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SHARP WELL-POSEDNESS AND ILL-POSEDNESS OF A HIGHER-ORDER MODIFIED CAMASSA–HOLM EQUATION

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Abstract. In this paper we consider the Cauchy problem for a higher-order modified Camassa–Holm equation. By using some dyadic bilinear estimates and the fixed-point theorem, we establish the local well-posedness of the higher-order modified Camassa–Holm equation for the small initial data in $H^{-n+\frac{5}{4}}(\mathbf{R})$, $n \geq 2$, $n \in \mathbf{N}$. We also prove that the Cauchy problem for the higher-order modified Camassa–Holm equation is ill-posed for the initial data in homogeneous Sobolev spaces $\dot{H}^s(\mathbf{R})$ with $s < -n + \frac{5}{4}$, $n \in \mathbf{N}$, $n \geq 2$. Our result partially answers the open problem which is proposed below in Theorem 1.2 by Erika A. Olson in the Journal of Differential Equations, 246 (2009), 4154–4172.

1. Introduction

This paper is devoted to studying the Cauchy problem for a higher-order modified Camassa–Holm equation

$$u_t + \partial_x^{2n+1} u + \tfrac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x \left[u^2 + \tfrac{1}{2}u_x^2\right] = 0,\ x,t \in \mathbf{R}, \quad (1.1)$$

$$u(x,0) = u_0(x), \quad (1.2)$$

where $n \in \mathbf{N}$. Omitting the second term in (1.1) yields the Camassa–Holm equation

$$u_t + \tfrac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x \left[u^2 + \tfrac{1}{2}u_x^2\right] = 0,$$

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which was derived by Camassa and Holm as a nonlinear model for water wave motion in shallow channels by using an asymptotic expansion directly in the Hamiltonian for Euler equations and has been extensively studied by many people; for instance, see [4, 6, 10, 7, 9, 8, 11, 19, 32, 33]. Xin and Zhang [32, 33] investigated the existence, uniqueness, and large-time behavior of the weak solution of the Cauchy problem associated to the Camassa–Holm equation. By using a regularization technique analogous to the one in Bona and Smith [2], Li and Olver [25] established local well-posedness associated with the nonperiodic initial value problem of the Camassa–Holm equation in the Sobolev space $H^s(\mathbf{R})$ with $s > 3/2$. Himonas and Misiolek [19] proved that the map data solution of the Cauchy problem for the Camassa–Holm equation fails to be uniformly continuous for the initial data in $H^s(\mathbf{R})$, $s < \frac{3}{2}$; that is to say, if we require that the data is uniformly continuous in the definition of local well-posedness, then the Cauchy problem for the Camassa–Holm equation is ill-posed. Thus the exponent $s = 3/2$ is the critical Sobolev exponent for the well-posedness of the Cauchy problem for the Camassa–Holm equation. Danchin [14] proved that the Cauchy problem for the Camassa–Holm equation is locally well-posed in Besov spaces $B^s_{2,r}$ with $s = 3/2$.

To obtain low regularity, some people considered the Cauchy problem for (1.1) which is the higher modification of the Camassa–Holm equation [5, 17, 18, 27, 31, 26]. Himonas and Misiolek [17] studied the periodic case of (1.1)–(1.2). By using the Fourier restriction norm method introduced in [3] and the I-method introduced in [12] and developed in [13], Wang and Cui [31] proved that the Cauchy problem for (1.1) in the case of $n = 1$ is globally well-posed in $H^s(\mathbf{R})$ with $\frac{5\sqrt{7}-10}{4} < s < 1$. By reconsidering the relation between the lifespan of the local solution and the initial data, Yang and Li [34] proved that the Cauchy problem for (1.1) in the case of $n = 1$ is globally well-posed in $H^s(\mathbf{R})$ with $\frac{6\sqrt{10}-17}{4} < s < 1$. Olson [28] proved that the Cauchy problem for (1.1) is locally well-posed for the initial data in the Sobolev space $H^s(\mathbf{R})$ with $s > s' = \frac{1}{2}\big(1 - \frac{1}{2n(2n-1)}\big)$ and $\frac{1}{4} \le s' < \frac{1}{2}$. Recently, by establishing some new bilinear estimates and using the Fourier restriction norm method, Li, Yan, and Yang [26] improved the result of Olson and proved that the Cauchy problem for (1.1) is locally well-posed for the initial data in $H^s(\mathbf{R})$ with $s > -n + \frac{5}{4}$, $n \in \mathbf{N}$, and that it is globally well-posed for the initial data in $H^1(\mathbf{R})$.

In this paper, motivated by [30, 1, 29, 22, 23, 15, 24], we prove that the Cauchy problem for (1.1) is locally well-posed in $H^{-n+\frac{5}{4}}(\mathbf{R})$, $n \in \mathbf{N}$, and that the Cauchy problem for (1.1) is ill-posed in $\dot{H}^s(\mathbf{R})$ with $s < -n + \frac{5}{4}$, $n \in \mathbf{N}$, $n \geq 2$.

Before stating the main results we give some notation and definitions. $\mathcal{F}u$ is the Fourier transform of u with respect to its all variables. $\mathcal{F}_x u$ is the Fourier transform of u with respect to its space variable. $\mathcal{F}^{-1}u$ is the Fourier inverse transform of u with respect to its all variables. $\mathcal{F}_x^{-1}u$ is the Fourier inverse transform of u with respect to its space variable. Let $\mathbf{Z}$ and $\mathbf{N}$ be the sets of integers and natural numbers, respectively. $\mathbf{Z}_+ = \mathbf{N} \bigcup \{0\}$. Let

$$I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\}, \quad I_0 = \{\xi : |\xi| \leq 2\},$$

