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TRAVELING WAVES OF A NON-LOCAL CONSERVATION LAW

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Abstract. We consider traveling waves related to shocks for a non-local scalar conservation law

$$u_t + (f(u))_x = K * u - u,$$

where f is an arbitrary convex function, $K * u$ stands for the convolution in the spatial variable x, and K is an arbitrary non-negative kernel with unit mass centered at the origin. Given a pair of values at $\pm\infty$ with $u_- > u_+$, we establish here the existence and uniqueness of a traveling-wave solution $u = \phi\,(x - ct)$ such that the wave speed

$$c = \frac{f\,(u_+) - f\,(u_-)}{u_+ - u_-}$$

and ϕ, satisfying $\phi\,(\pm\infty) = u_\pm$, is a smooth decreasing function that can have at most one jump discontinuity at the origin. Our approach here is to consider a family of "truncated" problems on $[-n, n]$ which can be solved using Schauder's fixed-point theorem and then sending $n \to \infty$.

1. Introduction

This paper is concerned with traveling waves related to shock waves for a class of non-local scalar conservation laws of the form

$$u_t + (f(u))_x = K * u - u, \qquad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

Here f is a convex function,

$$K * \phi(x) := \int_{\mathbb{R}} K(x-y)\phi(y)\,dy$$

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stands for convolution, and K is a non-negative integrable function with unit mass centered at the origin. More precisely,

$$\begin{cases} f \in C^2(\mathbb{R}), \quad f'' > 0 \text{ in } \mathbb{R}, \\ K \geq 0,\ \int_{\mathbb{R}} K(z)\,dz = 1,\ \int_{\mathbb{R}} |z|\,K(z)\,dz < \infty,\ \int_{\mathbb{R}} zK(z)\,dz = 0. \end{cases} \tag{1.2}$$

A traveling wave is a solution of the form $u(x,t) = \phi(z)$, where $z = x - ct$ is the coordinate in the moving frame whose origin moves with velocity c. Of concern are traveling waves that connect constant states u_- at $z = -\infty$ and u_+ at $z = \infty$. This translates to the following traveling-wave problem, for (c, ϕ):

$$\begin{cases} (\,f(\phi) - c\phi\,)_z + \phi = K * \phi \qquad \text{in } \mathbb{R}, \\ \phi(\pm\infty) = u_\pm. \end{cases} \tag{1.3}$$

Here to account for possible discontinuities, the differential equation is understood as

$$f(\phi(z)) - c\phi(z) = C + \int_0^z [K * \phi(y) - \phi(y)]\,dy \qquad \forall\, z \in \mathbb{R},$$

where C is a constant. Thus, $f(\phi) - c\phi$ is continuous across any discontinuities of ϕ. Sending z to $\pm\infty$ and taking the difference we find that

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} - c = \frac{1}{u_+ - u_-}\int_{\mathbb{R}} (K * \phi - \phi)\,dy = -\int_{\mathbb{R}} zK(z)\,dz, \tag{1.4}$$

where the second equation is to be shown in the next section. Hence, under the assumption $\int_{\mathbb{R}} zK(z)\,dz = 0$, the traveling speed is given uniquely by the Rankine–Hugoniot formula

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{1.5}$$

In this paper, we shall prove the following

Theorem 1. *Assume* (1.2). *For each pair* $(u_-, u_+) \in \mathbb{R}^2$ *satisfying* $u_- > u_+$, *there exists a solution* (c, ϕ) *to* (1.3), *where* c *is given by* (1.5), $\phi \in C^1(\mathbb{R} \setminus \{0\})$,

$$\phi' < 0 \text{ in } \mathbb{R} \setminus \{0\}, \quad \phi > K * \phi \text{ in } (-\infty, 0), \quad \phi < K * \phi \text{ in } (0, \infty).$$

Moreover, the solution has a jump at the origin (i.e., $\phi(0-) > \phi(0+)$*) if*

$$\max_{s \in \mathbb{R}} \frac{(s - u_+)f(u_-) + (u_- - s)f(u_+) - (u^- - u^+)f(s)}{(u_- - u_+)^2} > \int_{\mathbb{R}} \frac{|z|}{2} K(z)\,dz. \tag{1.6}$$

The traveling-wave problem was raised by Serre [25]. A prototype of the problem is the system

$$u_t + uu_x + q_x = 0, \qquad q_{xx} - q = u_x, \tag{1.7}$$

originally derived by Hamer [13] (see also [21]) from an Euler system about the conservations of mass, momentum, and energy for a radiating gas. Here q is the heat flux given by an integral equation, whereas the Hamer's equation (1.7) is a first-order approximation of a perturbation about an equilibrium. The equation $q_{xx} - q = u_x$ can be solved to give

$$q_x = u - K * u, \quad K(z) = \tfrac{1}{2}e^{-|z|}.$$

Thus, Hamer's system can be put into the general form of

$$u_t + (f(u))_x = K * u - u.$$

When $K(z) = \frac{1}{2}e^{-|z|}$, the function $p := K * u$ satisfies

$$p_{xx} = p - u = K * u - u,$$

so that the non-local traveling-wave problem (1.3) can be transferred to a second-order equation, for which a phase-plane analysis can be used. Schochet and Tadmor [24] first considered continuous solutions (i.e., small $u_- - u_+$), and lately Kawashima and Nishibata [14] proved that when $f = \frac{1}{2}u^2$ and $K(z) = \frac{1}{2}e^{-|z|}$, (1.3) admits a solution if and only if $u_- > u_+$, and the solution is continuous if and only if $u_- - u_+ \le \sqrt{2}$. Recently, Lattanzio, Mascia, and Serre [19] extended the result from a single conservation law (1.1) to more general hyperbolic-elliptic systems.

Non-local operators such as a convolution have been used to replace the classical differential operators, such as the Laplacian, in a variety of linear and non-linear models. In fact, many differential equations originate from approximations of non-local models. For example, van der Waals in 1893 [27] first derived a thermodynamical theory involving a non-local operator $K * u - u$, from which he derived local models using the approximation

$$K * u - u \approx a\, \Delta u,$$

where

$$a = \tfrac{1}{2}\int_{\mathbb{R}^n} |y|^2 K(|y|)\, dy.$$

Non-local models have recently gained popularity; see for example [1, 4, 5, 6, 7, 9, 10, 12, 20, 28] and the references therein.

When $f(u) = \frac{1}{2}u^2$, (1.1) is a non-local Burgers equation and was studied in [13, 14, 15, 16, 24]; for the particular Hamer equation, e.g., (1.1) with

$f(u) = \frac{1}{2}u^2$ and $K(z) = \frac{1}{2}e^{-|z|}$, see [18, 25, 26]. The Burgers equation has a special feature that distinguishes it from any other conservation law with non-quadratic f. In the Burgers equation, one of the physical meanings of u is the velocity of fluid or traffic. Hence, in a moving coordinate system whose origin moves with a velocity s, the corresponding velocity differs from that in the original coordinate system by an additive s. Thus, using a moving frame, one can reduce the general problem to the case $c = 0$ and $u_- = -u_+$, whereas the system of equations are unchanged. In particular, when K is even, the traveling-wave profile ϕ is an odd function. Based on this observation, recently Chmaj [8] established the existence of a traveling wave.

The method of Chmaj [8] is based on a monotonic iteration technique; see the original development in [3]. Note that when $\phi(x) = -\phi(-x)$ and $K(z) = K(-z)$,

$$K * \phi(x) := \int_{\mathbb{R}} K(x-y)\phi(y)\,dy = \int_0^\infty \{K(|x-y|) - K(x+y)\}\,\phi(y)\,dy.$$

For the map from $\phi \in C([0,\infty))$ to $K * \phi \in C([0,\infty))$ to be monotonic, it is necessary and sufficient to have

$$K(|x-y|) \geq K(x+y)$$

for all $x > 0$ and $y > 0$, i.e., $K(\cdot)$ non-increasing on $[0,\infty)$. Chmaj discovered that the map from ψ to ϕ, where ϕ solves

$$\psi\phi_x + \phi = K * \psi$$

in $(0,\infty)$ subject to the boundary condition $\phi(\infty) = u_+$, is monotonic. He cleverly introduced a notion of sub-/supersolution in the sense that

$$\phi\phi_x + \phi - K * \phi \lesseqgtr 0$$

in $(0,\infty)$ for odd ϕ, and demonstrated that $\underline{\phi}(x) = u_+x/|x|$ is a subsolution and $\bar{\phi}(x) = \frac{2u_+}{\pi}\arctan(\varepsilon x)$ is a supersolution for a sufficiently small positive ε, provided that $K(z) = o(z^{-4})$ as $z \to \infty$. Note that if $(\underline{\phi}, \bar{\phi})$ is a sub-/supersolution pair, then $\underline{\phi} \leq \bar{\phi}$ in $(0,\infty)$ and $\bar{\phi} \leq \underline{\phi}$ in $(-\infty,0)$ since $(\underline{\phi}, \bar{\phi})$ is odd. Hence, such a notion of sub-/supersolution is different from the classical one for parabolic equations.

For parabolic differential equations such as $u_t = u_{xx} + f(u)$, the existence, uniqueness, and asymptotic stability of traveling waves has been well-studied since the original 1937 work of Kolmogorov, Petrovsky, and Piskunov [17]; see for example [2, 11, 22, 23] and the references therein. For non-local problems, the first author developed in [7] a quite general framework for the traveling-wave problem of a bistable dynamics $u_t = \mathcal{A}[u]$, where $\mathcal{A}$ is a general non-linear non-local operator, with the fundamental requirement that the system satisfies a comparison principle: If

$$u_t \geq \mathcal{A}[u],\ v_t \geq \mathcal{A}[v], \text{ and } u(\cdot,0) \geq v(\cdot,0),$$

then $u(\cdot,t) \geq v(\cdot,t)$ for all $t \geq 0$.

For the traveling-wave problem (1.3), here we develop a technique totally different from that in [8]. We shall first solve a problem, obtained by truncation, in a bounded interval, and then take the limit in letting the bounded interval approach $\mathbb{R}$. The problem on the bounded interval can be solved using Chmaj's idea in his construction of a monotonic operator [8].

For the uniqueness of traveling waves, in general there are infinitely many non-monotonic solutions to (1.3). In certain admissible classes, Serre [25] proved that traveling waves are unique. For the completeness of the paper, we shall carry out in Section 6 Serre's uniqueness proof for the admissible class

$$\mathcal{A} := \left\{ \phi : \mathbb{R} \to \mathbb{R} : \lim_{y \nearrow x} \phi(x) \geq \lim_{y \searrow x} \phi(x),\ \forall x \in \mathbb{R} \right\}.$$

This admissible class, in which we have uniqueness, is a natural one. The condition that the jumps are decreasing (when the flux is convex) is the so-called "entropy condition." One finds it in many ways. For instance, in the viscous sense, one makes a viscous approximation by adding an ϵu_{xx} term on the right-hand side of (1.1); then the solutions are smooth and they converge, as ϵ tends to zero, towards those discontinuous solutions that satisfy an inequality in the distributional sense. When the limit, a solution of (1.1), is piecewise smooth, this inequality tells us that the jumps are negative. See the discussion in Serre's review paper [25].

