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k \le \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \frac{k}{2\sigma\|e_p\|_\infty^{p-1}}$$

$$= \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big)\Big]. \tag{2.6}$$

Now in Ω_δ we have $|\nabla z_\mu| \ge m$, and by (2.2)

$$\frac{1}{k_0^{p-1+\alpha}}(1 + \frac{kk_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}})] \le \frac{p^{p-1}(1-\alpha)(p-1)m^p}{(p-1+\alpha)^p}.$$

Hence,

$$\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k \le \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p} \text{ in } \Omega_\delta. \tag{2.7}$$

From (2.5) and (2.7) it can be seen that (2.4) holds in Ω_δ. We will now prove (2.4) holds also in $\Omega - \Omega_\delta$. Since $z_\mu \ge A$ in $\Omega - \Omega_\delta$ and by (2.2) and (2.6) we get

$$\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k \le \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu$$

$$= k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \text{ in } \Omega - \Omega_\delta. \tag{2.8}$$

From (2.5) and (2.8), (2.4) holds also in $\Omega - \Omega_\delta$. Thus ψ is a positive subsolution of (1.1) if $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. We can now choose $M_\lambda \gg 1$ such that $\psi \le Z$. Let $J(\Omega) = 2\|e_p\|_\infty^{p-1}\mu(\frac{p}{p-1+\alpha})^{p-1}$. If $\frac{\sigma_1}{\sigma} \ge J$ it is easy to see that $\underline{\lambda} \le \hat{\lambda}$, and for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$ we have a positive solution. This completes the proof of Theorem 1.1.

Remark 4. Note that in the proof the choice of k_0 can be adjusted easily to obtain a subsolution for all $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{\sigma})$ where $\underline{\lambda} = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\sigma_1}$. Further, for the case when $p = 2$ using the asymptotically linear condition at ∞, a large-enough supersolution can be created for all $\lambda \le \frac{\lambda_1}{\sigma}$ (see [8] for details). Hence in the case $p = 2$, a positive solution exists for all $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{\sigma})$.

3. PROOF OF THEOREM 1.2

We first construct a supersolution for the system (1.2) when

$$\lambda \leq \frac{1}{(2\sigma)^{\frac{p-1}{p-1+\tau}} \|e_p\|_\infty^{p-1}} = \lambda_{**}.$$

Let $(Z_1, Z_2) = (M_\lambda e_p, [\lambda f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}} e_p)$, where e_p is as before and M_λ is a large positive constant. Since $\lim_{s\to\infty} \frac{f_1(f_2(s)^{p-1})}{s^{p-1}} = \sigma$ we can choose $M_\lambda \gg 1$ such that

$$2\sigma \geq \frac{f_1([f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}})}{(M_\lambda \|e_p\|_\infty)^{p-1}}.$$

Then

$$-\Delta_p Z_1 = M_\lambda^{p-1} \geq \frac{f_1([f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}})}{\|e_p\|_\infty^{p-1} 2\sigma}.$$

Now since $\lambda \leq \lambda_{**}$ we have

$$\begin{aligned} -\Delta_p Z_1 &\geq \frac{\lambda^{\frac{p-1+\tau}{p-1}} \|e_p\|_\infty^{p-1+\tau} f_1([f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}})}{\|e_p\|_\infty^{p-1}} \\ &= \lambda \lambda^{\frac{\tau}{p-1}} \|e_p\|_\infty^{\tau} f_1([f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}}). \end{aligned}$$

Note that (H_4) implies $f_2(s) \to \infty$ as $s \to \infty$. Hence from (H_5) for $M_\lambda \gg 1$ we get

$$\begin{aligned} -\Delta_p Z_1 &\geq \lambda f_1(\lambda^{\frac{1}{p-1}} \|e_p\|_\infty [f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}}) \\ &\geq \lambda f_1([\lambda f_2(M_\lambda \|e_p\|_\infty)]^{\frac{1}{p-1}} \|e_p\|_\infty) \geq \lambda f_1(Z_2) - \frac{1}{Z_1^\alpha}. \end{aligned} \tag{3.1}$$

Also,

$$-\Delta_p Z_2 = \lambda f_2(M_\lambda \|e_p\|_\infty) \geq \lambda f_2(M_\lambda e_p) \geq \lambda f_2(Z_1) - \frac{1}{Z_2^\alpha}. \tag{3.2}$$