where $k \geq 1$ and $k \in \mathbf{N}$. Let $\eta_0 : \mathbf{R} \longrightarrow [0,1]$ denote an even smooth function supported in $[-\frac{8}{5}, \frac{8}{5}]$ and equal to 1 in $[-\frac{5}{4}, \frac{5}{4}]$. We define $\psi(t) = \eta_0(t)$. For $k \in \mathbf{Z}$, let $\eta_k(\xi) = \eta_0(\frac{\xi}{2^k}) - \eta_0(\frac{\xi}{2^{k-1}})$ if $k \geq 1$ and $\eta_k(\xi) = 0$ if $k \leq -1$. For $k \in \mathbf{Z}$, let $\chi_k(\xi) = \eta_0(\frac{\xi}{2^k}) - \eta_0(\frac{\xi}{2^{k-1}})$. We define

$$\mathcal{F}_x P_k u(\xi) = \eta_k(\xi)\mathcal{F}_x u(\xi), \quad \mathcal{F}P_k u(\xi,\tau) = \eta_k(\xi)\mathcal{F}u(\xi,\tau),$$
$$P_{\leq l} = \textstyle\sum_{k \leq l} P_k, \quad P_{\geq l} = \textstyle\sum_{k \geq l} P_k, \quad P_{\leq 0} = P_0.$$

We define

$$\|f\|_{L_t^q L_x^p} = \left(\left(\int_{\mathbf{R}} |f(x,t)|^p \, dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}, \;\; \|f\|_{L_x^q L_t^p} = \left(\left(\int_{\mathbf{R}} |f(x,t)|^p \, dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$

and define $L_I^q L_x^p$ and $L_x^p L_I^q$ by the norms

$$\|u\|_{L_I^q L_x^p} = \left\| \|u\|_{L_x^p} \right\|_{L_t^q(I)}, \quad \|u\|_{L_x^p L_I^q} = \left\| \|u\|_{L_t^q(I)} \right\|_{L_x^p}.$$

We define

$$W(t)u_0 = C \int_{\mathbf{R}} e^{ix\xi} e^{it(-1)^{n+1}\xi^{2n+1}} \mathcal{F}_x u_0(\xi) \, d\xi.$$

Then (1.1)–(1.2) are formally equivalent to the following integral equation:

$$u(t) = W(t)u_0 - \int_0^t W(t-\tau)\left(\tfrac{1}{2}(u^2)_x + (1-\partial_x^2)^{-1}\left[u^2 + \tfrac{1}{2}(u_x)^2\right]_x\right) d\tau. \quad (1.3)$$

For $k \in \mathbf{Z}_+$ we define the dyadic $X^{s,b}$-type normed spaces $X_k = X_k(\mathbf{R}^2)$,

$$X_k(\mathbf{R}^2) = \left\{ f \in L^2(\mathbf{R}^2) : \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \left\| \eta_j(\tau + (-1)^n \xi^{2n+1}) f \right\|_{L^2} \right\},$$

where $f(\xi,\tau)$ is supported in $I_k \times \mathbf{R}$. We define

$$\|u\|_{F^s}^2 = \sum_{k\geq 0}^{\infty} 2^{2sk} \left\|\eta_k(\xi)\mathcal{F}u(\xi,\tau)\right\|_{X_k}^2$$

and $\|u\|_{\bar{X}_0} = \|u\|_{L_x^2 L_t^\infty}$. Based on the the above preparations, we define the resolution spaces

$$\bar{F}^s = \Big\{u \in \mathcal{S}'(\mathbf{R}^2) : \|u\|_{\bar{F}^s} = \sum_{k\geq 1} 2^{2sk}\|\eta_k(\xi)\mathcal{F}u\|_{X_k}^2 + \|P_{\leq 0}(u)\|_{\bar{X}_0}^2 < \infty\Big\},$$

where $\mathcal{S}(\mathbf{R}^d)$ is the Schwartz space in $\mathbf{R}^d$ and $\mathcal{S}'(\mathbf{R}^d)$ is its dual space. For $s \in \mathbf{R}$, $H^s(\mathbf{R})$ and $\dot{H}^s(\mathbf{R})$ are the usual Sobolev spaces defined by

$$\begin{aligned} H^s(\mathbf{R}) &= \{u_0 \in \mathcal{S}'(\mathbf{R}) : \|u_0\|_{H^s(\mathbf{R})}^2 = \int_{\mathbf{R}} \langle\xi\rangle^{2s}|\mathcal{F}_x u_0(\xi)|^2\, d\xi < \infty\}, \\ \dot{H}^s(\mathbf{R}) &= \{u_0 \in \mathcal{S}'(\mathbf{R}) : \|u_0\|_{\dot{H}^s(\mathbf{R})}^2 = \int_{\mathbf{R}} |\xi|^{2s}|\mathcal{F}_x u_0(\xi)|^2\, d\xi < \infty\}. \end{aligned}$$

For $T \geq 0$, we define the time-localized spaces $\bar{F}^s(T)$:

$$\|u\|_{\bar{F}^s(T)} = \inf_{w\in\bar{F}^s}\Big\{\|P_{\leq 0}u\|_{L_x^2 L_{|t|\leq T}^\infty} + \|P_{\geq 1}w\|_{\bar{F}^s},\ w(t) = u(t)\ \text{on}\ [-T,T]\Big\}.$$

We use $N_m = 2^{k_m}$ and $L_m = 2^{j_m}$ to denote dyadic numbers, where $K_m, j_m \in \mathbf{Z}$ $(m = 1,2,3)$.

$$\begin{aligned} k_{\max} &= \text{maximum}\{k_1,k_2,k_3\}, \quad k_{\text{med}} = \text{median}\{k_1,k_2,k_3\}, \\ k_{\min} &= \text{minimum}\{k_1,k_2,k_3\}, \quad j_{\max} = \text{maximum}\{j_1,j_2,j_3\}, \\ j_{\text{med}} &= \text{median}\{j_1,j_2,j_3\}, \quad j_{\min} = \text{minimum}\{j_1,j_2,j_3\}. \end{aligned}$$

We use $C_1|X| \leq |Y| \leq C_2|X|$ to denote $X \sim Y$, where C_1 and C_2 are generic positive constants. We use $|X| \geq C_3|Y|$ to denote $X \gg Y$, where C_3 is a generic positive constant and bigger than 4. The positive constant C which may depend on n may vary from line to line.

