Differential and Integral Equations Volume 25, Numbers 11-12 (2012) , 1119–1134

POLYNOMIAL DECAY TO A CLASS OF ABSTRACT COUPLED SYSTEMS WITH PAST HISTORY

L.P.V. Matos, D.S.A. Júnior, and M.L. Santos
Faculdade de Matemática-Programa
de Pós-Graduação em Matemática e Estatística
Universidade Federal do Pará, Campus Universitario do Guamá
Rua Augusto Corrêa 01, Cep 66075-110, Pará, Brazil

(Submitted by: Tohru Ozawa)

Abstract. We consider a class of coupled systems with past history acting only in one equation. We show in the abstract setting that the dissipation given by the history term is not strong enough to produce exponential stability. We show that the solution decays polynomially to zero, with rates that can be improved depending on the regularity of the initial data. Some examples are given.

1. Introduction

Let us denote by $\mathcal{H}$ a Hilbert space. Let $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ be self-adjoint positive-definite operators with the domain $\mathcal{D}(\mathbb{A}_1) \subseteq \mathcal{D}(\mathbb{A}_2) \subset \mathcal{H}$ and $\mathcal{D}(\mathbb{B}) \subset \mathcal{H}$ with compact embeddings in $\mathcal{H}$. Let $g : [0, \infty) \to [0, \infty)$ be a smooth and summable function. We introduce a class of abstract models of linear coupled systems with past history acting only in one equation

$$\begin{cases} u_{tt} + \mathbb{A}_1 u - \displaystyle\int_0^\infty g(s)\mathbb{A}_2 u(t-s)\, ds + \beta v = 0 \\ v_{tt} + \mathbb{B}v + \beta u = 0 \qquad\qquad \text{in} \quad L^2(\mathbf{R}^+; \mathcal{H}) \end{cases} \tag{1.1}$$

satisfying the initial conditions

$$\begin{cases} u(-t) = u_0(t), \qquad \forall t \geq 0 \\ v(0) = v_0, \\ u_t(0) = u_1, v_t(0) = v_1, \end{cases} \tag{1.2}$$

where the initial data u_0, u_1, v_0, and v_1 belong to suitable spaces that we will define later and β is a small positive constant. Here the subscript $\cdot_t$ denotes

Accepted for publication: July 2012.
AMS Subject Classifications: 35B40, 35L70.

the time derivative. Following the approach of Dafermos [2], we consider $\eta = \eta^t(s)$, the relative history of u, defined as

$$\eta^t(s) = u(t) - u(t-s).$$

Hence, putting

$$\kappa = \int_0^\infty g(s)\,ds \tag{1.3}$$

the system (1.1) turns into the system

$$\begin{cases} u_{tt} + \mathbb{A}_1 u - \kappa \mathbb{A}_2 u + \displaystyle\int_0^\infty g(s)\mathbb{A}_2 \eta^t(s)\,ds + \beta v = 0, & \\ v_{tt} + \mathbb{B} v + \beta u = 0, & \text{in } L^2(\mathbb{R}^+;\mathcal{H}) \\ \eta_t^t + \eta_s^t - u_t(t) = 0. & \end{cases} \tag{1.4}$$

Accordingly, the initial conditions become

$$u(0) = u_0, \quad v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1, \quad \eta^0 = \eta_0, \tag{1.5}$$

having set $u_0 = u_0(0)$, $\eta_0(s) = u_0(0) - u_0(s)$, and

$$\eta^t(0) = \lim_{s\to 0} \eta^t(s) = 0, \quad t \geq 0. \tag{1.6}$$

The aim of this work is to analyze the decay properties of the system (1.4)–(1.6). Our main result characterizes the asymptotic properties of the semigroup in terms of the spectral properties of the operator. For this, let us suppose that

$$\mathbb{A}_2 = f(\mathbb{A}_1), \quad \mathbb{B} = h(\mathbb{A}_1) \ \text{ with } \ f(s) = o(s^\alpha), \quad h(s) = o(s^\gamma) \ \text{ as } \ s \to \infty.$$

We show that the semigroup associated with the system (1.4) is not exponentially stable when $\gamma > 0$ and $\alpha < 1$. However, in this case the solution decays polynomially to zero, in an appropriate norm, with rates that can be improved by taking more regular initial data.

The asymptotic stability of the C_0-semigroup associated with the initial-value problem (1.1)–(1.2) has been studied in [3, 4, 13] in the case u_t instead of $\int_0^\infty g(s)\mathbb{A}_2 u(t-s)\,ds$, where a class of linear dissipative evolution equations were considered. See also B.F. Alves et al. [1].

For more details on stability questions for dissipative systems and systems under compact perturbation, we refer the reader to [8].

2. NOTATION AND MATHEMATICAL TOOLS

In the next sections, we will use the semigroup approach to show existence and uniqueness of solutions to (1.4)–(1.6). So, for $r \in \mathbb{R}$, we introduce the scale of a Hilbert space, $\mathcal{D}(\mathbb{A}_1^{r/2})$, endowed with the usual inner products

$$\langle v_1,\ v_2\rangle_{\mathcal{D}(\mathbb{A}_1^{r/2})} = \langle \mathbb{A}_1^{r/2} v_1, \mathbb{A}_1^{r/2} v_2\rangle.$$

