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A CHARACTERIZATION OF THE MOUNTAIN PASS GEOMETRY FOR FUNCTIONALS BOUNDED FROM BELOW

Gabriele Bonanno
Department of Science for Engineering and Architecture (Mathematics Section)
Engineering Faculty, University of Messina, 98166, Messina, Italy

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper it is proved that, when a regular functional is bounded from below, the mountain pass geometry and the existence of at least two distinct local minima are equivalent conditions. As a consequence, the classical mountain pass theorem, under the additional assumption of boundedness from below of the functional, ensures actually three distinct critical points. Moreover, as application, the existence of three solutions to Hamiltonian systems is established.

1. Introduction

The classical mountain pass theorem of Ambrosetti and Rabinowitz [1, Theorem 2.1] establishes that a C^1-functional which satisfies (PS) and an appropriate geometry, called mountain pass geometry, admits at least one critical point. The aim of this note is to point out that, if this functional is, in addition, bounded from below, then it admits at least three critical points (see Theorem 2.2). To this end, we show that the mountain pass geometry, in its more general form (see Remark 2.1), is equivalent to the existence of at least two local minima (see Theorem 2.1). As an application of the previous three-critical-points theorem, we establish an existence result of three solutions (see Theorem 3.1) to the following Hamiltonian system:

$$\begin{cases} u'' = \nabla F(t,u) & \text{a.e. } t \in [0,T] \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $F : T \times \mathbb{R}^N \to \mathbb{R}$, $N \geq 1$, is a function and ∇F is the gradient of F with respect to u. In [4], Brézis and Nirenberg, by using a three-critical-points theorem in the presence of splitting, have obtained three solutions

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under a set of assumptions including a suitable behavior at the origin on the nonlinearity F, namely,

there exists $r > 0$ and an integer $k \geq 0$ such that

$$-\frac{1}{2}(k+1)^2\omega^2|u|^2 \leq F(t,u) - F(t,0) \leq -\frac{1}{2}k^2\omega^2|u|^2, \tag{1.2}$$

for all $|u| \leq r$ and almost every $t \in [0,T]$, where $\omega = 2\pi/T$.

Examples of Hamiltonian systems that admit three solutions owing to our results, and where the nonlinear term F does not satisfy (1.2), are presented (see Example 3.1 and Remark 3.2). For a complete overview on Hamiltonian systems we cite the exhaustive monograph [5], while for more recent results we refer to [3], [11], and references therein.

This note is arranged as follows. In Section 2, we present our main results, while Section 3 is devoted to Hamiltonian systems.

2. On the mountain pass geometry and a three-critical-points theorem

The main result of this paper is the following theorem, which shows that, if the functional is bounded from below, the mountain pass geometry is equivalent to the existence of two local minima.

Theorem 2.1. *Let X be a real Banach space, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux-differentiable function which satisfies (PS) and is bounded from below. Then, the following assertions are equivalent:*

(a) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(b) *I admits at least two distinct local minima.*

Proof. $(b) \to (a)$. Let u_0 and u_1 be two distinct local minima of I. To fix ideas, assume $I(u_1) \leq I(u_0)$. Fix ρ_1 such that $0 < \rho_1 < \|u_1 - u_0\|$ and $\rho_2 > 0$ such that $I(u_0) \leq I(u)$ for all $u \in X$ such that $\|u - u_0\| < \rho_2$, and put $r = \min\{\rho_1, \rho_2/2\}$. Therefore, $I(u) \geq I(u_0) \geq I(u_1)$ for all $u \in X$ such that $\|u - u_0\| = r$. Hence (a) true holds.