The main results of this paper are as follows:

Theorem 1.1. *Let $s = -n + \frac{5}{4}$, $n \in \mathbf{N}$, $n \geq 2$. Then the Cauchy problem for* (1.1) *is locally well-posed for sufficiently small initial data $u_0 \in H^s(\mathbf{R})$.*

Theorem 1.2. *Let $s < -n + \frac{5}{4}$, $n \in N$, $n \geq 2$. Then there does not exist any $T > 0$ such that* (1.1) *is well-posed in $\dot{H}^s(\mathbf{R})$ on the interval $[0,T]$ in the following sense. For any $T > 0$, the solution map $u_0 \longrightarrow u$, $t \in [0,T]$, is NOT C^2-differentiable at zero from $\dot{H}^s(\mathbf{R})$ to $C([0,T];\dot{H}^s(\mathbf{R}))$.*

Remark. Below Theorem 1.2 of [28], Olson noted that it is not known whether or not the initial-value problem (1.1)–(1.2) is locally well-posed for any $s < \frac{1}{2}\frac{m^2-3m+1}{m^2-3m+2}$ with $m = 2n+1$. Theorems 1.1 and 1.2 partially answer this question.

The rest of paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we give some crucial bilinear estimates. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2.

2. Preliminaries

Lemma 2.1. *Let $s \in \mathbf{R}$, $\phi \in H^s(\mathbf{R})$, $k \in \mathbf{Z}$, and $(i+\tau+(-1)^n\xi^{2n+1})^{-1}\mathcal{F}u \in X_k$. Then*

$$\|\psi(t)W(t)\phi\|_{\bar{F}^s} \leq C\|\phi\|_{H^s(\mathbf{R})}. \tag{2.1}$$

$$\left\|\mathcal{F}\left[\psi(t)\int_0^t W(t-t^{'})u(t^{'})\,dt^{'}\right]\right\|_{X_k} \leq C\|(i+\tau+(-1)^n\xi^{2n+1})^{-1}\mathcal{F}u\|_{X_k}. \tag{2.2}$$

Lemma 2.1 can be proved similarly to Proposition 4.1 of [15].

Lemma 2.2. *Let $I \subset \mathbf{R}$ be an interval with $I \leq C$ and $k \in \mathbf{Z}_+$. Then for all $\phi \in \mathcal{S}(\mathbf{R})$, we have*

$$\begin{aligned} \|W(t)\phi\|_{L_t^qL_x^r} &\leq C\|\phi\|_{L^2}, & (2.3)\\ \|W(t)P_k\phi\|_{L_x^2L_I^\infty} &\leq C2^{(2n+1)k/4}\|\phi\|_{L^2}, & (2.4)\\ \|W(t)\phi\|_{L_x^4L_t^\infty} &\leq C\|\phi\|_{\dot{H}^{1/4}(\mathbf{R})}, & (2.5)\\ \|W(t)\phi\|_{L_x^\infty L_t^2} &\leq C\|\phi\|_{\dot{H}^{-n}(\mathbf{R})}, & (2.6) \end{aligned}$$

where $2 \leq q, r \leq \infty$ and $\frac{1}{2} = \frac{2n+1}{q} + \frac{1}{r}$.

Proof. For the proof of (2.3), we refer the readers to Theorem 2.4 of [20]. For the proof of (2.4), we refer the readers to Corollary 2.9 of [20]. For the proof of (2.5), we refer the readers to Theorem 2.5 of [21]. For the proof of (2.6), we refer the readers to Lemma 2.1 of [20].

Lemma 2.3. *Let Y be any space-time Banach space that obeys the time modulation estimate,*

$$\|g(t)F(t,x)\|_Y \leq \|g\|_{L_t^\infty}\|F(t,x)\|_Y,$$

for any $F \in Y$ and $g \in L_t^\infty$. Let $T : (f_1, \ldots, f_m) \longrightarrow T(f_1, \ldots, f_m)$ be a spatial multilinear operator which satisfies that for all $u_{1,0}, \ldots, u_{m,0} \in L_x^2$,

$$\|T(W(t)u_{1,0}, \ldots, W(t)u_{m,0})\|_Y \leq C\prod_{j=1}^m \|u_{j,0}\|_{L_x^2}.$$

Then one also has the estimate that for all $k_1, \ldots, k_m \in Z_+$ *and* $u_1, \ldots, u_m \in F^0$,

$$\|T(P_{k_1}(u_1), \ldots, P_{k_m}(u_m)\|_Y \leq C \prod_{j=1}^{m} \|\mathcal{F}P_{k_j}(u_j)\|_{X_{k_j}}.$$

Lemma 2.3 can be proved similarly to the proof of Lemma 4.1 of [29].

Combining Lemma 2.2 with Lemma 2.3, we obtain

Lemma 2.4. *Let* $j \in \mathbf{Z}$, $k \in \mathbf{Z}_+$, $\frac{2n+1}{q} = \frac{1}{2} - \frac{1}{r}$, *and* $u \in F^0$. *Then*

$$\|P_k(u)\|_{L_t^q L_x^r} \leq C\|\mathcal{F}[P_k(u)]\|_{X_k}, \tag{2.7}$$

$$\|P_k(u)\|_{L_x^2 L_I^\infty} \leq C 2^{(2n+1)k/4}\|\mathcal{F}[P_k(u)]\|_{X_k}, \tag{2.8}$$

$$\|P_k u\|_{L_x^4 L_t^\infty} \leq C 2^{k/4}\|\mathcal{F}[P_k(u)]\|_{X_k}, \tag{2.9}$$

$$\|P_j(u)\|_{L_x^\infty L_t^2} \leq C 2^{-nj}\|\mathcal{F}P_j(u)\|_{X_j}. \tag{2.10}$$

In particular, in (2.7), *when* $r = 2$ *and* $q = \infty$, *we have*

$$\|P_k(u)\|_{L_t^\infty L_x^2} \leq C\|\mathcal{F}[P_k(u)]\|_{X_k}. \tag{2.11}$$

Lemma 2.5. *Let* $n \in \mathbf{N}^+$ *and* $\xi = \xi_1 + \xi_2$. *Then*

$$|\xi^{2n+1} - \xi_1^{2n+1} - \xi_2^{2n+1}| \sim |\xi_{\min}||\xi_{\max}|^{2n}, \tag{2.12}$$

where $|\xi_{\min}| := \min\{|\xi|, |\xi_1|, |\xi_2|\}$ *and* $|\xi_{\max}| := \max\{|\xi|, |\xi_1|, |\xi_2|\}$.

