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SCATTERING AND BLOWUP PROBLEMS FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the scattering and blowup problem for a class of nonlinear Schrödinger equations with general nonlinearities in the spirit of Kenig and Merle [17]. Our conditions on the nonlinearities allow us to treat a wider class of those than ever treated by several authors, so that we can prove the existence of a ground state (a standing-wave solution of minimal action) for any frequency $\omega > 0$. Once we get a ground state, a so-called potential-well scenario works well: for the nonlinear dynamics determined by the nonlinear Schrödinger equations, we define two invariant regions $A_{\omega,+}$ and $A_{\omega,-}$ for each $\omega > 0$ in $H^1(\mathbb{R}^d)$ such that any solution starting from $A_{\omega,+}$ behaves asymptotically free as $t \to \pm\infty$, one from $A_{\omega,-}$ blows up or grows up, and the ground state belongs to $\overline{A_{\omega,+}} \bigcap \overline{A_{\omega,-}}$. Our weaker assumptions as to the nonlinearities demand that we argue in a subtle way in proving the crucial properties of the solutions in the invariant regions.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation:

$$2i\frac{\partial \psi}{\partial t}(x,t) + \Delta\psi(x,t) + f(\psi(x,t)) = 0, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \tag{1.1}$$

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where $i := \sqrt{-1}$, $d \geq 1$, ψ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$, Δ is the Laplace operator on $\mathbb{R}^d$, and $f : \mathbb{C} \to \mathbb{C}$ is a continuously differentiable function in an $\mathbb{R}^2$ sense to be specified later.

This paper is in the spirit of authors Kenig and Merle [17]; Duyckaerts, Holmer, and Roudenko [8, 12]; Akahori and Nawa [1]; and Fang, Xie, and Cazenave [9]; who studied the scattering and blowup problems "below the ground state" in the case of the pure power nonlinearity of the form $f(\psi) = |\psi|^{p-1}\psi$ with $d \geq 1$ and $2 + 4/d < p + 1 < 2^*$, where

$$2^* = \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases} \tag{1.2}$$

Our purpose here is to extend these results to the nonlinear Schrödinger equation with more general nonlinearities than the pure power nonlinearity above.

Conducting such a study, we encounter the existence problem of the ground state as a matter of course. In the previous works of treating double-power or triple-power like nonlinearities, a so-called monotonicity condition (see (1.27) below) is always required (see, e.g., [4, 13, 18]). Our salient point is to relax this monotonicity condition, and we shall study all the properties of the existence of the ground state, the scattering problem, and the blow-up problem under the same conditions on f.

Now, we state our assumption on the nonlinearity f:

(N1) The origin is always a fixed point; that is, $f(0) = 0$. This allows (1.1) to have a trivial solution.

(N2) We impose an H^1-subcritical growth condition such that there exist p_1 and p_2 with $2 + \frac{4}{d} < p_1 + 1 < p_2 + 1 < 2^*$, and a constant $C_f > 0$, such that

$$\left|\frac{\partial f}{\partial z}(z)\right| + \left|\frac{\partial f}{\partial \bar{z}}(z)\right| \leq C_f\big(|z|^{p_1-1} + |z|^{p_2-1}\big) \qquad \text{for any } z \in \mathbb{C},$$

where

$$\frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big). \tag{1.3}$$

From this condition together with (N1), we have

$$|f(z)| \leq C_f\big(|z|^{p_1} + |z|^{p_2}\big) \qquad \text{for any } z \in \mathbb{C}. \tag{1.4}$$

(N3) L^2-norm conservation law is guaranteed by $\Im f(z)\bar{z} = 0$.

(N4) Our equation (1.1) is governed by a Hamiltonian under the condition that there exists a real-valued function $F \in C^2(\mathbb{C}, \mathbb{R})$ such that

$F(0)=0$ and $\frac{\partial F}{\partial \bar{z}}=f$. In this case, the Hamiltonian $\mathcal{H}\colon H^1(\mathbb{R}^d)\longrightarrow \mathbb{R}$ is given by

$$\mathcal{H}(u):=\|\nabla u\|_{L^2}^2-\int_{\mathbb{R}^d}F(u(x))\,dx \qquad \text{for } u\in H^1(\mathbb{R}^d). \tag{1.5}$$

For F, we have from (N1) and (N2) that

$$|F(z)|\le 2C_f\big(|z|^{p_1+1}+|z|^{p_2+1}\big) \qquad \text{for any } z\in\mathbb{C}. \tag{1.6}$$

(N5) F satisfies the so-called L^2-supercritical condition

$$\lim_{|z|\to\infty}\frac{F(z)}{|z|^{2+\frac{4}{d}}}=+\infty.$$

This gives us a lower growth estimate for F.

Besides these conditions above, we assume a structure condition on the functional $\mathcal{K}$ defined by

$$\mathcal{K}(u):=\frac{d}{d\lambda}\Big|_{\lambda=1}\mathcal{H}\left(T_\lambda u\right)=2\left\|\nabla u\right\|_{L^2}^2-d\int_{\mathbb{R}^d}\Big\{f(u(x))\overline{u}(x)-F(u(x))\Big\}\,dx, \tag{1.7}$$

where $T_\lambda u(x):=\lambda^{\frac{d}{2}}u(\lambda x)$, $\lambda>0$. We note that the condition (N3) has been used here, and that $\mathcal{K}$ is an $\mathbb{R}$-valued continuous functional on $H^1(\mathbb{R}^d)$. For $\mathcal{K}$, we impose the following mountain-pass-type structure condition:

(N6) For any $u\in H^1(\mathbb{R}^d)\setminus\{0\}$, there exists a unique number $\lambda(u)>0$ such that

$$\mathcal{K}\left(T_\lambda u\right)\begin{cases} >0 & \text{if} \quad 0<\lambda<\lambda(u),\\ =0 & \text{if} \quad \lambda=\lambda(u),\\ <0 & \text{if} \quad \lambda(u)<\lambda.\end{cases}$$

A typical example of our nonlinearity f satisfying these conditions (N1)–(N6) is

$$f(z)=\sum_{j=1}^{n}\mu_j|z|^{p_j-1}z, \quad z\in\mathbb{C}, \tag{1.8}$$

where $\mu_1,\dots,\mu_m\le 0$ for some $m<n$, $\mu_{m+1},\dots,\mu_n>0$, and $2+\frac{4}{d}<p_1+1<\cdots<p_n+1<2^*$. One can see that this type of nonlinearity does not satisfy the monotonicity condition (1.27) below.

We associate the equation (1.1) with an initial datum from $H^1(\mathbb{R}^d)$ at $t=0$. We put

$$\psi(\cdot,0)=\psi_0\in H^1(\mathbb{R}^d). \tag{1.9}$$

Here we summarize the basic properties of the Cauchy problem (1.1) and (1.9). Under the conditions (N1)–(N4), we know that for any $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique solution ψ in $C(I_{\max}; H^1(\mathbb{R}^d))$ for some interval $I_{\max} = (-T_{\max}^-, T_{\max}^+) \subset \mathbb{R}$, a maximal existence interval including 0. If $I_{\max} \subsetneqq \mathbb{R}$, then we have

$$\lim_{t \to *T_{\max}^*} \|\nabla \psi(t)\|_{L^2} = \infty \quad \text{(blowup)}, \tag{1.10}$$

provided that $T_{\max}^* < \infty$, where $*$ stands for $+$ or $-$. The solution ψ satisfies the following conservation laws of L^2-norm (or mass), the Hamiltonian (or energy), and the momentum in this order: for any $t \in I_{\max}$,

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \tag{1.11}$$

$$\mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0), \tag{1.12}$$

$$\mathcal{P}(\psi(t)) := \Im \int_{\mathbb{R}^d} \nabla \psi(x,t) \overline{\psi(x,t)}\, dx = \mathcal{P}(\psi_0) \tag{1.13}$$

(see, e.g., [6, 11, 14, 15, 16, 22]). If, in addition, $\psi_0 \in L^2(\mathbb{R}^d, |x|^2\, dx)$, then the corresponding solution ψ also belongs to $C(I_{\max}; L^2(\mathbb{R}^d, |x|^2\, dx))$ and satisfies the so-called virial identity (see [11]):

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 \left|\psi(x,t)\right|^2\, dx &= \int_{\mathbb{R}^d} |x|^2 \left|\psi_0(x)\right|^2\, dx + 2t\Im \int_{\mathbb{R}^d} x \cdot \nabla \psi_0(x) \overline{\psi_0(x)}\, dx \\ &\quad + \int_0^t \int_0^{t'} \mathcal{K}(\psi(t''))\, dt'' dt' \qquad \text{for any } t \in I_{\max}, \end{aligned} \tag{1.14}$$

which is expected to give us a kind of propagation or concentration estimate for solutions. We emphasize that we treat any $H^1(\mathbb{R}^d)$ function as initial data, so that we need some modification on (1.14) by truncating the weight functions $|x|^2$ and x. Thus, in the course of our analysis in this paper, we shall employ a generalized version of the virial identity (see Appendix A below).

Now, we consider standing waves to our equation (1.1). A standing wave Ψ is a solution to (1.1) of the form

$$\Psi(x,t) = e^{\frac{i}{2}\omega t} Q(x) \tag{1.15}$$

for some $Q \in H^1(\mathbb{R}^d) \setminus \{0\}$ and $\omega > 0$. Thus, Q should solve the semilinear elliptic equation

$$\Delta Q - \omega Q + f(Q) = 0. \tag{1.16}$$

The standing waves are neither asymptotically free nor singular. Therefore, considering the scattering problem or blowup problem, we have to rule out such solutions. To this end, we introduce the following variational value:

$$m_\omega := \inf \left\{ \mathcal{S}_\omega(u) \colon u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{K}(u) = 0 \right\} \qquad \text{for } \omega > 0, \tag{1.17}$$

where $\mathcal{S}_\omega$ is the action functional on $H^1(\mathbb{R}^d)$ defined by

$$\mathcal{S}_\omega(u) := \omega \|u\|_{L^2}^2 + \mathcal{H}(u) \qquad \text{for } u \in H^1(\mathbb{R}^d). \tag{1.18}$$

We remark that $S'_\omega(Q_\omega) = 0$ in $H^{-1}(\mathbb{R}^d)$ if and only if $Q_\omega \in H^1(\mathbb{R}^d)$ is a weak solution to the equation (1.16). Moreover, we can easily verify that

$$\mathcal{K}(u) = \frac{d}{d\lambda}\Big|_{\lambda=1} \mathcal{S}_\omega(T_\lambda u) \qquad \text{for any } u \in H^1(\mathbb{R}^d) \text{ and } \omega > 0. \tag{1.19}$$

Under our assumptions on f of (N1)–(N6), we can find a minimizer of (1.17):

Theorem 1.1 (Ground State)**.** *Assume that $d \geq 1$ and* (N1)–(N6) *hold. Then, for any $\omega > 0$, there exists a non-trivial function $Q_\omega \in H^1(\mathbb{R}^d)$ such that*

$$\mathcal{S}_\omega(Q_\omega) = m_\omega, \quad \mathcal{K}(Q_\omega) = 0. \tag{1.20}$$

In particular, we have $m_\omega > 0$. Furthermore, Q_ω is a ground state of the equation (1.16) *in the sense of Q_ω being a minimal action solution of* (1.16)*; i.e., $\mathcal{S}_\omega(Q_\omega) \leq \mathcal{S}_\omega(u)$ for any non-trivial solution u to* (1.16).

Now we introduce our potential well(s): for any $\omega > 0$, we define a set A_ω by

$$A_\omega := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega \right\}, \tag{1.21}$$

and we divide this set A_ω according to the sign of the functional $\mathcal{K}$:

$$A_{\omega,+} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega,\ \mathcal{K}(u) > 0 \right\}, \tag{1.22}$$

$$A_{\omega,-} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega,\ \mathcal{K}(u) < 0 \right\}. \tag{1.23}$$

It is worthwhile noting here that the following two facts hold true:

(i) It follows from the definition of m_ω (see (1.17)) that any function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ with $\mathcal{K}(u) = 0$ obeys $\mathcal{S}_\omega(u) \geq m_\omega$. Thus, $A_\omega = A_{\omega,+} \cup A_{\omega,-}$.

(ii) Each of $A_{\omega,+}$ and $A_{\omega,-}$ is invariant under the flow defined by (1.1) (see Lemmata 3.2 and 7.1).

Then we have the following:

Theorem 1.2 (Scattering and Blowup). *Assume that $d \geq 1$ and* (N1)–(N6) *hold. Then, for any $\omega > 0$, we have*

(i) *If $\psi_0 \in A_{\omega,+}$, then the corresponding solution ψ exists globally in time; i.e., $I_{\max} = \mathbb{R}$. Furthermore, there exist $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that*

$$\lim_{t\to+\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta}\phi_+ \right\|_{H^1} = \lim_{t\to-\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta}\phi_- \right\|_{H^1} = 0. \tag{1.24}$$

(ii) *If $\psi_0 \in A_{\omega,-}$, then the corresponding solution ψ satisfies that*

$$\limsup_{t\to T_{\max}^+} \|\nabla\psi(t)\|_{L^2} = \limsup_{t\to -T_{\max}^-} \|\nabla\psi(t)\|_{L^2} = \infty. \tag{1.25}$$

In particular, if $T_{\max}^{\pm} = \infty$, then we have

$$\limsup_{t\to\pm\infty} \int_{|x|>R} |\nabla\psi(t)|^2\, dx = \infty \qquad \text{for any } R > 0. \tag{1.26}$$

Since we have $T_\lambda Q_\omega \in A_{\omega,+}$ for $0 < \lambda < 1$ and $T_\lambda Q_\omega \in A_{\omega,-}$ for $\lambda > 1$ (see Lemma 2.1 below), we obtain the following result from Theorem 1.2:

Theorem 1.3. *Assume that $d \geq 1$, $\omega > 0$, and* (N1)–(N6) *hold. Let Q_ω be the ground state of the equation* (1.16) *found in Theorem* 1.1. *Then the standing wave $e^{i\omega t}Q_\omega$ has two unstable directions in the sense that $Q_\omega \in \overline{A_{\omega,+}} \cap \overline{A_{\omega,-}}$.*

We are working under a classical potential-well scenario. Our potential well $A_\omega = A_{\omega,+} \cup A_{\omega,-}$ is defined through a ground state Q_ω. Here, as to the existence of a ground state, we discuss some relations between previous works and our results. Most of the previous results seem to require the so-called monotonicity condition,

$$h(s) := \frac{sf(s) - F(s)}{s^{2+\frac{4}{d}}} \text{ is strictly increasing on } (0, +\infty), \tag{1.27}$$

together with

$$\lim_{s\to 0} h(s) = 0. \tag{1.28}$$

See, e.g., Berestycki and Cazenave [4] and Le Coz [18]. Under these conditions, they proved the existence of a minimizer for m_ω. We should note that the conditions (1.27) and (1.28) together with (N1)–(N4) imply (N6) (see Lemma 2.3 in [18]). In this paper, such a monotonicity condition on the nonlinearities is not assumed, however. We use (N6) instead of (1.27) and (1.28). Then, it is hard to show the tightness of the minimizing sequence for m_ω. To overcome this difficulty, we distinguish several cases according to the sign of the functional $\mathcal{K}$ of the weak limit for the minimizing sequence;

making use of a Brezis–Lieb-type lemma (see Lemma 2.3 below) effectively for each case, we shall show that the weak limit of the minimizing sequence becomes a ground state.