The rest of the paper is organized as follows. In Section 2, we show the second equation in (1.4); hence the condition

$$\int_{\mathbb{R}} zK(z)\,dz = 0$$

is necessary and sufficient for the validity of the Rankine–Hugoniot formula. In Section 3, we study the traveling-wave problem on a finite interval $[-n, n]$, imposing the "boundary" condition

$$\phi = u_- \text{ in } (-\infty, -n] \text{ and } \phi = u_+ \text{ in } [n, \infty).$$

In Section 4, we take the limit as $n \to \infty$ to establish the existence of a traveling wave to (1.3). The central difficulty in this approach is to prove that the limit is non-trivial. In Section 5, we show for Hamer's equation (1.7) with boundary condition

$$u(\pm\infty) = \mp M \text{ with } M > 0,$$

that there is a two-parameter family (not including translation invariance) of non-monotonic traveling-wave solutions. Finally, in Section 6, we show that solutions of (1.3) in the class $\mathcal{A}$ are unique (up to a translation).

2. The Wave Speed

The formula (1.5) for the wave speed c follows by integrating the differential equation $[f(\phi) - c\phi]_x = K * \phi - \phi$ over (a, b), sending $a \to -\infty$ and $b \to \infty$, and using the following:

Lemma 2.1. *Assume that*

$$\int_{\mathbb{R}} (1 + |z|)|\,K(z)|\,dz < \infty \ \textit{and} \ \int_{\mathbb{R}} K(z)\,dz = 1.$$

Then for every $\psi \in L^\infty(\mathbb{R})$ having limits

$$\lim_{x\to\pm\infty} \psi(x) =: \psi(\pm\infty),$$

there holds

$$\lim_{a\to-\infty,\, b\to\infty} \int_a^b \{\psi(x) - K * \psi(x)\}\,dx = [\psi(\infty) - \psi(-\infty)] \int_{\mathbb{R}} zK(z)\,dz.$$

Proof. Since $\int_{\mathbb{R}} K(z)\,dz = 1$, for every $a, b \in \mathbb{R}$,

$$\int_a^b \{\psi - K * \psi\}\,dx = \int_{\mathbb{R}} K(z) \left\{ \int_a^b \psi(x)\,dx - \int_a^b \psi(x - z)\,dx \right\} dz$$
$$= \int_{\mathbb{R}} K(z) \left\{ \int_{b-z}^b \psi(x)\,dx - \int_{a-z}^a \psi(x)\,dx \right\} dz$$
$$= [\psi(\infty) - \psi(-\infty)] \int_{\mathbb{R}} zK(z)\,dz + J_1(b) - J_2(a),$$

where

$$J_1(b) := \int_{\mathbb{R}} K(z) \int_{b-z}^b \{\psi(x) - \psi(\infty)\}\,dx\,dz$$
$$= \int_{\mathbb{R}} zK(z) \int_0^1 \{\psi(b - tz) - \psi(\infty)\}\,dt\,dz,$$
$$J_2(a) := \int_{\mathbb{R}} zK(z) \int_0^1 \{\psi(a - tz) - \psi(-\infty)\}\,dt\,dz.$$

It is easy to estimate, when $b > 0$,

$$|J_1(b)| \le \int_{|z|<b/2} |zK(z)|dz \sup_{|x|\le b/2} |\psi(b-x)-\psi(\infty)| + 2\|\psi\|_{L^\infty} \int_{|z|>b/2} |zK(z)|\, dz.$$

Hence $\lim_{b\to\infty} J_1(b) = 0$. Similarly, $\lim_{z\to-\infty} J_2(a) = 0$. This completes the proof. □

3. The Truncated Problem

In the sequel, conditions (1.2) are always assumed. Also u_- and u_+ are fixed constants with $u_- > u_+$. Denote

$$c := \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \qquad g(s) := f(s) - f(u_-) - c[s - u_-];$$

see Figure 1. Since $g'' = f'' > 0$ on $\mathbb{R}$, there exists a unique $u_0 \in (u_+, u_-)$ such that

$$g(u_-) = g(u_+) = g'(u_0) = 0, \quad g' < 0 \text{ in } (-\infty, u_0),\ g' > 0 \text{ in } (u_0, \infty).$$

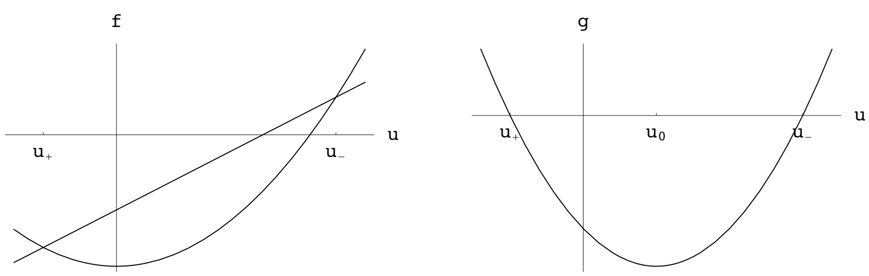

FIGURE 1. Sketch of f and g.

For every constant $n > 0$, consider the following free-boundary problem: Find $b_n \in (-n, n)$ and $u_n \in C((-\infty, b_n-] \cup [b_n+, \infty))$, i.e., u_n is allowed to have a single jump at b_n, such that

$$\begin{cases} g(u_n(x)) = \int_{-n}^{x} (K * u_n(y) - u_n(y))\, dy \qquad \forall\, x \in [-n, n], \\ u_n = u_- \text{ in } (-\infty, -n], \qquad u_n = u_+ \text{ in } [n, \infty), \\ u_- \ge u_n \ge u_0 \text{ in } (-\infty, b_n), \qquad u_0 \ge u_n \ge u_+ \text{ in } (b_n, \infty). \end{cases} \tag{3.1}$$

Theorem 2. *For each $n > 0$,* (3.1) *admits a solution (b_n, u_n) satisfying $(u_n)_x \le 0$ in $(-n, b_n) \cup (b_n, n)$, $u_n \ge K * u_n$ in $(-\infty, b_n)$, and $u_n \le K * u_n$ in (b_n, ∞). The solution to* (3.1) *is unique in the sense that if $(\tilde{b}, \tilde{u})$ is another solution, then $\tilde{u} \equiv u_n$.*

Although the uniqueness assertion is not used in the subsequent analysis and its proof is quite complicated, we present it here for the sake of the completeness of the theorem itself.

The existence of a solution is proven by using Schauder's fixed-point theorem for a mapping $\mathbf{T} : v \to u$, where u is the solution to the ordinary differential equation $(g(u))_x + u = K * v$ with appropriate boundary conditions. It is first proven for the case $K > 0$ and then by approximation for the general case. The uniqueness is shown by a sliding method, where complication arises when K is not everywhere positive.

Proof. For the reader's convenience, the proof is divided into several steps. First, assume that $K > 0$ in $\mathbb{R}$. This condition will be removed in Step 5.

1. The Banach space $L^2((-n, n))$ is taken as the base space for the mapping $\mathbf{T}$, which is defined on the set

$$D = \left\{ v \in L^2((-n,n)) : \begin{array}{c} u_+ \le v(\cdot) \le u_- \text{ on } (-n,n), \\ v(x) \ge v(y) \ \forall\ x < y \end{array} \right\}.$$

It is easy to see that D is a bounded, closed, and convex subset of $L^2\,(-n, n)$. For every $v \in D$, the definition domain of v is extended from $(-n, n)$ to $\mathbb{R}$ by $v := u_-$ on $(-\infty, -n]$, $v := u_+$ on $[n, \infty)$. Then $\{K * v\}_{v \in D}$ is a family of equicontinuous functions:

$$\sup_{|x-y| \le \delta} |K * v(x) - K * v(y)|$$
$$\le [u_- - u^+] \sup_{0 < h \le \delta} \int_{\mathbb{R}} |K(z-h) - K(z)|\, dz \quad \forall\, \delta > 0.$$

In addition, for every $v \in D$ and $x < y$, since $K > 0$ on $\mathbb{R}$ and $v(x - \cdot) - v(y - \cdot)$ is non-negative and not identically equal to zero,

$$K * v(x) - K * v(y) = \int_{\mathbb{R}} K(z)\{v(x-z) - v(y-z)\}\, dz > 0.$$

Thus, $K * v$ is strictly decreasing and continuous on $\mathbb{R}$. Also,

$$u_+ < K * v < u_- \text{ on } \mathbb{R}.$$

2. Motivated by Chmaj's construction [8] of his monotonic operator, here we consider the initial-value problem, for w, of the ordinary differential equation in its maximal existence interval $[-n, b^*]$ to

$$\begin{cases} g'(w)w_x + w = K * v \text{ in } (-n, b^*), \\ w(-n) = u_-, \qquad w > u_0 \text{ in } (-n, b^*), \qquad w(b^*) = u_0. \end{cases}$$

Since $g \in C^2(\mathbb{R})$ and $g'(s) > 0$ for all $s > u_0$, the ordinary differential equation

$$w_x = (K * v - w)/g'(w)$$

with the initial value

$$w(-n) = u_- > u_0$$

has a unique solution in $[-n, -n + \delta)$ for some small $\delta > 0$. The solution can be uniquely extended as long as $w > u_0$. The slope field of the ordinary differential equation shows that $w \le u_-$ on any existence interval of w. Hence, there exists a maximal existence interval $[-n, b^*)$ of the ordinary differential equation on which the solution satisfies $w > u_0$. Since $v \equiv u_+$ in $[n, \infty)$,

$$\lim_{x\to\infty} K * v(x) = u_+.$$

It is then easy to verify that $b^* < \infty$ and

$$w(b^*) := \lim_{x \nearrow b^*} w(x) = u_0.$$

Next we show that $w > K * v$ in $[-n, b^*)$. Fix any $x \in (-n, b^*)$. Integrating

$$g'(w(y))[u_- - w(y)]_y + (u_- - w(y)) = u_- - K * v(y)$$

multiplied by $e^{\int_x^y \frac{1}{g'(w(z))}\,dz}/g'(w(y))$ over $[-n, x]$ gives

$$\begin{aligned} u_- - w(x) &= \int_{-n}^{x} \frac{e^{\int_x^y \frac{1}{g'(w(z))}\,dz}}{g'(w(y))} \{u_- - K * v(y)\}\,dy \\ &< (u_- - K * v(x)) \int_{-n}^{x} \frac{e^{\int_x^y \frac{1}{g'(w(z))}\,dz}}{g'(w(y))}\,dy \qquad (3.2) \\ &= \{u_- - K * v(x)\}\left\{1 - e^{\int_x^{-n} \frac{1}{g'(w(z))}\,dz}\right\} < u_- - K * v(x). \end{aligned}$$

This implies that $w > K * v$ and

$$w_x = (K * v - w)/g'(w) < 0 \text{ in } (-n, b^*).$$

Analogously, since $g \in C^2$ and $g'(s) < 0$ for all $s < u_0$, there exists a unique solution to

$$\begin{cases} g'(\hat{w})\hat{w}_x + \hat{w} = K * v \text{ in } (b_*, n), \\ \hat{w} = u_+ \text{ on } [n, \infty), \qquad \hat{w} < u_0 \text{ in } (b_*, n), \qquad \hat{w}(b_*) = u_0. \end{cases}$$

The solution satisfies $\hat{w} < K * v$ and $\hat{w}_x < 0$ in (b_*, n).