Hence, from (3.1) and (3.2) we see that (Z_1, Z_2) is a supersolution of (1.2) when $\lambda \leq \frac{1}{(2\sigma)^{\frac{p-1}{p-1+\tau}} \|e_p\|_\infty^{p-1}}$. Next we let $\psi_1 = \psi_2 = k_0 z_\mu^{\frac{p}{p-1+\alpha}}$, where k_0 is as in (2.2) with $k = \max\{k_1, k_2\}$. Setting $s^* = k_0 \|z_\mu^{\frac{p}{p-1+\alpha}}\|_\infty$ and following the steps in the proof of Theorem 1.1 it is now easy to see that (ψ_1, ψ_2) is a subsolution of (1.2) when $\lambda \in [\lambda_*, \lambda_{**}]$, where λ_{**} is as defined above and $\lambda_* = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\min(\sigma_1, \sigma_2)}$. We now choose $M_\lambda \gg 1$ such that $\psi_1 \leq Z_1$ and

$\psi_2 \le Z_2$. Let $J^*(\Omega) = 2^{\frac{p-1}{p-1+\tau}}\mu(\frac{p}{p-1+\alpha})^{p-1}\|e_p\|_\infty^{p-1}$. If $\frac{\min(\sigma_1,\sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \ge J^*$, then the interval of λ for which we have positive solution is nonempty. Thus we have proven Theorem 1.2.

4. Proofs of Theorem 1.3 and Theorem 1.4

4.1. **Proof of Theorem 1.3.** We begin the proof by constructing the supersolution. Let $Z = M_\lambda e_p$, where $M_\lambda \gg 1$ and e_p is the unique positive solution of

$$\begin{cases} -(|e_p'|^{p-2}e_p')' = h(t) & \text{in } (0,1), \\ e_p(0) = 0 = e_p(1). \end{cases}$$

As in the proof of Theorem (1.1) it can be seen that Z is a supersolution of (1.4) when $\lambda \le \hat{\lambda} = \frac{1}{2\sigma\|e_p\|_\infty^{p-1}}$. Now consider the boundary-value problem

$$\begin{cases} -(|z'|^{p-2}z')' - \mu|z|^{p-2}z = -1 & \text{in } (0,1), \\ z(0) = 0 = z(1). \end{cases} \tag{4.1}$$

By the anti-maximum principle established in [10], there exists a $\xi > 0$ such that if $\mu \in (\mu_1, \mu_1 + \xi)$, where μ_1 is the principal eigenvalue of

$$\begin{cases} -(|z'|^{p-2}z')' = \mu|z|^{p-2}z & \text{in } (0,1), \\ z(0) = 0 = z(1), \end{cases}$$

then the solution z of (4.1) is positive in $(0,1)$ and $|z'| > 0$ at $s = 0, 1$. Now fix a $\mu \in (\mu_1, \mu_1 + \xi)$ and let z_μ denote the solution of (4.1). Since $z_\mu > 0$ in $(0,1)$ and $|z_\mu'| > 0$ at $s = 0, 1$, there exist $m > 0$, $A > 0$, and $\epsilon > 0$ such that $|z_\mu'| \ge m$ in $(0, \epsilon] \cup [1-\epsilon, 1)$ and $z_\mu \ge A$ in $(\epsilon, 1-\epsilon)$, where $\epsilon < \epsilon_1$ (ϵ_1 is as in (1.5)). Also note that there exists a $c > 0$ such that $0 < z_\mu(s) \le cs(1-s)$ for all $s \in (0,1)$. Define $\psi = k_0 z_\mu^{\frac{p-\beta}{p-1+\alpha}}$, where $k_0 > 0$ is such that

$$\begin{aligned} &\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^\alpha z_\lambda *^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big) \\ &\le \min\Big\{\frac{(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)m^p}{(p-1+\alpha)^p dc^\beta}, \Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\frac{A^{1-\beta}}{\bar{c}}\Big\}, \end{aligned} \tag{4.2}$$

where $\bar{c}$ is such that $h(s) \le \bar{c}$ for all $s \in (\epsilon, 1-\epsilon)$ and d and β are as in (1.5). Now,

$$-(|\psi'|^{p-2}\psi')' = k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} \tag{4.3}$$

$$-k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha-\beta)(p-1)}{p-1+\alpha}} - \frac{k_0^{p-1}(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)|z_\mu'|^p}{z_\mu^{\frac{\alpha p+\beta p-\beta}{p-1+\alpha}}(p-1+\alpha)^p}$$

and let $s^*(\sigma,\Omega) = k_0\|z_\mu^{\frac{p-\beta}{p-1+\alpha}}\|_\infty$. If we can prove

$$-(|\psi'|^{p-2}\psi')' \le h(s)\Big[\lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} - \lambda k - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big], \tag{4.4}$$

then by (H_1), ψ will be a subsolution of (1.7). Now we compare the terms in (4.3) and (4.4) to see that (4.4) holds in $(0,1)$. Let $\underline{\tilde{\lambda}} = \frac{\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}}{\sigma_1\hat{h}}$, where $\hat{h} = \inf_{s\in(0,1)} h(s) > 0$. For $\lambda \ge \underline{\tilde{\lambda}}$,

$$k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} \le h(s)\lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}}. \tag{4.5}$$