Lack of the monotonicity condition as in (1.27) cast a shadow over the arguments in proving Theorem 1.2:

- A key step to prove the scattering result (i) of Theorem 1.2 is to show that a solution ψ belongs to an appropriate function space $X(\mathbb{R})$ over the space-time $\mathbb{R}^d \times \mathbb{R}$ (see (4.14) below) such that

$$\|\psi\|_{X(\mathbb{R})} < \infty. \tag{1.29}$$

  We shall show this fact by contradiction. Suppose to the contrary that (1.29) fails to be valid. Then, employing the idea of Kenig and Merle [17], we can find a solution called a critical element, which behaves like a one-soliton. However, with the aid of the fact that $\inf_{t>0} \mathcal{K}(\psi(t)) > 0$ for any solution ψ with $\psi(0) = \psi_0 \in A_{\omega,+}$ (see Lemma 3.3), the generalized virial identity does not allow a critical element to exist.
- In order to prove the blow-up result (ii) of Theorem 1.2, we first show that $\sup_{t>0} \mathcal{K}(\psi(t)) < 0$ for any solution ψ with $\psi(0) = \psi_0 \in A_{\omega,-}$ (see Lemma 7.1). Once we get this upper estimate, we obtain the desired result by using the generalized virial identity as in [1] and [20, 21].

In the proofs of both the scattering result and blow-up one (in Theorem 1.2), our weaker assumptions demand that we argue in a subtle way in proving crucial properties such as a uniform lower bound (respectively upper bound) of $\mathcal{K}(\psi(t))$ for any solution ψ starting from $A_{\omega,+}$ (respectively $A_{\omega,-}$) (see Lemmata 3.3 and 7.1 below). If we impose the monotonicity condition (1.27) as well on f, the well-known standard convexity argument enables us to obtain such uniform bounds of $\mathcal{K}(\psi(t))$ for the solution ψ staring from both $A_{\omega,+}$ and $A_{\omega,-}$ (see, e.g., Berestycki and Cazenave [4]). However, since we don't assume the monotonicity condition (1.27) on our nonlinearity, we cannot employ the convexity argument.

We should mention the results of Ibrahim, Masmoudi, and Nakanishi [13]. They prove the corresponding results of Theorem 1.2 for a class of nonlinear Klein–Gordon equations assuming a kind of monotonicity on their nonlinearities.

For the scattering problem, their nonlinearities in [13] are allowed to include the Sobolev exponent 2^* as $p+1$, which is the so-called energy-critical case. Some of our results as to both the scattering and blow-up problems can be extended to the nonlinear Schrödinger equations with such nonlinearities as well (see [2] and [3]).

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we discuss basic properties of the invariant set $A_{\omega,+}$. In Section 4, introducing several function spaces where the Strichartz-type estimates work well, we give a sufficient condition for the scattering. In Section 5, we give a proof of Theorem 1.2 (i), admitting the existence of the critical element. In Section 6, we show the existence of a critical element in a contradiction argument. Section 7 is devoted to the proof of Theorem 1.2 (ii). Appendix A is devoted to an auxiliary result.

Notation.

(1) Let A and B be two positive quantities. Then, the symbol $A \lesssim B$ means that there exists a constant $C > 0$, which depends only on d, p_1, p_2, and C_f (see (N2)), such that $A \le CB$.
(2) $|\nabla|^s$ and $\langle \nabla \rangle$ are the Fourier multiplier operators associated to the symbols $|\xi|^s$ and $\sqrt{1+|\xi|^2}$, respectively.
(3) For $\lambda > 0$, we define a scaling operator T_λ by
$$(T_\lambda u)(x) := \lambda^{\frac{d}{2}} u(\lambda x) \quad \text{for a function } u \text{ on } \mathbb{R}^d. \tag{1.30}$$
(4) For a point $Q \in [0,1] \times [0,1]$ with the coordinate $(\frac{1}{q}, \frac{1}{r})$ and an interval I, $L(Q;I)$ denotes the Bochner space of $L^r(I, L^q(\mathbb{R}^d))$.
(5) For $q \in [1,\infty]$, q' denotes the Hölder conjugate of q; i.e., $\frac{1}{q'} = 1 - \frac{1}{q}$.
(6) For a point $Q = (\frac{1}{q}, \frac{1}{r}) \in [0,1] \times [0,1]$, we put $Q' := (\frac{1}{q'}, \frac{1}{r'})$.

2. Existence of minimizer

This section is devoted to a proof of Theorem 1.1. We first give several preliminary results.

Lemma 2.1. *Assume that $d \ge 1$ and* (N1)–(N6) *hold. Let $\omega > 0$. Then, we have the following:*

(i) *For any function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ and any $\lambda > 0$, it holds that*
$$\frac{d}{d\lambda}\mathcal{S}_\omega(T_\lambda u) = \frac{d}{d\lambda}\mathcal{H}(T_\lambda u) = \frac{1}{\lambda}\mathcal{K}(T_\lambda u). \tag{2.1}$$
(ii) *For any $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ with $\mathcal{K}(u) = 0$ and any $\lambda > 0$ with $\lambda \neq 1$, one has*
$$\mathcal{S}_\omega(u) > \mathcal{S}_\omega(T_\lambda u). \tag{2.2}$$

(iii) *For any* $u \in H^1(\mathbb{R}^d)$ *with* $\mathcal{K}(u) > 0$ *and any* $0 < \lambda < 1$*, one has*

$$\mathcal{S}_\omega(u) > \mathcal{S}_\omega(T_\lambda u). \tag{2.3}$$

(iv) *For any function* $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ *with* $\mathcal{K}(u) \geq 0$*, one has*

$$\mathcal{H}(u) > 0. \tag{2.4}$$

(v) *We have*

$$m_\omega \geq 0. \tag{2.5}$$

Proof. A direct calculation shows the claim (i). Once we obtain the claim (i), it together with the assumption (N6) gives us the claims (ii) and (iii).

We shall prove the claim (iv). Note first that (1.6) shows that for any $u \in H^1(\mathbb{R}^d)$,

$$\lim_{\lambda \to 0} \mathcal{H}(T_\lambda u) = 0. \tag{2.6}$$

Let $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ satisfy $\mathcal{K}(u) \geq 0$. Then, it follows from the assumption (N6) that $\mathcal{K}(T_\lambda u) > 0$ for any $0 < \lambda < 1$, so that

$$\int_{\lambda_0}^{1} \frac{1}{\lambda} \mathcal{K}(T_\lambda u)\, d\lambda > \int_{\frac{1}{2}}^{1} \mathcal{K}(T_\lambda u)\, d\lambda > 0 \qquad \text{for any } 0 < \lambda_0 < \tfrac{1}{2}. \tag{2.7}$$

Hence, we see from the claim (i) and (2.6) that

$$\mathcal{H}(u) = \lim_{\lambda_0 \to 0} \left\{ \mathcal{H}(T_{\lambda_0} u) + \int_{\lambda_0}^{1} \frac{1}{\lambda} \mathcal{K}(T_\lambda u)\, d\lambda \right\} > \int_{\frac{1}{2}}^{1} \mathcal{K}(T_\lambda u)\, d\lambda > 0, \tag{2.8}$$

which gives the desired result.

Finally, using the claim (iv) and the constraint $\mathcal{K}(u) = 0$, we easily verify the claim (v). □

Lemma 2.2. *Assume* $d \geq 1$ *and* (N1)–(N6) *hold. Let* $\omega > 0$*, and define*

$$B_\omega := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) \leq m_\omega + 1,\ \mathcal{K}(u) \geq 0 \right\}.$$

Then, there exists a constant $C_\omega > 0$ *such that*

$$\sup_{u \in B_\omega} \|u\|_{H^1} \leq C_\omega. \tag{2.9}$$

Proof. We see from Lemma 2.1 (iv) that

$$\omega \|u\|_{L^2}^2 \leq \mathcal{S}_\omega(u) \leq m_\omega + 1 \qquad \text{for any } u \in B_\omega. \tag{2.10}$$

Therefore, for the conclusion (2.9), it suffices to show that

$$\sup_{u \in B_\omega} \|\nabla u\|_{L^2} < \infty. \tag{2.11}$$

Now, suppose to the contrary that (2.11) fails. Then, we can take a sequence $\{u_n\}$ in B_ω such that

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} = \infty. \tag{2.12}$$

Put

$$\mu_n := \frac{1}{\|\nabla u_n\|_{L^2}}, \qquad h_n := T_{\mu_n} u_n, \tag{2.13}$$

so that

$$\lim_{n\to\infty} \mu_n = 0, \tag{2.14}$$

$$\|\nabla h_n\|_{L^2} = 1, \tag{2.15}$$

$$\|h_n\|_{L^2} = \|u_n\|_{L^2} \le \sqrt{\frac{m_\omega + 1}{\omega}} \qquad \text{(by (2.10))}. \tag{2.16}$$

In particular, $\{h_n\}$ is bounded in $H^1(\mathbb{R}^d)$.

We shall show that

$$\inf_{n\in\mathbb{N}} \|h_n\|_{L^{2+\frac{4}{d}}} > 0. \tag{2.17}$$

We see from (2.14) that for any $s > 0$, there exists $N_s \in \mathbb{N}$ such that

$$s\mu_n < 1 \qquad \text{for any } n \ge N_s. \tag{2.18}$$

Hence, it follows from (2.10), Lemma 2.1 (ii), and (1.6) that

$$\begin{aligned} m_\omega + 1 &> \mathcal{S}_\omega(u_n) > \mathcal{S}_\omega(T_{s\mu_n} u_n) = \mathcal{S}_\omega(T_s h_n) \\ &\ge s^2 - C\left(s^{\frac{d}{2}(p_1-1)} \|h_n\|_{L^{p_1+1}}^{p_1+1} + s^{\frac{d}{2}(p_2-1)} \|h_n\|_{L^{p_2+2}}^{p_2+1}\right) \end{aligned} \tag{2.19}$$

for any $s > 0$ and $n \ge N_s$, where $C > 0$ is some constant independent of n and s.

Supposing that (2.17) fails, we can extract a subsequence of $\{h_n\}$ (still denoted by the same symbol) such that

$$\lim_{n\to\infty} \|h_n\|_{L^{2+\frac{4}{d}}} = 0. \tag{2.20}$$

This together with (2.15) also gives us that

$$\lim_{n\to\infty} \|h_n\|_{L^{p_1+1}} = \lim_{n\to\infty} \|h_n\|_{L^{p_2+1}} = 0. \tag{2.21}$$

We choose $s > 2\sqrt{m_\omega + 1}$ first, and then take $n \ge N_s$ so large that the second term on the right-hand side of (2.19) is less than $2m_\omega + 2$. Then, we obtain an absurd conclusion $m_\omega + 1 \ge 2(m_\omega + 1)$. Thus, (2.17) must hold under the hypothesis (2.12).

Combining (2.15) with (2.16) and (2.17), we can take constants $\eta_0 > 0$ and $C_0 > 0$, both independent of n, such that

$$\mathcal{L}^d\left([|h_n| > \eta_0]\right) > C_0 \qquad \text{for any } n \in \mathbb{N} \tag{2.22}$$

(see, e.g., [10, Lemma 2.1]). Moreover, the assumption (N5) shows that for any $L > 0$, we can take a constant $M_L > 0$ such that

$$F(z) \geq L|z|^{2+\frac{4}{d}} \qquad \text{for any } z \in \mathbb{C} \text{ with } |z| \geq M_L. \tag{2.23}$$

Then, it follows from (2.14) that for any $L > 0$, there exists $n_L \in \mathbb{N}$ such that

$$\mu_n^{\frac{d}{2}} M_L < \eta_0 \qquad \text{for any } n \geq n_L. \tag{2.24}$$

This together with (2.22) shows

$$\mathcal{L}^d\Big(\Big[\mu_n^{-\frac{d}{2}}|h_n(x)| \geq M_L\Big]\Big) \geq \mathcal{L}^d\Big([|h_n| > \eta_0]\Big) > C_0 \quad \text{for any } n \geq n_L. \tag{2.25}$$

Now, we shall derive a contradiction from the hypothesis (2.12), which completes the proof. We see from Lemma 2.1 (iv), (2.10), and (2.14) that

$$0 = \lim_{n\to\infty} \mu_n^2 \mathcal{H}(u_n). \tag{2.26}$$

On the other hand, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mu_n^2 \mathcal{H}(u_n) = \lim_{n\to\infty} \Big(1 - \int_{\mathbb{R}^d} F(\mu_n^{-\frac{d}{2}} h_n(x))\mu_n^{d+2}\,dx\Big) \qquad (2.27)\\
&= 1 - \lim_{n\to\infty} \Big\{ \int_{[\mu_n^{-\frac{d}{2}}|h_n|<M_L]} F(\mu_n^{-\frac{d}{2}} h_n(x))\mu_n^{d+2}\,dx \\
&\qquad + \int_{[\mu_n^{-\frac{d}{2}}|h_n|\geq M_L]} F(\mu_n^{-\frac{d}{2}} h_n(x))\mu_n^{d+2}\,dx \Big\} \quad \text{for any } L > 0.
\end{aligned}$$

Here, it follows from (1.6) and (2.16) that

$$\begin{aligned}
&\Big|\int_{[\mu_n^{-\frac{d}{2}}|h_n|<M_L]} F(\mu_n^{-\frac{d}{2}} h_n(x))\mu_n^{d+2}\,dx\Big| \qquad (2.28)\\
&\lesssim \int_{[\mu_n^{-\frac{d}{2}}|h_n|<M_L]} \Big(|\mu_n^{-\frac{d}{2}} h_n(x)|^{p_1+1} + |\mu_n^{-\frac{d}{2}} h_n(x)|^{p_2+1}\Big)\mu_n^{d+2}\,dx \\
&\leq \int_{[\mu_n^{-\frac{d}{2}}|h_n|<M_L]} \Big(M_L^{p_1-1}|h_n(x)|^2 + M_L^{p_2-1}|h_n(x)|^2\Big)\mu_n^2\,dx \\
&\leq (M_L^{p_1-1} + M_L^{p_2-1})\frac{(m_\omega+1)}{\omega}\mu_n^2 \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

We also have by (2.22), (2.23), and (2.25) that

$$\begin{aligned}
&\int_{[\mu_n^{-\frac{d}{2}}|h_n|\ge M_L]} F(\mu_n^{-\frac{d}{2}}h_n(x))\mu_n^{d+2}\,dx \qquad (2.29)\\
&\ge L\int_{[\mu_n^{-\frac{d}{2}}|h_n|\ge M_L]} \left|\mu_n^{-\frac{d}{2}}h_n(x)\right|^{2+\frac{4}{d}}\mu_n^{d+2}\,dx\\
&\ge L\int_{[|h_n|>\eta_0]} \left|\mu_n^{-\frac{d}{2}}h_n(x)\right|^{2+\frac{4}{d}}\mu_n^{d+2}\,dx = L\int_{[|h_n|>\eta_0]} |h_n(x)|^{2+\frac{4}{d}}\,dx\\
&\ge LC_0\eta_0^{2+\frac{4}{d}}.
\end{aligned}$$

We choose L to be $LC_0\eta_0^{2+\frac{4}{d}} = 2$. Then, it follows from (2.26), (2.27), (2.28), and (2.29) that $0 \le -1$. This is a contradiction. Thus, we have completed the proof. □

Lemma 2.3. *Let $u \in H^1(\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} u_n = u \qquad \text{weakly in } H^1(\mathbb{R}^d). \tag{2.30}$$

Then, passing to some subsequence, we have that

$$\lim_{n\to\infty} u_n = u \qquad \text{almost everywhere in } \mathbb{R}^d, \tag{2.31}$$

and that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} \left|F(u_n(x))\,dx - F(u_n(x)-u(x)) - F(u(x))\right|dx = 0, \tag{2.32}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} \Big|f(u_n(x))\overline{u_n(x)} - f(u_n(x)-u(x))(\overline{u_n(x)-u(x)}) - f(u(x))\overline{u(x)}\Big|\,dx = 0. \tag{2.33}$$

See page 1227 of [7] for the proof of Lemma 2.3.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two steps: Proof of the existence of a non-trivial function Q_ω satisfying (1.20) and proof of Q_ω being a ground state.

(Step 1). Let $\{u_n\}$ be a minimizing sequence of the variational problem for m_ω. We see from Lemma 2.2 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$.