3. Note that $K * v(b^*) \le w(b^*) = u_0 = \hat{w}(b_*) \le K * v(b_*)$. This implies $b_* \le b^*$ since $K * v$ is strictly decreasing. Also, $g(w)$ is strictly decreasing and $g(\hat{w})$ is strictly increasing on $[b_*, b^*]$. As

$$g(u_0) = \min_{s \in \mathbb{R}} \{g(s)\}, g(w) - g(\hat{w})|_{x=b^*} = g(u_0) - g(\hat{w}(b^*)) \ge 0$$

and

$$g(w) - g(\hat{w})|_{x=b_*} = g(w(b_*)) - g(u_0) \le 0,$$

there exists a unique $b \in [b_*, b^*]$ such that $g(w(b)) = g(\hat{w}(b))$. Since $g(u_\pm) =$

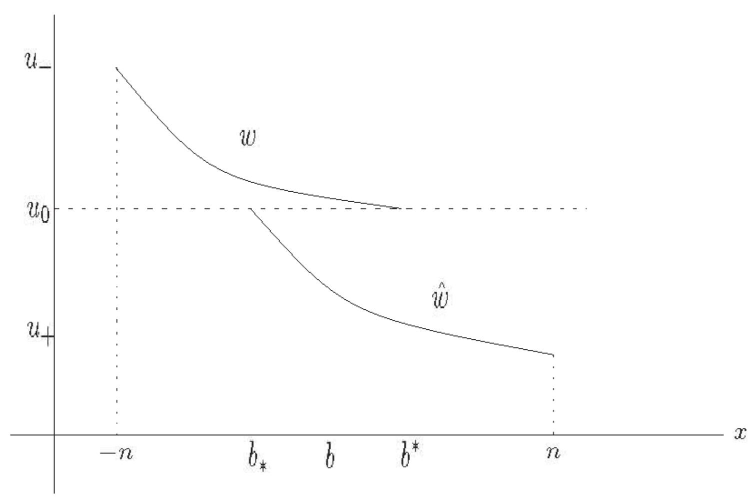

FIGURE 2. Sketch of w and $\hat{w}$.

0 and $g(s) < 0$ for all $s \in (u_+, u_-)$, neither $b \le -n$ nor $b \ge n$ is possible, so $b \in (-n, n)$. Finally, define $u = w$ on $[-n, b-]$ and $u = \hat{w}$ on $[b+, n]$. Then

(b,u) is the solution, where b is the free boundary, to

$$\begin{cases} g(u(x)) = \int_{-n}^{x}\{K*v(y)-u(y)\}\,dy \qquad \forall\, x\in[-n,n], \\ u>u_0 \text{ in } [-n,b), \qquad u<u_0 \text{ in } (b,n]. \end{cases} \tag{3.3}$$

Solutions to this problem are unique since if (b,u) is a solution, then

$$g(u)_x = K*v-u \text{ on } [-n,b)\cup(b,n],$$

so $u=w$ on $[-n,b-]$ and $u=\hat{w}$ on $[b+,n]$ with $g(w(b))=g(\hat{w}(b))$.

4. Now for each $v\in D$, define $\mathbf{T}[v]:=u$, where (b,u) is the unique solution to (3.3). We claim that $\{\mathbf{T}[v]\}_{v\in D}$ is a precompact subset of $D\subset L^2((-n,n))$. To see this, let $u_j=Tv_j$, $j\in\mathbb{N}$ be a sequence in $\{\mathbf{T}[v]\}_{v\in D}$ and $\{b_j\}$ be the sequence of corresponding free boundary. Since for each j, $g(u_j(\cdot))$ is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $2[u_- - u_+]$, there exists a subsequence $\{j_k\}$ such that as $k\to\infty$,

$$g(u_{j_k}(\cdot))\to\tilde{h} \text{ uniformly on } [-n,n] \text{ and } b_{j_k}\to\tilde{b}.$$

Noticing that for each k, $g(u_{j_k}(\cdot))$ has a maximum at b_{j_k}, and is monotone decreasing in $[-n,b_{j_k})$ and monotone increasing in $(b_{j_k},n]$, $\tilde{h}$ has a maximum in $\tilde{b}\in[-n,n]$, and is monotone decreasing in $[-n,\tilde{b})$ and monotone increasing in $(\tilde{b},n]$. Moreover, $\tilde{h}(-n)=\tilde{h}(n)=0$ and $\tilde{h}\,(b^*)\le g(u_0)$. Let $\tilde{u}\in D\cap C[-n,\tilde{b})\cap C(\tilde{b},n]$ be the unique function such that $g(\tilde{u})=\tilde{h}$ with $\tilde{u}(-n)=u_-$ and $\tilde{u}(n)=u_+$. One can show that as $k\to\infty$, $u_{j_k}\to\tilde{u}$ locally uniformly in $[-n,\tilde{b})\cup(\tilde{b},n]$, which together with the uniform bounds of $\{u_{j_k}\}$ and $\tilde{u}$ implies that $u_{j_k}\to\tilde{u}$ in $L^2((-n,n))$. Hence $\{\mathbf{T}[v]\}_{v\in D}$ is a precompact subset of D. To show that $\mathbf{T}$ is continuous, let $v\in L^2((-n,n))$ and $\{v_j\}\in D$ satisfy

$$\lim_{j\to\infty} v_j = v \text{ in } L^2((-n,n)).$$

Then $v\in D$ and

$$\lim_{n\to\infty} K*v_n = K*v \text{ uniformly in } \mathbb{R}.$$

One then derives from the construction of $u_n=\mathbf{T}[v_n]$ and $u=\mathbf{T}[v]$ in the previous steps that

$$\lim_{n\to\infty}\mathbf{T}[v_n]=\mathbf{T}[v] \text{ in } L^2((-n,n)).$$

Hence, $\mathbf{T}$ is a continuous operator. By Schauder's fixed-point theorem, $\mathbf{T}$ admits a fixed point in D, which is a solution to (3.1).

We remark that the extra assumption $K > 0$ helps tremendously in the above arguments. It ensures that $K * v$, where $v \in D$, is strictly decreasing, so $b^* \geq b_*$ and b in (3.3) is unique. The uniqueness of the solution to (3.3) ensures that $\mathbf{T}$ is continuous.

5. Now we remove the extra assumption $K > 0$.

Let K be as in (1.2). Set $K_\varepsilon(z) = \varepsilon e^{-|z|} + (1 - 2\varepsilon)K(z)$. For each $\varepsilon \in (0, 1/2)$, there exists a solution $(b^\varepsilon, u^\varepsilon)$ to (3.1) with K replaced by K_ε. Since $g(u^\varepsilon)$ is Lipschitz continuous with Lipschitz constant $2[u_- - u_+]$, the family $\{g(u^\varepsilon)\}_{0<\varepsilon<1/2}$ is uniformly bounded and equicontinuous. Along a sequence $\varepsilon \searrow 0$, $b^\varepsilon \to b$ and $g(u^\varepsilon) \to G$ uniformly on $[-n, n]$ for some $b \in [-n, n]$ and Lipschitz-continuous function G.

Let $s = h^-(g)$ be the inverse of $s \in [u_0, \infty) \to g = g(s)$ and by $s = h^+(g)$ denote the inverse of $s \in (-\infty, u_0] \to g = g(s)$. Define $u = h^-(G)$ in $[-n, b-]$ and $u = h^+(G)$ in $[b+, n]$. Since $h^\pm$ is strictly monotonic, $g(u_\varepsilon) \to G$ implies $u_\varepsilon \to u$ uniformly in any compact subset of $[-n, b) \cup (b, n]$. Consequently, $K_\varepsilon * u_\varepsilon \to K * u$ uniformly on $[-n, n]$. It follows from the integral equation for $g(u_\varepsilon)$ that

$$g(u(x)) = G(x) = \int_{\pm n}^{x} \{K * u(y) - u(y)\}\, dy \quad \forall\, x \in [-n, n].$$

Also in $(-n, b)$, $u \geq u_0$, $u_x \leq 0$, and $u \geq K * u$, whereas in (b, n), $u \leq u_0$, $u_x \leq 0$, and $u \leq K * u$. Setting $(b_n, u_n) = (b, u)$ completes the existence proof of the theorem.

6. Before proving the uniqueness, let's first look at an example illustrating the complication that may be caused by the non-positiveness of K. Consider the situation

$$g(u) = \tfrac{1}{2}(u^2 - 1), \qquad K(z) = \begin{cases} 1 & \text{when } \frac{9}{2} < |z| < 5, \\ 0 & \text{when } |z| \leq \frac{9}{2} \text{ or } |z| \geq 5. \end{cases}$$

Consider problem (3.1) with $u_- = 1$, $u_+ = -1$, and $n = 2$. For every $v \in D$, $K * v \equiv 0$ in $[-2, 2]$, so the equation for u is $u(u_x + 1) = 0$. It is easy to compute the solution

$$u = \begin{cases} 1 & \text{when } x \leq -2, \\ -1 - x & \text{when } -2 < x < -1, \\ 0 & \text{when } -1 \leq x \leq 1, \\ 1 - x & \text{when } 1 < x < 2, \\ -1 & \text{when } x \geq 2. \end{cases}$$

For any $b \in [-1, 1]$, (b, u) is a solution to (3.1). Here b is not unique, but u is unique.

Note that when $K > 0$ on $\mathbb{R}$, the set $\Gamma(u) := \{x : g(u(x)) = g(u_0)\}$ is at most a singleton. However, as the above example demonstrate, when K vanishes in a large neighborhood of the origin, the set $\Gamma(u)$ could contain a whole interval.

7. Now we show that solutions to (3.1) are unique. Let (b, u) be the solution from the existence proof. Then $u_x \le 0$ on $[-n, b) \cup (b, n]$. Let

$$b_* = \inf\{x : u(x) \le u_0\}, \quad b^* := \sup\{x : u(x) \ge u_0\}.$$

Then $u > u_0$ in $[-n, b_*)$, $u < u_0$ in $(b^*, n]$, $b \in [b_*, b^*]$, and $u = u_0$ in $[b_*, b^*]$ if $b^* > b_*$. Following the same proof as that after (3.2), we can show that $u_x < 0$ in $[-n, b_*) \cup (b^*, n]$. Let $(\tilde{b}, \tilde{u})$ be any solution to (3.1); we want to show that $\tilde{u} \equiv u$.

First we show that

$$u_- > \tilde{u}(x) > u_+, \quad u_- > u(x) > u_+ \qquad \forall\, x \in (-n, n).$$

For this, let $x_* = \max\{x : \tilde{u}(x) = u_-\}$. Then $\tilde{u} < u_-$ in (x_*, ∞) so that

$$K * \tilde{u}(x_*) - u_- = \int_{\mathbb{R}} K(z)[\tilde{u}(x_* - z) - u_-]\, dz < 0$$

since $\int_{-\infty}^0 K(z)\, dz > 0$. If $x_* > -n$, we then would have

$$(g(\tilde{u}))_x|_{x=x_*-} = \tilde{u}(x_*-) - K * \tilde{u}(x_*) > 0,$$

and hence $\tilde{u} > u_-$ in $(x_* - \varepsilon, x_*)$ for small positive ε, which is impossible. Therefore, $x_* = -n$ and $\tilde{u} < u_-$ in $(-n, \infty)$. Similarly, we can show that $\tilde{u} > u_+$ in $(-\infty, n)$. The same conclusion holds also for u. This also implies

$$u_- > K * u > u_+ \text{ and } u_- > K * \tilde{u} > u_+ \text{ in } [-n, n].$$

Next we define

$$\xi := \inf\{t \in [0, 2n] : u(\cdot) \ge \tilde{u}(\cdot + t) \text{ on } [-n, n - t]\},$$
$$\hat{u}(x) = \tilde{u}(x + \xi), \qquad \hat{b} = \tilde{b} - \xi.$$

Since $u(x) \approx u_-$ for all x close to $-n$ and $\tilde{u}(x) \approx u_+$ for all x close to $-n$, $\xi \in [0, 2n)$ is well-defined. Also $u \ge \hat{u}$ on $\mathbb{R}$ since $u \in C([-n, b-] \cup [b+, n])$, $\hat{u} \in C([-n, \hat{b}-] \cup [\hat{b}+, n])$, $u = u_-$ on $(-\infty, -n]$, and $\hat{u} = u_+$ on $[n - \xi, \infty)$. Consequently, $K * u \ge K * \hat{u}$ in $\mathbb{R}$.