$(a) \to (b)$. Put $E = \{u \in X : \|u - u_0\| \leq r\}$. E is a complete metric space, and I is lower semicontinuous and bounded from below on E. Put $\alpha = \inf_E I$, let $\{u_n\} \subseteq E$ be a sequence such that $\lim\limits_{n\to+\infty} I(u_n) = \alpha$, and let $\{\varepsilon_n\}$ be a sequence of positive real numbers defined as follows: $\varepsilon_n = I(u_n) - \alpha$ if $I(u_n) > \alpha$, and $\varepsilon_n = \frac{1}{n}$ if $I(u_n) = \alpha$. From the Ekeland variational principle (see, for instance, [5, Theorem 4.1]) there is a sequence $\{v_n\} \subseteq E$ such that, for all $n \in \mathbb{N}$, one has $I(v_n) \leq I(u_n)$, $\|u_n - v_n\| < \sqrt{\varepsilon}_n$, and $I(w) > I(v_n) - \sqrt{\varepsilon}_n\|w - v_n\|$ for all $w \in E$, with $w \neq v_n$. Now, we have the two following alternatives: either there is a strictly increasing sequence of positive integers $\{n_k\}$ such that $v_{n_k} \in S = \{u \in X : \|u - u_0\| = r\}$ for all $k \in \mathbb{N}$, or there is $\bar{n} \in \mathbb{N}$ such that $v_n \in E \setminus S$ for all $n > \bar{n}$. In the first case, taking into account (a), one has $I(u_0) \leq I(v_{n_k}) \leq I(u_{n_k}) \leq \alpha + \varepsilon_{n_k}$ for all $k \in \mathbb{N}$. Hence, $I(u_0) = \alpha$; that is, u_0 is a global minimum for I in E and, in particular, it is a local minimum for I in X. In the second case, by choosing $w = v_n + th$, $n > \bar{n}$, $\|h\| = 1$, and for all positive t small enough in a such way that $w \in E \setminus S$, from $I(v_n + th) > I(v_n) - \sqrt{\varepsilon}_n\|th\|$, by a standard computation, one has $\|I'(v_n)\|_{X^*} \leq \sqrt{\varepsilon}_n$ for all $n > \bar{n}$; that is, $\lim_{n\to+\infty} \|I'(v_n)\|_{X^*} = 0$. Moreover, one has $\lim_{n\to+\infty} I(v_n) = \alpha$ and $\lim_{n\to+\infty} \|u_n - v_n\| = 0$. Since I satisfies (PS) there is $\{v_{n_k}\} \subseteq E \setminus S$ such that $\lim_{k\to+\infty} v_{n_k} = v^*$, with $v^* \in E$. Therefore, one has $\lim_{k\to+\infty} I(v_{n_k}) = I(v^*)$; that is, $I(v^*) = \alpha$. If $v^* \in E \setminus S$, v^* is the local minimum of our conclusion; otherwise. if $v^* \in S$, since $I(u_0) \leq I(v^*)$, the local minimum is u_0.

Now, put $F = \{u \in X : \|u - u_0\| \geq r\}$. F is a complete metric space, and I is lower semicontinuous and bounded from below on F. Put $\beta = \inf_F I$, let $\{u_n\} \subseteq F$ be a sequence such that $\lim\limits_{n\to+\infty} I(u_n) = \beta$, and let $\{\varepsilon_n\}$ be a sequence of positive real numbers defined as follows: $\varepsilon_n = I(u_n) - \beta$ if $I(u_n) > \beta$, and $\varepsilon_n = \frac{1}{n}$ if $I(u_n) = \beta$. From the Ekeland variational principle, arguing exactly as before, we obtain the existence of a local minimum of I belonging to $F \setminus S$. □

An immediate consequence of preceding result is the following three-critical-point theorem.

Theorem 2.2. *Let X be a real Banach space, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux-differentiable function which satisfies (PS) and is bounded from below. Assume that*

(a) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

Then, I admits at least three distinct critical points.

Proof. Theorem 2.1 and the mountain pass theorem as given by Pucci and Serrin [7, Corollary 1] guarantee the conclusion. □

Remark 2.1. In the classical mountain pass theorem of Ambrosetti and Rabinowitz [1, Theorem 2.1] the condition (a) is expressed by a strict inequality. The limiting case has been subsequently studied by many authors, and we refer to [6], [8], and [9] for a complete overview on this subject.

Remark 2.2. Recently, three-critical-points theorems have been widely investigated when the structure of the functional is of the type $I = \Phi + \lambda\Psi$ where $\Phi, \Psi : X \to \mathbb{R}$ and $\lambda > 0$, assuming, among other hypotheses, Φ coercive (see, for instance, [2] and [10]). We observe that this type of structure is not very appropriate for some problems such as the Hamiltonian systems investigated in the next section, while it is fruitful, for instance, for Hamiltonian systems with a weight (see Remark 3.3).