Now, the constraint $\mathcal{K}(u_n) = 0$ together with the assumption (N2) and the Gagliardo–Nirenberg inequality shows that

$$\|\nabla u_n\|_{L^2}^2 \lesssim \|u_n\|_{L^{p_1+1}}^{p_1+1} + \|u_n\|_{L^{p_2+1}}^{p_2+1} \tag{2.34}$$

$$\lesssim \|u_n\|_{L^2}^{p_1+1-\frac{d}{2}(p_1-1)} \|\nabla u_n\|_{L^2}^{\frac{d}{2}(p_1-1)} + \|u_n\|_{L^2}^{p_2+1-\frac{d}{2}(p_2-1)} \|\nabla u_n\|_{L^2}^{\frac{d}{2}(p_2-1)},$$

which together with the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^d)$ also gives us that

$$\inf_{n\in\mathbb{N}} \|\nabla u_n\|_{L^2} > 0. \tag{2.35}$$

Moreover, using (2.34), (2.35), and the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^d)$, we can verify that

$$\inf_{n\in\mathbb{N}} \|u_n\|_{L^{2+\frac{4}{d}}} > 0. \tag{2.36}$$

Thus, we find (see, e.g., [10, Lemma 2.1] and [19, Lemma 6]) that there exist a subsequence of $\{u_n\}$ (still denoted by the same symbol), a sequence $\{y_n\}$ in $\mathbb{R}^d$, and $Q_\omega \in H^1(\mathbb{R}^d) \setminus \{0\}$ such that, putting $\widetilde{u}_n := u_n(\cdot + y_n)$, we have

$$\lim_{n\to\infty} \widetilde{u}_n = Q_\omega \qquad \text{weakly in } H^1(\mathbb{R}^d). \tag{2.37}$$

Note here that Lemma 2.3 together with $\mathcal{K}(\widetilde{u}_n) = \mathcal{K}(u_n) = 0$ shows that

$$\lim_{n\to\infty} \mathcal{K}(\widetilde{u}_n - Q_\omega) = -\mathcal{K}(Q_\omega). \tag{2.38}$$

We shall show, by contradiction, that $\mathcal{K}(Q_\omega) \le 0$: Suppose to the contrary that $\mathcal{K}(Q_\omega) > 0$. Then, it follows from (2.38) that

$$\lim_{n\to\infty} \mathcal{K}(\widetilde{u}_n - Q_\omega) = -\mathcal{K}(Q_\omega) < 0. \tag{2.39}$$

Hence, we find from the assumption (N6) and (2.39) that for any sufficiently large $n \in \mathbb{N}$, there exists $\lambda_n \in (0,1)$ such that

$$\mathcal{K}(T_{\lambda_n}(\widetilde{u}_n - Q_\omega)) = 0, \tag{2.40}$$

so that the definition of m_ω shows

$$m_\omega \le \mathcal{S}_\omega(T_{\lambda_n}(\widetilde{u}_n - Q_\omega)). \tag{2.41}$$

Moreover, it follows from (2.39), (1.4), (1.6), the Gagliardo-Nirenberg inequality, and the boundedness of $\{\widetilde{u}_n\}$ in $L^2(\mathbb{R}^d)$ that

$$\begin{aligned} 0 &> \mathcal{K}(\widetilde{u}_n - Q_\omega) \\ &\ge \|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^2 - C_0\Big(\|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^{\frac{d}{2}(p_1-1)} + \|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^{\frac{d}{2}(p_2-1)}\Big) \end{aligned} \tag{2.42}$$

for any sufficiently large $n \in \mathbb{N}$, where $C_0 > 0$ is some constant independent of n. Since $\frac{d(p_2-1)}{2} > \frac{d(p_1-1)}{2} > 2$, this estimate shows that there exists a constant $C_1 > 0$ independent of n such that

$$\|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2} \ge C_1 \qquad \text{for any sufficiently large } n \in \mathbb{N}. \tag{2.43}$$

We also obtain in a way similar to the estimate (2.42) that

$$0 = \mathcal{K}(T_{\lambda_n}(\widetilde{u}_n - Q_\omega)) \geq \lambda_n^2 \|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^2 \\ - C_0\Big(\lambda_n^{\frac{d}{2}(p_1-1)}\|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^{\frac{d}{2}(p_1-1)} + \lambda_n^{\frac{d}{2}(p_2-1)}\|\nabla(\widetilde{u}_n - Q_\omega)\|_{L^2}^{\frac{d}{2}(p_2-1)}\Big). \tag{2.44}$$

This together with (2.43) and the boundedness of $\{\widetilde{u}_n\}$ in $H^1(\mathbb{R}^d)$ gives us that

$$0 \geq \lambda_n^2 C_1^2 - C_2\Big(\lambda_n^{\frac{d}{2}(p_1-1)} + \lambda_n^{\frac{d}{2}(p_2-1)}\Big) \tag{2.45}$$

for some constant $C_2 > 0$ independent of n.

Since $\{\lambda_n\}$ is a sequence in $(0,1)$, we can take a convergent subsequence (still denoted by the same symbol) such that $\lim_{n\to\infty} \lambda_n = \lambda_\infty$ for some $\lambda_\infty \in [0,1]$. In particular, (2.45) shows $\lambda_\infty > 0$, and therefore

$$\lim_{n\to\infty} T_{\lambda_n} Q_\omega = T_{\lambda_\infty} Q_\omega \qquad \text{strongly in } H^1(\mathbb{R}^d). \tag{2.46}$$

Then, we can find from Lemma 2.3 and (2.46) that

$$\lim_{n\to\infty} \{\mathcal{S}_\omega(T_{\lambda_n}\widetilde{u}_n) - \mathcal{S}_\omega(T_{\lambda_n}(\widetilde{u}_n - Q_\omega))\} = \mathcal{S}_\omega(T_{\lambda_\infty} Q_\omega), \tag{2.47}$$

$$\lim_{n\to\infty} \{\mathcal{K}(T_{\lambda_n}\widetilde{u}_n) - \mathcal{K}(T_{\lambda_n}(\widetilde{u}_n - Q_\omega))\} = \mathcal{K}(T_{\lambda_\infty} Q_\omega). \tag{2.48}$$

Here, it follows from the assumption (N6) together with the hypothesis $\mathcal{K}(Q_\omega) > 0$ and $\lambda_\infty \leq 1$ that

$$\mathcal{K}(T_{\lambda_\infty} Q_\omega) > 0, \tag{2.49}$$

so that Lemma 2.1 (ii) shows

$$\mathcal{S}_\omega(T_{\lambda_\infty} Q_\omega) > 0. \tag{2.50}$$

Combining (2.47) with (2.41) and (2.50), we obtain

$$\liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_n}\widetilde{u}_n) > m_\omega. \tag{2.51}$$

However, Lemma 2.1 (ii) together with $\mathcal{K}(\widetilde{u}_n) = 0$ and $\lambda_n < 1$ shows that

$$\limsup_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_n}\widetilde{u}_n) \leq \lim_{n\to\infty} \mathcal{S}_\omega(\widetilde{u}_n) = \lim_{n\to\infty} \mathcal{S}_\omega(u_n) = m_\omega, \tag{2.52}$$

which contradicts (2.51). Thus, we have proved that

$$\mathcal{K}(Q_\omega) \leq 0. \tag{2.53}$$

Now, we see from the assumption (N6) and (2.53) that there exists $\lambda_0 \in (0,1]$ such that

$$\mathcal{K}(T_{\lambda_0} Q_\omega) = 0. \tag{2.54}$$

Then, it follows from the definition of m_ω that

$$\mathcal{S}_\omega(T_{\lambda_0}Q_\omega) \ge m_\omega. \tag{2.55}$$

We also verify that

$$\lim_{n\to\infty} T_{\lambda_0}\widetilde{u}_n = T_{\lambda_0}Q_\omega \qquad \text{weakly in } H^1(\mathbb{R}^d), \tag{2.56}$$

so that Lemma 2.3 shows

$$\lim_{n\to\infty} \{\mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n) - \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega)\} = \mathcal{S}_\omega(T_{\lambda_0}Q_\omega), \tag{2.57}$$

$$\lim_{n\to\infty} \{\mathcal{K}(T_{\lambda_0}\widetilde{u}_n) - \mathcal{K}(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega)\} = 0. \tag{2.58}$$

Moreover, since $\mathcal{K}(\widetilde{u}_n) = \mathcal{K}(u_n) = 0$, Lemma 2.1 (ii) gives us that

$$\limsup_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n) \le \lim_{n\to\infty} \mathcal{S}_\omega(\widetilde{u}_n) = \lim_{n\to\infty} \mathcal{S}_\omega(u_n) = m_\omega. \tag{2.59}$$

If we had $\liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega) > 0$, then (2.57) together with (2.55) and (2.59) would give us that

$$m_\omega \le \mathcal{S}_\omega(T_{\lambda_0}Q_\omega) < \limsup_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n) \le m_\omega. \tag{2.60}$$

Thus, passing to some subsequence, we have that

$$\lim_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega) = 0. \tag{2.61}$$

Now, we suppose that $\mathcal{K}(Q_\omega) < 0$. Then, returning to (2.38), we find that

$$\lim_{n\to\infty} \mathcal{K}(\widetilde{u}_n - Q_\omega) = -\mathcal{K}(Q_\omega) > 0, \tag{2.62}$$

which together with the assumption (N6) also gives us that

$$\liminf_{n\to\infty} \mathcal{K}(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega) \ge 0. \tag{2.63}$$

Moreover, Lemma 2.1 (iv) together with (2.63) shows that

$$\liminf_{n\to\infty} \mathcal{H}(T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega) \ge 0. \tag{2.64}$$

Combining (2.61) with (2.64), we obtain the strong convergence:

$$\lim_{n\to\infty} \|T_{\lambda_0}\widetilde{u}_n - T_{\lambda_0}Q_\omega\|_{L^2} = \lim_{n\to\infty} \|\widetilde{u}_n - Q_\omega\|_{L^2} = 0, \tag{2.65}$$

which together with the boundedness in $H^1(\mathbb{R}^d)$ gives us the following: For any $2 \le q < 2^*$,

$$\lim_{n\to\infty} \widetilde{u}_n = Q_\omega \qquad \text{strongly in } L^q(\mathbb{R}^d). \tag{2.66}$$

Since $\mathcal{K}(\widetilde{u}_n) = 0$, the hypothesis $\mathcal{K}(Q_\omega) < 0$ together with (2.66) shows that

$$\|\nabla Q_\omega\|_{L^2} < \liminf_{n\to\infty} \|\nabla \widetilde{u}_n\|_{L^2}\,, \tag{2.67}$$

$$\|\nabla T_{\lambda_0} Q_\omega\|_{L^2} < \liminf_{n\to\infty} \|\nabla T_{\lambda_0} \widetilde{u}_n\|_{L^2}\,. \tag{2.68}$$

Then, we can derive from (2.55), (2.66), (2.59), and (2.68) that

$$m_\omega \le \mathcal{S}_\omega(T_{\lambda_0} Q_\omega) < \liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_0}\widetilde{u}_n) \le m_\omega. \tag{2.69}$$

This is a contradiction; thus, we have proved that

$$\mathcal{K}(Q_\omega) = 0. \tag{2.70}$$

Finally, we show that

$$\mathcal{S}_\omega(Q_\omega) = m_\omega, \tag{2.71}$$

which together with (2.70) shows that Q_ω is a minimizer.

Note that (2.70) implies $\lambda_0 = 1$, so that (2.61) becomes

$$\lim_{n\to\infty} \mathcal{S}_\omega(\widetilde{u}_n - Q_\omega) = 0. \tag{2.72}$$

Hence, we see from (2.57) with $\lambda_0 = 1$ that

$$m_\omega = \lim_{n\to\infty} \mathcal{S}_\omega(\widetilde{u}_n) = \mathcal{S}_\omega(Q_\omega), \tag{2.73}$$

which completes the proof of (2.71).

(Step 2). We shall show that the function Q_ω found above solves the equation (1.16).

Suppose that $\mathcal{S}_\omega{}'(Q_\omega) \neq 0$ in $H^{-1}(\mathbb{R}^d)$. Then, we can take $v_0 \in H^1(\mathbb{R}^d)$ such that $\langle \mathcal{S}_\omega{}'(Q_\omega), v_0\rangle = -2$. Moreover, it follows from the continuity of the mapping $\mathcal{S}_\omega{}'(Q_\omega)$ that there exists $\delta > 0$ such that

$$\langle \mathcal{S}_\omega{}'(T_\lambda Q_\omega + \theta\eta_\delta(\lambda)v_0), v_0\rangle \le -1 \tag{2.74}$$

for any $\lambda > 0$ with $|\lambda - 1| < 2\delta$ and $\theta \in [0,1]$, where η_δ is a smooth cut-off function with the property that

$$0 \le \eta_\delta(\lambda) \le \delta \quad \text{for any } \lambda > 0, \tag{2.75}$$
$$\eta_\delta(\lambda) > 0 \quad \text{if } |\lambda - 1| < \frac{\delta}{2}, \qquad \eta_\delta(\lambda) = 0 \quad \text{if } |\lambda - 1| \ge \frac{\delta}{2}.$$

Put $U(\lambda) := T_\lambda Q_\omega + \eta_\delta(\lambda)v_0$. Since $\mathcal{K}(Q_\omega) = 0$, we have

$$\mathcal{K}\Big(U\big(1+\frac{\delta}{2}\big)\Big) = \mathcal{K}\Big(T_{1+\frac{\delta}{2}}Q_\omega + \eta_\delta\big(1+\frac{\delta}{2}\big)v_0\Big) = \mathcal{K}\big(T_{1+\frac{\delta}{2}}Q_\omega\big) < 0, \tag{2.76}$$

$$\mathcal{K}\Big(U\big(1-\frac{\delta}{2}\big)\Big) = \mathcal{K}\Big(T_{1-\frac{\delta}{2}}Q_\omega + \eta_\delta\big(1-\frac{\delta}{2}\big)v_0\Big) = \mathcal{K}\big(T_{1-\frac{\delta}{2}}Q_\omega\big) > 0. \tag{2.77}$$

Hence, we find that $\mathcal{K}(U(\lambda_0)) = 0$ for some $\lambda_0 \in (1-\delta/2, 1+\delta/2)$. Then, it follows from the definition of m_ω, (2.74), (2.75), and Lemma 2.1 (ii) that

$$\begin{aligned} m_\omega \le \mathcal{S}_\omega\left(U(\lambda_0) \right) &= \mathcal{S}_\omega\left(T_{\lambda_0} Q_\omega \right) + \int_0^1 \frac{d}{d\theta} \mathcal{S}_\omega\big(T_{\lambda_0}Q_\omega + \theta\eta_\delta(\lambda_0)v_0\big)\, d\theta \\ &= \mathcal{S}_\omega(T_{\lambda_0}Q_\omega) + \int_0^1 \langle \mathcal{S}_\omega'\big(T_{\lambda_0}Q_\omega + \theta\eta_\delta(\lambda_0)v_0\big), \eta_\delta(\lambda_0)v_0\rangle\, d\theta \\ &\le \mathcal{S}_\omega(T_{\lambda_0}Q_\omega) - \eta_\delta(\lambda_0) < \mathcal{S}_\omega(T_{\lambda_0}Q_\omega) \le m_\omega, \end{aligned} \tag{2.78}$$

which is a contradiction. Thus, we find that ${\mathcal{S}_\omega}'(Q_\omega) = 0$ and Q_ω solves (1.16). This completes the proof. □

3. Analysis on $A_{\omega,+}$

In this section, we give fundamental properties of $A_{\omega,+}$ and solutions starting there.