Proof. Let $f(\beta) = (1+\beta)^{2n+1} - \beta^{2n+1} - 1$. Thus, we have $f(0) = 0$ and $f'(\beta) = (1+2n)((1+\beta)^{2n} - \beta^{2n}) > 0$ as well as $f''(\beta) = (2n+1)2n((1+\beta)^{2n-1} - \beta^{2n-1}) > 0$ for $\beta > 0$. Thus $f(\beta)$ and $f'(\beta)$ are monotone increasing with respect to $\beta \geq 0$. Thus, we have $f(\beta) = f(\beta) - f(0) = f'(\theta\beta)\beta \leq (4^n - 1)(2n+1)\beta$ when $0 \leq \beta \leq 1$, where θ is a function between 0 and β. Since $f(\beta) = \beta^{2n+1} f(\frac{1}{\beta})$ for all $\beta > 0$, we have $f(\beta) = \beta^{2n+1} f(\frac{1}{\beta}) = \beta^{2n+1}\left(f(\frac{1}{\beta}) - f(0)\right) \leq \beta^{2n}(4^n - 1)(2n+1)$ for $\beta \geq 1$. From Lemma 4.2.6. of [16], we know that

$$\begin{aligned} f(\beta) &\geq (2n+1)\beta \quad \text{for } \beta \in [0,1], \\ f(\beta) &\geq (2n+1)\beta^{2n} \quad \text{for } \beta > 1. \end{aligned}$$

Thus we have

$$\begin{aligned} f(\beta) &\sim (2n+1)\beta \quad \text{for } \beta \in [0,1], \\ f(\beta) &\sim (2n+1)\beta^{2n} \quad \text{for } \beta > 1. \end{aligned}$$

From the process of the proof of Lemma 4.2.6 of [16], we have (2.12).

The proof of Lemma 2.5 is completed.

In order to introduce Lemma 2.6, we fix some notation.

$$\sum_{L_{\max}\sim H} = \sum_{L_1,\, L_2,\, L_3\geq C:\, L_{\max}\sim H}, \quad \sum_{N_{\max}\sim N_{\rm med}\sim N} = \sum_{N_1,\, N_2,\, N_3>0:\, N_{\max}\sim N_{\rm med}\sim N},$$

$$\sum_{j=1}^{3}\xi_j = 0, \quad \sum_{j=1}^{3}\tau_j = 0, \quad \lambda_1 = \tau_1 - \phi(\xi_1), \lambda_2 = \tau_2 - \phi(\xi_2), \lambda_3 = \tau_3 - \phi(\xi_3),$$

$$h(\xi) = \sum_{j=1}^{3}\phi(\xi_j) = -\sum_{j=1}^{3}\lambda_j, \tag{2.13}$$

$$X_{N_1,\, N_2, N_3;\, H;\, L_1,\, L_2,\, L_3} = \chi_{|h(\xi)|\sim H}\prod_{j=1}^{3}\chi_{|\xi_j|\sim N_j}\chi_{|\lambda_j|\sim L_j}, \tag{2.14}$$

$$|\xi_j|\sim N_j,\ |\lambda_j|\sim L_j (j=1,2,3), |h(\xi)|\sim H.$$

Following the idea of $[k;Z]$ multiplier norm method in [29], by dyadic decomposition of the variables ξ_j and λ_j as well as the function $h(\xi)$, one is led to consider

$$\|X_{N_1,\, N_2, N_3;\, H;\, L_1,\, L_2,\, L_3}\|_{[3,\, \mathbf{R}\times\mathbf{R}]}\,. \tag{2.15}$$

By using the identities

$$\sum_{j=1}^{3}\xi_j = 0 \ \text{ and } \ \sum_{j=1}^{3}\lambda_j + h(\xi) = 0$$

on the support of the multiplier, we know that $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{\max}\sim N_{\rm med} \tag{2.16}$$

and

$$L_{\max}\sim \max(H, L_{med}). \tag{2.17}$$

Since

$$|\phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3)|\sim |\xi_{\min}||\xi_{\max}|^{2n}, \tag{2.18}$$

when $N_{\max}\sim N_{\rm med}\geq 100$ we know

$$\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}\geq CN_{\max}^{2n}N_{\min}. \tag{2.19}$$

When $N_{\max}\sim N_{\rm med}\geq 100$, we may assume that

$$H\sim N_{\max}^{2n}N_{\min} \tag{2.20}$$

since the multiplier (2.13) vanishes otherwise.

Lemma 2.6. *Let* H, N_1, N_2, N_3, *and* $L_1, L_2, L_3 > 0$ *obey* (2.16), (2.17), *and* (2.20).

$$2^{j_{\max}} \sim \max(2^{j_{\rm med}}, 2^{2nk_{\max}+k_{\min}}). \tag{2.21}$$

Then
(i) *If* $N_{\max} \sim N_{\min}$ *and* $L_{\max} \sim N_{\max}^{2n} N_{\min}$, *then*

$$(2.15) \leq CL_{\min}^{\frac{1}{2}} N_{\max}^{-\frac{2n-1}{4}} L_{\rm med}^{1/4}. \tag{2.22}$$

(ii) *If* $N_2 \sim N_3 \gg N_1$ *and* $N_{\max}^{2n} N_{\min} \sim L_1 \geq CL_2, CL_3$, *then*

$$(2.15) \leq CL_{\min}^{\frac{1}{2}} N_{\max}^{-n} \min\Big(H, \frac{N_{\max}}{N_{\min}} L_{\rm med}\Big)^{\frac{1}{2}}, \tag{2.23}$$

similarly for permutations.
(iii) *In all other cases, we have*

$$(2.15) \leq CL_{\min}^{\frac{1}{2}} N_{\max}^{-n} \min\left(H, L_{\rm med}\right)^{\frac{1}{2}}. \tag{2.24}$$

Proof. When $L_{\max} \sim L_{\rm med} \gg H$, we have $(2.15) \leq CL_{\min}^{1/2} N_{\min}^{1/2}$; we obtain Lemma 2.6. It remains to consider the case $H \sim L_{\max}$.