Also, since $\lambda \le \hat{\lambda}$,

$$\begin{aligned} h(s)\Big[\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] &\le h(s)\Big[\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \frac{k}{2\sigma\|e_p\|_\infty^{p-1}}\Big] \\ &= \frac{h(s)k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big)\Big]. \end{aligned} \tag{4.6}$$

Now in $(0,\epsilon]$ we have $h(s) \le \frac{d}{t^\beta}$ and $z_\mu \le ct$. Hence,

$$\begin{aligned} h(s)\Big[\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] &\le \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big)\Big]\frac{d}{t^\beta} \\ &\le \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma\|e_p\|_\infty^{p-1}}\Big)\Big]\frac{dc^\beta}{z_\mu^\beta}. \end{aligned}$$

Also in $(0,\epsilon]$, $|{z_\mu}'| \ge m$, and thus by (4.2) we have

$$h(s)\Big[\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \le \frac{k_0^{p-1}(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)|z_\mu'|^p}{z_\mu^{\frac{\alpha p+\beta p-\beta}{p-1+\alpha}}(p-1+\alpha)^p}. \tag{4.7}$$

From (4.5) and (4.7) we see that (4.4) holds in $(0,\epsilon]$. Proving that (4.4) holds in $[1-\epsilon,1)$ is easier since h is not singular at $s=1$. Next we prove

(4.4) holds also in $(\epsilon, 1-\epsilon)$. Since $z_\mu \geq A$, $h(s) \leq \bar{c}$ for all $s \in (\epsilon, 1-\epsilon)$, and by (4.2) and (4.6) we get

$$h(s)\Big[\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \leq k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha-\beta)(p-1)}{p-1+\alpha}}. \tag{4.8}$$

Thus, (4.4) holds also in $(\epsilon, 1-\epsilon)$ and ψ is a subsolution of (1.4). Now we can choose $M_\lambda \gg 1$ such that $\psi \leq Z$. Hence (1.3) has a positive solution when $\lambda \in [\underline{\tilde{\lambda}}, \hat{\lambda}]$. Let $\bar{J}(\Omega) = \frac{2\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}\|e_p\|_\infty^{p-1}}{\hat{h}}$. It is clear that if $\frac{\sigma_1}{\sigma} \geq \bar{J}$ we have a nonempty interval of λ where (1.3) has a positive solution.

4.2. **Proof of Theorem 1.4.** The proof of Theorem 1.4 follows using arguments similar to those in the proof of Theorem 1.2 with the necessary adjustments to overcome the singularity from $h(s)$ (as done in the proof of Theorem 1.3). Here, $s^* = k_0\|z_\mu^{\frac{p-\beta}{p-1+\alpha}}\|_\infty$, $\bar{J}^*(\Omega) = \frac{2^{\frac{p-1}{p-1+\tau}}\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}\|e_p\|_\infty^{p-1}}{\hat{h}}$, $\tilde{\lambda}_* = \frac{\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}}{\min(\sigma_1,\sigma_2)\hat{h}}$, and $\lambda_{**} = \frac{1}{\sigma^{\frac{p-1}{p-1+\tau}}\|e_p\|_\infty^{p-1}}$.

5. Examples and computational results

5.1. **Single-equation case.** Here we give a simple example of a function satisfying our hypotheses for Theorem 1.1. Note that the same example satisfies the hypotheses of Theorem 1.3. Consider the function $f(s, m_0) = \sigma s^{p-1} + m_0 s^\gamma - k$, where $\sigma > 0$, $m_0 > 0$, $p > 1$, $\gamma \in (0, p-1)$, and k is a real number. Now let $s_0 = (\frac{m_0}{m_0^\nu - \sigma})^{\frac{1}{p-1-\gamma}}$ for some $\nu \in (0,1)$. Then for every $0 \leq s \leq s_0$, $m_0 \geq m_0^\nu s^{p-1-\gamma} - \sigma s^{p-1-\gamma}$. Multiplying by s^γ we see that