We want to show that $\xi = 0$. Suppose to the contrary that $\xi > 0$.

We claim that

$$u > \hat{u} \qquad \text{in } (-n - \xi, b_*) \cup (b^*, n). \tag{3.4}$$

This inequality is trivial when $x \in (-n-\xi,-n] \cup [n-\xi,n)$ since

$$\hat{u} < u_- = u \text{ in } (-n-\xi,n] \text{ and } u > u_+ = \hat{u} \text{ in } [n-\xi,n).$$

In $(-n,n-\xi)$, we use

$$\frac{d}{dx}\{g(u)-g(\hat{u})\} + \{u-\hat{u}\} = K*u - K*\hat{u} \geq 0 \quad \text{in } (-n,n-\xi).$$

Let $W(x) := g(u(x)) - g(\hat{u}(x))$, and

$$L(x) = \begin{cases} \frac{u(x)-\hat{u}(x)}{g(u(x))-g(\hat{u}(x))} & \text{if } W(x) \neq 0, \\ \frac{1}{g'(u(x))} & \text{if } W(x) = 0. \end{cases}$$

On the interval $I = [-n, \min\{b_*, \hat{b}\})$, $u > u_0$ and $\hat{u} \geq u_0$, so that L is continuous and positive in I. Using $W(-n) = u(-n) - \hat{u}(-n) = u_- - \tilde{u}(\xi - n) > 0$ and integrating

$$\left(e^{\int_{-n}^{x} L(z)\,dz} W(x)\right)_x \geq 0$$

over $[-n,x)$ we obtain

$$W(x) \geq W(-n) e^{-\int_{-n}^{x} L(z)\,dz} > 0 \quad \forall\, x \in [-n, \min\{b_*, \hat{b}\}). \tag{3.5}$$

As $g' > 0$ in (u_0,∞), this implies that $u > \hat{u}$ in I. Also since

$$\hat{u} \leq u_0 < u \text{ on } [\min\{b_*, \hat{b}\}, b_*),$$

we conclude that $u > \hat{u}$ in $[-n, b_*)$. In a similar manner, we can show that $u > \hat{u}$ in $(b^*, n-\xi]$. Hence, (3.4) holds.

There are only three possibilities:

$$\text{(i)} \quad u(b-) > u_0 > u(b+), \quad \text{(ii)} \quad u(b\pm) = u_0,\ b_* = b^*, \quad \text{(iii)} \quad b_* < b^*.$$

Case (i): $u(b-) > u_0 > u(b+)$. Then $\hat{u}(b+) \leq u(b+) < u_0$. As $\hat{u} \geq u_0$ for all $x < \hat{b}$, we must have $\hat{b} \leq b$. Then $L(x)$ defined above is bounded in $[-n, \hat{b}-]$ so (3.5) implies $u > \hat{u}$ on $[-n, \hat{b}-]$. Similarly, we have $u > \hat{u}$ on $[b+, n-\xi]$. This excludes the possibility $\hat{b} = b$ since if $b = \hat{b}$, then $u_0 > u(b+) > \hat{u}(b+)$ implies $g(\hat{u}(b+)) > g(u(b+))$, whereas $u(b-) > \hat{u}(b-)$ implies $g(u(b-)) > g(\hat{u}(b-))$; this contradicts the continuity of $g(u)$ and $g(\hat{u})$. Thus $\hat{b} < b$ and $\hat{u} < u$ in $(-n-\xi, n)$. Therefore there exists $\varepsilon \in (0,\xi)$ such that $u(\cdot) > \hat{u}(\cdot - \varepsilon)$ on $[-n, n-\xi+\varepsilon]$. But this would imply $u(\cdot) > \tilde{u}(\cdot + t)$ on $[-n, n-t]$ with $t = \xi - \varepsilon$, contradicting the definition of ξ. Thus, case (i) does not happen.

Case (ii): $u(b\pm) = u_0$ and $b_* = b^* = b$. This implies that $u \in C(\mathbb{R})$ and $g(u(\cdot)) \in C^1((-n,n))$. If $\hat{u}(b+) < u_0$, then as in case (i) above, we have

$\hat{b} < b$, and there exists $\varepsilon > 0$ such that $u(x) > \hat{u}(x-\varepsilon)$ on $[-n, n-\xi+\varepsilon]$, which contradicts the definition of ξ. Hence, we must have $u(b) = \hat{u}(b\pm) = u_0$ so $g(\hat{u}) \in C^1((-n-\xi, n-\xi))$. As $g(u_0) = \min_{s\in\mathbb{R}} g(s)$, we must have

$$g(u)_x|_{x=b} = g(\hat{u})_x|_{x=b} = 0.$$

This implies $K * u(b) = u_0$, $K * \hat{u}(b) = u_0$, and

$$0 = K * u(b) - K * \hat{u}(b) = \int_{\mathbb{R}} K(b-z)[u(z) - \hat{u}(z)]\, dz.$$

Since $u > \hat{u}$ on $(-n-\xi, b)\cup(b, n)$, we must have $K(b-z) = 0\ \forall z \in (-n-\xi, n)$. Consequently, for any $\eta \in [0, \xi]$,

$$K * u(b) - K * u(b+\eta) = \int_{-n-\eta}^{n} K(b-z)[u(z) - u(z+\eta)]\, dz = 0.$$

Therefore, $K * u \equiv u_0$ in $[b, b+\xi]$, so that

$$[g(u)]_x = K * u - u = u_0 - u > 0$$

in $(b, b+\xi]$. Note that $b + \xi = b + (\tilde{b} - \hat{b}) \le \tilde{b} < n$. Now

$$1 = \frac{g(u)_x}{u_0 - u} = \frac{g'(u)}{u_0 - u} u_x = \Psi(u)_x \qquad \forall\, x \in (b, b+\xi),$$

where

$$\Psi(v) := \int_{u_0}^{v} \frac{g'(s)}{u_0 - s}\, ds = \int_{u_0}^{v} \frac{g'(s) - g'(u_0)}{u_0 - s}\, ds \quad \forall\, v \in \mathbb{R}.$$

As $\hat{u}(\cdot) < u(\cdot) < u_0$ in $(b, n-\xi)$, we obtain

$$[\Psi(\hat{u})]_x = \frac{g(\hat{u})_x}{u_0 - \hat{u}} = 1 + \frac{K * \hat{u} - u_0}{u_0 - \hat{u}} \quad \text{in } (b, n-\xi).$$

It follows that for $x \in (b, b+\xi] \cap (b, n-\xi)$,

$$\Psi(u(x)) - \Psi(\hat{u}(x)) = \int_{b}^{x} \frac{u_0 - K * \hat{u}}{u_0 - \hat{u}} \ge 0.$$

This implies, since

$$\Psi'(s) = g'(s)/(u_0 - s) < 0 \text{ in } (-\infty, u_0),$$

that $u(x) \le \hat{u}(x)$ in $[b, b+\xi) \cap (b, n-\xi)$. But this contradicts our earlier conclusion that $u > \hat{u}$ in (b, n). This contradiction shows that case (ii) cannot happen.

Case (iii): $b_* < b^*$. Then $g(u)_x \equiv 0$ in $[b_*, b^*]$, which implies that for every $\varepsilon \in (0, b^* - b_*]$ and $x \in [b_*, b^* - \varepsilon]$

$$0 = K * u(x) - K * u(x + \varepsilon) = \int_{\mathbb{R}} K(x - z)[u(z) - u(z + \varepsilon)]\, dz.$$

As $u_x < 0$ in $(-n, b_*) \cup [b^*, n)$, this implies that $K(x - z) = 0$ for all $z \in (-n, b_*) \cup (b^*, n)$ and all $x \in (b_*, b^*)$. Thus, $K(y) = 0$ in $(x - n, x - b^*) \cup (x - b_*, x + n)$ for all $x \in (b_*, b^*)$, so that $K(y) = 0$ for all $y \in (b_* - n, b^* + n)$ and

$$K(b^* - z) = 0 \quad \forall\, z \in (-n, n + b^* - b_*),$$
$$K(b_* - z) = 0 \quad \forall\, z \in (b_* - b^* - n, n).$$

We now show that $\hat{u}(b^*) < u_0$. Suppose not. Then $\hat{u}(b^*) = u_0$. This implies that $g(\hat{u})_x|_{x=b^*} = 0$ and $K * [u - \hat{u}](b^*) = 0$, so that $K(b^* - z) = 0$ for all $z \in (-n - \xi, b_*]$. Together with the earlier conclusion, we have $K(b^* - z) = 0$ for all $z \in (-n - \xi, b^* - b_* + n)$. Hence $K * u$ is a constant in $[b_*, b^* + \xi]$ and using the same argument as in the second part of the proof of case (ii) we derive a contradiction. Hence, we must have $\hat{u}(b^*) < 0$.

Next we show that there exists $\varepsilon \in (0, \xi)$ such that $\hat{u}(b_* - \varepsilon) \le u_0$. Suppose not. Then $\hat{u} > u_0$ in $(-\infty, b_*)$ and $\hat{u}(b_*) = u_0$. Then $g(\hat{u})_x|_{x=b_*} = 0$, so that $K * [u - \hat{u}](b_*) = 0$. Then $K(b_* - z) = 0$ for all $z \in [-n - \xi, b_*]$. Combining with the previous estimate, we have $K(b_* - z) = 0$ for all $y \in [-n - \xi, n]$. Thus, for all $\eta \in [0, \xi]$,

$$K * \hat{u}(b_* - \eta) - K * \hat{u}(b_*) = \int_{-n-\xi}^{n} K(b_* - z)[\tilde{u}(z + \xi - \eta) - \tilde{u}(z + \xi)]\, dz = 0.$$

Therefore, $K * \hat{u}(x) = K * \hat{u}(b_*) = u_0$ for all $x \in [b_* - \xi, b_*]$. Using the same Ψ as before, we have, when $x \in (b_* - \xi, b_*)$,

$$\Psi(u) - \Psi(\hat{u}) = \int_{b_*}^{x} \frac{K * u - u_0}{u_0 - u} = \int_{x}^{b_*} \frac{K * u - u_0}{u - u_0} \ge 0.$$

As

$$\Psi'(s) = g'(s)/(u_0 - s) < 0 \text{ for } s \in (u_0, \infty),$$

this implies $\hat{u} \ge u$ in $(b_* - \xi, b_*)$, which is impossible. Thus there exists $\varepsilon \in (0, \xi)$ such that $\hat{u}(b_* - \varepsilon) \le u_0$. Consequently, $\hat{u} \le u_0$ in $(b_* - \varepsilon, \infty)$. Finally, from $\hat{u} > \hat{u}$ in $(n - \xi, b_*) \cup (b^*, n)$, $\hat{u} \le u_0 \le \hat{u}$ on $(b_* - \varepsilon, b^*]$, and $\hat{u}(b^*) < u_0 = u(b^*)$ we conclude that there exists $\delta \in (0, \varepsilon)$ such that $\hat{u}(x - \delta) \le u(x)$ in $[-n, n]$, which again contradicts the definition of ξ. Thus case (iii) also cannot happen.

In conclusion, we must have $\xi = 0$, so that $u(x) \geq \tilde{u}(x)$ for all $x \in [-n, n]$. Similarly we can show that $u(x) \leq \tilde{u}(x)$ for all $x \in [-n, n]$. Thus, $u = \tilde{u}$. This completes the proof. □

4. THE LIMIT PROCESS

For each $n > 0$, let (b_n, u_n) be a solution (the uniqueness part of Theorem 2 is not needed) to (3.1) that satisfies $(u_n)_x \leq 0$ in $\mathbb{R}$ in the distribution sense. Set $v_n(\cdot) = u_n(\cdot + b_n)$, $B_n = n - b_n$, and $A_n = n + b_n = 2n - B_n$.