The embedding $\mathcal{D}(\mathbb{A}_1^{r_1/2}) \subset \mathcal{D}(\mathbb{A}_1^{r_2/2})$ is compact whenever $r_1 > r_2$. Analogously, with respect to the operators $\mathbb{A}_2$ and $\mathbb{B}$ we can introduce the related scale of Hilbert spaces and the compact embedding. We suppose also that

$$\begin{cases} \|\mathbb{B}^{1/2}u\|^2 \leq C_0\|\mathbb{A}_1^{1/2}u\|^2, & \forall u \in \mathcal{D}(\mathbb{A}_1^{1/2}) \\ \|\mathbb{A}_2^{1/2}u\|^2 \leq C_1\|\mathbb{A}_1^{1/2}u\|^2, & \forall u \in \mathcal{D}(\mathbb{A}_1^{1/2}) \\ \|\mathbb{A}_1^{1/2}u\|^2 - \kappa\|\mathbb{A}_2^{1/2}u\|^2 > C_2\|\mathbb{A}_1^{1/2}u\|^2, & \forall u \in \mathcal{D}(\mathbb{A}_1^{1/2}), \end{cases} \tag{2.1}$$

where C_i, $i = 0, 1, 2$, are positive constants, and the coefficient k introduced in (1.3) will be assumed positive in (2.2). The inner product and the norm on $\mathcal{H}$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, without subscript.

Concerning the constitutive memory kernel g, recalling (1.3), we assume the following set of hypotheses:

$$\begin{cases} g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\ g(s) \geq 0, \qquad g'(s) < 0, \qquad \forall s \in \mathbb{R}^+, \\ 0 < \kappa = \int_0^\infty g(s)\, ds < +\infty, \\ \exists a \in \mathbb{R}^+ : g'(s) \leq -a g(s) \quad \forall s \in \mathbb{R}^+. \end{cases} \tag{2.2}$$

Condition $(2.2)_3$ assures that g is not identically zero. In fact, the dissipative nature of the system is due also to the presence of the memory kernel.

These hypotheses allow us to introduce the weighted L^2-space with respect to the measure $g(s)\,ds$,

$$\begin{aligned} \mathcal{M}_j &= L^2_g\big(\mathbb{R}^+, \mathcal{D}(\mathbb{A}_2^{j/2})\big), \\ \mathcal{N}_j &= L^2_g\big(\mathbb{R}^+, \mathcal{D}(\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{j/2})\big),\ j = 1, 2\ldots, \\ \Lambda_j &= L^2_g\big(\mathbb{R}^+, \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}\mathbb{A}_2^{j/2})\big) \end{aligned}$$

endowed with the usual inner products, for some $\alpha < 1$. Besides (see [5]), let $T \equiv -\partial_s$ be the linear operator on $\mathcal{M}_1$ with domain

$$\mathcal{D}(T) = \{\eta \in \mathcal{M}_1 : \eta_s \in \mathcal{M}_1,\ \eta(0) = 0\},$$

where η_s is the distributional derivative of η with respect to the internal variable s.

To give an accurate formulation of the evolution problem we introduce the product Hilbert space

$$\mathcal{S} = \mathcal{D}(\mathbb{A}_1^{1/2}) \times \mathcal{H} \times \mathcal{D}(\mathbb{B}^{1/2}) \times \mathcal{H} \times \mathcal{M}_1,$$

endowed with the following inner product (cf. $(2.1)_2$):

$$\begin{aligned}\langle \xi, \zeta \rangle_{\mathcal{S}} = \langle \xi_1, \zeta_1 \rangle_{\mathcal{D}(\mathbb{A}_1^{1/2})} + \langle \xi_2, \zeta_2 \rangle - \kappa \langle \xi_1, \zeta_1 \rangle_{\mathcal{D}(\mathbb{A}_2^{1/2})} + \beta \langle \xi_1, \zeta_3 \rangle \\ + \langle \xi_3, \zeta_3 \rangle_{\mathcal{D}(\mathbb{B}^{1/2})} + \langle \xi_4, \zeta_4 \rangle + \beta \langle \xi_3, \zeta_1 \rangle + \langle \xi_5, \zeta_5 \rangle_{\mathcal{M}_1},\end{aligned} \tag{2.3}$$

where $\xi = [\xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \xi_5]^T$ and $\zeta = [\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_5]^T \in \mathcal{S}$.

3. Existence and uniqueness results

Following the notation and the assumptions on the operators $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ as introduced in Section 2, putting $\varphi = u_t$, $\psi = v_t$, and

$$U(t) = [u,\ \varphi,\ v,\ \psi,\ \eta^t]^T, \quad U_0 = [u_0,\ u_1,\ v_0,\ v_1,\ \eta_0]^T \in \mathcal{S},$$

the problem (1.4)–(1.5) can be rewritten as the abstract linear evolution equation in the Hilbert space $\mathcal{S}$,

$$\begin{cases} \dfrac{d}{dt} U(t) = \mathcal{A} U(t), & t > 0, \\ U(0) = U_0, \end{cases} \tag{3.1}$$

where the linear operator $\mathcal{A}$ is defined as

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ -\mathbb{A}_1 u + \kappa \mathbb{A}_2 u - \beta v - \mathbb{D}\eta^t \\ \psi \\ -\beta u - \mathbb{B} v \\ \varphi + T\eta \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ U \in \mathcal{S} : \begin{array}{c} u \in \mathcal{D}(\mathbb{A}_1),\ \varphi \in \mathcal{D}(\mathbb{A}_1^{1/2}),\ v \in \mathcal{D}(\mathbb{B}), \\ \psi \in \mathcal{D}(\mathbb{B}^{1/2}),\ \eta \in \mathcal{D}(T),\ \mathbb{D}\eta^t \in \mathcal{H} \end{array} \right\},$$

where the operator $\mathbb{D}\eta^t$ is given by

$$\mathbb{D}\eta^t = \int_0^\infty g(s) \mathbb{A}_2 \eta^t(s)\, ds$$

and the embedding $\mathcal{D}(\mathcal{A}) \subset \mathcal{S}$ is compact. First, we will show that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t) = e^{t\mathcal{A}}$ on $\mathcal{S}$. Then, we will look for the conditions on the operators $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ for which there does not exist exponential stability. In the latter case we will find that the solutions decays polynomially in appropriate norms.