$$\sigma s^{p-1} + m_0 s^\gamma \geq m_0^\nu s^{p-1}.$$

This implies $f(s) \geq \sigma_1 s^{p-1} - k$ for every $0 \leq s \leq s_0$ where $\sigma_1 = m_0^\nu$. Hence (H_1) is satisfied. Also, f satisfies (H_2) since $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \sigma$. Clearly, when m_0 is large s_0 and $\frac{\sigma_1}{\sigma}$ are also large, and hence Theorem 1.1 holds. In particular, if $\lambda \in [\frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{m_0^\nu}, \frac{1}{2\sigma\|e_p\|_\infty^{p-1}}]$, (1.1) has a positive solution. Note that $\frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{m_0^\nu} \to 0$ as $m_0 \to \infty$, and hence this interval is nonempty when the constant m_0 in f is large enough. In fact, given a $\lambda \in (0, \frac{1}{2\sigma\|e_p\|_\infty^{p-1}}]$, there exists $m^*(\lambda)$ such that if $m_0 > m^*(\lambda)$, (1.1) has a positive solution.

5.2. **Numerical results when $\Omega = (0,1)$ and $p = 2$.** Here we consider the boundary-value problem

$$\begin{cases} -u''(x) = \lambda f(u) - \frac{1}{u^\alpha}, & x \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{5.1}$$

where $f(s) = s^{p-1} + m_0 s^{\frac{1}{2}} - 2$, $m_0 > 0$, and $\alpha \in (0,1)$. Using the quadrature method (see [4]), it follows that the bifurcation diagram of positive solutions of (5.1) is given by

$$G(\rho, \lambda) = \int_0^\rho \frac{ds}{\sqrt{[2\lambda(F(\rho) - F(s)) - (\frac{\rho^{1-\alpha} - s^{1-\alpha}}{1-\alpha})]}} = \frac{1}{2}, \tag{5.2}$$

where $F(s) := \int_0^s f(t)\,dt$ and $\rho = u(\frac{1}{2}) = \|u\|_\infty$. Now we use Mathematica to plot (5.2) and provide the exact bifurcation diagrams when $m_0 = 10$ and $m_0 = 5000$ (see Figure 3).

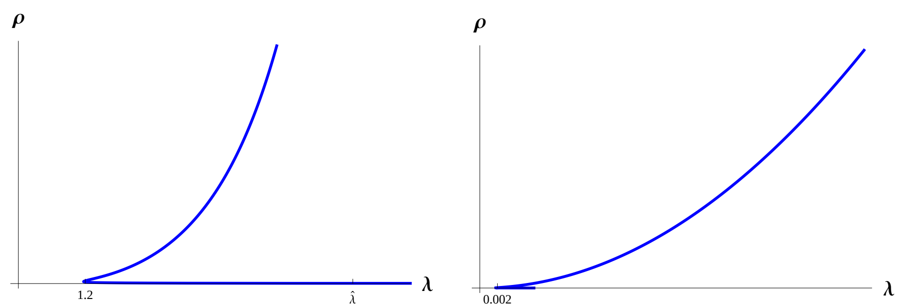

FIGURE 3. Bifurcation diagrams with $m_0 = 10$, $m_0 = 5000$ respectively.

5.3. **System case.** We now give examples of functions satisfying our hypotheses for Theorem 1.2. Here again we note that the same examples satisfy the hypotheses for Theorem 1.4. Consider $f_1(s) = s^{p-1}$ and $f_2(s,a,b) = as^{\frac{1}{p-1}} + bs^\gamma - k$, where $p > 1$, $a, b > 0$, $0 < \gamma < \frac{1}{p-1}$, and k is a real number. Clearly f_1 satisfies (H_3) and (H_5) with $\sigma_1 = 1$, $s_1 = \infty$, and $\tau = p - 1$. Now, set $s_2 = (b^{1-\nu})^{\frac{1}{p-1-\gamma}}$, for some $\nu \in (0,1)$. This implies for $s \leq s_2$, $bs^\gamma \geq b^\nu s^{p-1}$. Thus, f_2 satisfies (H_3) with $\sigma_2 = b^\nu$. Also, $\lim_{s\to\infty} \frac{f_1(f_2(s)^{p-1})}{s^{p-1}} = a^{(p-1)^2}$. Next, when $b \gg 1$, $\min\{s_1, s_2\} = s_2$ is large and $\frac{\min\{\sigma_1,\sigma_2\}}{\sigma^{\frac{p-1}{p-1+\tau}}} = \frac{1}{a^{\frac{(p-1)^3}{p-1+\tau}}}$. Hence when b is large and a is small the hypotheses of Theorem 1.2 hold and we obtain a nonempty interval of λ where a positive solution exists.

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