3. Hamiltonian systems

Consider the second-order Hamiltonian system (1.1).

Here and in the sequel, F satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for almost every $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \qquad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and for almost every $t \in [0,T]$.

Moreover, for simplicity, we also assume $F(t,0) = 0$ for almost every $t \in [0,T]$.

Now, denote by H_T^1 the Sobolev space of functions $u \in L^2([0,T], \mathbb{R}^N)$ having weak derivative $u' \in L^2([0,T], \mathbb{R}^N)$, endowed with the norm

$$\|u\| := \Big(\int_0^T |u(t)|^2\, dt + \int_0^T |u'(t)|^2\, dt \Big)^{\frac{1}{2}}$$

for all $u \in H_T^1$ (see [5, pages 6–7] for more details). Arguing as in the proof of [5, Proposition 1.1], one has

$$\|u\|_\infty \le k\|u\| \tag{3.2}$$

for all $u \in H_T^1$, where

$$k = \sqrt{2} \max\Big\{ \frac{1}{\sqrt{T}}, \sqrt{T} \Big\}. \tag{3.3}$$

Define $I : H^1_T \to \mathbb{R}$ as follows:

$$I(u) := \frac{1}{2}\int_0^T |u'(t)|^2\,dt + \int_0^T F(t,u(t))\,dt \tag{3.4}$$

for all $u \in H^1_T$. The critical points of I are weak solutions of the problem (1.1) (see [5, Corollary 1.1]).

The main result of this Section is the following.

Theorem 3.1. *Assume that*

(*i*) $\lim\limits_{|x|\to\infty} F(t,x) = \infty$ *uniformly in* t;

(*ii*) *there are* $d_0, d_1 \in \mathbb{R}^N$, *with* $d_0 \neq d_1$, *such that*

$(ii)_1$ $\displaystyle\int_0^T \min_{|\xi-d_0|\le\delta} F(t,\xi)\,dt = \int_0^T F(t,d_0)\,dt$ *for some* $\delta > 0$;

$(ii)_2$ $\displaystyle\int_0^T F(t,d_0)\,dt \ge \int_0^T F(t,d_1)\,dt.$

Then, the problem (1.1) *admits at least three weak solutions.*

Proof. Our aim is to apply Theorem 2.2. To this end, let X be the Sobolev space H^1_T and let I be the functional defined in (3.4). Owing to (i), standard computations show that I is bounded from below and that it satisfies (PS). Now, put $r = \frac{\delta}{k}$, where δ and k are given by (ii) and (3.3) respectively, $u_0(t) = d_0$, and $u_1(t) = d_1$ for all $t \in [0,T]$. Clearly, $u_0, u_1 \in X$,

$$I(u_0) = \int_0^T F(t,d_0)\,dt \quad \text{and} \quad I(u_1) = \int_0^T F(t,d_1)\,dt.$$

Moreover, fix $u \in X$ such that $\|u - u_0\| = r$. Taking (3.2) into account, one has

$$\|u - u_0\|_\infty \le k\|u - u_0\| = kr = \delta.$$

Therefore,

$$I(u) \ge \int_0^T F(t,u(t))\,dt \ge \int_0^T \min_{|\xi-d_0|\le\delta} F(t,\xi)\,dt$$

for all $u \in X$ such that $\|u - u_0\| = r$; that is,

$$\inf_{\|u-u_0\|=r} I(u) \ge \int_0^T \min_{|\xi-d_0|\le\delta} F(t,\xi)\,dt.$$

From $(ii)_1$ and $(ii)_2$, and owing to our setting, one has

$$\inf_{\|u-u_0\|=r} I(u) \ge I(u_0) \ge I(u_1).$$

Hence, Theorem 2.2 ensures the conclusion. □

Remark 3.1. The following condition,

(ii') *there are* $c \in \mathbb{R}$ *and* $d_0, d_1 \in \mathbb{R}^N$*, with* $|d_0| < c < |d_1|$*, such that*

$$\int_0^T \min_{|\xi| \le c} F(t,\xi)\,dt = \int_0^T F(t,d_0)\,dt \ge \int_0^T F(t,d_1)\,dt,$$

implies assumption (ii) of Theorem 3.1. In fact, setting $\delta = c - |d_0|$, one has

$$\left\{\xi \in \mathbb{R}^N : |\xi - d_0| \le \delta\right\} \subseteq \left\{\xi \in \mathbb{R}^N : |\xi| \le c\right\}.$$