The energy functional is given by

$$\mathcal{E}_1(t) = \frac{1}{2}\|U(t)\|_{\mathcal{S}}^2 = \frac{1}{2}[\|\mathbb{A}_1^{1/2}u(t)\|^2 + \|\varphi(t)\|^2 - k\|\mathbb{A}_2^{1/2}u(t)\|^2 + 2\beta\langle u(t), v(t)\rangle + \|\mathbb{B}v(t)\|^2 + \|\psi(t)\|^2 + \|\eta^t\|_{\mathcal{M}_1}^2].$$

Under the above notation we can establish the following theorem.

Theorem 3.1. *The operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup $S(t)$ of contraction in $\mathcal{H}$.*

Proof. We will use the Lumer–Phillips theorem (see [11]). First note that the operator $\mathcal{A}$ is of dissipative type. In fact, for every $U \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{S}} &= \langle u_t, u\rangle_{\mathcal{D}(\mathbb{A}_1^{1/2})} + \langle -\mathbb{A}_1 u + \kappa\mathbb{A}_2 u - \beta v - \mathbb{D}\eta, u_t\rangle \\ &\quad - \kappa\langle u_t, u\rangle_{\mathcal{D}(\mathbb{A}_2^{1/2})} + \beta\langle u_t, v\rangle + \langle v_t, v\rangle_{\mathcal{D}(\mathbb{B}^{1/2})} + \langle -\beta u - \mathbb{B}v, v_t\rangle \\ &\quad + \beta\langle v_t, u\rangle + \langle u_t + T\eta, \eta\rangle_{\mathcal{M}_1} = \langle T\eta, \eta\rangle_{\mathcal{M}_1}.\end{aligned}$$

Integrating by parts, we find

$$\langle T\eta, \eta\rangle_{\mathcal{M}_1} = \frac{1}{2}\int_0^\infty g'(s)\|\mathbb{A}_2^{1/2}\eta(s)\|^2\, ds, \quad \forall \eta \in \mathcal{D}(T).$$

It follows that

$$\langle \mathcal{A}U, U\rangle_{\mathcal{S}} = \frac{1}{2}\int_0^\infty g'(s)\|\mathbb{A}_2^{1/2}\eta(s)\|^2\, ds \le 0, \tag{3.2}$$

provided $g'(s) \le 0$, for any $s \ge 0$. Thus, $\mathcal{A}$ is a dissipative operator.

Next, we show that the operator $I - \mathcal{A}$ is onto. Let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{S}$, and consider the equation

$$(I - \mathcal{A})U = F, \tag{3.3}$$

or equivalently

$$u - \varphi = f_1, \tag{3.4}$$

$$\varphi + \mathbb{A}_1 u - \kappa\mathbb{A}_2 u + \beta v + \int_0^\infty g(s)\mathbb{A}_2\eta(s)\, ds = f_2, \tag{3.5}$$

$$v - \psi = f_3, \tag{3.6}$$

$$\psi + \mathbb{B}v + \beta u = f_4, \tag{3.7}$$

$$\eta - \varphi - T\eta = f_5. \tag{3.8}$$

From equations (3.4) and (3.6) we have that $\varphi \in \mathcal{D}(\mathbb{A}_1^{1/2})$ and $\psi \in \mathcal{D}(\mathbb{B}^{1/2})$. Integrating the equation (3.8) we obtain

$$\eta(s) = (1 - e^{-s})\varphi + \int_0^s e^{y-s} f_5(y)\, dy. \tag{3.9}$$

Substituting the equation (3.9) and $\varphi = u - f_1$ into (3.5) we arrive at

$$u + \mathbb{A}_1 u - (\kappa - \kappa_1)\mathbb{A}_2 u + \beta v = \Phi_1, \tag{3.10}$$

where $0 < \kappa_1 = \int_0^\infty (1 - e^{-s}) g(s)\, ds < \kappa$ and

$$\Phi_1 = f_2 + f_1 + \kappa_1 \mathbb{A}_2 f_1 - \int_0^s e^{y-s} f_5(y)\, dy.$$

On the other hand, substituting $\psi = v - f_3$ into (3.7) we arrive at

$$v + \mathbb{B}v + \beta u = \Phi_2, \tag{3.11}$$

where $\Phi_2 = f_4 + f_3$.

Finally, from (3.10) and (3.11) we obtain the system

$$\begin{cases} \widetilde{\mathbb{A}}u + \beta v = \Phi_1, \\ \widetilde{\mathbb{B}}v + \beta u = \Phi_2, \end{cases} \tag{3.12}$$

where $\widetilde{\mathbb{A}} = I + \mathbb{A}_1 - (\kappa - \kappa_1)\mathbb{A}_2$ and $\widetilde{\mathbb{B}} = I + \mathbb{B}$. Since the operators $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$ are self-adjoint and positive definite, and Φ_1 and Φ_2 belong to $\mathcal{H}$, there exists only the solution $(u,\ v) \in \mathcal{D}(\mathbb{A}_1^{1/2}) \times \mathcal{D}(\mathbb{B}^{1/2})$ of the system (3.12). From (3.9) we can see that $\eta(s) \in \mathcal{M}_1$ and $\eta(0) = 0$. Furthermore, from (3.5) and (3.8) we get $\int_0^\infty g(s)\mathbb{A}_2\eta(s)\, ds \in \mathcal{H}$, and as a consequence $T\eta \in \mathcal{M}_1$. Hence $U = (u, \varphi, v, \psi, \eta)^T \in \mathcal{D}(\mathcal{A})$ solves equation (3.3), and our conclusion follows. $\square$

4. LACK OF EXPONENTIAL DECAY

Here we use a result due to Prüss [12] (see also [6], [7], [9], and [10]), which gives a necessary and sufficient condition to get exponential stability. That is, the semigroup $S(t) = e^{\mathcal{A}t}$ is exponentially stable if only if

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\|(i\lambda - \mathcal{A})^{-1}\| \le M, \quad \forall\, \mathrm{Re}\,\lambda \ge 0, \tag{4.1}$$

for some fixed $M > 0$. To this aim, for $m \in \mathbb{N}$, let us denote by λ_m the eigenvalues of $\mathbb{A}_1$ (recall that $\lambda_m > 0$ and $\lambda_m \to \infty$), and by e_m the corresponding sequence of eigenvectors, with $\|e_m\| = 1$.