From [29], we know that

$$\begin{aligned}(2.15) \quad \leq \quad & CL_{min}^{1/2} |\{\xi_2 : \ |\xi_2 - \xi_2^0| \ll N_{min};\ |\xi - \xi_2 - \xi_3^0| \ll N_{min}; \\ & \phi(\xi_2) + \phi(\xi - \xi_2) = \tau + O(L_2)\}|^{1/2}\end{aligned} \tag{2.25}$$

for some $\tau \in \mathbf{R}$, ξ, ξ_1^0, ξ_2^0, and ξ_3^0 satisfying

$$\begin{aligned}|\xi_j^0| \quad &\sim \quad N_j (j=1,2,3);\ |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min};\ |\xi + \xi_1^0| \ll N_{min}, \\ |\xi_2 - \xi_2^0| \quad &\ll \quad N_{\min};\ |\xi - \xi_2 - \xi_3^0| \ll N_{\min}.\end{aligned}$$

To estimate the right-hand side of the expression (2.25), we use the following identity:

$$\phi(\xi_2) + \phi(\xi - \xi_2) = \phi(\xi) + p(\xi, \xi_2), \tag{2.26}$$

where $p(\xi, \xi_2) \sim N_{\max}^{2n} N_{\min}$. By using (2.25) and (2.26), we have

$$\phi(\xi) + p(\xi, \xi_2) = \tau + O(L_2). \tag{2.27}$$

We need only consider three cases, namely, $N_1 \sim N_2 \sim N_3$, $N_1 \sim N_2 \gg N_3$, and $N_2 \sim N_3 \gg N_1$, since the case $N_1 \sim N_3 \gg N_2$ follows by symmetry.

If $N_1 \sim N_2 \sim N_3$, by using (2.26) and (2.27), we have that the ξ_2 variable is contained in one interval of length $O(N_{\max}^{-(2n-1)/2} L_2^{1/2})$. Thus we obtain (i).

If $N_1 \sim N_2 \gg N_3$, (2.26) and (2.27) show that ξ_2 is contained in two intervals of length $O(N_1^{-2n} L_2)$, and (2.24) follows.

If $N_2 \sim N_3 \gg N_1$, (2.26) and (2.27) show that ξ_2 is contained in two intervals of length $O(N_1^{-1} N_2^{-(2n-1)} L_2)$. But ξ_2 is also contained in an interval of length $\ll N_1$.

Consequently, we complete the proof of Lemma 2.6.

3. Dyadic bilinear estimates

In this section, we will prove some dyadic bilinear estimates.

Firstly, we consider the case *low* $\times$ *low* $\longrightarrow$ *low* interaction.

Lemma 3.1. *Let* $0 \le k_1, k_2, k_3 \le 100$. *Then*

$$\left\| \left(i + \tau + (-1)^n \xi^{2n+1}\right)^{-1} \eta_{k_1}(\xi) \frac{i\xi}{1+\xi^2} \mathcal{F}(\psi(t) P_{k_2} \partial_x u) * \mathcal{F}(\psi(t) P_{k_3} \partial_x v) \right\|_{X_{k_1}}$$
$$\le C \|P_{k_2} u\|_{L_t^\infty L_x^2} \|P_{k_3} v\|_{L_t^\infty L_x^2},$$

where $u, v \in F^s$.

Proof. Combining the definition of X_{k_1} with Plancherel's equality and Bernstein's inequality, we obtain

$$\left\| (i + \tau + (-1)^n \xi^{2n+1})^{-1} \eta_{k_1}(\xi) \frac{i\xi}{1+\xi^2} \mathcal{F}(\psi(t) P_{k_2} \partial_x u) * \mathcal{F}(\psi(t) P_{k_3} \partial_x v) \right\|_{X_{k_1}}$$
$$\le C 2^{k_1+k_2+k_3} \sum_{j_3 \ge 0} 2^{-j_3/2} \|\psi(t) P_{k_2} u P_{k_3} u\|_{L_t^2 L_x^2}$$
$$\le C \|P_{k_2} u\|_{L_t^\infty L_x^2} \|P_{k_3} v\|_{L_t^\infty L_x^2}.$$

Thus, we complete the proof of Lemma 3.1.

Now we consider high-low interactions.

Lemma 3.2. (a) *If* $k \ge 10$ *and* $|k - k_2| \le 5$, *then*

$$\left\| \left(i + \tau + (-1)^n \xi^{2n+1}\right)^{-1} \eta_k(\xi) \frac{i\xi}{1+\xi^2} \mathcal{F}(P_{\le 0} \partial_x u) * \mathcal{F}(P_{k_2} \partial_x v) \right\|_{X_k}$$
$$\le C 2^{-nk} \|P_{\le 0} u\|_{L_x^2 L_t^\infty} \|\mathcal{F} P_{k_2} v\|_{X_{k_2}}, \tag{3.1}$$

where $u, v \in \bar{F}^s$.

(b)*If* $k \geq 10$, $|k-k_2| \leq 5$, *and* $1 \leq k_1 \leq k-9$, *then*

$$\left\| \left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1} \eta_k(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_{k_1}\partial_x u) * \mathcal{F}(P_{k_2}\partial_x v)\right\|_{X_k} \leq Ck^3 2^{(-2n+\frac{1}{2})k}\|\mathcal{F}P_{k_1}v\|_{X_{k_1}}\|\mathcal{F}P_{k_2}v\|_{X_{k_2}}, \tag{3.2}$$

where $u, v \in \bar{F}^s$.