The first property is the boundedness of $A_{\omega,+}$ in $H^1(\mathbb{R}^d)$:

Lemma 3.1. *Assume that $d \ge 1$ and* (N1)–(N6) *hold. Then, for any $\omega > 0$, there exists a constant $C_\omega > 0$ such that*

$$\sup_{u \in A_{\omega,+}} \|u\|_{H^1} \le C_\omega. \tag{3.1}$$

Proof of Lemma 3.1. This lemma is a direct consequence of Lemma 2.2. □

The second property is the invariance of $A_{\omega,+}$ under the flow defined by (1.1):

Lemma 3.2. *Assume that $d \ge 1$ and* (N1)–(N6) *hold. Let $\omega > 0$ and $\psi_0 \in A_{\omega,+}$, and let ψ be the corresponding solution to* (1.1). *Then, ψ exists globally in time* (*i.e., $I_{\max} = \mathbb{R}$*) *and satisfies the following:*

$$\psi(t) \in A_{\omega,+} \quad \text{for any } t \in \mathbb{R}, \tag{3.2}$$

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1} < \infty. \tag{3.3}$$

Proof of Lemma 3.2. In view of the conservation laws (1.11), (1.12), and Lemma 3.1, it suffices to show that

$$\mathcal{K}(\psi(t)) > 0 \qquad \text{for any } t \in I_{\max}. \tag{3.4}$$

If (3.4) fails, then we can take $t_0 \in I_{\max}$ such that

$$\mathcal{K}(\psi(t_0)) = 0. \tag{3.5}$$

This together with the definition of m_ω (see (1.17)) implies that

$$m_\omega \le \mathcal{S}_\omega(\psi(t_0)). \tag{3.6}$$

However, since $\psi_0 \in A_{\omega,+}$, we have by the conservation law of the action $\mathcal{S}_\omega$ that

$$\mathcal{S}_\omega(\psi(t_0)) = \mathcal{S}_\omega(\psi_0) < m_\omega, \tag{3.7}$$

which contradicts (3.6). Hence, (3.4) holds. □

Finally, we show the uniform positivity of the functional $\mathcal{K}$ of a solution starting from $A_{\omega,+}$.

Lemma 3.3. *Assume that $d \ge 1$ and* (N1)–(N6) *hold. Let $\omega > 0$ and $\psi_0 \in A_{\omega,+}$, and let ψ be the corresponding solution to* (1.1)*. Then, we have that*

$$\inf_{t\in\mathbb{R}} \mathcal{K}(\psi(t)) > 0. \tag{3.8}$$

Proof of Lemma 3.3. We suppose to the contrary that (3.8) fails. Then, there exists a sequence $\{t_n\}$ in $\mathbb{R}$ such that

$$\lim_{n\to\infty} t_n \in \{\pm\infty\}, \tag{3.9}$$

$$\lim_{n\to\infty} \mathcal{K}(\psi(t_n)) = 0. \tag{3.10}$$

We find by Lemmata 2.1 (iv), 3.1, and 3.2 that the sequence $\{\psi(t_n)\}$ satisfies that

$$\psi(t_n) \in A_{\omega,+} \qquad \text{for any } n \in \mathbb{N}, \tag{3.11}$$

$$\sup_{n\in\mathbb{N}} \|\psi(t_n)\|_{H^1} < K_1 \qquad \text{for some constant } K_1 > 0, \tag{3.12}$$

$$\mathcal{H}(\psi(t_n)) = \mathcal{H}(\psi_0) > 0 \qquad \text{for any } n \in \mathbb{N}. \tag{3.13}$$

Now, we suppose a dispersive situation

$$\liminf_{n\to\infty} \|\psi(t_n)\|_{L^{2+\frac{4}{d}}} = 0. \tag{3.14}$$

Then, passing to some subsequence, we see from (3.12) that

$$\lim_{n\to\infty} \|\psi(t_n)\|_{L^{p_1+1}} = \lim_{n\to\infty} \|\psi(t_n)\|_{L^{p_2+1}} = 0. \tag{3.15}$$

Combining (1.6) with (3.15), we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |F(\psi(x,t_n))| \, dx = 0. \tag{3.16}$$

Similarly, we find that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\left|f(\psi(x,t_n))\overline{\psi(x,t_n)}\right|\,dx=0. \tag{3.17}$$

Using (3.13) and (3.16), we have that

$$\begin{aligned}0<\mathcal{H}(\psi_0)&=\lim_{n\to\infty}\mathcal{H}(\psi(t_n)) \qquad (3.18)\\ &=\lim_{n\to\infty}\|\nabla\psi(t_n)\|_{L^2}^2-\lim_{n\to\infty}\int_{\mathbb{R}^d}F(\psi(x,t_n))\,dx=\lim_{n\to\infty}\|\nabla\psi(t_n)\|_{L^2}^2.\end{aligned}$$

On the other hand, (3.10) together with (3.16) and (3.17), gives us that

$$\begin{aligned}&\lim_{n\to\infty}2\|\nabla\psi(t_n)\|_{L^2}^2 \qquad (3.19)\\ &=\lim_{n\to\infty}2\|\nabla\psi(t_n)\|_{L^2}^2-\lim_{n\to\infty}d\int_{\mathbb{R}^d}\left\{f(\psi(x,t_n))\overline{\psi(x,t_n)}-F(\psi(x,t_n))\right\}dx\\ &=\lim_{n\to\infty}\mathcal{K}(\psi(t_n))=0,\end{aligned}$$

which contradicts (3.18). Thus, the dispersive situation (3.14) never happens, and therefore we have that

$$\liminf_{n\to\infty}\|\psi(t_n)\|_{L^{2+\frac{4}{d}}}>0. \tag{3.20}$$

Then, combining with the result we find by Lemmata 2.1 of [10] and 6 of [19] that there exists a subsequence of $\{\psi(t_n)\}$ (still denoted by the same symbol) with the following property: There exists a sequence $\{y_n\}$ in $\mathbb{R}^d$ and a nontrivial function $u_\infty\in H^1(\mathbb{R}^d)$ such that, putting $u_n(x):=\psi(x+y_n,t_n)$, we have that

$$\lim_{n\to\infty}u_n=u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d), \tag{3.21}$$

$$\lim_{n\to\infty}u_n=u_\infty \qquad \text{almost everywhere in } \mathbb{R}^d, \tag{3.22}$$

$$u_n\in A_{\omega,+} \qquad \text{for any } n\in\mathbb{N}, \tag{3.23}$$

$$\sup_{n\in\mathbb{N}}\|u_n\|_{H^1}<K_1, \tag{3.24}$$

$$\inf_{n\in\mathbb{N}}\mathcal{L}^d([|u_n|>\eta])\geq C \qquad \text{for some constants } \eta>0 \text{ and } C>0. \tag{3.25}$$

Moreover, it follows from Lemma 2.3 that passing to subsequence, we have

$$\lim_{n\to\infty}\{\mathcal{K}(u_n)-\mathcal{K}(u_n-u_\infty)-\mathcal{K}(u_\infty)\}=0, \tag{3.26}$$

$$\lim_{n\to\infty}\{\mathcal{S}_\omega(u_n)-\mathcal{S}_\omega(u_n-u_\infty)-\mathcal{S}_\omega(u_\infty)\}=0. \tag{3.27}$$

In particular, we can verify that

$$\lim_{n\to\infty} \mathcal{K}(u_n - u_\infty) = -\mathcal{K}(u_\infty) \qquad \text{(by (3.10))}, \tag{3.28}$$

$$\lim_{n\to\infty} \mathcal{S}_\omega(u_n - u_\infty) = \mathcal{S}_\omega(\psi_0) - \mathcal{S}_\omega(u_\infty). \tag{3.29}$$

Using this sequence $\{u_n\}$, we shall derive a contradiction; hence, the Proposition 3.3 holds. We divide the proof into three cases: Case 1: $\mathcal{K}(u_\infty) < 0$, Case 2: $\mathcal{K}(u_\infty) > 0$, and Case 3: $\mathcal{K}(u_\infty) = 0$.

Case 1: Suppose that

$$\mathcal{K}(u_\infty) < 0. \tag{3.30}$$

Then, it follows from the assumption (N6) that there exists a unique $\lambda^\infty \in (0, 1)$ such that

$$\mathcal{K}(T_{\lambda^\infty} u_\infty) = 0, \tag{3.31}$$

so that we have by the definition of m_ω (see (1.17)) that

$$\mathcal{S}_\omega(T_{\lambda^\infty} u_\infty) \geq m_\omega. \tag{3.32}$$

We also verify that

$$\lim_{n\to\infty} T_{\lambda^\infty} u_n = T_{\lambda^\infty} u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d), \tag{3.33}$$

$$\lim_{n\to\infty} T_{\lambda^\infty} u_n = T_{\lambda^\infty} u_\infty \qquad \text{almost everywhere in } \mathbb{R}^d, \tag{3.34}$$

so that Lemma 2.3 give us that

$$\lim_{n\to\infty} \{\mathcal{S}_\omega(T_{\lambda^\infty} u_n) - \mathcal{S}_\omega(T_{\lambda^\infty} u_n - T_{\lambda^\infty} u_\infty) - \mathcal{S}_\omega(T_{\lambda^\infty} u_\infty)\} = 0. \tag{3.35}$$

Here, since $\lambda^\infty < 1$, it follows from (3.23), Lemma 2.1 (iii), and the conservation law of the action that

$$\mathcal{S}_\omega(T_{\lambda^\infty} u_n) < \mathcal{S}_\omega(u_n) = \mathcal{S}_\omega(\psi(t_n)) = \mathcal{S}_\omega(\psi_0) < m_\omega. \tag{3.36}$$

On the other hand, we have by (3.28), (3.30), and (N6) that

$$\mathcal{K}(T_{\lambda^\infty} u_n - T_{\lambda^\infty} u_\infty) > 0 \qquad \text{for any sufficiently large } n \in \mathbb{N}, \tag{3.37}$$

so that Lemma 2.1 (ii) gives us that

$$\liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda^\infty} u_n - T_{\lambda^\infty} u_\infty) \geq 0. \tag{3.38}$$

This together with (3.32), (3.35), (3.36), and (3.38) shows that

$$m_\omega \leq \mathcal{S}_\omega(T_{\lambda^\infty} u_\infty) \leq \liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda^\infty} u_n) < m_\omega, \tag{3.39}$$

which is a contradiction. Thus, the case $\mathcal{K}(u_\infty) < 0$ never happens.

Case 2: Suppose that

$$\mathcal{K}(u_\infty) > 0. \tag{3.40}$$

In this case, (3.28) shows that, extracting some subsequence of $\{u_n\}$ (still denoted by the same symbol), we have that

$$\mathcal{K}(u_n - u_\infty) \le -\frac{1}{2}\mathcal{K}(u_\infty) < 0 \qquad \text{for any } n \in \mathbb{N}. \tag{3.41}$$

Hence, we can take a sequence $\{\lambda_n\}$ in the open interval $(0,1)$ and $\lambda_\infty \in [0,1]$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda_\infty, \tag{3.42}$$

$$\mathcal{K}(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{3.43}$$

Here, we find by (1.4), (1.6), the Gagliardo–Nirenberg inequality, and (3.24) that

$$\begin{aligned}\mathcal{K}(u_n - u_\infty) &\ge \|\nabla(u_n - u_\infty)\|_{L^2}^2 \\ &\quad - K_1'\Big(\|\nabla(u_n - u_\infty)\|_{L^2}^{\frac{d}{2}(p_1-1)} + \|\nabla(u_n - u_\infty)\|_{L^2}^{\frac{d}{2}(p_2-1)} \Big)\end{aligned} \tag{3.44}$$

for some constant $K_1' > 0$ depending only on d, p_1, p_2, C_f, and K_1. Hence, if $\{\nabla u_n\}$ converges to ∇u_∞ strongly in $L^2(\mathbb{R}^d)$, then we have that

$$\liminf_{n\to\infty} \mathcal{K}(u_n - u_\infty) \ge 0. \tag{3.45}$$

However, this contradicts (3.41). Thus, we can extract a subsequence of $\{u_n\}$ (still denoted by the same symbol) such that

$$\inf_{n\in\mathbb{N}} \|\nabla(u_n - u_\infty)\|_{L^2} > K_2 \qquad \text{for some constant } K_2 > 0. \tag{3.46}$$

We shall show that

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n > 0. \tag{3.47}$$

Indeed, in an estimate similar to (3.44), we have that

$$\begin{aligned} 0 &= \mathcal{K}(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) \\ &\ge \lambda_n^2 \|\nabla(u_n - u_\infty)\|_{L^2}^2 - K_1''\Big(\lambda_n^{\frac{d}{2}(p_1-1)} \|\nabla(u_n - u_\infty)\|_{L^2}^{\frac{d}{2}(p_1-1)} \\ &\qquad\qquad + \lambda_n^{\frac{d}{2}(p_2-1)} \|\nabla(u_n - u_\infty)\|_{L^2}^{\frac{d}{2}(p_2-1)}\Big) \\ &> K_2^2 \lambda_n^2 - K_1''\Big(K_1^{\frac{d}{2}(p_1-1)} \lambda_n^{\frac{d}{2}(p_1-1)} + K_1^{\frac{d}{2}(p_2-1)} \lambda_n^{\frac{d}{2}(p_2-1)}\Big)\end{aligned} \tag{3.48}$$

for any $n \in \mathbb{N}$, where $K_1'' > 0$ is some constant depending only on d, p_1, p_2, C_f, and K_1. Since $2 < \frac{d}{2}(p_1 - 1) \le \frac{d}{2}(p_2 - 1)$, the estimate (3.48) shows (3.47).

Now, using (3.21) and (3.47), we can verify that

$$\lim_{n\to\infty} T_{\lambda_n} u_n = T_{\lambda_\infty} u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d), \tag{3.49}$$

$$\lim_{n\to\infty} T_{\lambda_n} u_\infty = T_{\lambda_\infty} u_\infty \qquad \text{strongly in } H^1(\mathbb{R}^d). \tag{3.50}$$

Hence, we obtain by Lemma 2.3 that

$$\lim_{n\to\infty} \{\mathcal{K}(T_{\lambda_n} u_n) - \mathcal{K}(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) - \mathcal{K}(T_{\lambda_\infty} u_\infty)\} = 0, \tag{3.51}$$

$$\lim_{n\to\infty} \{\mathcal{S}_\omega(T_{\lambda_n} u_n) - \mathcal{S}_\omega(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) - \mathcal{S}_\omega(T_{\lambda_\infty} u_\infty)\} = 0. \tag{3.52}$$

Since $\lambda_\infty \le 1$ and $\mathcal{K}(u_\infty) \ge 0$ (see (3.40)), the assumption (N6) shows that $\mathcal{K}(T_{\lambda_\infty} u_\infty) \ge 0$. Hence, we have by Lemma 2.1 (iv) that $\mathcal{S}_\omega(T_{\lambda_\infty} u_\infty) > 0$. Moreover, since $\mathcal{K}(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) = 0$ (see (3.43)), we have by the definition of m_ω (see (1.17)) that $\mathcal{S}_\omega(T_{\lambda_n} u_n - T_{\lambda_n} u_\infty) \ge m_\omega$. These allow us to conclude that

$$\liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_n} u_n) > m_\omega. \tag{3.53}$$

On the other hand, since $\lambda_n < 1$, it follows from (3.23), Lemma 2.1 (iii), and the conservation law of the action that

$$\liminf_{n\to\infty} \mathcal{S}_\omega(T_{\lambda_n} u_n) \le \lim_{n\to\infty} \mathcal{S}_\omega(u_n) = \mathcal{S}_\omega(\psi(t_n)) = \mathcal{S}(\psi_0) < m_\omega, \tag{3.54}$$

which contradicts (3.53). Thus, the case $\mathcal{K}(u_\infty) > 0$ never happens.
Case 3: Suppose finally that

$$\mathcal{K}(u_\infty) = 0. \tag{3.55}$$

Then, we have by the definition of m_ω that

$$\mathcal{S}_\omega(u_\infty) \ge m_\omega. \tag{3.56}$$

Moreover, (3.29) together with $\mathcal{S}_\omega(\psi_0) < m_\omega$ and (3.56) gives us that

$$\lim_{n\to\infty} \mathcal{S}_\omega(u_n - u_\infty) < 0 \tag{3.57}$$

so that $\limsup_{n\to\infty} \mathcal{H}(u_n - u_\infty) < 0$. Hence, we can extract a subsequence of $\{u_n\}$ (still denoted by the same symbol) such that

$$\mathcal{H}(u_n - u_\infty) < 0 \qquad \text{for any } n \in \mathbb{N}. \tag{3.58}$$

Here, Lemma 2.1 (iv) together with (3.58) shows that

$$\mathcal{K}(u_n - u_\infty) < 0 \qquad \text{for any } n \in \mathbb{N}. \tag{3.59}$$

Then by an argument similar to that of Case 2, we can derive a contradiction, which implies that the case where $\mathcal{K}(u_\infty) = 0$ never happens. Thus, this completes the proof. □

4. Sufficient condition for scattering

In this section, we shall give a sufficient condition for a solution to scatter in $H^1(\mathbb{R}^d)$. To this end, we first introduce Strichartz-type spaces, which are convenient to control the long-time behavior of a solution to (1.1).