To study the lack of exponential decay of the semigroup associated with (3.1), we assume that the operators $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ have the same eigenvectors e_m and that the eigenvalues satisfy

$$\mathbb{A}_1 e_m = \lambda_m e_m, \quad \mathbb{A}_2 e_m = \lambda_m^{\alpha} e_m, \quad \mathbb{B} e_m = \lambda_m^{\gamma} e_m.$$

Indeed, there holds

Theorem 4.1. *Let $S(t)$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Assuming that $\gamma > 0$ and $\alpha < 1$, it follows that $S(t)$ is not exponentially stable.*

Proof. To prove that the semigroup $S(t)$ on $\mathcal{S}$ is not exponentially stable, we will find a sequence of bounded functions $F_m = [F_{1,m}, F_{2,m}, F_{3,m}, F_{4,m}, F_{5,m}]^T \in \mathcal{S}$ for which the corresponding solutions of the resolvent equations are not bounded. This will prove that the resolvent operator is not uniformly bounded. Let us consider the resolvent equation

$$(i\omega I - \mathcal{A})U_m = F_m.$$

To simplify the notation we will omit the subindex m. The equation reads

$$\begin{cases} i\omega u - \varphi = F_1 \\ i\omega\varphi + \mathbb{A}_1 u - \kappa\mathbb{A}_2 u + \beta v + \mathbb{D}\eta^t = F_2 \\ i\omega v - \psi = F_3 \\ i\omega\psi + \beta u + \mathbb{B}v = F_4 \\ i\omega\eta - \varphi - T\eta^t = F_5. \end{cases} \tag{4.2}$$

Let us take

$$\mathbf{F}_m = \Big[\frac{\lambda_m^{-1} e_m}{4},\ \frac{e_m}{4},\ \frac{\lambda_m^{-\gamma} e_m}{4},\ \frac{e_m}{4},\ 0\Big]^T.$$

We look for solutions of the form

$$u = ae_m, \quad \varphi = be_m, \quad v = ce_m, \quad \psi = de_m, \quad \eta(s) = \phi(s)e_m,$$

with $a, b, c, d \in \mathbb{C}$ and $\phi \in L^2_g(\mathbb{R}^+)$. Then the system (4.2) becomes

$$\begin{cases} i\omega a - b = \frac{\lambda_m^{-1}}{4}, \\ i\omega b + \lambda_m a - \kappa\lambda_m^{\alpha} a + \beta c + \displaystyle\int_0^\infty \lambda_m^{\alpha}\phi(s)g(s)\,ds = \frac{1}{4}, \\ i\omega c - d = \frac{\lambda_m^{-\gamma}}{4}, \\ i\omega d + \beta a + \lambda_m^{\gamma} c = \frac{1}{4}, \\ i\omega\phi(s) - b + \phi_s(s) = 0. \end{cases} \tag{4.3}$$

From the equations $(4.3)_3$ and $(4.3)_4$ we get

$$-\omega^2 c + i\omega\frac{\lambda_m^{-\gamma}}{4} + \beta a + \lambda_m^{\gamma} c = \frac{1}{4}.$$

Choosing $\omega^2 = \lambda_m^{\gamma}$ we have

$$a = \frac{1}{\beta}\Big(\frac{1}{4} - \frac{\lambda_m^{-\gamma/2}}{4}\Big). \tag{4.4}$$

Substituting (4.4) into $(4.3)_1$ we arrive at

$$b = i\lambda_m^{\gamma/2}\Big(\frac{1}{4} - i\frac{\lambda_m^{-\gamma/2}}{4}\Big)\beta^{-1} - \frac{\lambda_m^{-1}}{4}.$$

Recalling that $\varphi = be_m$, we conclude that

$$\|U_m\|_{\mathcal{S}}^2 \geq \|be_m\|_{\mathcal{H}}^2 = |b|^2 \to \infty, \quad \text{as} \quad \lambda_m \to \infty$$

so that (4.1) is violated when $\gamma > 0$ and $\alpha < 1$. □

5. Polynomial decay

In this section we will study the asymptotic behavior of the solution for problem (1.1) with initial data (1.2) when $\gamma \geq 1$ and $\alpha < 1$, provided that the memory kernel g decays exponentially as time goes to infinity.

Let us introduce the energy functionals defined by

$$\mathcal{E}_2(t) = \frac{1}{2}\|U_{tt}(t)\|_{\mathcal{S}}^2, \quad \mathcal{E}_3(t) = \frac{1}{2}\|\mathbb{A}_1^{(1-\alpha)/2}U(t)\|_{\mathcal{S}}^2 \quad \mathcal{E}_4(t) = \frac{1}{2}\|\mathbb{B}^{(1-\alpha)/2}U(t)\|_{\mathcal{S}}^2,$$
$$\mathcal{E}_5(t) = \frac{1}{2}\|\mathbb{A}_1^{(1-\alpha)/2}U_{tt}(t)\|_{\mathcal{S}}^2, \quad \mathcal{E}_6(t) = \frac{1}{2}\|\mathbb{B}^{(1-\alpha)/2}U_{tt}(t)\|_{\mathcal{S}}^2.$$

We will prove first some lemmas which will play an important role in the proof of the main result (see Theorem 5.1) of this section. For all the following lemmas, let us assume that initial data

$$(u_0, u_1, v_0, v_1, \eta^0) \in \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}\mathbb{A}^{1/2}) \times \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}) \times \mathcal{D}(\mathbb{B}^{1-\alpha/2}) \times \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}) \times \Lambda_1$$

and that the memory kernel g satisfies the conditions (2.2).