Proof. Without loss of generality we assume $k = k_2$. For part (a), by using the definition of X_k and (2.10), we have

$$\begin{aligned}&\left\| \left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1} \eta_k(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_{\leq 0}\partial_x u) * \mathcal{F}(P_{k_2}\partial_x v)\right\|_{X_k} \\ \leq\ & C\sum_{j\geq 0} 2^{-j/2}\|\mathcal{F}(P_{\leq 0}u) * \mathcal{F}(P_{k_2}v)\|_{L^2_{\xi\tau}} \\ \leq\ & C\|P_{\leq 0}u\|_{L^2_xL^\infty_t}\|P_k v\|_{L^\infty_x L^2_t} \leq C\|P_{\leq 0}u\|_{L^2_xL^\infty_t}2^{-nk}\|\mathcal{F}P_k v\|_{X_k}.\end{aligned}$$

Thus, we complete the proof of part (a).

For part (b), by using the definition of X_k, we have

$$\left\| \left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1} \eta_k(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_{k_1}\partial_x u) * \mathcal{F}(P_k\partial_x v)\right\|_{X_k} \leq C2^{k_1}\sum_{j_1,j_2,j_3\geq 0} 2^{-j_3/2}\left\| I_{D_{k,j_3}} \cdot u_{k_1,j_1}v_{k,j_2}\right\|_{L^2_{\xi\tau}}, \tag{3.3}$$

where

$$\begin{aligned} u_{k_1,j_1} &= \eta_{k_1}(\xi)\eta_{j_1}(\tau+(-1)^n\xi^{2n+1})\mathcal{F}u(\xi,\tau), \\ v_{k,j_2} &= \eta_k(\xi)\eta_{j_2}(\tau+(-1)^n\xi^{2n+1})\mathcal{F}v(\tau,\xi). \end{aligned} \tag{3.4}$$

From (2.21), we may assume $j_{\max} \geq 2nk + k_1 - 10$ in the summation on the right-hand side of (3.3). We may also assume $j_1, j_2, j_3 \leq 10k$; otherwise, we will apply the trivial estimates

$$\left\| I_{D_{k_3,j_3}} u_{k_1,j_1} * v_{k,j_2}\right\|_{L^2_{\xi\tau}} \leq C2^{j_{\min}/2}2^{k_{\min}/2}\|u_{k_1,j_1}\|_{L^2_{\xi\tau}}\|v_{k,j_2}\|_{L^2_{\xi\tau}};$$

then there is a 2^{-5k} to spare, which suffices to control (3.2). Thus, by applying (2.23), we have

$$\begin{aligned}&2^{k_1}\sum_{j_1,j_2,j_3\geq 0} 2^{-j_3/2}\left\| I_{D_{k,j_3}} u_{k_1,j_1}v_{k,j_2}\right\|_{L^2_{\xi\tau}} \\ &\leq C2^{k_1}\sum_{j_1,j_2,j_3\geq 0} 2^{-j_3/2}2^{j_{\min}/2}2^{(-n+\frac{1}{2})k}2^{-k_1/2}2^{j_{\rm med}/2}\|u_{k_1,j_1}\|_{L^2_{\xi\tau}}\|v_{k,j_2}\|_{L^2_{\xi\tau}}\end{aligned}$$

$$\le C2^{k_1} \sum_{j_{max} \ge 2nk+k_1-10} k^3 2^{(-n+\frac{1}{2})k} 2^{-k_1/2} 2^{-j_{\max}/2} \|\mathcal{F}P_{k_1}u\|_{X_{k_1}} \|\mathcal{F}P_k v\|_{X_k}$$
$$\le Ck^3 2^{(-2n+\frac{1}{2})k} \|\mathcal{F}P_{k_1}u\|_{X_{k_1}} \|\mathcal{F}P_k v\|_{X_k}. \qquad (3.5)$$

Thus, we complete the proof of part (b).

When the low frequency is comparable to the high frequency, then we have

Lemma 3.3. *If* $k \ge 10$, $|k-k_2| \le 10$, *and* $k-9 \le k_1 \le k+10$, *then for any* $u, v \in F^{-n+\frac{5}{4}}$,

$$\left\| \left(i+\tau+(-1)^n \xi^{2n+1}\right)^{-1} \eta_{k_1}(\xi) \frac{i\xi}{1+\xi^2} \mathcal{F}(P_k \partial_x u) * \mathcal{F}(P_{k_2} \partial_x v) \right\|_{X_{k_1}}$$
$$\le C2^{(3-6n)/4} \|\mathcal{F}P_{k_1}u\|_{X_k} \|\mathcal{F}P_k v\|_{X_{k_2}}.$$

Proof. Without loss of generality we assume $k = k_2$, and it follows from the definition of X_{k_1} that

$$\left\| \left(i+\tau+(-1)^n \xi^{2n+1}\right)^{-1} \eta_{k_1}(\xi) \frac{i\xi}{1+\xi^2} \mathcal{F}(P_k \partial_x u) * \mathcal{F}(P_{k_2} \partial_x v) \right\|_{X_{k_1}}$$
$$\le C2^{k_1} \sum_{j_1, j_2, j_3 \ge 0} 2^{-j_1/2} \left\| I_{D_{k,j_1}} u_{k_1,j_2} v_{k,j_3} \right\|_{L^2_{\xi\tau}},$$

where u_{k,j_1} and v_{k,j_2} are as in (3.4). Without loss of generality, we assume $j_{\max} \ge (2n+1)k - 20$ and $j_1, j_2, j_3 \le 10k$ in the summation. By using (2.22), we have

$$2^{k_1} \sum_{j_1, j_2, j_3 \ge 0} 2^{-j_1/2} \left\| I_{D_{k_1,j_1}} u_{k,j_2} v_{k,j_3} \right\|_{L^2_{\xi\tau}} \le C\left(I + II + III\right),$$

where

$$I = \sum_{j_1 = j_{\max}} 2^{-j_1/2} 2^{(5-2n)/4} 2^{j_{\min}/2} 2^{j_{\text{med}}/4} \|u_{k,j_2}\|_{L^2_{\xi\tau}} \|u_{k,j_3}\|_{L^2_{\xi\tau}},$$
$$II = \sum_{j_2 = j_{\max}} 2^{-j_1/2} 2^{(5-2n)/4} 2^{j_{\min}/2} 2^{j_{\text{med}}/4} \|u_{k,j_2}\|_{L^2_{\xi\tau}} \|u_{k,j_3}\|_{L^2_{\xi\tau}},$$
$$III = \sum_{j_3 = j_{\max}} 2^{-j_1/2} 2^{(5-2n)/4} 2^{j_{\min}/2} 2^{j_{\text{med}}/4} \|u_{k,j_2}\|_{L^2_{\xi\tau}} \|u_{k,j_3}\|_{L^2_{\xi\tau}}.$$