We recall the integral equation associated with (1.1):

$$\psi(t) = e^{\frac{i}{2}t\Delta}\psi(t_0) + \frac{i}{2}\int_{t_0}^{t} e^{\frac{i}{2}(t-t')\Delta} f(\psi(t'))\, dt'. \tag{4.1}$$

We also recall the Strichartz estimates: For any interval I and any admissible pairs[1] (q,r) and (q_1,r_1), we have

$$\left\| e^{\frac{i}{2}t\Delta} u \right\|_{L^r(I,L^q)} \lesssim \|u\|_{L^2}, \tag{4.2}$$

$$\left\| \int_0^t e^{\frac{i}{2}(t-t')\Delta} v(t')\, dt' \right\|_{L^r(I,L^q)} \lesssim \|v\|_{L^{r_1'}(I,L^{q_1'})}, \tag{4.3}$$

where the implicit constants are independent of I.

Let p_1 and p_2 be numbers found in the assumption (N2), and fix a number σ satisfying $p_2+1<\sigma<2^*$. For $p>1$, we put

$$s_p := \frac{d}{2} - \frac{2}{p-1}. \tag{4.4}$$

It is easy to verify that $0 < s_{p_1} \le s_{p_2} < 1$.

We define $Q_0 = (\frac{1}{q_0}, \frac{1}{r_0})$ by

$$q_0 := \sigma, \qquad \frac{1}{r_0} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{\sigma}\Big), \tag{4.5}$$

so that the pair (q_0, r_0) is admissible. For any interval I, we define a usual Strichartz space $S(I)$ by

$$S(I) := L^\infty(I, L^2(\mathbb{R}^d)) \cap L(Q_0; I). \tag{4.6}$$

Next, we shall introduce a Strichartz-type space $X(I)$. Define $Q_1 = (\frac{1}{q_1}, \frac{1}{r_1})$ and $\widetilde{Q}_1 = (\frac{1}{\widetilde{q}_1}, \frac{1}{\tilde{r}_1})$ as points such that

$$q_1 := \sigma, \qquad \frac{1}{r_1} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{\sigma} - \frac{s_{p_1}}{d}\Big), \tag{4.7}$$

$$\frac{p_1-1}{\widetilde{q}_1} = 1 - \frac{2}{\sigma}, \qquad \frac{1}{\tilde{r}_1} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{\widetilde{q}_1} - \frac{s_{p_1}}{d}\Big). \tag{4.8}$$

[1] A pair (q,r) is said to be admissible if $2 \le q < 2^*$ and $\frac{1}{r} = \frac{d}{2}(\frac{1}{2} - \frac{1}{q})$.

We also define two points $Q_2 = (\frac{1}{q_2}, \frac{1}{r_2})$ and $\widetilde{Q}_2 = (\frac{1}{\widetilde{q}_2}, \frac{1}{\widetilde{r}_2})$ by

$$q_2 := \sigma, \qquad \frac{1}{r_2} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{\sigma} - \frac{s_{p_2}}{d}\Big), \tag{4.9}$$

$$\frac{p_2 - 1}{\widetilde{q}_2} = 1 - \frac{2}{\sigma}, \qquad \frac{1}{\widetilde{r}_2} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{\widetilde{q}_2} - \frac{s_{p_2}}{d}\Big). \tag{4.10}$$

Besides these points, we define Q_1^* and Q_2^* by

$$Q_1^* := Q_1 + (p_1 - 1)\widetilde{Q}_1, \qquad Q_2^* := Q_2 + (p_2 - 2)\widetilde{Q}_2. \tag{4.11}$$

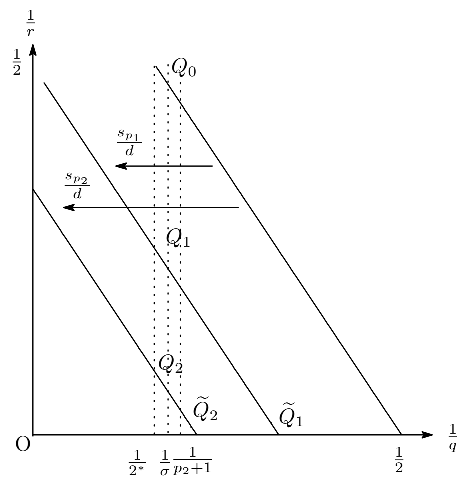

FIGURE 1. Strichartz-type spaces

It is worthwhile noting here that

$$\frac{p_2 - 1}{p_1 - 1}\widetilde{Q}_2 = \widetilde{Q}_1 \tag{4.12}$$

and

$$Q_0' = Q_0 + (p_1 - 1)\widetilde{Q}_1 = Q_0 + (p_2 - 1)\widetilde{Q}_2. \tag{4.13}$$

We finally define our Strichartz-type space $X(I)$ by

$$X(I) := X_1(I) \cap X_2(I), \tag{4.14}$$

where

$$X_j(I) := L(Q_j; I) \cap L(\widetilde{Q}_j; I), \qquad j = 1, 2. \tag{4.15}$$

Then, we see from the Sobolev embedding and the Strichartz estimate (4.2) that

$$\|e^{\frac{i}{2}t\Delta}u\|_{X(\mathbb{R})} \lesssim \||\nabla|^{s_{p_1}}u\|_{L^2} + \||\nabla|^{s_{p_2}}u\|_{L^2}. \tag{4.16}$$

Moreover, we have

Proposition 4.1. *Assume that $d \geq 1$ and* (N1)–(N4) *hold. Let ψ be a global solution to* (1.1). *Suppose that*

$$\|\psi\|_{L^\infty(\mathbb{R},H^1)}, \|\psi\|_{X(\mathbb{R})} < \infty. \tag{4.17}$$

Then, there exist functions $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|\psi(t) - e^{\frac{i}{2}t\Delta}\phi_\pm\|_{H^1} = 0. \tag{4.18}$$

Proof of Proposition 4.1. We can prove this proposition in a way similar to [12]. □

The following proposition gives us a sufficient condition for a solution to be bounded in $X(I)$ and $\langle\nabla\rangle^{-1}S(I)$ for any given interval I:

Proposition 4.2 (Small data theory)**.** *Assume that $d \geq 1$ and* (N1)–(N4) *hold. Let I be an interval, and let $t_0 \in I$. Then, for any $A > 0$, there exists $\delta > 0$ with the following property: For any $\psi_0 \in H^1(\mathbb{R}^d)$ with*

$$\|\psi_0\|_{H^1} \leq A, \qquad \big\|e^{\frac{i}{2}(t-t_0)\Delta}\psi_0\big\|_{X(I)} \leq \delta, \tag{4.19}$$

there exists a unique solution ψ in $C(I, H^1(\mathbb{R}^d))$ to (1.1) *with $\psi(t_0) = \psi_0$ such that*

$$\|\psi\|_{X(I)} < 2\big\|e^{\frac{i}{2}(t-t_0)\Delta}\psi_0\big\|_{X(I)}, \qquad \|\langle\nabla\rangle\psi\|_{S(I)} \lesssim \|\psi_0\|_{H^1}, \tag{4.20}$$

where the implicit constant is independent of t_0 and I.

This proposition is essentially known. Hence, we omit the proof.

As a corollary of Proposition 4.2, we have

Corollary 4.1. *There exists $\varepsilon > 0$ such that for any $\psi_0 \in H^1(\mathbb{R}^d)$ with $\|\psi_0\|_{H^1} < \varepsilon$, the corresponding solution ψ to* (1.1) *satisfies*

$$\|\psi\|_{X(\mathbb{R})}, \|\langle\nabla\rangle\psi\|_{S(\mathbb{R})} \lesssim \varepsilon \tag{4.21}$$

and has asymptotic states in $H^1(\mathbb{R}^d)$ at $\pm\infty$.

Proof of Corollary 4.1. Propositions 4.1 and 4.2 together with the inequality (4.16) immediately show the conclusion. □

5. PROOF OF THEOREM 1.2 (i)

This section is devoted to the proof of the scattering part in Theorem 1.2. In view of Lemma 3.2 and Proposition 4.1, it is sufficient to show that any solution ψ to (1.1) starting from $A_{\omega,+}$ satisfies $\|\psi\|_{X(\mathbb{R})} < \infty$. To this end, we introduce the set

$$A_{\omega,+}(m) := \Big\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m,\ \mathcal{K}(u) > 0 \Big\} \tag{5.1}$$

for $\omega > 0$ and $m > 0$. Moreover, for $\omega > 0$, we define a number m_ω^* by

$$\begin{aligned} m_\omega^* :=& \sup \big\{ m > 0 : \|\psi\|_{X(\mathbb{R})} < \infty \text{ for all } \psi_0 \in A_{\omega,+}(m) \big\} \\ =& \inf \big\{ m > 0 : \|\psi\|_{X(\mathbb{R})} = \infty \text{ for some } \psi_0 \in A_{\omega,+}(m) \big\}, \end{aligned} \tag{5.2}$$

where ψ denotes the solution to (1.1) with $\psi(0) = \psi_0$.

Clearly, we have $A_{\omega,+} = A_{\omega,+}(m_\omega)$. Hence, our task is to show that

$$m_\omega^* \ge m_\omega. \tag{5.3}$$

Here, we can derive $m_\omega^* > 0$ from our small data theory (Proposition 4.2); indeed, it follows from Lemma 2.1 (iv) and Lemma 3.1 that for any $\psi_0 \in A_{\omega,+}(m)$ with $m \le m_\omega$ and $0 < s < 1$,

$$\| |\nabla|^s \psi_0 \|_{L^2} \le \|\psi_0\|_{L^2}^{1-s} \|\nabla \psi_0\|_{L^2}^s \lesssim m^{1-s} \to 0 \qquad \text{as } m \to 0, \tag{5.4}$$

which together with the inequality (4.16) shows that

$$\big\| e^{it\Delta} \psi_0 \big\|_{X(\mathbb{R})} \to 0 \qquad \text{as } m \to 0. \tag{5.5}$$

Hence, the small data theory shows $m_\omega^* > 0$.

We shall show (5.3) by contradiction. Suppose to the contrary that $m_\omega^* < m_\omega$. In this undesirable situation, we can find the so-called critical element:

Proposition 5.1 (Existence of a critical element in $A_{\omega,+}$)**.** *Suppose $m_\omega^* < m_\omega$. Then, there exists a global solution $\Psi \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ to (1.1) with the following property:*

(i)

$$\Psi(t) \in A_{\omega,+} \quad \text{for any } t \in \mathbb{R}, \tag{5.6}$$

$$\|\Psi\|_{X(\mathbb{R})} = \infty, \tag{5.7}$$

$$\mathcal{S}_\omega(\Psi(t)) = m_\omega^* \quad \text{for any } t \in \mathbb{R}; \tag{5.8}$$

(ii)

$$\Im \int_{\mathbb{R}^d} \overline{\Psi}(x,t) \nabla \Psi(x,t)\, dx = 0 \qquad \text{for any } t \in \mathbb{R}; \tag{5.9}$$

(iii) $\{\Psi(t)\}_{t\geq 0}$ *is tight in* $H^1(\mathbb{R}^d)$ *in the following sense: For any* $\varepsilon > 0$*, there exists* $R_\varepsilon > 0$ *and a continuous path* $\gamma_\varepsilon \in C([0,+\infty),\mathbb{R}^d)$ *with* $\gamma_\varepsilon(0) = 0$ *such that*

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} |\Psi(x,t)|^2 \, dx > \|\Psi(0)\|_{L^2}^2 - \varepsilon \quad \text{for any } t \in [0,+\infty) \tag{5.10}$$

and

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} |\nabla\Psi(x,t)|^2 \, dx > \|\nabla\Psi(t)\|_{L^2}^2 - \varepsilon \quad \text{for any } t \in [0,+\infty). \tag{5.11}$$

We shall give the proof of Proposition 5.1 in the next section.

Since the momentum of the critical element is zero, we can expect that its center of mass does not move. Unfortunately, we just work in $H^1(\mathbb{R}^d)$, and therefore the notion of center of mass no longer has the meaning. However, we can find the following:

Lemma 5.2 (Almost center of mass)**.** *Let* Ψ *be a global solution to* (1.1) *found in Proposition* 5.1*, and let* R_ε *denote the radius found in Proposition* 5.1 *for each* $\varepsilon > 0$*. We define the "almost center of mass" by*

$$\gamma_{\varepsilon,R}^{ac}(t) := \int_{\mathbb{R}^d} \nabla W_{20R}(x) \frac{|\Psi(x,t)|^2}{\|\Psi(0)\|_{L^2}^2} \, dx \tag{5.12}$$

for any $\varepsilon \in (0, \frac{1}{100})$ *and* $R > R_\varepsilon$*, where* W_R *is the function defined by* (A.7)*. Then, we have* $\gamma_{\varepsilon,R}^{ac} \in C^1([0,\infty),\mathbb{R}^d)$*. Furthermore, there exists a constant* $\alpha > 0$*, depending only on* d*,* p_1*, and* p_2*, such that*

$$\left|\gamma_{\varepsilon,R}^{ac}(t)\right| \leq 20R, \tag{5.13}$$

$$\int_{|x-\gamma_{\varepsilon,R}^{ac}(t)|\leq 4R} |\Psi(x,t)|^2 + |\nabla\Psi(x,t)|^2 \, dx \geq \|\Psi(t)\|_{H^1}^2 - \varepsilon \tag{5.14}$$

for any $t \in [0, \alpha\frac{R}{\sqrt{\varepsilon}}]$*.*

The proof of Lemma 5.2 is the same as that of Lemma 4.2 in [1] (see also the proof of Lemma D.4 in [21], while it has typos). Hence, we omit it.