Lemma 5.1. *Let us suppose that $U(t) = [u, u_t, v, v_t, \eta]^T$ is a solution of (1.4) with initial data (1.5) and (1.6). Then the energy $\mathcal{E}_1(t)$ associated the system (1.4) satisfies*

$$\frac{d}{dt}\mathcal{E}_1(t) \leq 0.$$

Proof. Multiplying the equations $(1.4)_1$ and $(1.4)_2$ by u_t and v_t, respectively, and recalling the definition of $\mathcal{E}_1(t)$ and hypotheses (2.2) on g, we get that

$$\frac{d}{dt}\mathcal{E}_1(t) = \frac{1}{2}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g'(s)\,ds \le -\frac{a}{2}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds \le 0 \tag{5.1}$$

$\forall t \in \mathbb{R}^+$, and consequently the problem (1.4) is dissipative. □

By the same procedure applied in Lemma 5.1 we can conclude that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) &\le -\frac{a}{2}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta_{tt}^t(s)\|^2 g(s)\,ds,\\
\frac{d}{dt}\mathcal{E}_3(t) &\le -\frac{a}{2}\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,\\
\frac{d}{dt}\mathcal{E}_4(t) &\le -\frac{a}{2}\int_0^\infty \|\mathbb{B}^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,\\
\frac{d}{dt}\mathcal{E}_5(t) &\le -\frac{a}{2}\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta_{tt}^t(s)\|^2 g(s)\,ds,\\
\frac{d}{dt}\mathcal{E}_6(t) &\le -\frac{a}{2}\int_0^\infty \|\mathbb{B}^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta_{tt}^t(s)\|^2 g(s)\,ds.
\end{aligned}$$

Let us introduce the functionals

$$\Phi_1(t) = \langle u_t, u\rangle, \tag{5.2}$$

$$\Phi_2(t) = -\Big\langle u_t, \int_0^\infty \eta^t(s)g(s)\,ds\Big\rangle, \tag{5.3}$$

$$\Phi_3(t) = \langle v_t, v\rangle, \tag{5.4}$$

$$\begin{aligned}
\Phi_4(t) &= \langle u_{tt}, v_t\rangle + \langle \mathbb{A}_1^{1/2}u_t, \mathbb{A}_1^{1/2}v\rangle - \kappa\langle \mathbb{A}_2^{1/2}u_t, \mathbb{A}_2^{1/2}v\rangle\\
&\quad + \Big\langle \int_0^\infty \mathbb{A}_2^{1/2}\eta_t^t(s)g(s)\,ds, \mathbb{A}_2^{1/2}v\Big\rangle.
\end{aligned} \tag{5.5}$$

Lemma 5.2. *Let us suppose that $U(t) = [u, u_t, v, v_t, \eta]^T$ is a solution of problem* (1.4) *with initial conditions* (1.5) *and* (1.6)*. Then the functional Φ_1 satisfies*

$$\begin{aligned}
\frac{d}{dt}\Phi_1(t) \le \|u_t\|^2 - \|\mathbb{A}_1^{1/2}u\|^2 + \Big(\kappa + \frac{C}{2\epsilon}\Big)\|\mathbb{A}_2^{1/2}u\|^2\\
+ \frac{\epsilon}{2}\|\eta\|_{\mathcal{M}_1}^2 - \beta\langle v, u\rangle,
\end{aligned}$$

where ϵ is a small positive constant and C is a positive constant.

Proof. Multiplying the first equation of (1.4) by u we find

$$\frac{d}{dt}\Phi_1 = \|u_t\|^2 - \|\mathbb{A}_1^{1/2}u\|^2 + \kappa\|\mathbb{A}_2^{1/2}u\|^2 - \Big\langle \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds, \mathbb{A}_2^{1/2}u\Big\rangle - \beta\langle v, u\rangle. \tag{5.6}$$

Using the Cauchy–Schwarz and Young inequalities we have that

$$\Big\langle \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds, \mathbb{A}_2^{1/2}u\Big\rangle \le C\|\eta\|_{\mathcal{M}_1}\|\mathbb{A}_2^{1/2}u\| \le \frac{\epsilon}{2}\|\eta\|^2_{\mathcal{M}_1} + \frac{C}{2\epsilon}\|\mathbb{A}_2^{1/2}u\|^2. \tag{5.7}$$

Substituting the inequality (5.7) into the equation (5.6) our conclusion follows. □

The following lemma gives estimates for u_t.