From Lemma 2.5, we have

$$I = \sum_{j_1 = j_{\max}} 2^{-j_1/2} 2^{(5-2n)/4} 2^{j_{\min}/2} 2^{j_{\text{med}}/4} \|u_{k,j_2}\|_{L^2_{\xi\tau}} \|u_{k,j_3}\|_{L^2_{\xi\tau}}$$

$$\leq C\sum_{j_2,j_3\geq 0} 2^{(3-6n)/4}2^{j_{\min}/2}2^{j_{\mathrm{med}}/4}\|u_{k,j_2}\|_{L^2_{\xi\tau}}\|u_{k,j_3}\|_{L^2_{\xi\tau}}$$
$$\leq C2^{(3-6n)/4}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_k v\|_{X_k}.$$

Because of the symmetry between II and III, we only need to bound II:

$$\begin{aligned} II \quad &\leq \quad C\Big(\sum_{j_2=j_{\max},j_1\leq j_3} + \sum_{j_2=j_{\max},j_1\geq j_3}\Big) \\ &\qquad \times 2^{-j_1/2}2^{(5-2n)/4}2^{j_{\min}/2}2^{j_{\mathrm{med}}/4}\|u_{k,j_2}\|_{L^2_{\xi\tau}}\|u_{k,j_3}\|_{L^2_{\xi\tau}} \\ &\leq \quad C\sum_{j_2\geq(2n+1)k-20,j_3\geq 0} 2^{(5-2n)/4}2^{j_3/2}\|u_{k,j_2}\|_{L^2_{\xi\tau}}\|u_{k,j_3}\|_{L^2_{\xi\tau}} \\ &\leq \quad C2^{(3-6n)/4}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_k v\|_{X_k}. \end{aligned}$$

Thus we complete the proof of Lemma 3.3.

Now we consider the case *high* $\times$ *high* $\longrightarrow$ *low* interactions.

Lemma 3.4. (a) *If* $k\geq 10$ *and* $|k-k_2|\leq 10$, *then for any* $u,v\in F^s$,

$$\begin{aligned} &\Big\|\left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1}\eta_0(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_k\partial_x u)*\mathcal{F}(P_{k_2}\partial_x v)\Big\|_{X_0} \\ &\leq Ck2^{-2nk+\frac{5}{2}k}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_k v\|_{X_{k_2}}. \end{aligned} \tag{3.6}$$

(b) *If* $k\geq 10$, $|k-k_2|\leq 10$, *and* $1\leq k_1\leq k-9$, *then for any* $u,v\in F^s$,

$$\begin{aligned} &\Big\|\left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1}\eta_{k_1}(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_k\partial_x u)*\mathcal{F}(P_{k_2}\partial_x v)\Big\|_{X_{k_1}} \\ &\leq C\left(2^{-2k_1}2^{(-2n+\frac{5}{2})k}+k2^{-\frac{3k_1}{2}}2^{(-2n+2)k}\right)\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_k v\|_{X_{k_2}}. \end{aligned} \tag{3.7}$$

Proof. To prove part (a), without loss of generality we assume $k=k_2$. By using the definition of X_k, we know that the left-hand side of (3.6) is dominated by

$$\sum_{k_3=-\infty}^{0} 2^{k_3}2^{k_1}2^{k_2}\sum_{j_1,j_2,j_3\geq 0} 2^{-j_3/2}\left\|D_{k_3,j_3}u_{k,j_1}*v_{k,j_2}\right\|_{L^2_{\xi\tau}},$$

where u_{k,j_1} and v_{k,j_2} are as in (3.4). Without loss of generality, we can assume $k_3\geq -10k$ and $j_1,j_2,j_3\leq 10k$. Just as in the process of the proof of Proposition 3.8 in [15], we only consider the worst case $|j_3-2nk-k_1|\leq 10$. By using (2.23), we have

$$\Big\|\left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1}\eta_0(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_k\partial_x u)*\mathcal{F}(P_{k_2}\partial_x v)\Big\|_{X_{k_1}}$$

$$\le C\sum_{k_3=-10k}^{0}\sum_{j_1,j_2\ge 0}2^{-kn}2^{-k_3/2}2^{k_3}2^{k_1}2^{k_2}2^{(-n+\frac{1}{2})k}2^{-k_3/2}\|u_{k,j_1}\|_{L^2_{\xi\tau}}\|v_{k,j_2}\|_{L^2_{\xi\tau}}$$
$$\le k2^{-2nk+\frac{5}{2}k}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_kv\|_{X_{k_2}}.$$

Thus, we complete the proof of part (a). To prove part (b), without loss of generality, we assume $k=k_2$; by using the definition of X_k, we have

$$\Big\|\left(i+\tau+(-1)^n\xi^{2n+1}\right)^{-1}\eta_{k_1}(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}(P_k\partial_xu)*\mathcal{F}(P_{k_2}\partial_xv)\Big\|_{X_{k_1}}$$
$$\le C2^{-k_1}2^{k_2}2^{k_3}\sum_{j_1,j_2,j_3\ge 0}2^{-j_1/2}\left\|I_{D_{k_1,j_1}}u_{k,j_2}*v_{k,j_2}\right\|_{L^2_{\xi\tau}}. \tag{3.8}$$