Now, we give the proof of Theorem 1.2 (i):

Proof of Theorem 1.2 (i). The generalized virial identity (A.14) together with Lemma 3.3 and (A.10) yields that

$$\int_{\mathbb{R}^d} W_R|\Psi(t)|^2 \geq \int_{\mathbb{R}^d} W_R|\Psi(0)|^2 + t\Im\int_{\mathbb{R}^d} \nabla W_R \cdot \nabla\Psi(0)\overline{\Psi(0)} + C_1 t^2 \tag{5.15}$$

$$- C_2 \int_0^t \int_0^{t'} \int_{|x|\ge R} |\nabla \Psi(x,t'')|^2 \, dx \, dt'' dt'$$
$$- C_3 \int_0^t \int_0^{t'} \int_{|x|\ge R} |\Psi(x,t'')|^{p_1+1} + |\Psi(x,t'')|^{p_2+1} \, dx \, dt' dt'' - C_4 \frac{t^2}{R^2} \|\Psi(0)\|_{L^2}^2$$

for any $R > 0$, where $C_1 := \frac{1}{2} \inf_{t\in\mathbb{R}} \mathcal{K}(\Psi(t)) > 0$, and C_2, C_3, and C_4 are some constants depending only on d, p_1, p_2, C_f, and $\|\rho\|_{W^{2,\infty}}$. Using Lemma 5.2, for any $\varepsilon \in (0, \frac{1}{100})$, we can take $R_\varepsilon > 0$ with the property that for any $R \ge R_\varepsilon$ there exists $\gamma_{\varepsilon,R}^{ac} \in C^1([0,\infty), \mathbb{R}^d)$ such that

$$\left|\gamma_{\varepsilon,R}^{ac}(t)\right| \le 20R, \tag{5.16}$$
$$\int_{|x-\gamma_{\varepsilon,R}^{ac}(t)|\ge 4R} |\nabla \Psi(x,t)|^2 \, dx < \varepsilon, \tag{5.17}$$
$$\int_{|x-\gamma_{\varepsilon,R}^{ac}(t)|\ge 4R} |\Psi(x,t)|^{p_1+1} + |\Psi(x,t)|^{p_2+1} \, dx < \varepsilon, \tag{5.18}$$

for any $t \in \left[\, 0, \, \alpha \frac{R}{\sqrt{\varepsilon}} \,\right]$, where α is some constant depending only on p_1, p_2, and d. We see from (5.16) that

$$|x - \gamma_{\varepsilon,R}^{ac}(t)| \ge 4R \quad \text{for any } x \in \mathbb{R}^d \text{ with } |x| \ge 24R \text{ and } t \in \left[\, 0, \, \alpha \tfrac{R}{\sqrt{\varepsilon}} \,\right]. \tag{5.19}$$

Hence, (5.15) together with (5.17) and (5.18) shows that

$$\begin{aligned} &\int_{\mathbb{R}^d} W_{50R} |\Psi(t)|^2 \\ &\ge \int_{\mathbb{R}^d} W_{50R} |\Psi(0)|^2 + t\Im \int_{\mathbb{R}^d} \nabla W_{50R} \cdot \nabla \Psi(0) \overline{\Psi(0)} + C_1 t^2 - (C_2 + C_3)\varepsilon t^2 \\ &\qquad - C_4 \frac{t^2}{(50R)^2} \|\Psi(0)\|_{L^2}^2 \qquad \text{for any } R \ge R_\varepsilon \text{ and } t \in \left[\, 0, \, \alpha \tfrac{R}{\sqrt{\varepsilon}} \,\right]. \end{aligned} \tag{5.20}$$

Choosing ε and R satisfying

$$\varepsilon < \min\left\{ \frac{1}{100}, \; \frac{C_1}{4(C_2 + C_3)} \right\}, \quad R \ge \max\left\{ R_\varepsilon, \; \|\Psi(0)\|_{L^2} \sqrt{\frac{C_4}{C_1}} \right\}, \tag{5.21}$$

we obtain from (5.20) that

$$\int_{\mathbb{R}^d} W_{50R} |\Psi(t)|^2 \ge \int_{\mathbb{R}^d} W_{50R} |\Psi(0)|^2 + t\Im \int_{\mathbb{R}^d} \nabla W_{50R} \cdot \nabla \Psi(0) \overline{\Psi(0)} + \frac{C_1}{2} t^2 \tag{5.22}$$

for any $t \in [\,0,\, \alpha\frac{R}{\sqrt{\varepsilon}}\,]$. Moreover, choosing $t = \alpha\frac{R}{\sqrt{\varepsilon}}$ in (5.22), we see from (A.8) and (A.9) that

$$\begin{aligned}&(50R)^2 \|\Psi(0)\|_{L^2}^2 \\ &\gtrsim -(50R)^2 \|\Psi(0)\|_{L^2}^2 - 50\frac{\alpha}{\sqrt{\varepsilon}}R^2 \|\Psi(0)\|_{L^2} \|\nabla\Psi(0)\|_{L^2} + \frac{C_1}{2}\alpha^2\frac{R^2}{\varepsilon}.\end{aligned} \tag{5.23}$$

However, this leads us to a contradiction when ε tends to 0. Thus, we find $m_\omega^* \geq m_\omega$, so that the proof of Theorem 1.2 (i) is completed. □

6. Construction of a critical element

In this section, we sketch the proof of Proposition 5.1. The proof is essentially the same as [17, Section 4]. However, since we consider general nonlinearities, we need some additional argument (see Lemma 6.2).

We see from the hypothesis $m_\omega^* < m_\omega$ that there exists a sequence $\{\psi_n\}$ of solutions to (1.1) such that

$$\psi_n(t) \in A_{\omega,+} \qquad \text{for any } t \in \mathbb{R} \text{ and } n \in \mathbb{N}, \tag{6.1}$$

$$\|\psi_n\|_{X(\mathbb{R})} = \infty, \tag{6.2}$$

$$\lim_{n\to\infty} \mathcal{S}_\omega(\psi_n(t)) = m_\omega^* \ (< m_\omega) \qquad \text{for any } t \in \mathbb{R}. \tag{6.3}$$

Put $u_n := \psi_n(0)$. Then, the sequence $\{u_n\}$ satisfies the following:

Proposition 6.1. *Assume that $d \geq 1$ and (N1)–(N4) hold. Then, we can extract a subsequence of $\{u_n\}$ (still denoted by the same symbol) with the following properties: There exists*

(i) *a family of non-trivial functions $\{\widetilde{u}^1, \widetilde{u}^2, \ldots\}$ in $H^1(\mathbb{R}^d)$, and*
(ii) *a family of sequences $\big\{\{(x_n^1, t_n^1)\}_{n\in\mathbb{N}}, \{(x_n^2, t_n^2)\}_{n\in\mathbb{N}}, \ldots, \big\}$ in $\mathbb{R}^d \times \mathbb{R}$ with*

$$\lim_{n\to\infty} t_n^k = t_\infty^k \in \mathbb{R} \cup \{\pm\infty\} \qquad \textit{for any } k \geq 1, \tag{6.4}$$

and

$$\lim_{n\to\infty} \Big\{|t_n^l - t_n^k| + |x_n^l - x_n^k|\Big\} = \infty \qquad \textit{for any } 1 \leq k < l \tag{6.5}$$

such that, putting

$$u_n^0 := u_n, \quad u^0 := 0, \quad t_n^0 := 0, \quad x_n^0 := 0, \tag{6.6}$$

$$u_n^k := e^{\frac{i}{2}(t_n^k - t_n^{k-1})\Delta} e^{(x_n^k - x_n^{k-1})\cdot\nabla}(u_n^{k-1} - \widetilde{u}^{k-1}), \tag{6.7}$$

we have, for any $k \geq 1$,

$$\lim_{n\to\infty} u_n^k = \widetilde{u}^k \quad \textit{weakly in } H^1(\mathbb{R}^d) \textit{and strongly in } L^q_{loc}(\mathbb{R}^d) \quad \textit{for any } q \in [2, 2^*), \tag{6.8}$$

$$\lim_{n\to\infty} \left\{ \||\nabla|^s u_n\|_{L^2}^2 - \||\nabla|^s (u_n^k - \widetilde{u}^k)\|_{L^2}^2 \right\} = \sum_{j=1}^{k} \||\nabla|^s \widetilde{u}^j\|_{L^2}^2, \tag{6.9}$$

$$\lim_{n\to\infty} \Big\{ \mathcal{H}(T_\lambda u_n) - \mathcal{H}\left(T_\lambda e^{-\frac{i}{2}t_n^k\Delta}(u_n^k - \widetilde{u}^k) \right) - \sum_{j=1}^{k} \mathcal{H}\left(T_\lambda e^{-\frac{i}{2}t_n^j\Delta}\widetilde{u}^j \right) \Big\} = 0, \tag{6.10}$$

$$\lim_{n\to\infty} \Big\{ \mathcal{K}(T_\lambda u_n) - \mathcal{K}\left(T_\lambda e^{-\frac{i}{2}t_n^k\Delta}(u_n^k - \widetilde{u}^k) \right) - \sum_{j=1}^{k} \mathcal{K}\left(T_\lambda e^{-\frac{i}{2}t_n^j\Delta}\widetilde{u}^j \right) \Big\} = 0, \tag{6.11}$$

$$\lim_{n\to\infty} \Big\{ \mathcal{S}_\omega(T_\lambda u_n) - \mathcal{S}_\omega\left(T_\lambda e^{-\frac{i}{2}t_n^k\Delta}(u_n^k - \widetilde{u}^k) \right) - \sum_{j=1}^{k} \mathcal{S}_\omega\left(T_\lambda e^{-\frac{i}{2}t_n^j\Delta}\widetilde{u}^j \right) \Big\} = 0, \tag{6.12}$$

uniformly in $\lambda \in (0,1]$. *Furthermore, putting* $M := \#\{\widetilde{u}^1, \widetilde{u}^2, \dots\}$, *we have the following alternatives: for any* $q \in (2, 2^*)$,
if M *is finite, then*

$$\lim_{n\to\infty} \|e^{\frac{i}{2}t\Delta}(u_n^M - \tilde{u}^M)\|_{X(\mathbb{R})\cap L^\infty(\mathbb{R},L^q)} = 0; \tag{6.13}$$

if $M = \infty$, *then*

$$\lim_{k\to\infty}\lim_{n\to\infty} \|e^{\frac{i}{2}t\Delta}(u_n^k - \widetilde{u}^k)\|_{X(\mathbb{R})\cap L^\infty(\mathbb{R},L^q)} = 0. \tag{6.14}$$

We can prove Proposition 6.1 in a way similar to [12, 13], and therefore we omit the proof.

We find that the limits $\{\widetilde{u}^1, \widetilde{u}^2, \dots\}$ possess the following properties:

Lemma 6.2. *Assume that* $d \geq 1$ *and* (N1)–(N6) *hold. Let* $\{\widetilde{u}^1, \widetilde{u}^2, \dots\}$ *be the family of functions found in Proposition* 6.1 *and* $M := \#\{\widetilde{u}^1, \widetilde{u}^2, \dots\}$. *Then, for any* $k \in \{1, 2, \dots, M\}$, *there exists* $n_k \in \mathbb{N}$ *such that*

$$\mathcal{K}(e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k) > 0 \qquad \textit{for any} \quad n \geq n_k, \tag{6.15}$$

$$\mathcal{S}_\omega(e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k) \le \frac{m_\omega + 3m_\omega^*}{4} \qquad \textit{for any} \quad n \ge n_k. \tag{6.16}$$

Proof. It follows from (6.12) together with (6.13) or (6.14) that there exists a number $k_0 \in \{1, 2, \dots, M\}$ for which we have the following: For any $k \ge k_0$, there exists $N_k \in \mathbb{N}$ such that

$$\begin{aligned}&\Big|\mathcal{S}_\omega(T_\lambda u_n) - \Big\|\nabla T_\lambda e^{-\frac{i}{2}t_n^k\Delta}(u_n^k - \widetilde{u}^k)\Big\|_{L^2}^2 \\ &\quad - \omega\Big\|T_\lambda e^{-\frac{i}{2}t_n^k\Delta}\left(u_n^k - \widetilde{u}^k\right)\Big\|_{L^2}^2 - \sum_{j=1}^{k}\mathcal{S}_\omega\left(T_\lambda e^{-\frac{i}{2}t_n^j\Delta}\widetilde{u}^j\right)\Big| < \frac{m_\omega - m_\omega^*}{4}\end{aligned} \tag{6.17}$$

for any $n \ge N_k$ and $\lambda \in (0,1]$.

Now, suppose to the contrary that (6.15) fails. Then, we can find an index $\widetilde{k}_0 \in \{1, 2, \dots, M\}$ such that the following holds: For any $n \in \mathbb{N}$, there exists $N \ge n$ such that

$$\mathcal{K}(e^{-\frac{i}{2}t_N^{\widetilde{k}_0}\Delta}\widetilde{u}^{\widetilde{k}_0}) \le 0. \tag{6.18}$$

Put $l_0 := \max\{k_0, \widetilde{k}_0\}$. Then, by (6.17) and (6.18), we can take $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\Big|\mathcal{S}_\omega(T_\lambda u_{n_0}) - \Big\|\nabla T_\lambda e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0})\Big\|_{L^2}^2 \\ &\quad - \omega\Big\|T_\lambda e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0})\Big\|_{L^2}^2 - \sum_{j=1}^{l_0}\mathcal{S}_\omega\left(T_\lambda e^{-\frac{i}{2}t_{n_0}^j\Delta}\widetilde{u}^j\right)\Big| < \frac{m_\omega - m_\omega^*}{4}\end{aligned} \tag{6.19}$$

and

$$\mathcal{K}(e^{-\frac{i}{2}t_{n_0}^{\widetilde{k}_0}\Delta}u^{\widetilde{k}_0}) \le 0. \tag{6.20}$$

Put $\lambda_* := \min\limits_{1\le k\le l_0}\lambda(e^{-\frac{i}{2}t_{n_0}^k\Delta}u^k)$, where $\lambda = \lambda(e^{-\frac{i}{2}t_{n_0}^k\Delta}u^k)$ is the number such that

$$\mathcal{K}(T_\lambda e^{-\frac{i}{2}t_{n_0}^k\Delta}u^k) = 0 \tag{6.21}$$

(see (N6)). Define $k_* \in \{1, 2, \dots, l_0\}$ as a number satisfying

$$\lambda(e^{-\frac{i}{2}t_{n_0}^{k_*}\Delta}u^{k_*}) = \lambda_*. \tag{6.22}$$

Here, we see from assumption (N6) and (6.20) that

$$0 < \lambda_* \le 1, \tag{6.23}$$

$$\mathcal{K}(T_{\lambda_*}e^{-\frac{i}{2}t_{n_0}^k\Delta}u^k) \ge 0 \qquad \text{for any } 1 \le k \le l_0, \tag{6.24}$$

$$\mathcal{K}(T_{\lambda_*}e^{-\frac{i}{2}t_{n_0}^{k_*}\Delta}u^{k_*}) = 0. \tag{6.25}$$

Since $\lambda_* \le 1$, we find from assumption (N1), Lemma 2.1, (1.19), and (6.3) that

$$0 \le \mathcal{S}_\omega(T_{\lambda_*} u_{n_0}) \le \mathcal{S}_\omega(u_{n_0}) \le \frac{m_\omega^* + m_\omega}{2}. \tag{6.26}$$

Moreover, since $\mathcal{S}_\omega(T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{k_*}\Delta} u^{k_*}) \ge m_\omega$ (by the definition of m_ω; see (1.18)) and $\mathcal{S}_\omega(T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{k}\Delta} u^{k}) \ge 0$ for any $1 \le k \le l_0$ (by Lemma 2.1 (iii) together with (6.24)), we have by (6.26) that

$$\begin{aligned} &\Big| \mathcal{S}_\omega(T_{\lambda_*} u_{n_0}) - \Big\| \nabla T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0}) \Big\|_{L^2}^2 \\ &\qquad - \omega \Big\| T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0}) \Big\|_{L^2}^2 - \sum_{j=1}^{l_0} \mathcal{S}_\omega(T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{j}\Delta} \widetilde{u}^j) \Big| \\ &\ge \sum_{j=1}^{l_0} \mathcal{S}_\omega(T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{j}\Delta} \widetilde{u}^j) + \Big\| \nabla T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0}) \Big\|_{L^2}^2 \\ &\qquad + \omega \Big\| T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{l_0}\Delta}(u_{n_0}^{l_0} - \widetilde{u}^{l_0}) \Big\|_{L^2}^2 - \mathcal{S}_\omega(T_\lambda u_{n_0}) \\ &\ge \mathcal{S}_\omega(T_{\lambda_*} e^{-\frac{i}{2}t_{n_0}^{k_*}\Delta} \widetilde{u}^{k_*}) - \mathcal{S}_\omega(T_{\lambda_*} u_{n_0}) \ge m_\omega - \frac{m_\omega + m_\omega^*}{2} = \frac{m_\omega - m_\omega^*}{2}, \end{aligned} \tag{6.27}$$

which contradicts (6.19). Thus, (6.15) must hold.

Next, we shall show (6.16). It follows from Lemma 2.1 together with (6.15) that for any $k \in \{1, 2, \dots, M\}$, there exists $n_k \in \mathbb{N}$ such that $\mathcal{S}_\omega(u_n^k) > 0$ for any $n \ge n_k$. Then, the desired result (6.16) follows from (6.3) and (6.17) with $\lambda = 1$. □

Using Lemma 6.2, we can prove the following lemmata:

Lemma 6.3. *Assume that* $d \ge 1$ *and* (N1)–(N4) *hold. Let* $\{\widetilde{u}^1, \widetilde{u}^2, \dots\}$ *be as in Lemma* 6.1. *It follows that if* $t_\infty^k \in \mathbb{R}$, *we have* $e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k \in A_{\omega,+}$; *if* $t_\infty^k = \pm\infty$, *we have* $\|\nabla \widetilde{u}^k\|_{L^2}^2 + \omega \|\widetilde{u}^k\|_{L^2}^2 < m_\omega$.