Lemma 5.3. *Let us suppose that* $U(t) = [u, u_t, v, v_t, \eta]^T$ *is a solution of problem* (1.4) *with initial conditions* (1.5) *and* (1.6). *Then*

$$\begin{aligned}\frac{d}{dt}\Phi_2(t) \le &-\Big(\kappa - \frac{\epsilon}{2}\Big)\|u_t\|^2 + \frac{\epsilon}{2}\|\mathbb{A}_1^{1/2}u\|^2 + \frac{\epsilon}{2}\|\mathbb{A}_2^{1/2}u\|^2 \\ &+ \frac{C}{2\epsilon}\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds \\ &+ \frac{\epsilon}{2}\|\mathbb{B}^{1/2}v\|^2 + \Big(1 + \frac{Ca^2 + \kappa^2 + C\beta^2}{2\epsilon}\Big)\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,\end{aligned}$$

where ϵ *is a small positive constant and* C *is a positive constant.*

Proof. Multiplying the first equation of (1.4) by $\int_0^\infty \eta^t(s)g(s)\,ds$ we obtain

$$\begin{aligned}\frac{d}{dt}\Phi_2 = &-\kappa\|u_t\|^2 - \Big\langle u_t, \int_0^\infty \eta^t(s)g'(s)\,ds\Big\rangle \\ &+ \Big\langle \mathbb{A}_1^{1/2}u, \int_0^\infty \mathbb{A}_1^{1/2}\eta^t(s)g(s)\,ds\Big\rangle - \kappa\Big\langle \mathbb{A}_2^{1/2}u, \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds\Big\rangle \\ &+ \Big\langle \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds, \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds\Big\rangle + \beta\Big\langle v, \int_0^\infty \eta^t(s)g(s)\,ds\Big\rangle.\end{aligned} \tag{5.8}$$

Using hypotheses (2.2) on g, the Cauchy–Schwarz and Young inequalities, and the positivity of the operators, we have

$$\Big|\Big\langle u_t, \int_0^\infty \eta^t(s)g'(s)\,ds\Big\rangle\Big| \le \frac{\epsilon}{2}\|u_t\|^2 + \frac{Ca^2}{2\epsilon}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,$$

$$\begin{aligned}&\left|\left\langle \mathbb{A}_1^{1/2}u, \int_0^\infty \mathbb{A}_1^{1/2}\eta^t(s)g(s)\,ds\right\rangle\right|\\&\qquad\le \frac{\epsilon}{2}\|\mathbb{A}_1^{1/2}u\|^2 + \frac{C}{2\epsilon}\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,\\&\left|\kappa\left\langle \mathbb{A}_2^{1/2}u, \int_0^\infty \mathbb{A}_2^{1/2}\eta^t(s)g(s)\,ds\right\rangle\right|\\&\qquad\le \frac{\epsilon}{2}\|\mathbb{A}_2^{1/2}u\|^2 + \frac{\kappa^2}{2\epsilon}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds,\\&\left|\beta\left\langle v, \int_0^\infty \eta^t(s)g(s)\,ds\right\rangle\right| \le \frac{\epsilon}{2}\|\mathbb{B}^{1/2}v\|^2 + \frac{C\beta^2}{2\epsilon}\int_0^\infty \|\mathbb{A}_2^{1/2}\eta^t(s)\|^2 g(s)\,ds.\end{aligned}$$

Substituting the above inequalities into (5.8), our conclusion follows. □

Lemma 5.4. *Let us suppose that* $U(t) = [u, u_t, v, v_t, \eta]^T$ *is a solution of problem* (1.4) *with the initial conditions* (1.5). *Then the functional* Φ_3 *satisfies*

$$\frac{d}{dt}\Phi_3(t) = \|v_t\|^2 - \|\mathbb{B}^{1/2}v\|^2 - \beta\langle u, v\rangle.$$

Proof. Multiply the second equation of (1.4) by v, and after simple calculations the result follows. □

The following lemma plays an important role because it will connect the dissipative properties of the first equation of (1.4) with the second equation of (1.4).

Lemma 5.5. *Let us suppose that* $U(t) = [u, u_t, v, v_t, \eta]^T$ *is a solution of problem* (1.4) *with initial conditions* (1.5). *There holds*

$$\begin{aligned}&\frac{d}{dt}\Phi_4(t) \le -\beta\|v_t\|^2 + \frac{\beta^2 C(\Omega)}{2\epsilon}\|\mathbb{A}_1^{1/2}u\|^2 + C\|\mathbb{B}^{1/2}v\|^2\\&+ C\int_0^\infty \|\mathbb{B}^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta_{tt}(s)\|^2 g(s)\,ds + C\int_0^\infty \|\mathbb{B}^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta(s)\|^2 g(s)\,ds\\&+ C\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta_{tt}(s)\|^2 g(s)\,ds + C\int_0^\infty \|\mathbb{A}_1^{(1-\alpha)/2}\mathbb{A}_2^{1/2}\eta(s)\|^2 g(s)\,ds\\&+ C\int_0^\infty \|\mathbb{A}_2^{1/2}\eta_{tt}(s)\|^2 g(s)\,ds + C\int_0^\infty \|\mathbb{A}_2^{1/2}\eta(s)\|^2 g(s)\,ds, \qquad (5.9)\end{aligned}$$

where ϵ *is a small positive constant and* C *is a positive constant.*

Proof. Taking the time derivative in the first equation of (1.4) we get

$$\frac{d}{dt}\Phi_4(t) = -\beta\|v_t\|^2 - \beta\langle u_{tt}, u\rangle - \langle \mathbb{B}^{1/2}u_{tt}, \mathbb{B}^{1/2}v\rangle + \langle \mathbb{A}_1^{1/2}u_{tt}, \mathbb{A}_1^{1/2}v\rangle - \kappa\langle \mathbb{A}_2^{1/2}u_{tt}, \mathbb{A}_2^{1/2}v\rangle + \Big\langle \int_0^\infty \mathbb{A}_2^{1/2}\eta_{tt}(s)g(s)\,ds, \mathbb{A}_2^{1/2}v\Big\rangle.$$

Using the relation $\eta_{tt} = \eta_{ss} + u_{tt}$, the inequality motivated by the compact embedding $\mathcal{D}(\mathbb{A}_1^\gamma) \subset \mathcal{D}(\mathbb{A}_1)$ $(\gamma > 1)$, and the positivity of the operators, our result follows. $\square$

The main result of this section is given by the following theorem.