Hence,

$$\int_0^T \min_{|\xi - d_0| \le \delta} F(t,\xi)\,dt \ge \int_0^T \min_{|\xi| \le c} F(t,\xi)\,dt = \int_0^T F(t,d_0)\,dt.$$

Example 3.1. Fix $N = 1$, $T = 1$, and $b(t) = e^t$ for all $t \in [0,1]$. Let $G_1 : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$G_1(x) = \begin{cases} x^2 & x \le 0 \\ -\sqrt{x^3} & 0 < x \le 1 \\ x^2 - \frac{7}{2}x + \frac{3}{2} & 1 < x \le 3 \\ -x^2 + \frac{17}{2}x - \frac{33}{2} & 3 < x \le 7 \\ x^2 - \frac{39}{2}x + \frac{163}{2} & x > 7. \end{cases}$$

Simple computations ensure that all assumptions of Theorem 3.1 are satisfied. In particular, $(ii)'$ in Remark 3.1 is satisfied by choosing, for instance, $d_0 = \frac{7}{4}$, $c = 3$, and $d_1 = 7$. Hence, the problem

$$\begin{cases} u'' = e^t G_1'(u) & \text{a.e. } t \in [0,1] \\ u(0) - u(1) = u'(0) - u'(1) = 0, \end{cases} \tag{3.7}$$

admits at least three classical solutions.

Now, let $G_2 : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$G_2(x) = \begin{cases} x^2 & x \le 1 \\ -\frac{2}{\pi}\sin(\pi x) + 1 & 1 < x \le 2 \\ (x-3)^2 & x > 2. \end{cases}$$

Simple computations ensure that all assumptions of Theorem 3.1 are satisfied. In particular, $(ii)'$ in Remark 3.1 is stisfied by choosing, for instance, $d_0 = 0$, $c = 1$, and $d_1 = 3$. Hence, the problem

$$\begin{cases} u'' = e^t G_2'(u) & \text{a.e. } t \in [0,1] \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases} \tag{3.8}$$

admits at least three classical solutions.

Remark 3.2. In [4], Brézis and Nirenberg have obtained the existence of three solutions for (1.1) assuming (i), (1.2), and

$$\int_0^T F(t, d^*)\, dt < 0 \tag{3.9}$$

for some $d^* \in \mathbb{R}^N$.

Since $\liminf_{x\to 0^+} \frac{G_1(x)}{|x|^2} = -\infty$, where G_1 is the function defined in Example 3.1, the function G_1 does not satisfy condition (1.2). Hence, [4, Theorem 7] cannot be applied to problem (3.7).

Since $G_2(x) \geq 0$ for all $x \in \mathbb{R}^N$, where G_2 is the function defined in Example 3.1, the function G_2 does not satisfy condition (3.9). Hence, [4, Theorem 7] cannot be applied to problem (3.8).

Remark 3.3. Very recently, the existence of three solutions for Hamiltonian systems of the type

$$\begin{cases} u'' - A(t)u = \lambda b(t)\nabla G(u) & \text{a.e. } t \in [0,T] \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

where $A : [0,T] \to \mathbb{R}^{N\times N}$ is a matrix-valued function fulfilling appropriate technical assumptions and $\lambda > 0$, has been widely investigated (see [3] and references therein). In this case, the structure of the energy functional is

$$I = \Phi + \lambda\Psi,$$

with

$$\Phi(u) = \frac{1}{2}\Big(\int_0^T |u'(t)|^2\, dt + \int_0^T (A(t)u(t), u(t))\, dt\Big)$$

coercive, and

$$\Psi(u) = \int_0^T F(t, u(t))\, dt,$$

for all $u \in H^1_T$. On the contrary, in our case, the functional I is given by (3.4), where $\frac{1}{2}\int_0^T |u'(t)|^2\, dt$ is not coercive.

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