Proof. We first consider the case where $t_\infty \in \mathbb{R}$. It follows from Lemma 6.2 that

$$\begin{aligned} &\mathcal{K}(e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k) \\ &= \lim_{n\to\infty} \Big\{ \mathcal{K}(e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k) - \Big(\mathcal{K}(e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k) - \mathcal{K}(e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k) \Big) \Big\} \ge 0. \end{aligned} \tag{6.28}$$

Suppose that $\mathcal{K}(e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k) = 0$. Then we have by the definition of m_ω that $\mathcal{S}_\omega(e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k) \ge m_\omega$. However, this is impossible by virtue of (6.16).

Moreover, we can easily see from (6.16) that $\mathcal{S}_\omega(e^{-\frac{i}{2}t_\infty^k\Delta}\widetilde{u}^k) < m_\omega$. Hence, the claim holds.

Next, we consider the other case. Using (6.16) and the decay estimate for $e^{\frac{i}{2}t\Delta}$, we obtain that

$$\left\|\nabla\widetilde{u}^k\right\|_{L^2}^2 + \omega\left\|\widetilde{u}^k\right\|_{L^2}^2 < m_\omega. \qquad \square$$

Lemma 6.4. *Assume that $d \geq 1$ and (N1)–(N4) hold. Then, for each $k \in \{1, 2\dots, M\}$, there exists a global solution ψ^k to (1.1) such that $\psi^k(t) \in A_{\omega,+}$ for any $t \in \mathbb{R}$ and*

$$\lim_{n\to\infty} \|\psi^k(-t_n^k) - e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k\|_{H^1} = 0. \tag{6.29}$$

Proof. When $t_\infty^k \in \mathbb{R}$, let ψ^k be the solution to (1.1) with $\psi^k(-t_\infty^k) = e^{-\frac{i}{2}t_n^k\Delta}\widetilde{u}^k$. Then, we see from the continuity of the solution in $H^1(\mathbb{R}^d)$ and Lemmata 3.2 and 6.2 that ψ^k is a global solution satisfying (6.29) and $\psi^k(t) \in A_{\omega,+}$ for any $t \in \mathbb{R}$. On the other hand, when $t_\infty^k = \pm\infty$, Proposition B.1 together with Lemma 6.3 shows the existence of the desired solution ψ^k. $\square$

We shall show that $M = 1$. Then, putting $\Psi = \psi^1$, we see that Ψ satisfies the properties in Proposition 5.1. To this end, we need the following proposition.

Proposition 6.5 (Long time perturbation theory)**.** *Assume that $d \geq 1$ and (N1)–(N4) hold. Let I be an interval, $t_1 \in I$, and let ψ be a solution to (1.1) on I. Furthermore, let $A > 0$, and let u be a function on $\mathbb{R}^d \times I$ satisfying that*

$$\|u\|_{X(I)} \leq A, \tag{6.30}$$

$$2i\partial_t u + \Delta u + f(u) = e \text{ for some function } e \in L(Q_1^*; I) + L(Q_2^* : I). \tag{6.31}$$

Then, there exists $\varepsilon > 0$ depending only on d, p_1, p_2, C_f, σ, and A such that if

$$\left\|e^{\frac{i}{2}(t-t_1)\Delta}(\psi(t_1) - u(t_1))\right\|_{X(I)} \leq \varepsilon, \tag{6.32}$$

$$\|e\|_{L(Q_1^*;I)\cap L(Q_2^*;I)} \leq \varepsilon, \tag{6.33}$$

then we have $\|\psi\|_{X(I)} \leq C$ for some constant C depending only on A.

You can find the proof of Proposition 6.5 in [12]. So, we omit the proof.

Now, we give a sketch of the proof of Proposition 5.1.

Proof of Proposition 5.1. Suppose to the contrary that $M > 2$. We first show that

$$\|\psi^k\|_{X(\mathbb{R})} < \infty \qquad \text{for any} \quad 1 \le k \le M \quad \text{if } M \ge 2. \tag{6.34}$$

Then, the formula (6.12) together with (6.3), (6.15), and (6.29) gives us that

$$\begin{aligned} m_\omega^* &= \lim_{n\to\infty} \mathcal{S}_\omega(u_n) > \lim_{n\to\infty} \mathcal{S}_\omega(e^{-\frac{i}{2}t_n^l\Delta}\widetilde{u}^l) \\ &= \lim_{n\to\infty} \mathcal{S}_\omega(\psi^l(-t_n^l)) = \mathcal{S}_\omega(\psi^l(0)) \qquad \text{for any } 1 \le l \le M. \end{aligned} \tag{6.35}$$

Hence, from the definition of m_ω^* (see (5.2)), (6.34) holds.

We put

$$\psi_n^{app}(x,t) := \sum_{k=1}^{L} \psi^k(x - x_n^k, t - t_n^k), \tag{6.36}$$

where $L = M$ when $M < \infty$ and L is a sufficiently large number when $M = \infty$. We know by (6.34) that

$$\sup_{n\in\mathbb{N}} \|\psi_n^{app}\|_{X(\mathbb{R})} < \infty. \tag{6.37}$$

On the other hand, we see that ψ_n^{app} satisfies the following:

$$2i\frac{\partial \psi_n^{app}}{\partial t} + \Delta\psi_n^{app} + f(\psi_n^{app}) = e_n, \tag{6.38}$$

where

$$e_n = f(\psi_n^{app}) - \sum_{k=1}^{L} f(\psi^k(x - x_n^k, t - t_n^k)). \tag{6.39}$$

We can verify (see [1, proof of Lemma 4.4] and [13, proof of Lemma 5.5]) that for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that

$$\|e^{\frac{i}{2}t\Delta}(\psi_n(0) - \psi_n^{app}(0))\|_{X(\mathbb{R})} \le \epsilon \qquad \text{for any } n \ge N_\varepsilon \tag{6.40}$$

and

$$\|e_n\|_{L(Q_1^*;\mathbb{R})\cap L(Q_2^*;\mathbb{R})} \le \epsilon. \tag{6.41}$$

Then, we can apply Proposition 6.5, which yields that $\|\psi_n\|_{X(\mathbb{R})} < \infty$. This is a contradiction. Thus, we obtain $M = 1$.

Put $\Psi := \psi^1$; then we see that Ψ satisfies (i)–(iii) in Proposition 5.1 by an argument similar to that in [1]. □

7. Analysis on $A_{\omega,-}$ and proof of Theorem 1.2 (ii)

In this section, we give basic properties of solutions starting from $A_{\omega,-}$ and prove Theorem 1.2 (ii).

The following lemma is a key for the proof of Theorem 1.2.

Lemma 7.1. *Assume that $d \geq 1$ and* (N1)–(N6) *hold. Let $\omega > 0$ and $\psi_0 \in A_{\omega,-}$, and let ψ be the corresponding solution to* (1.1). *Then, there exists $\epsilon_0 > 0$ so that*

$$\psi(t) \in A_{\omega,-} \quad \textit{for any } t \in I_{\max}, \tag{7.1}$$

$$\sup_{t \in I_{\max}} \mathcal{K}(\psi(t)) < -\epsilon_0. \tag{7.2}$$

Proof of Lemma 7.1. We can obtain (7.1) in the same proof as for (3.2) in Lemma 3.2. Thus, we omit it.

It remains to prove (7.2). We suppose to the contrary that

$$\sup_{t \in I_{\max}} \mathcal{K}(\psi(t)) = 0.$$

Then, we can take a sequence $\{t_n\}$ in $I_{\max}$ such that

$$\lim_{n \to \infty} \mathcal{K}(\psi(t_n)) = 0. \tag{7.3}$$

We shall show that

$$\inf_{n \in \mathbb{N}} \|\nabla \psi(t_n)\|_{L^2} > 0. \tag{7.4}$$

We have $\mathcal{K}(\psi(t_n)) < 0$ for any $n \in \mathbb{N}$. This together with (1.4), (1.6), the Gagliardo–Nirenberg inequality, and the L^2-conservation law (1.11) gives us that

$$\begin{aligned} 0 > \mathcal{K}(\psi(t_n)) \geq 2 \|\nabla \psi(t_n)\|_{L^2}^2 - C\Big(& \|\psi_0\|_{L^2}^{p_1+1-\frac{d(p_1-1)}{2}} \|\nabla \psi(t_n)\|_{L^2}^{\frac{d(p_1-1)}{2}} \\ & + \|\psi_0\|_{L^2}^{p_2+1-\frac{d(p_2-1)}{2}} \|\nabla \psi(t_n)\|_{L^2}^{\frac{d(p_2-1)}{2}} \Big) \end{aligned} \tag{7.5}$$

for any $n \in \mathbb{N}$, where $C > 0$ is some constant depending only on d, p_1, p_2, and C_f. Since $2 < \frac{d}{2}(p_1 - 1) \leq \frac{d}{2}(p_2 - 1)$, the claim (7.4) follows from (7.5).

Put

$$a_n := 2 \|\nabla \psi(t_n)\|_{L^2}^2, \tag{7.6}$$

$$b_n := d \int_{\mathbb{R}^d} \Big\{ f(\psi(x,t_n))\overline{\psi(x,t_n)} - F(\psi(x,t_n)) \Big\} \, dx. \tag{7.7}$$

Here, we have by (7.3) that

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}\mathcal{K}(\psi(t_n)) = 0. \tag{7.8}$$

We also put

$$\nu_n := \left(\frac{b_n}{a_n}\right)^{\frac{1}{2}} > 1. \tag{7.9}$$

Then, it follows from (7.4) and (7.8) that $\lim_{n\to\infty}\nu_n = 1$. Indeed,

$$|1-\nu_n^2| = \frac{1}{a_n}|a_n - b_n| \le \frac{1}{\inf\limits_{n\in\mathbb{N}} a_n}|a_n - b_n| \to 0 \qquad \text{as } n\to\infty. \tag{7.10}$$

Now, we put $u_n := \psi(\nu_n x, t_n)$. Since $\mathcal{K}(u_n) = 0$, the definition of m_ω (see (1.17)) implies that

$$m_\omega \le \mathcal{S}_\omega(u_n) \qquad \text{for any } n\in\mathbb{N}. \tag{7.11}$$

However, taking $n\in\mathbb{N}$ so large that

$$(\nu_n^2 - 1)\,\|\nabla\psi(t_n)\|_{L^2}^2 = \tfrac{1}{2}(b_n - a_n) < \tfrac{1}{2}\left(m_\omega - \mathcal{S}_\omega(\psi_0)\right), \tag{7.12}$$

we find by $\nu_n > 1$ (see (7.9)) and the conservation law of the action that

$$\begin{aligned}\mathcal{S}_\omega(u_n) &= \nu_n^{-d}\left(\nu_n^2\,\|\nabla\psi(t_n)\|_{L^2}^2 - \int_{\mathbb{R}^d}F(\psi(x,t_n))\,dx + \omega\,\|\psi(t_n)\|_{L^2}^2\right)\\ &\le (\nu_n^2-1)\,\|\nabla\psi(t_n)\|_{L^2}^2 + \mathcal{S}_\omega(\psi_0) < \tfrac{1}{2}(m_\omega + \mathcal{S}_\omega(\psi_0)) < m_\omega.\end{aligned} \tag{7.13}$$

This contradicts (7.11). Thus, we have proved the lemma. □

We are now in position to prove Theorem 1.2 (ii).

Proof of Theorem 1.2 (ii). We consider the case $t>0$ only because the other cases are treated in a similar way. If $T_{\max}^{\pm}$ is finite, we are done. Thus, we assume that $T_{\max}^{\pm} = \infty$. Suppose that (1.26) fails. Then there exists $R_0 > 0$ such that

$$M_0 = \sup_{t\in[0,\infty)}\int_{|x|\ge R_0}|\nabla\psi(x,t)|^2\,dx < \infty. \tag{7.14}$$

We shall derive a contradiction in three steps.
(Step 1). We first claim that there exists a constant $m_0 > 0$ such that

$$\begin{aligned} m_0 < \inf\Big\{ \int_{|x|\ge R}|v(x)|^2dx : v\in H^1(\mathbb{R}^d),\ \mathcal{K}^R(v) \le -\epsilon_0/4,\\ \|\nabla v\|_{L^2(|x|\ge R)}^2 \le M_0,\ \|v\|_{L^2} \le \|\psi_0\|_{L^2}\Big\}\end{aligned} \tag{7.15}$$

for all $R > 0$, where $\epsilon_0 > 0$ is given by (7.2) and

$$\mathcal{K}^R(v) := \int_{\mathbb{R}^d} \kappa_1(x)|\nabla v|^2 dx - \int_{\mathbb{R}^d} \kappa_2(x)\Big\{ f(u(x))\overline{u}(x) - F(u(x)) \Big\}\, dx. \tag{7.16}$$

Here,

$$\kappa_1(x) = 2\int_0^{\frac{|x|^2}{R^2}} \rho(r)\, dr + \frac{4|x|^2}{R^2}\rho(\frac{|x|^2}{R^2}), \tag{7.17}$$

$$\kappa_2(x) = d\int_0^{\frac{|x|^2}{R^2}} \rho(r)\, dr + \frac{2|x|^2}{R^2}\rho(\frac{|x|^2}{R^2}). \tag{7.18}$$

Then, we can verify that $\ker \kappa_j \subset \{|x| \geq R\}$ for $j = 1, 2$. Take any $v \in H^1(\mathbb{R}^d)$ with the following properties:

$$\mathcal{K}^R(v) \leq -\epsilon_0/4, \quad \|\nabla v\|^2_{L^2(|x|\geq R)} \leq M_0, \quad \|v\|_{L^2} \leq \|\psi_0\|_{L^2}. \tag{7.19}$$

Then, we have

$$\|v\|^2_{H^1(|x|\geq R)} = \|\nabla v\|^2_{L^2(|x|\geq R)} + \|v\|^2_{L^2(|x|\geq R)} \leq M_0 + \|\psi_0\|^2_{L^2}. \tag{7.20}$$

We also have by the first property in (7.19) that

$$\begin{aligned}
\frac{\epsilon_0}{4} &\leq -\mathcal{K}^R(v) \leq -\int_{\mathbb{R}^d} \kappa_2(x)\Big\{ f(u(x))\overline{u}(x) - F(u(x)) \Big\}\, dx \qquad (7.21)\\
&\leq C_1\|\kappa_2\|_{L^\infty}\int_{|x|\geq R} |v|^{p_1+1}\, dx + C_2\|\kappa_2\|_{L^\infty}\int_{|x|\geq R} |v|^{p_2+1}\, dx\\
&\leq C\|\kappa_2\|_{L^\infty}\Big\{ \|v\|_{L^2(|x|\geq R)}^{p_1+1-\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}} \|v\|_{L^{p_{1,*}+1}(|x|\geq R)}^{\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}}\\
&\qquad\qquad + \|v\|_{L^2(|x|\geq R)}^{p_2+1-\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}} \|v\|_{L^{p_{2,*}+1}(|x|\geq R)}^{\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}} \Big\}\\
&\leq C\|\kappa_2\|_{L^\infty}\Big\{ \|v\|_{L^2(|x|\geq R)}^{p_1+1-\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}} \|v\|_{H^1(|x|\geq R)}^{\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}}\\
&\qquad\qquad + \|v\|_{L^2(|x|\geq R)}^{p_2+1-\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}} \|v\|_{H^1(|x|\geq R)}^{\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}} \Big\}.
\end{aligned}$$

Here, we define $p_{i,*}$ $(i = 1, 2)$ by

$$p_{i,+} = \begin{cases} 2p_i - 1 & \text{if } d = 1, 2, \\ 2^* - 1 & \text{if } d \geq 3. \end{cases} \tag{7.22}$$