Theorem 5.1. *Suppose that the initial data satisfies*

$$(u_0, u_1, v_0, v_1, \eta^0) \in \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}\mathbb{A}^{1/2}) \times \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}) \times \mathcal{D}(\mathbb{B}^{1-\alpha/2}) \times \mathcal{D}(\mathbb{B}^{(1-\alpha)/2}) \times \Lambda_1$$

and the memory kernel g satisfies the conditions (2.2). *Then, the solution of* (1.4) *decays polynomially to zero. That is, there exists a positive constant ρ such that*

$$\mathcal{E}_1(t) \le \frac{\rho}{t}\sum_{i=1}^{6}\mathcal{E}_i(0).$$

Proof. From the inequalities proved in the previous lemmas, we can choose positive constants M_i and N_j such that the functional

$$\mathcal{L}(t) = \sum_{i=1}^{6} M_i\mathcal{E}_i(t) + \sum_{j=1}^{4} N_j\Phi_j(t)$$

satisfies

$$\frac{d}{dt}\mathcal{L}(t) \le -C\mathcal{E}_1(t), \tag{5.10}$$

where C is a positive constant. Integrating the inequality (5.10) on the interval $[0, t]$, it follows that

$$\mathcal{L}(t) + C\int_0^t \mathcal{E}_1(s)\,ds \le \mathcal{L}(0) \le \sum_{i=1}^{6} M_i\mathcal{E}_i(0).$$

Thus, there is a positive constant M such that

$$\int_0^t \mathcal{E}_1(s)\,ds \le \frac{M}{C}\sum_{i=1}^{6}\mathcal{E}_i(0).$$

Since $\frac{d}{dt}[t\mathcal{E}_1(t)] = \mathcal{E}_1(t) + t\frac{d}{dt}\mathcal{E}_1(t) \le \mathcal{E}_1(t)$ we have that

$$t\mathcal{E}_1(t) \le \int_0^t \mathcal{E}_1(s)\,ds \le \frac{M}{C}\sum_{i=1}^{6}\mathcal{E}_i(0).$$

Finally choosing $\rho = \frac{M}{C}$ we obtain

$$\mathcal{E}_1(t) \le \frac{\rho}{t}\sum_{i=1}^{6}\mathcal{E}_i(0). \qquad \square$$

Corollary 5.1. *Under the same hypotheses as those of Theorem* 5.1, *there holds*

$$\|S(t)\mathcal{A}^{-1+\alpha/2}\| \le \frac{\rho}{t},\ \forall t > 0. \tag{5.11}$$

Proof. The polynomial decay implies that

$$\|S(t)U_0\|_{\mathcal{S}} \le \frac{\rho}{t}\|\mathcal{A}^{-1+\alpha/2}U_0\|_{\mathcal{S}}.$$

Setting $F_0 = \mathcal{A}^{-1+\alpha/2}U_0$, we find

$$\|S(t)\mathcal{A}^{-1+\alpha/2}F_0\|_{\mathcal{S}} \le \frac{\rho}{t}\|F_0\|_{\mathcal{S}},$$

and then

$$\|S(t)\mathcal{A}^{-1+\alpha/2}\| \le \frac{\rho}{t},\ \forall t > 0. \qquad \square$$

Finally, if $U_0 := (u_0, u_1, v_0, v_1, \eta) \in D(\mathcal{A}^k)$, then we use Prüss's results [12] to obtain

$$\|S(t)\|_{\mathcal{S}} \le \frac{C_k}{t^k}\|\mathcal{A}^k U_0\|_{\mathcal{S}}. \tag{5.12}$$

6. Applications

Finally, we list some examples belonging to class system (1.1) where we assume hypotheses (2.2) for the kernel g introduced in Section 2.

Example 6.1. Let $\mathcal{H}$ be a Hilbert space and A a self-adjoint positive-definite operator with domain $\mathcal{D}(A) \subset \mathcal{H}$ with compact embedding in $\mathcal{H}$. Choosing $\mathbb{A}_1 = \mathbb{B} = A$ and $\mathbb{A}_2 = A^\delta$, $\delta < 1$, and $\gamma > 1$, our problem becomes

$$\begin{cases} u_{tt} + Au + \displaystyle\int_0^\infty g(s)A^\delta u(t-s)\,ds + \beta v = 0 & \\ v_{tt} + Av + \beta u = 0 & \text{in } L^2(\mathbb{R}^+;\ \mathcal{H}), \end{cases}$$

satisfying the initial conditions

$$u(-t) = u_0(t), \text{ with } t \geq 0, \ u_t(0) = u_1, \ v(0) = v_0, \ v_t(0) = v_0.$$

By Theorem 4.1, the related semigroup $S(t)$ on $\mathcal{S} = \mathcal{D}(A^{1/2}) \times \mathcal{H} \times \mathcal{D}(A^{1/2}) \times \mathcal{H} \times L^2_g(\mathbb{R}^+, \ \mathcal{D}(A^{\delta/2}))$ is not exponentially stable. However, by Theorem 5.1 the energy decays polynomially.

Example 6.2. The system

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)\,ds + \beta v = 0 & \\ v_{tt} - \Delta v + \beta u = 0 & \text{in } \mathbb{R}^+ \times \Omega \end{cases}$$

satisfying $u = v = 0$ on $\mathbb{R}^+ \times \partial\Omega$ and the initial condition

$$\begin{cases} u(-t) = u_0(t), \qquad \forall t \geq 0, \qquad v(0) = v_0, \\ u_t(0) = u_1, \ v_t(0) = v_1 \end{cases}$$

is a particular application of the abstract formulation given in this work. In this case, we have $\mathcal{H} = L^2(\Omega)$. Moreover, $\mathbb{A}_1 : \mathcal{D}(\mathbb{A}_1) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ with $\mathbb{A}_1 = -\Delta$ and $\mathcal{D}(\mathbb{A}_1) = H^2(\Omega) \cap H^1_0(\Omega)$, $\mathbb{A}_2 \equiv \mathbb{B} \equiv \mathbb{A}_1$ with $\mathcal{D}(\mathbb{A}_2) = \mathcal{D}(\mathbb{B}) = \mathcal{D}(\mathbb{A}_1)$, and $\mathcal{D}(\mathbb{A}_1^{1/2}) = \mathcal{D}(\mathbb{A}_2^{1/2}) = \mathcal{D}(\mathbb{B}^{1/2}) = H^1_0(\Omega)$.