Consider the family $\{v_n\}_{n=1}^{\infty}$. It is bounded and decreasing. Hence, there exist constants A and B and a decreasing function ϕ on $\mathbb{R}$ such that along a sequence $\{n_j\}_{j=1}^{\infty}$,

$$\lim_{j\to\infty} n_j = \infty,$$
$$\lim_{j\to\infty} A_{n_j} = A \in (0, \infty], \quad \lim_{j\to\infty} B_{n_j} = B \in (0, \infty], \quad A + B = \infty,$$

and for any $M > 0$,

$$\lim_{j\to\infty} v_{n_j} = \phi \qquad \text{uniformly on } [-M, 0) \cup (0, M],$$
$$\lim_{n_j\to\infty} g(v_{n_j}) = g(\phi) \quad \text{uniformly on } [-M, M],$$
$$\lim_{n_j\to\infty} K * v_{n_j} = K * \phi \quad \text{uniformly on } [-M, M].$$

Hence, using the integral formulation

$$g(v_{n_j}(x)) = g(v_{n_j}(0)) + \int_0^x [K * v_{n_j} - v_{n_j}]\, dy$$

we can derive that ϕ satisfies

$$\begin{cases} (g(\phi))_x + \phi = K * \phi \text{ in } (-A, 0) \cup (0, B), \\ u_- \geq \phi \geq u_0 \text{ in } (-\infty, 0), \qquad u_0 \geq \phi \geq u_+ \text{ in } (0, \infty), \\ g(\phi) \in C(\mathbb{R}) \cap C^1((-A, 0) \cup (0, B)), \qquad \phi' \leq 0 \text{ in } (-A, 0) \cup (0, B). \end{cases}$$

Lemma 4.1. *$A = \infty$ and $B = \infty$.*

Proof. Suppose, for the sake of contradiction, that $B < \infty$. Then $A = \infty$ and $\phi = u_+$ in $[B, \infty)$. Since $(g(v_{n_j}))_x + v_{n_j} = K * v_{n_j}$ in the distribution sense in $[-A_{n_j}, B_{n_j}]$, for any $x < B$ and $j \gg 1$, integrating the equation over $[x, B_{n_j}]$ we have

$$g(u_+) - g(v_{n_j}(x)) = \int_x^{B_{n_j}} \left\{K * v_{n_j}(y) - v_{n_j}(y)\right\} dy.$$

Using $g(u_+) = 0$ and sending $j \to \infty$ we derive that

$$g(\phi(x)) = \int_x^B \{\phi(y) - K * \phi(y)\}\, dy \quad \forall\, x < B.$$

Since ϕ is decreasing in $(-\infty, 0)$, $\phi(-\infty) = \lim_{x\to-\infty} \phi(x) \in [u_0, u_-]$ exists. Hence, sending $x \to -\infty$ in the above equation we obtain

$$\begin{aligned} g(\phi(-\infty)) &= \int_{-\infty}^B \{\phi(y) - K * \phi(y)\}\, dy \\ &= \int_{-\infty}^{\infty} \{\phi(y) - K * \phi(y)\}\, dy - \int_B^{\infty} \{\phi(y) - K * \phi(y)\}\, dy \\ &= \int_B^{\infty} \{K * \phi(y) - u_+\}\, dy. \end{aligned}$$

Here $\int_{\mathbb{R}} (K * \phi - \phi)\, dy = 0$ since $\int_{\mathbb{R}} zK(z)\, dz = 0$ and $\phi(\pm\infty)$ exist.

Since ϕ is decreasing and $\phi \ge u_0$ on $(-\infty, 0]$, the function $K * \phi$ is also decreasing and

$$\lim_{x\to-\infty} K * \phi(x) \ge u_0 > u_+.$$

Hence, we can define $\hat{B} = \sup\{x \in \mathbb{R} : K * \phi(x) > u_+\}$.

Case (i): $\hat{B} > B$. Then

$$g(\phi(-\infty)) = \int_B^{\infty} \{K * \phi - u_+\}\, dy > 0.$$

As $g \le 0$ in $[u_0, u_-]$, this would imply $\phi(-\infty) > u_-$, which is impossible.

Case (ii): $\hat{B} \le B$. Then

$$0 = K * \phi(\hat{B}) - u_+ = \int_{\mathbb{R}} K(z)(\phi(\hat{B} - z) - u_+)\, dz.$$

Since $\int_0^\infty K(z)\, dz > 0$ there exists $z_0 > 0$ such that $K(z_0) > 0$. As $\phi \ge u_+$ and $K \ge 0$, we must have $\phi(\hat{B} - z_0) = u_+$. The monotonicity of ϕ then implies that $\phi \equiv u_+$ on $[\hat{B} - z_0, \infty)$. However, on $(\hat{B} - z_0, \hat{B})$, $(g(\phi))_x = K * \phi - \phi = K * \phi - u_+ > 0$, a contradiction.

As both cases (i) and (ii) cannot happen, we must have $B = \infty$. In a similar manner, we can show that $A = \infty$. This completes the proof. ☐

Lemma 4.2.

$$\lim_{x\to\pm\infty} \phi(x) =: \phi(\pm\infty) = u_\pm.$$

In addition,

$$g(\phi(x)) = \int_{\mathbb{R}} zK(z) \int_0^1 \{h - \phi(x - tz)\}\, dt\, dz \qquad \forall\, x, h \in \mathbb{R}. \tag{4.1}$$

Proof. Fix any $x \in \mathbb{R}$. For each large-enough j such that $x \in (-A_{n_j}, B_{n_j})$, integrating $(g(v_{n_j}))_x = K * v_{n_j} - v_{n_j}$ over $[x, B_{n_j}]$ we obtain, since

$$g(v_{n_j}(B_{n_j})) = g(u_+) = 0,$$

for any $h \in \mathbb{R}$

$$\begin{aligned}
-g(v_{n_j}(x)) &= \int_x^{B_{n_j}} \left\{K * v_{n_j} - v_{n_j}\right\} dy \\
&= \int_x^{\infty} \left\{K * v_{n_j} - v_{n_j}\right\} dy + \int_{B_{n_j}}^{\infty} \left\{u^+ - K * v_{n_j}\right\} dy \\
&\le \int_x^{\infty} \left\{K * v_{n_j}(y) - v_{n_j}(y)\right\} dy \\
&= \int_{\mathbb{R}} zK(z) \int_0^1 \left\{v_{n_j}(x - tz) - h\right\} dt\, dz.
\end{aligned}$$

Here in the first inequality, we used the fact that $K * v_{n_j} \ge u_+$ on $\mathbb{R}$, and in the last equality we have used the fact that $\int_{\mathbb{R}} zK(z)\, dz = 0$ and a calculation demonstrated in the proof of Lemma 2.1.

Sending $j \to \infty$ and using the Lebesgue's dominated convergence theorem, we derive that

$$-g(\phi(x)) \le \int_{\mathbb{R}} zK(z) \int_0^1 \{\phi(x - tz) - h\}\, dt\, dz \quad \forall\, x, h \in \mathbb{R}.$$

Finally, setting $h = \phi(x)$ and sending $x \to \infty$ we obtain $-g(\phi(\infty)) \le 0$. Since $\phi(\infty) \in [u_+, u_0]$ and $g(u_\pm) = 0 > g(s)$ for all $s \in (u_+, u_-)$, we must have $\phi(\infty) = u_+$.

Similarly, we can show that $\phi(-\infty) = u_-$.

Finally, integrating the equation for ϕ over (x, b) and sending $b \to \infty$ we obtain the integral identity (4.1). □

Lemma 4.3. $\phi' < 0$ *in* $(-\infty, 0) \cup (0, \infty)$, $\phi > K * \phi$ *in* $(-\infty, 0)$, *and* $\phi < K * \phi$ *in* $(0, \infty)$.

Proof. First we show that $\phi > u_+$ on $\mathbb{R}$. For this, define $B := \sup\{x \in \mathbb{R} : \phi(x) > u_+\}$. If $B = \infty$, then $\phi > u_+$ on $\mathbb{R}$. Suppose to the contrary that $B < \infty$. Then by continuity $g(\phi(\hat{B})) = g(u_+)$, so $\phi(B) = u_+$ and $\phi > u_+$

on $(-\infty, B)$. As $K \not\equiv 0$ in $(0,\infty)$, we see that $K*\phi(B) > u_+$. This implies that

$$g(\phi)_x|_{x=B} = K*\phi(B) - \phi(B) > 0,$$

contradicting the fact that $g(\phi) \equiv g(u_+)$ on $[B,\infty)$. Hence, we must have $B = \infty$ and $\phi > u_+$ on $\mathbb{R}$. Similarly, we can show that $\phi < u_-$ on $\mathbb{R}$.

Next we show that $K*\phi < \phi$ on $(-\infty, 0)$. For this, let $x_* = \sup\{x < 0 : u(x) > u_0\}$. Let $x < x_*$ be any number. Fix $x_1 \in (-\infty, x)$ such that $K*\phi(x_1) > K*\phi(x)$. Then $\phi(x) > u_0$ and

$$g'(\phi(y)) \geq g'(\phi(x)) > 0$$

for all $y \in [x_1, x]$. Integrating

$$\frac{d}{dy}\left\{e^{\int_x^y \frac{1}{g'(\phi(z))}\,dz}(\phi(x_1) - \phi(y))\right\} = \frac{e^{\int_x^y \frac{1}{g'(\phi(z))}\,dz}}{g'(w(y))}\left\{\phi(x_1) - K*\phi(y)\right\}$$

over $y \in [x_1, x]$ gives

$$\begin{aligned}
\phi(x_1) - \phi(x) &= \int_{x_1}^{x} \frac{e^{\int_x^y \frac{1}{g'(\phi(z))}\,dz}}{g'(w(y))}\left\{\phi(x_1) - K*\phi(y)\right\} dy \\
&< \left\{\phi(x_1) - K*\phi(x)\right\} \int_{x_1}^{x} \frac{e^{\int_x^y \frac{1}{g'(\phi(z))}\,dz}}{g'(w(y))}\, dy \\
&= \left\{\phi(x_1) - K*\phi(x)\right\}\left\{1 - e^{\int_x^{x_1} \frac{1}{g'(\phi(z))}\,dz}\right\} < \phi(x_1) - K*\phi(x).
\end{aligned}$$

This implies that $\phi(x) > K*\phi(x)$. Thus, $\phi > K*\phi$ on $(-\infty, x_*)$ and $\phi_x = (K*\phi - \phi)/g'(\phi)) < 0$ on $(-\infty, x_*)$. As $\int_0^\infty K(z)\,dz > 0$, we then see that $K*\phi(x_*) > K*\phi(y)$ for all $y > x_*$. It then follows that $x_* = 0$, since $x_* < 0$ would imply $\phi \equiv u_0$ on $[x_*, 0]$, which would further imply

$$K*\phi = (g(\phi))_x + \phi \equiv u_0 \quad \text{on } [x_*, 0).$$

In conclusion, $\phi > K*\phi$ and $\phi_x < 0$ in $(-\infty, 0)$. In a similar manner, we can show that $\phi < K*\phi$ and $\phi_x < 0$ in $(0,\infty)$. This completes the proof. $\square$

Finally, taking $x = 0$ and $h = \frac{1}{2}(u_+ + u_-)$ in (4.1) we see that

$$-g(\phi(0\pm)) = \int_{\mathbb{R}} zK(z)\int_0^1 [\phi(-tz) - h]\,dt\,dz \leq \frac{[u_- - u_+]}{2}\int_{\mathbb{R}} |zK(z)|.$$

Recall that

$$-g(u_0) = \max_{s\in\mathbb{R}}\{-g(s)\} = \max_{s\in(u_+,u_+)}\left\{c[s - u_-] - [f(s) - f(u_-)]\right\}.$$

We see that if (1.6) holds, then $-g(\phi(0\pm)) < -g(u_0)$ so that $\phi(0-) > u_0 > \phi(0_+)$. This completes the proof of Theorem 1.