Set

$$\bar{\rho} = \min\left\{p_i + 1 - \frac{(p_i - 1)(p_{i,*} + 1)}{p_{i,*} - 1}\ ;\ i = 1, 2\right\}. \tag{7.23}$$

Then, we have

$$\frac{\epsilon_0}{4} \le C\|\kappa_2\|_{L^\infty}\|v\|_{L^2(|x|\ge R)}^{\bar{\rho}}\left\{\|v\|_{H^1(|x|\ge R)}^{\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}} + \|v\|_{H^1(|x|\ge R)}^{\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}}\right\}. \tag{7.24}$$

This yields that

$$\frac{\epsilon_0}{4C\|\kappa_2\|_{L^\infty}\left\{\|v\|_{H^1(|x|\ge R)}^{\frac{(p_1-1)(p_{1,*}+1)}{p_{1,*}-1}} + \|v\|_{H^1(|x|\ge R)}^{\frac{(p_2-1)(p_{2,*}+1)}{p_{2,*}-1}}\right\}} \le \|v\|_{L^2(|x|\ge R)}^{\bar{\rho}}. \tag{7.25}$$

Thus, we obtain the desired result.
(Step 2). Let m_0 be a constant in Step 1. Then we prove that

$$\sup_{0\le t<\infty}\int_{|x|\ge R}|\psi(x,t)|^2\,dx \le m_0$$

for all R satisfying the following properties:

$$R > R_0, \tag{7.26}$$

$$\frac{10d^2K}{R^2}\|\psi_0\|_{L^2} < \epsilon_0 \tag{7.27}$$

$$\int_{|x|\ge R}|\psi_0(x)|^2dx < m_0, \tag{7.28}$$

$$\frac{1}{R^2}\left(1 + \tfrac{8}{\epsilon_0}\|\nabla\psi_0\|_{L^2}^2\right)(W_R, |\psi_0|^2) < m_0. \tag{7.29}$$

For $R > 0$ satisfying (7.26)–(7.29), we put

$$T_R = \sup\left\{T > 0\colon \sup_{0\le t<T}\int_{|x|\ge R}|\psi(x,t)|^2\,dx \le m_0\right\}.$$

From (7.28), we see that $T_R > 0$. It is enough to show that $T_R = \infty$. We suppose to the contrary that $T_R < \infty$. It follows from $\psi \in C([0, T_{\max}); L^2(\mathbb{R}^d))$ that

$$\int_{|x|\ge R}|\psi(x, T_R)|^2\,dx = m_0. \tag{7.30}$$

From the mass conservation law, we have $\|\psi(T_R)\|_{L^2} = \|\psi_0\|_{L^2}$. Thus, by (7.15), we have $-\frac{\epsilon_0}{4} \le \mathcal{K}^R(\psi(T_R))$. Applying this inequality to the generalized virial identity (A.14), we obtain

$$(W_R, |\psi(T_R)|^2) < (W_R, |\psi_0|^2) + 2T_R(\nabla W_R\cdot\nabla\psi_0, \psi_0) - \epsilon_0 T_R^2 \tag{7.31}$$

$$+\frac{\epsilon_0}{4}T_R^2 - \frac{1}{2}\int_0^{T_R}\int_0^{t'}(\Delta^2 W_R, |\psi(t'')|^2)\,dt''\,dt'$$
$$= (W_R, |\psi_0|^2) + 2T_R(\nabla W_R\cdot\nabla\psi_0, \psi_0) - \frac{3\epsilon_0^2}{4}T_R^2$$
$$-\frac{1}{2}\int_0^{T_R}\int_0^{t'}(\Delta^2 W_R, |\psi(t'')|^2)\,dt''\,dt'.$$

Then, we have by (7.27) that

$$-\frac{1}{2}\int_0^{T_R}\int_0^{t'}(\Delta^2 W_R, |\psi(t'')|^2)\,dt''\,dt' \le \frac{1}{2}\int_0^{T_R}\int_0^{t'}\frac{10d^2K}{R^2}\|\psi(t'')\|_{L^2}^2\,dt''\,dt'$$
$$\le \frac{\epsilon_0}{4}T_R^2. \tag{7.32}$$

This together with the Schwarz and Hölder inequalities implies that

$$(W_R, |\psi(T_R)|^2) < (W_R, |\psi_0|^2) + 2T_R\Im(\nabla W_R\cdot\nabla\psi_0, \psi_0) - \frac{T_R^2\epsilon_0}{2} \tag{7.33}$$
$$= (W_R, |\psi_0|^2) - \frac{\epsilon_0}{2}\left\{T_R - \frac{2}{\epsilon_0}\Im(\nabla W_R\cdot\nabla\psi_0, \psi_0)\right\}^2 + \frac{2}{\epsilon_0}|(\nabla W_R\cdot\nabla\psi_0, \psi_0)|^2$$
$$\le (W_R, |\psi_0|^2) + \frac{2}{\epsilon_0}|(\nabla W_R\cdot\nabla\psi_0, \psi_0)|^2$$
$$\le (W_R, |\psi_0|^2) + \frac{2}{\epsilon_0}\|\nabla\psi_0\|_{L^2}^2\int|\nabla W_R|^2|\psi_0|^2\,dx.$$

Thus, it follows from (7.29), (7.33), and (A.11) that

$$(W_R, |\psi(T_R)|^2) \le (1 + \frac{8}{\epsilon_0}\|\nabla\psi_0\|_{L^2}^2)(W_R, |\psi_0|^2) < R^2 m_0. \tag{7.34}$$

Since $W_R(x) \ge R^2$ for $|x| > R$, we have

$$\int_{|x|\ge R}|\psi(x, T_R)|^2\,dx = \frac{1}{R^2}\int_{|x|\ge R}R^2|\psi(x, T_R)|^2\,dx \tag{7.35}$$
$$\le \frac{1}{R^2}(W_R, |\psi(x, T_R)|^2) < m_0,$$

which contradicts (7.30).
(Step 3). It follows from the definition of m_0 that $-\frac{\epsilon_0}{4} \le \mathcal{K}^R(\psi(t))$. We obtain the following estimate as well as Step 2:

$$(W_R, |\psi(t)|^2) \le (W_R, |\psi_0|^2) + 2\Im(\nabla W_R\cdot\nabla\psi_0, \psi_0) - \frac{\epsilon_0 t^2}{2} \tag{7.36}$$

for all $t \geq 0$. This inequality implies that $(W_R, |\psi(t)|^2)$ becomes negative in a finite time, so that $T_{\max} < \infty$. However, this contradicts $T_{\max} = \infty$. Hence, (7.14) derives an absurd conclusion. Thus, (1.26) holds. □

Appendix A. Generalized virial identity

Let ρ be a smooth function on $\mathbb{R}$ such that

$$\rho(x) = \rho(4 - x) \qquad \text{for all } x \in \mathbb{R}, \tag{A.1}$$

$$\rho(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}, \tag{A.2}$$

$$\int_{\mathbb{R}} \rho(x)\, dx = 1, \tag{A.3}$$

$$\operatorname{supp} \rho \subset (1, 3), \tag{A.4}$$

$$\rho'(x) \geq 0 \qquad \text{for all } x < 2. \tag{A.5}$$

We put

$$w(r) := r - \int_0^r (r - s)\rho(s)\, ds \qquad \text{for } r \geq 0, \tag{A.6}$$

$$W_R(x) := R^2 w\left(\tfrac{|x|^2}{R^2}\right) \qquad \text{for } x \in \mathbb{R}^d \text{ and } R > 0. \tag{A.7}$$

We easily verify that

$$\|W_R\|_{L^\infty} \lesssim R^2, \tag{A.8}$$

$$\|\nabla W_R\|_{L^\infty} \lesssim R, \tag{A.9}$$

$$\left\|\Delta^2 W_R\right\|_{L^\infty} \lesssim \frac{1}{R^2} \|\rho\|_{W^{2,\infty}(\mathbb{R})}, \tag{A.10}$$

$$|\nabla W_R(x)|^2 \leq 4W_R(x) \qquad \text{for any } R > 0 \text{ and } x \in \mathbb{R}^d. \tag{A.11}$$

For a sufficiently regular solution ψ of the equation (1.1), we verify that

$$\frac{d}{dt} \int_{\mathbb{R}^d} W_R |\psi(t)|^2\, dx = \Im \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla\psi(t)\overline{\psi(t)}\, dx, \tag{A.12}$$

$$\begin{aligned} \frac{d^2}{dt^2} \int_{\mathbb{R}^d} W_R |\psi(t)|^2\, dx &= \mathcal{K}(\psi(t)) - \int_{\mathbb{R}^d} \left(2 - H(W_R)\right) \nabla\psi(t) \cdot \overline{\nabla\psi(t)}\, dx \\ &\quad + \int_{\mathbb{R}^d} \left(d - \tfrac{1}{2}\Delta W_R\right) \left\{ f(\psi(t))\overline{\psi(t)} - F(\psi(t)) \right\} dx \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 W_R |\psi(t)|^2\, dx, \end{aligned} \tag{A.13}$$

where $H(W_R)$ denotes the Hessian of W_R. These identities (A.12) and (A.13) together with a regularization argument give us the following generalized version of the virial identity:

Lemma A.1 (Generalized virial identity and center of mass)**.** *Assume that $d \geq 1$. Let $\psi_0 \in H^1(\mathbb{R}^d)$, and let ψ be the solution of the equation* (1.1) *with $\psi(t_0) = \psi_0$ for some $t_0 \in I_{\max}$. Then, we have* (i) *the generalized virial identity,*

$$\begin{aligned}\int_{\mathbb{R}^d} W_R|\psi(t)|^2\,dx &= \int_{\mathbb{R}^d} W_R|\psi_0|^2\,dx + (t-t_0)\Im\int_{\mathbb{R}^d}\nabla W_R\cdot\nabla\psi_0\,\overline{\psi_0}\,dx\\ &+\int_{t_0}^{t}\int_{t_0}^{t'}\mathcal{K}(\psi(t''))\,dt''dt' - \int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\Big\{2\int_0^{\frac{|x|^2}{R^2}}\rho(r)\,dr|\nabla\psi(t'')|^2\\ &\qquad + \frac{4|x|^2}{R^2}\rho\Big(\frac{|x|^2}{R^2}\Big)\Big|\frac{x}{|x|}\cdot\nabla\psi(t'')\Big|^2\Big\}\,dx\,dt''\,dt' \qquad \text{(A.14)}\\ &+\int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\Big\{d\int_0^{\frac{|x|^2}{R^2}}\rho(r)\,dr + \frac{2|x|^2}{R^2}\rho\Big(\frac{|x|^2}{R^2}\Big)\Big\}\\ &\qquad\times\Big\{f(\psi(t))\overline{\psi(t)} - F(\psi(t))\Big\}\,dx\,dt''\,dt'\\ &-\frac{1}{4}\int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\Delta^2 W_R\,|\psi(t'')|^2\,dx\,dt''\,dt' \qquad \text{for any } R>0,\end{aligned}$$

and (ii) *the motion of the "almost center of mass,"*

$$\int_{\mathbb{R}^d}\frac{\partial W_R}{\partial x_j}|\psi(t)|^2\,dx = \int_{\mathbb{R}^d}\frac{\partial W_R}{\partial x_j}|\psi_0|^2\,dx + \Im\sum_{k=1}^{d}\int_{\mathbb{R}^d}\frac{\partial^2 W_R}{\partial x_j\partial x_k}\frac{\partial\psi}{\partial x_k}(t)\overline{\psi(t)}\,dx \quad \text{for any } 1\leq j\leq d \text{ and } R>0. \qquad \text{(A.15)}$$

Appendix B. Wave operator

In this appendix, we state the following proposition, which tells us the wave operators are well-defined on $A_{\omega,+}$.

Proposition B.1. *Assume that $d \geq 1$ and* (N1)–(N6) *hold. Let $\omega > 0$. Then, we have*
(i) *For any non-trivial function $\phi_+ \in H^1(\mathbb{R}^d)$ with*

$$\|\nabla\phi_+\|_{L^2}^2 + \omega\,\|\phi_+\|_{L^2}^2 < m_\omega, \qquad \text{(B.1)}$$

there exists a unique $\psi_0 \in A_{\omega,+}$ *such that the corresponding solution* ψ *to* (1.1) *with* $\psi(0) = \psi_0$ *exists globally in time and satisfies the following:*

$$\psi \in X([0, +\infty)), \tag{B.2}$$

$$\lim_{t \to +\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta}\phi_+ \right\|_{H^1} = 0, \tag{B.3}$$

$$\mathcal{H}(\psi(t)) = \|\nabla \phi_+\|_{L^2}^2 \qquad \textit{for any } t \in \mathbb{R}. \tag{B.4}$$

Furthermore, if $\|\phi_+\|_{H^1}$ *is sufficiently small, then we have*

$$\|\psi\|_{X(\mathbb{R})} \lesssim \|\phi_+\|_{H^1}, \tag{B.5}$$

where the implicit constant depends only on d, p_1, p_2, *and* σ.
(ii) *For any non-trivial function* $\phi_- \in H^1(\mathbb{R}^d)$ *with*

$$\|\nabla \phi_-\|_{L^2}^2 + \omega \|\phi_-\|_{L^2}^2 < m_\omega, \tag{B.6}$$

there exists a unique $\psi_0 \in A_{\omega,+}$ *such that the corresponding solution* ψ *to* (1.1) *with* $\psi(0) = \psi_0$ *exists globally in time and satisfies the following:*

$$\psi \in X((-\infty, 0]), \tag{B.7}$$

$$\lim_{t \to -\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta}\phi_- \right\|_{H^1} = 0, \tag{B.8}$$

$$\mathcal{H}(\psi(t)) = \|\nabla \phi_-\|_{L^2}^2 \qquad \textit{for any } t \in \mathbb{R}. \tag{B.9}$$

Furthermore, if $\|\phi_-\|_{H^1}$ *is sufficiently small, then we have*

$$\|\psi\|_{X(\mathbb{R})} \lesssim \|\phi_-\|_{H^1}, \tag{B.10}$$

where the implicit constant depends only on d, p_1, p_2, *and* σ.

Outline of the proof of Proposition B.1. We can solve the following final value problem for (1.1) on some interval $[T, +\infty)$:

$$\psi(t) = e^{\frac{i}{2}t\Delta}\phi_+ - \frac{i}{2}\int_t^{+\infty} e^{\frac{i}{2}(t-t')\Delta} f(\psi(t'))\, dt'. \tag{B.11}$$

In particular, the solution ψ exists globally in time and satisfies (B.5), provided that $\|\phi_+\|_{H^1}$ is sufficiently small. Moreover, we can verify that the solution ψ becomes a solution to (1.1) on $[T, +\infty)$ and satisfies that

$$\lim_{t \to +\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta}\phi_+ \right\|_{H^1} = 0. \tag{B.12}$$

Here, (B.12) together with (1.4), (1.6), and the decay estimate for the free solution yields that

$$\lim_{t\to+\infty} \mathcal{S}_\omega(\psi(t)) = \lim_{t\to+\infty} \mathcal{S}_\omega(e^{\frac{i}{2}t\Delta}\phi_+) = \|\nabla\phi_+\|_{L^2}^2 + \omega\,\|\phi_+\|_{L^2}^2 < m_\omega, \tag{B.13}$$

$$\lim_{t\to+\infty} \mathcal{K}(\psi(t)) = \lim_{t\to+\infty} \mathcal{K}(e^{\frac{i}{2}t\Delta}\phi_+) = 2\,\|\nabla\phi_+\|_{L^2}^2\,, \tag{B.14}$$

so that

$$\psi(t) \in A_{\omega,+} \qquad \text{for any sufficiently large } t \geq T. \tag{B.15}$$

Hence, we see from Lemma 3.2 that ψ exists globally in time; i.e., $I_{\max} = \mathbb{R}$. Put $\psi_0 = \psi(0)$. Then, this ψ_0 is what we want, and we can prove the uniqueness of such a ψ_0 as in a manner similar to proving the uniqueness of the initial-value problem. □

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