By Theorem 4.1, the related semigroup $S(t)$ on $\mathcal{S} = H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, \ H^1_0(\Omega))$ is not exponentially stable. However, by Theorem 5.1 the energy decays polynomially. This problem was considered in [14].

Example 6.3. Let us study the problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)\,ds + \beta(u-v) = 0 \quad \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v - \beta(u-v) = 0 \quad \text{in } \Omega \times (0,\infty), \\ u = v = 0 \quad \text{on } \Gamma \times (0,\infty) \end{cases}$$

and the initial condition

$$\begin{cases} u(-t) = u_0(t), \qquad \forall t \geq 0, \qquad v(0) = v_0, \\ u_t(0) = u_1, \ v_t(0) = v_1. \end{cases}$$

By Theorem 4.1, the related semigroup $S(t)$ on $\mathcal{S} = H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, \ H^1_0(\Omega))$ is not exponentially stable. However, by Theorem 5.1 the energy decays polynomially.

Example 6.4. The system

$$\begin{cases} u_{tt} + \Delta^2 u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)\,ds + \beta v = 0 \\ v_{tt} - \Delta v + \beta u = 0 & \text{in } \mathbb{R}^+ \times \Omega \end{cases}$$

satisfying $u = v = \Delta u = 0$ on $\mathbb{R}^+ \times \partial\Omega$ and the initial conditions

$$\begin{cases} u(-t) = u_0(t), \qquad \forall t \geq 0, \qquad v(0) = v_0, \\ u_t(0) = u_1,\ v_t(0) = v_1 \end{cases}$$

is also a particular case. Here, we have $\mathcal{H} = L^2(\Omega)$ and the operator $\mathbb{A}_1 : \mathcal{D}(\mathbb{A}_1) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ such that $\mathbb{A}_1 = \Delta^2$ is positive definite and self-adjoint with domain $\mathcal{D}(\mathbb{A}_1) = H^4(\Omega) \cap W$, where $W = \{u \in H^2(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$. The operator $\mathbb{B} : \mathcal{D}(\mathbb{B}) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ is such that $\mathbb{B} = -\Delta$ and $\mathcal{D}(\mathbb{B}) = H^2(\Omega) \cap H_0^1(\Omega)$. In this case $\mathbb{A}_2 = \mathbb{B}$, $\mathcal{D}(\mathbb{A}_1^{1/2}) = W$, and $\mathcal{D}(\mathbb{B}^{1/2}) = H_0^1(\Omega)$. By Theorem 4.1, the related semigroup $S(t)$ on $\mathcal{S} = W \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L_g^2(\mathbb{R}^+,\ H_0^1(\Omega))$ is not exponentially stable. However, by Theorem 5.1 the energy decays polynomially.

Acknowledgments. The authors are grateful to the referee for valuable suggestions that improved this paper.

References

[1] B.F. Alves, W.D. Bastos, and C.A. Raposo, *Loss of exponential stability for a thermoelastic system with memory,* Elect. J. Diff. Eq., Vol. 2010, No. 132 (2010), pp. 1–5.

[2] C. Dafermos, *Asymptotic stability in viscoelasticity,* Arch. Ration. Mech. Anal., 37 (1970), 297–308.

[3] F. Alabau, P. Cannarsa, and V. Komornik, *Indirect internal stabilization of weakly coupled evolution equations,* J. Evol. Equ., 2 (2002), 127–150.

[4] F. Alabau, *Stabilisation frontière indirecte de systèmes faiblement couplés,* C. R. Acad. Sci. Paris, Sér. I, 328 (1999), 1015–1020.

[5] M. Grasselli and V. Pata, *Uniform attractors of nonautonomous dynamical systems with memory,* in "Evolution Equations, Semigroups and Functional Analysis," Milano, 2000, in: Progr. Nonlinear Differential Equations Appl. 50, 155–178, Birkhäuser, Basel, 2002.

[6] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, in "CRC Reseach Notes in Mathematics 398," Chapman & Hall, 1999.

[7] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces,* Trans. Amer. Math. Soc., 236 (1978), 385–394.

[8] G.Z. Guo, *On the exponential stability of C_0-semigroups on Banach spaces with compact perturbations,* Semigroup Forum, 59 (1999), 190–196.

[9] A. Wiler, *Stability of wave equations with dissipative bounded conditions in boundend domain,* Diff. and Integral Eqs., 7 (1994), 345–366.

[10] F. Huang, *Characteristic condition for exponential stability of linear dynamical systems in Hilbert space,* Ann. of Diff. Eqs., 1 (1985), 43–56.

[11] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

[12] J. Prüss, *On the spectrum of C_0-semigroups,* Trans. Amer. Math. Soc., 28 (1984), 847–857.

[13] M.L. Santos, M.P.C. Rocha, and S.C. Gomes, *Polynomial stability of a coupled system of waves equations weakly dissipative,* Applicable Analysis, 86 (2007), 1293–1302.

[14] R.G.C. Almeida and M.L. Santos, *Lack of exponential decay of a coupled system of wave equations with memory,* Nonlinear Analysis: Real World Applications, 12 (2011), 1023–1032.