5. Non-Monotonic Traveling Waves

Without any restriction on the class of functions for the solutions to (1.3), there is no uniqueness of solutions in general. Here we provide one such example.

Consider the Hamer system (1.7) with boundary condition $u(\pm\infty) = \pm M$:

$$\begin{cases} \{u^2 + 2q\}_x = 0, \qquad \{q_x - u\}_x = q \qquad \text{in } \mathbb{R}, \\ \lim_{x\to\pm\infty} u(x) = \pm M. \end{cases} \tag{5.1}$$

Here $M > 0$ is an arbitrary constant. By a weak solution it is meant that at points of discontinuity, both $u^2 + 2q$ and $q_x - u$ are continuous. This in particular implies that q is Lipschitz continuous and $u^2 + 2q$ is a constant. Then q is bounded so the equation for q implies that $q(\pm\infty) = 0$. Hence, $u^2 + 2q = M^2$ on $\mathbb{R}$. Introduce $p = q_x - u$. Then

$$q_x = u + p, \qquad p_x = q, \qquad u^2 + 2q = M^2.$$

By the scaling change $(U, P, Q, \varepsilon) := (u, p, q, 2)/M$, the system (5.1) is equivalent to

$$\begin{cases} Q_x = U + P, \quad P_x = Q, \quad U^2 = 1 - \varepsilon Q \quad \text{on } \mathbb{R}, \\ \qquad\qquad U(\pm\infty) = \mp 1. \end{cases} \tag{5.2}$$

The special case $\varepsilon = 0$.

When $\varepsilon = 0$, the system (5.2) reduces to

$$\begin{cases} P_x = Q, \qquad Q_x - P = U, \qquad U^2 = 1, \quad \text{on } \mathbb{R}, \\ \qquad\qquad U(\pm\infty) = \mp 1. \end{cases} \tag{5.3}$$

Let l_1 and l_2 be positive constants. We seek a solution (U, P, Q) such that

$$U(x) = \begin{cases} 1 & \text{if } x \in (-\infty, -l_1) \cup (0, l_2), \\ -1 & \text{if } x \in (-l_1, 0) \cup (l_2, \infty). \end{cases}$$

Given such a U, the unknown P and Q can be obtained by solving

$$P_{xx} - P = U \text{ on } \mathbb{R},$$

resulting in

$$P(x) = \begin{cases} -1 + \alpha e^{x+l_1} & \text{if } x \le -l_1, \\ 1 + \alpha\, e^{x+l_1} - 2\cosh(x + l_1) & \text{if } -l_1 < x \le 0, \\ -1 - \beta\, e^{l_2 - x} + 2\cosh(l_2 - x) & \text{if } 0 < x \le l_2, \\ 1 - \beta e^{l_2 - x} & \text{if } x > l_2, \end{cases}$$

$$Q(x) = P_x(x) = \begin{cases} \alpha\, e^{x+l_1} & \text{if } x \le -l_1, \\ \alpha\, e^{x+l_1} - 2\sinh(x+l_1) & \text{if } -l_1 < x \le 0, \\ \beta\, e^{l_2-x} - 2\sinh(l_2-x) & \text{if } 0 < x \le l_2, \\ \beta e^{l_2-x} & \text{if } x > l_2, \end{cases}$$

$$\alpha = 1 + e^{-l_1-l_2} - e^{-l_1}, \qquad \beta = 1 + e^{-l_1-l_2} - e^{-l_2}.$$

This specific choice of (α, β) ensures that all P, P_x, and Q are Lipschitz continuous, and therefore are strong solutions to (5.3).

When ε is sufficiently small, i.e., when $M = 2/\varepsilon$ is sufficiently large, (5.2) is a regular perturbation of (5.3), and can be solved for each fixed $l_1 > 0$ and $l_2 > 0$ and all sufficiently small positive ε. Instead of presenting the details on this aspect, we shall consider the general case.

The general case $\varepsilon > 0$. Let $\varepsilon = 2/M > 0$ be fixed. For some constants $l_1, l_2 > 0$ to be determined, we seek a solution to (5.2) satisfying

$$U > 0 \quad \text{in } (-\infty, -l_1) \cup (0, l_2), \qquad U < 0 \quad \text{in } (-l_1, 0) \cup (l_2, \infty).$$

This is equivalent to finding P such that

$$\begin{cases} P \in C^1(\mathbb{R}), \qquad P(\pm\infty) = \pm 1, \\ P_{xx} - P = \sqrt{1 - \varepsilon P_x} \quad \text{in } (-\infty, -l_1) \cup (0, l_2), \\ P_{xx} - P = -\sqrt{1 - \varepsilon P_x} \qquad \text{in } (-l_1, 0) \cup (l_2, \infty). \end{cases} \tag{5.4}$$

Indeed, if P is a solution of this system, defining $Q = P_x$ on $\mathbb{R}$ and defining $U = \sqrt{1 - \varepsilon Q}$ in $(-\infty, l_1) \cup (0, l_2)$ and $U = -\sqrt{1 - \varepsilon Q}$ in $(-l_1, 0) \cup (0, \infty)$, we find that Q is Lipschitz continuous, $U^2 = 1 - \varepsilon Q$, $Q_x - P = U$, and $P_x = Q$, so (U, P, Q) solves (5.2).

To solve (5.4), it is convenient to use the (P, Q) phase plane. Referring to Figure 2(c), a solution to (5.4) can be obtained by integrating $P_x = Q(P)$ along the arcs $\overset{\frown}{AB} \cup \overset{\frown}{BC} \cup \overset{\frown}{CD} \cup \overset{\frown}{DE}$, where $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are trajectories to $P_x = Q$ and $Q_x - P = \sqrt{1 - \varepsilon Q}$, whereas $\overset{\frown}{BC}$ and $\overset{\frown}{DE}$ are trajectories to $P = Q_x$ and $Q_x = P - \sqrt{1 - \varepsilon Q}$. Now we provide details.

First we consider the problem

$$P' = Q, \qquad Q' = P + \sqrt{1 - \varepsilon Q}.$$

In the (P, Q) plane, $(-1, 0)$ is a saddle point, associated with which the unstable manifold in the second quadrant can be locally represented as a graph $Q = Q^\varepsilon(P)$. Using the slope field, we see that the stable manifold in the second quadrant can neither touch the P-axis nor the parabola $Q =$

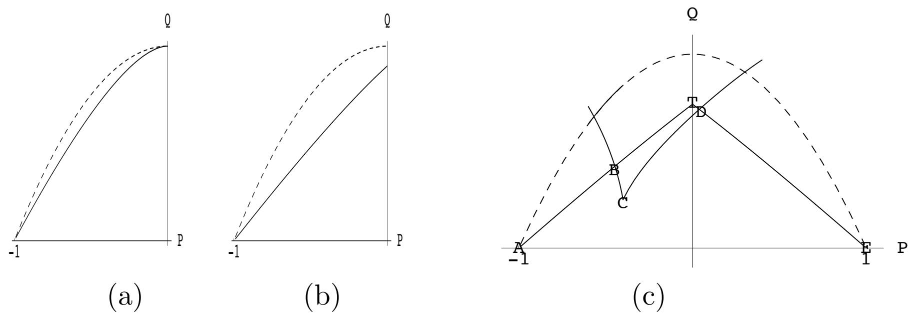

FIGURE 3. Figures (a) and (b): dashed curves represent $Q = (1-P^2)/\varepsilon$; solid curves are the stable manifold of $P' = Q$ and $Q' = P + \sqrt{1-\varepsilon Q}$; in (a), $\varepsilon \geqslant \sqrt{8}$, whereas in (b), $\varepsilon \in (0, \sqrt{8})$. Figure (c): the solid curve $A\widehat{B}T$ and $\widehat{CD}$ are trajectories of $P' = Q$ and $Q' = P + \sqrt{1-\varepsilon Q}$, whereas the solid curve $T\widehat{D}E$ and $\widehat{BC}$ are trajectories of $P' = Q$ and $Q' = P - \sqrt{1-\varepsilon Q}$. Integrating $P_x = Q(P)$ along the combined curve $AB\widehat{C}DE$ then gives a solution to (5.4).

$(1-P^2)/\varepsilon$, and therefore it can only touch the Q-axis. Thus, the stable manifold can be expressed as $Q = Q^\varepsilon(P)$ for $-1 \le P \le 0$, where Q^ε has the property that

$$0 < Q^\varepsilon(P) < \frac{1-P^2}{\varepsilon}, \quad \frac{dQ^\varepsilon(P)}{dP} = P + \sqrt{1-\varepsilon Q^\varepsilon(P)} > 0 \quad \forall\, P \in (-1, 0).$$

Two numerical (with 32 digits) solutions of Q^ε are plotted in Figure 3 (a) and (b), where $\varepsilon \ge \sqrt{8}$ in (a), and $\varepsilon \in (0, \sqrt{8})$ in (b). Observe the following:

$$\varepsilon\, Q^\varepsilon(0) = 1 \text{ if } \varepsilon \ge \sqrt{8}, \qquad \varepsilon\, Q^\varepsilon(0) \in (0,1) \text{ in } 0 < \varepsilon < \sqrt{8}.$$

Here the first equation is obtained as follows. When $\varepsilon \ge \sqrt{8}$, $Q(p) = \frac{1}{\varepsilon}(1 - kP^2)$ with $k = \frac{\varepsilon^2}{8}(\varepsilon^2 - 4 + \varepsilon\sqrt{\varepsilon^2 - 8})$ a subsolution, so that

$$1 - kP^2 < \varepsilon\, Q^\varepsilon(P) < 1 - P^2$$

for all $P \in (-1, 0)$, from which, $\varepsilon\, Q^\varepsilon(0) = 1$ follows. On the other hand, when $\varepsilon \in (0, \sqrt{8})$, $\varepsilon\, Q^\varepsilon(0) < 1$ since there is no solution, in any interval $(-\delta, 0)$ with $\delta > 0$, to the initial-value problem

$$\frac{d}{dP}[Q^2/2] = P + \sqrt{1-\varepsilon Q}$$

subject to the initial condition $Q(0) = 1/\varepsilon$. Since we are not going to use the distinction between $\varepsilon \geq 8$ and $\varepsilon < \sqrt{8}$, we omit the details.

We denote by $P_0(x)$ the solution that corresponds to the stable manifold

$$x = \int_0^{P_0(x)} \frac{dp}{Q^\varepsilon(p)} \qquad \forall\, x \leq 0.$$

Then

$$\begin{aligned} &P_0'' - P_0 = \sqrt{1 - \varepsilon P_0'}, \quad P_0' > 0, \quad P_0'' > 0 \quad \text{on } (-\infty, 0], \\ &P_0(0) = 0, \quad \lim_{x \to -\infty} P_0(x) = -1. \end{aligned}$$

By symmetry, the unstable manifold associated with $(1, 0)$ to the problem

$$P' = Q, \qquad Q' = P - \sqrt{1 - \varepsilon Q}$$

is given by $(P(x), Q(x)) = (-P_0(-x), P_0'(-x))$ for all $x \geq 0$. In particular, defining

$$\begin{aligned} P(x) &= \begin{cases} P_0(x) & \text{if } x \leq 0, \\ -P_0(-x) & \text{if } x > 0, \end{cases} \\ Q &= P_x, \\ U &= \begin{cases} \sqrt{1 - \varepsilon Q} & \text{in } (-\infty, 0), \\ -\sqrt{1 - \varepsilon Q} & \text{in } [0, \infty), \end{cases} \end{aligned}$$

we obtain a solution to (5.2). This solution is continuous when $\varepsilon \geq \sqrt{8}$ (since $\varepsilon\, Q(0) = 1$) and discontinuous when $\varepsilon \in (0, \sqrt{8})$ (since $\varepsilon Q(0) < 1$). This is a known result; see [14].

Now we construct non-monotonic (in U) solutions. For this, we denote

$$\Omega = \{(P, Q) : |P| < 1,\, 0 \leq Q < Q^\varepsilon(|P|)\}.$$

Let $C = (a, b) \in \Omega$ be an arbitrary point. Without loss of generality, we assume that $a \leq 0$; see Figure 3(c). Consider the initial-value problem, for $(P_1(a, b, \cdot), Q_1(a, b, \cdot))$, to

$$\begin{aligned} P_1' = Q_1, \quad Q_1' = P_1 - \sqrt{1 - \varepsilon Q_1}, \\ P_1(a, b, 0) = a, \quad Q_1(a, b, 0) = b. \end{aligned}$$

Let $(-l_1(a, b), 0)$ be the maximal interval such that the corresponding trajectory on the (P, Q) phase plane lies in Ω. Then $P_1' = Q_1 > 0$, $P_1 \leq a$, and $Q_1' = P_1 - \sqrt{1 - \varepsilon Q_1} < 0$ in $(-l_1(a, b), 0)$. Hence, the trajectory $\{(P_1, Q_1)\}$

at $x = -l_1(a,b)$ intersects the boundary of Ω, say at B, which is on the trajectory $\{(P_0, Q_0)\}$ at some point $x = -\hat{l}_1(a,b)$. Hence,

$$P_0(-\hat{l}_1(a,b)) = P_1(a,b,-l_1(a,b)),$$
$$Q_0(-\hat{l}_1(a,b)) = Q_1(a,b,-l_1(a,b)).$$

Similarly, denote by $(P_2(a,b,\cdot), Q_2(a,b,\cdot))$ the solution in Ω to the initial-value problem

$$P_2' = Q_2, \quad Q_2' = P_2 + \sqrt{1-\varepsilon Q_2},$$
$$P_2(a,b,0) = a, \quad Q_2(a,b,0) = b.$$

Since Ω is bounded by $|P| = \sqrt{1-\varepsilon Q}$, the trajectory in Ω satisfies $P_2' > 0$ and $P_2'' = Q_2' > 0$. Hence, the trajectory $\{(P_2, Q_2)\}$ intersects the top boundary of Ω, say at D. By comparison, the trajectory $\{(P_2, Q_2)\}$ in the second quadrant is strictly below the curve $Q = Q^\varepsilon(P)$. Hence, for some $l_2(a,b) > 0$ and $\hat{l}_2(a,b) \geq 0$,

$$P_2(a,b,l_2(a,b)) = -P_0(-l_2(a,b)) \geq 0,$$
$$Q_2(a,b,l_2(a,b)) = P_0'(-l_2(a,b)) > 0.$$

We remark that if $\varepsilon \in (0, \sqrt{8})$, then $\hat{l}_2(a,b) > 0$. On the other hand, when $a < 0$ and $\varepsilon \geq 8$, it is possible that $D = T$; i.e.,

$$\hat{l}_2(a,b) = 0, \ \varepsilon Q_2(a,b,l_2(a,b)) = 1,$$

and $P_2(a,b,l_2(a,b)) = 0$. For example, this happens when

$$(1-ka^2)/\varepsilon < b < Q^\varepsilon(a),$$

where $k = \frac{\varepsilon^2}{8}(\varepsilon^2 - 4 + \varepsilon\sqrt{\varepsilon^2 - 8})$.

Finally, a solution to (5.4) with $l_1 = l_1(a,b)$ and $l_2 = l_2(a,b)$ can be obtained by integrating $P_x = Q(P)$ along the arcs $\overset{\frown}{AB} \cup \overset{\frown}{BC} \cup \overset{\frown}{CD} \cup \overset{\frown}{DE}$; see Figure 3 (c). More precisely,

$$P(x) := \begin{cases} P_0(x - \hat{l}_1(a,b) + l_1(a,b)) & \text{if } x < -l_1(a,b), \\ P_1(x) & \text{if } -l_1(a,b) \leq x < 0, \\ P_2(x) & \text{if } 0 \leq x < l_2(a,b), \\ -P_0(-x + l_2(a,b) - \hat{l}_2(a,b)) & \text{if } x \geq l_2(a,b). \end{cases}$$

Now we transfer our result back to $(u,p,q) = M(U,P,Q)$. Denote by $Q^M(P)$ the function $\varepsilon\, Q^\varepsilon(P)$. Then $(a,\tilde{b}) \in \Omega$ is equivalent to $a \in (-1,1)$ and $b := \varepsilon\tilde{b} \in [0, Q^M(a))$. Also note that $p_{xx} - p = u$, so $p = -K * u$, where $K(z) = \frac{1}{2}e^{-|z|}$. Hence, we have the following:

Theorem 3. *For every $M > 0$, there exists an even positive function $Q^M(\cdot)$ defined on $(-1, 1)$ such that for every $a \in (-1, 1)$ and $b \in [0, Q^M(a))$, Problem (5.1) admits a solution satisfying*

$$\begin{aligned} &u(0-) = -M\sqrt{1-b},\ u(0+) = M\sqrt{1-b},\ K * u(0) = aM, \\ &M > u > 0 \ \textit{in}\ (-\infty, -l_1) \cup (0, l_2), \\ &0 > u > -M \ \textit{in}\ (-l_1, 0) \cup (0, \infty), \end{aligned}$$

where l_1 and l_2 are positive constants that depend continuously on (a, b).

6. Uniqueness

The proof of uniqueness and monotonicity of solutions to (1.3) was shown by Serre in [25]. For completeness, here we use Serre's idea providing a self-contained proof. The uniqueness is shown for solutions in the class

$$\mathcal{A} := \left\{ u : \mathbb{R} \to \mathbb{R} : \lim_{y \nearrow x} u(y) \geq \lim_{y \searrow x} u(y),\ \forall\, x \in \mathbb{R} \right\}.$$

Theorem 4. *Assume (1.2). Then up to a translation, the traveling-wave problem (1.3) admits a unique solution in the admissible class $\mathcal{A}$.*

Proof. We use sgn to denote the sign function:

$$\text{sgn}(z) = 1 \text{ if } z \geq 0, \qquad \text{sgn}(z) = -1 \text{ if } z < 0.$$

Also we denote by $h^{\pm}(g)$ the inverse of $g = g(h)$ in (u_0, ∞) and $(-\infty, u_0)$ respectively:

$$g(h^+(s)) = s = g(h^-(s)), \quad h^+(s) \leq u_0 \leq h^-(s) \qquad \forall\, s \in [g(u_0), \infty).$$

Let ϕ be the solution given in Theorem 1. Let $\psi \in \mathcal{A}$ be any solution to (1.3). By translation, we assume that $\psi(0+) \leq u_0 \leq \psi(0-)$. We want to show that $\psi \equiv \phi$ on $\mathbb{R} \setminus \{0\}$.

1. First we investigate the continuity of ψ. Let $G(x) = g(\psi(x))$. Suppose x_0 is a point of discontinuity of ψ. Then $\psi(x_0-) > \psi(x_0+)$. As $G(\cdot)$ is continuous on $\mathbb{R}$, we must have $\psi(x_0-) = h^-(G(x_0)) > u_0$ and $\psi(x_0+) = h^+(G(x_0)) < u_0$. Hence, there exists $\varepsilon > 0$ such that $\psi = h^-(G)$ in $[x_0 - \varepsilon, x_0)$ and $\psi = h^+(G)$ in $(x_0, x_0 + \varepsilon]$, so ψ is continuous in $[x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon]$.

For convenience, we assume that both ψ and ϕ are right-continuous:

$$\phi(0) = \phi(0+) := \lim_{y \searrow 0} \phi(y), \qquad \psi(x) = \psi(x+) := \lim_{y \searrow x} \psi(y) \quad \forall\, x \in \mathbb{R}.$$

Since $\lim_{x\to\pm\infty}\psi(x) = u_\pm$, there exists a positive constant M such that

$$\psi(x) > \phi(0-) \quad \forall x \le -M+1, \qquad \psi(x) < \phi(0) \quad \forall\, x \ge M-1.$$

Then $\psi = h^-(G)$ on $(-\infty, -M+1]$ and $\psi = h^+(G)$ on $[M-1,\infty)$. As G is Lipschitz continuous on $\mathbb{R}$, ψ is Lipschitz continuous in $(-\infty,-M+1] \cup [M-1,\infty)$.

2. Let

$$\xi := \inf\{t > 0 : \phi(x-t) \ge \psi(x) \quad \forall\, x \in [-M, M]\},$$
$$\Phi(x) := \phi(x-\xi), \qquad \zeta(x) := \Phi(x) - \psi(x) \qquad \forall\, x \in \mathbb{R}.$$

Note that $\zeta(x+) = \zeta(x)$ for all $x \in \mathbb{R}$. By continuity,

$$\lim_{t\searrow\xi} \phi(x-t) = \phi(x-\xi)$$

for every $x \ne \xi$ so $\zeta \ge 0$ in $[-M,M]\backslash\{\xi\}$. If $\xi \in [0, M)$, then $\zeta(\xi) = \zeta(\xi+) \ge 0$. Finally, if $\xi \ge M$, then $\phi(x-\xi) \ge \phi(0) > \psi(x)$ for all $x \in [M-1, \xi]$ so

$$\zeta \ge 0 \text{ in } [-M, \max\{M, \xi\}].$$

We claim that there exists $\hat{x} \in [-M, M]$ such that either $\zeta(\hat{x}) = 0$ or $\zeta(\hat{x}-) = 0$. Suppose otherwise. Then for every $x_0 \in [-M, M]$,

$$\min\{\zeta(x_0), \zeta(x_0-)\} > 0.$$

If both ψ and Φ are continuous at x_0, then $\zeta(x_0) > 0$ implies that $\Phi(x+\varepsilon) > \psi(x)$ for all $x \in (x_0-\varepsilon, x_0+\varepsilon)$ for some $\varepsilon > 0$. If x_0 is a point of discontinuity of ψ, then $\Phi(x_0-) > \psi(x_0-) > u_0$ so that $g(\Phi(x_0)) > G(x_0)$. This implies that x_0 cannot be a point of discontinuity of Φ since otherwise we would have

$$\Phi(x_0) = h^+(g(\Phi(x_0))) < h^+(G(x_0)) = \psi(x_0).$$

Thus, by the continuity, $\Phi(x_0+\varepsilon) > \psi(x)$ for all $x \in (x_0-\varepsilon, x_0+\varepsilon)$ and some $\varepsilon > 0$. Finally, if x_0 is a discontinuity of Φ, then $u_0 > \Phi(x_0) > \psi(x_0)$ so that $g(\Phi(x_0)) < G(x_0)$. As above, x_0 cannot be a point of discontinuity of ψ. Hence, $\psi = h^+(G)$ in a neighborhood of x_0 so there exists $\varepsilon > 0$ such that $\Phi(x+\varepsilon) > \psi(x)$ for all $(x_0-\varepsilon, x+\varepsilon)$. In conclusion, for every $x_0 \in [-M, M]$, there exists $\varepsilon = \varepsilon(x_0) > 0$ such that $\Phi(x+\varepsilon) > \psi(x)$ in $(x_0-\varepsilon, x_0+\varepsilon)$. As $\{(x-\varepsilon(x), x+\varepsilon(x))\}_{x\in[-M,M]}$ is an open covering of $[-M, M]$, there is a finite subcovering. Denote by ε_* the minimum of all the ε in the subcovering; we conclude, as Φ is decreasing, that $\Phi(x+\varepsilon_*) > \psi(x)$ in $[-M-\varepsilon_*, M+\varepsilon_*]$, so that $\phi(x-[\xi-\varepsilon_*]) > \psi$ for all $x \in [-M, M]$. But this contradicts the definition of ξ. Hence, there exists $\hat{x} \in [-M, M]$ such that

$$\min\{\zeta(\hat{x}), \zeta(\hat{x}-)\} = 0 \leq \zeta(x) \quad \forall\, x \in [-M, \max\{M, \xi\}]. \tag{6.1}$$

3. Consider the sets

$$\Omega^- = \{x \in \mathbb{R} : \zeta(x) < 0\},$$
$$\Omega^+ := \{x \in \mathbb{R} : \zeta(x) > 0\} \cup (-M, \max\{M, \xi\}).$$

Since $\zeta \geq 0$ on $[-M, \max\{M, \xi\}]$ and ζ is continuous on $(-\infty, -M) \cup (\max\{M, \xi\}, \infty)$, both Ω^- and Ω^+ are open. We can express $\Omega^\pm$ as an at-most-countable union of non-overlapping open intervals: $\Omega^+ = \bigcup_i (a_i, b_i)$ and $\Omega^- = \bigcup_i (c_i, d_i)$. Since $g(\Phi)$ and $g(\psi)$ are Lipschitz continuous, both $|g(\Phi)_x|$ and $|g(\psi)_x|$ are integrable, and $g(\Phi)_x = g(\psi)_x$ almost everywhere on $\mathbb{R} \setminus (\Omega^- \cup \Omega^+)$. Hence, using $[g(\Phi) - g(\psi)]_x = K * \zeta - \zeta$ we obtain, for every $L > M$,

$$\begin{aligned}
&\int_{-L}^{L} \{\operatorname{sgn}(\zeta)\, K * \zeta - |\zeta|\}\, dx = \int_{-L}^{L} \operatorname{sgn}(\zeta)\, [K * \zeta - \zeta]\, dx \\
&= \int_{-L}^{L} \operatorname{sgn}(\zeta)\ [g(\Phi) - g(\psi)]_x\, dx \\
&= \int_{\Omega^+ \cap (-L,L)} [g(\Phi) - g(\psi)]_x\, dx - \int_{\Omega^- \cap (-L,L)} [g(\phi) - g(\psi)]_x\, dx \\
&= \sum_i \int_{(a_i,b_i) \cap (-L,L)} [g(\Phi) - g(\phi)]_x\, dx - \sum_i \int_{(c_i,d_i) \cap (-L,L)} [g(\phi) - g(\phi)]_x\, dx \\
&= \operatorname{sgn}(\zeta(L))\, [g(\Phi(L)) - g(\psi(L))] - \operatorname{sgn}(\zeta(-L))\, [g(\Phi(-L)) - g(\psi(L))]\,.
\end{aligned}$$

Here we have used the fact that $g(\Phi(c)) = g(\psi(c))$ if $c \in (-L, L)$ and $c = a_i$ or $c = b_i$ or $c = c_i$ or $c = d_i$. Sending $L \to \infty$ we then obtain

$$\lim_{L \to \infty} \int_{-L}^{L} (\operatorname{sgn}(\zeta)\ K * \zeta - |\zeta|)\, dx = 0.$$

Finally, since $\lim_{x \to \pm\infty} |\zeta(x)| = 0$ and $\int_{\mathbb{R}} zK(z)\, dz = 0$, we have

$$\begin{aligned}
0 &= \lim_{L \to \infty} \int_{-L}^{L} (K * |\zeta| - |\zeta|)\, dx \\
&= \lim_{L \to \infty} \int_{-L}^{L} (K * |\,\zeta| - \operatorname{sgn}(\zeta)\, K * \zeta) + \lim_{L \to \infty} \int_{-L}^{L} (\operatorname{sgn}(\zeta) K * \zeta - |\zeta|)\, dx \\
&= \int_{\mathbb{R}} (K * |\zeta| - \operatorname{sgn}(\zeta) K * \zeta)\, dx.
\end{aligned}$$

Since $K * |\zeta| \geq |K * \zeta|$, we see that

$$K * |\zeta| \equiv \operatorname{sgn}(\zeta)\ K * \zeta, \quad \operatorname{sgn}(\zeta) K * |\zeta| = K * \zeta. \tag{6.2}$$

We have to point out that the above identity (6.2), the key ingredient of the uniqueness proof, was first shown by Serre [25]. The identity takes care of the behavior of ζ in $(-\infty, -M] \cup [M, \infty)$. The rest of the proof follows more or less the same argument as that presented in Step 7 of the proof of Theorem 2.

4. We now claim that $\zeta \geq 0$ in $\mathbb{R}$. Note that if $K > 0$ on $\mathbb{R}$, the assertion is trivially true. Hence, we consider the general case $K \geq 0$.

Suppose the claim is not true. Then there exists

$$x_0 \in (-\infty, M) \cup (\max\{M, \xi\}, \infty)$$

such that $\zeta(x_0) < 0$. First consider the case $x_0 < -M$. Let

$$x_1 = \min\{x \in [x_0, -M] : \zeta(x) = 0\}.$$

Since ζ is continuous on $(-\infty, -M]$ and $\zeta(-M) \geq 0$, x_1 is well-defined. Then $\zeta < 0$ in $[x_0, x_1)$ and $g(\Phi(x_1)) = G(x_1)$. By (6.2),

$$K * \zeta = \operatorname{sgn}(\zeta) K * |\zeta| \leq 0$$

on $[x_0, x_1]$.

Now on $[x_0, x_1]$, we have

$$\frac{d}{dx}\{g(\psi) - g(\Phi)\} = -K * \zeta + \Phi - \psi \geq \Phi - \psi = L(x)[g(\Phi) - g(\psi)],$$

where

$$L(x) = \begin{cases} \frac{\Phi(x) - \psi(x)}{g(\Phi(x)) - g(\psi(x))} & \text{if } \Phi(x) \neq \psi(x) \\ \frac{1}{g'(\Phi(x))} & \text{if } \Phi(x) = \psi(x). \end{cases}$$

Note that L is uniformly bounded on $(-\infty, -M] \cup [\max\{M, \xi\}, \infty)$. Gronwall's inequality then implies that

$$g(\psi(x_1)) - g(\Phi(x_1)) \geq [g(\psi(x_0)) - g(\Phi(x_0))] e^{\int_{x_0}^{x_1} L(s)\, ds} > 0.$$

This contradicts the earlier conclusion that $g(\Phi(x_1)) = g(\psi(x_1)$. This contradiction shows that $x_0 < -M$ is impossible. Similarly, we can show that $x_0 > \max\{M, \xi\}$ is also impossible. Thus, $\zeta \geq 0$ in $\mathbb{R}$; that is, $\Phi \geq \psi$ on $\mathbb{R}$. Also, we have $K * \zeta \geq 0$ on $\mathbb{R}$.

5. Finally, we show that $\Phi \equiv \psi$ in $\mathbb{R}$. Suppose this is not true. Then there exists $y_0 \in \mathbb{R}$ such that $\zeta(y_0) = \Phi(y_0) - \psi(y_0) > 0$.

Let $z_0 < 0$ be a Lebesgue point of K at which $K(z_0) > 0$. Then

$$K * \zeta(y_0 + z_0) = \int_{\mathbb{R}} K(y_0 + z_0 - y)\zeta(y)\, dy > 0.$$

This implies from the differential equation

$$[g(\psi + \zeta) - g(\psi)]_x + \zeta = K * \zeta$$

that ζ cannot be identically zero near y_0+z_0, so $\zeta(y_0+z_0) > 0$. By induction, we conclude that $\zeta(y_0 + nz_0) > 0$ for all positive integers n. Since $L(\cdot)$ (defined in the previous step) is bounded in $(-\infty, -M]$, a proof similar to that in Step 4 leads to the conclusion that $\zeta > 0$ in $[y_0 + nz_0, -M]$ for all large integers n. Hence, $\zeta > 0$ in $(-\infty, -M]$. Now set

$$x_* = \sup\{x : \min\{\zeta(y), \zeta(y-)\} > 0 \quad \forall\, y \text{ in } (-\infty, x)\}.$$

In view of (6.1), we see that $x_* \in (-M, \hat{x}]$.

Note that $\zeta > 0$ in $(-\infty, x_*)$ and $\zeta \geq 0$ in $[x_*, \infty)$. As

$$\int_0^\infty K(z)\, dz > 0 \text{ and } \int_{-\infty}^0 K(z)\, dz > 0,$$

this implies that $K * \zeta > 2\delta$ in $(x_* - \delta, x_* + \delta)$ for some $\delta > 0$. Also, either $\zeta(x_*) = 0$ or $\zeta(x_*-) = 0$, so $G(x_*) = g(\Phi(x_*))$. There are only four cases:

(i) $\Phi(x_*) = \psi(x_*) \leq u_0$, (ii) $\Phi(x_*) = \psi(x_*) > u_0$,

(iii) $\Phi(x_*-) = \psi(x_*-) \geq u_0$, (iv) $\Phi(x_*-) = \psi(x_*-) < u_0$.

Suppose (i) $\Phi(x_*) = \psi(x_*) \leq u_0$. Then $\zeta(x_*) = 0$, so $\zeta(x) \leq \delta$ for all $x \in (x_*, x_* + \varepsilon)$ and some $\varepsilon \in (0, \delta)$. Integrating $[g(\Phi) - G]_x = K * \zeta - \zeta$ over $(x_*, x_* + \varepsilon)$ and using $G(x_*) = g(\Phi(x_*))$ we obtain

$$g(\Phi(x_* + \varepsilon)) - g(\psi(x_* + \varepsilon)) = \int_{x_*}^{x_*+\varepsilon} \{K * \zeta - \zeta\}\, dy \geq \varepsilon\delta > 0.$$

As $\Phi < \Phi(x_*) \leq u_0$ in (x_*, ∞) and $g(\cdot)$ is strictly decreasing on $(-\infty, u_0]$, the above inequality implies that $\Phi(x_*+\varepsilon) < \psi(x_*+\varepsilon)$, which is impossible.

Suppose (ii) $\Phi(x_*) = \psi(x_*) > u_0$. Then x_* is a point of continuity of both Φ and ψ, so $\zeta < \delta$ in $[x_* - \varepsilon, x_*]$ for some $\varepsilon \in (0, \delta)$. Consequently,

$$g(\Phi(\xi - \varepsilon)) - g(\psi(\xi - \varepsilon)) = \int_{x_*}^{x_*-\varepsilon} \{K * \zeta - \zeta\}\, dy \leq -\delta\varepsilon < 0.$$

This also implies that $\psi(x_* - \varepsilon) > \Phi(x_* - \varepsilon)$, which is impossible.

Similarly, we can exclude possibilities (iii) and (iv). Thus, we must have $\Phi \equiv \psi$. This completes the uniqueness proof. ∎

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