Just as in Lemma 3.2 we assume $j_{\max}\ge 2nk+k_1-0$ and $j_1,j_2,j_3\le 10k$. To give the bound of (3.8), we firstly consider the case $j_1=j_{\max}$. By using (2.23), we have

$$2^{-k_1}2^{k}2^{k_2}\sum_{j_1,j_2,j_3\ge 0}2^{-j_1/2}\left\|I_{D_{k_1,j_1}}u_{k,j_2}*v_{k,j_3}\right\|_{L^2_{\xi\tau}}$$
$$\le C2^{-k_1}2^{k}2^{k_2}\sum_{j_1\ge 2nk+k_1-10}\sum_{j_2,j_3\ge 0}2^{-j_1/2}2^{(-n+\frac{1}{2})k}2^{-k_1/2}2^{(j_2+j_3)/2}$$
$$\times\|u_{k,j_2}\|_{L^2_{\xi\tau}}\|v_{k,j_3}\|_{L^2_{\xi\tau}}$$
$$\le C2^{-2k_1}2^{(-2n+\frac{5}{2})k}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_kv\|_{X_{k_2}}.$$

When $j_2=j_{\max}$, by using (2.24) and $j_1\le 10k$, we have

$$2^{-k_1}2^{k}2^{k_2}\sum_{j_1,j_2,j_3\ge 0}2^{-j_1/2}\left\|I_{D_{k_1,j_1}}u_{k,j_2}*v_{k,j_3}\right\|_{L^2_{\xi\tau}}$$
$$\le C2^{-k_1}2^{k}2^{k_2}\sum_{j_2\ge 2nk+k_1-10}\sum_{j_1,j_3\ge 0}2^{-j_1/2}2^{-nk}2^{(j_1+j_3)/2}\|u_{k,j_2}\|_{L^2_{\xi\tau}}\|v_{k,j_3}\|_{L^2_{\xi\tau}}$$
$$\le Ck2^{-\frac{3k_1}{2}}2^{(-2n+2)k}\|\mathcal{F}P_{k_1}u\|_{X_k}\|\mathcal{F}P_kv\|_{X_{k_2}}.$$

The case $j_3=j_{\max}$ is identical to the case $j_2=j_{\max}$ due to the symmetry between j_2 and j_3. The proof of part (b) is completed.

We complete the proof of Lemma 3.4.

Lemma 3.5. *Let $|k_1-k_2|\le 5$ and $k_1\ge 10$, $u=W(t)u_0$, $v=W(t)v_0$, $u_0\in L^2$, and $v_0\in L^2$. Then*

$$\left\|\psi(t)\int_0^t W(t-s)P_0\partial_x(1-\partial_x^2)^{-1}[(P_{k_1}\partial_xu(s))(P_{k_2}\partial_xv(s))]\,ds\right\|_{L^2_xL^\infty_t}$$

$$\leq C2^{k(-2n+\frac{5}{2})}\|u_0\|_{L^2}\|v_0\|_{L^2}. \tag{3.9}$$

For $u, v \in \bar{F}^0$, we have

$$\Big\|\psi(t)\int_0^t W(t-s)P_0\partial_x(1-\partial_x^2)^{-1}[(P_{k_1}\partial_x u(s))(P_{k_2}\partial_x v(s))]\,ds\Big\|_{L_x^2L_t^\infty}$$
$$\leq C2^{(-2n+\frac{5}{2})k}\|\mathcal{F}P_{k_1}u\|_{X_{k_1}}\|\mathcal{F}P_{k_2}v\|_{X_{k_2}}. \tag{3.10}$$

Proof. Since

$$\mathcal{F}_x\Big[\psi(t)\int_0^t W(t-s)P_0\partial_x(1-\partial_x^2)^{-1}[(P_{k_1}\partial_x u(s))(P_{k_2}\partial_x v(s))]\,ds\Big](\xi) = F_xI(\xi),$$

where

$$F_xI(\xi) = -\psi(t)\eta_0(\xi)e^{it(-1)^{n+1}\xi^{2n+1}}$$
$$\times\int_{\xi=\xi_1+\xi_2}\frac{\xi\xi_1\xi_2}{1+\xi^2}\frac{1-e^{i(-1)^n t(\xi_1^{2n+1}+\xi_2^{2n+1}-\xi^{2n+1})}}{\xi_1^{2n+1}+\xi_2^{2n+1}-\xi^{2n+1}}\mathcal{F}_xP_{k_1}u_0(\xi_1)\mathcal{F}_xP_{k_2}v_0(\xi_2)d\xi_1,$$

by using Lemma 2.5 and Minkowski's inequality as well as Hölder's inequality, we obtain

$$\|I\|_{L_x^2L_t^\infty} \leq C2^{k(-2n+2)}$$
$$\times\Big\|\int_{\xi=\xi_1+\xi_2}(1-e^{it(-1)^n(\xi_1^{2n+1}+\xi_2^{2n+1}-\xi^{2n+1})})\mathcal{F}_xP_{k_1}u_0(\xi_1)\mathcal{F}_xP_{k_2}v_0(\xi_2)\,d\xi_1\Big\|_{L_\xi^2}$$
$$\leq C2^{k(-2n+2)}\|\mathcal{F}_xP_{k_1}u_0\|_{L_\xi^1}\|\mathcal{F}_xP_{k_2}v_0\|_{L_\xi^2} \leq C2^{k(-2n+\frac{5}{2})}\|u_0\|_{L_x^2}\|v_0\|_{L_x^2}.$$

Thus the proof of Lemma 3.5 is completed.

Lemma 3.6. *Let $u, v \in \bar{F}^{-n+\frac{5}{4}}$, $n \geq 2$, $n \in N$, and*

$$B(u,v) = \psi\left(\tfrac{t}{4}\right)\int_0^t W(t-\tau)\partial_x(1-\partial_x^2)^{-1}((\psi(\tau)\partial_x u(\tau))(\psi(\tau)\partial_x v(\tau)))\,d\tau.$$

Then

$$\|B(u,v)\|_{\bar{F}^{-n+\frac{5}{4}}} \leq C\|u\|_{\bar{F}^{-n+\frac{5}{4}}}\|v\|_{\bar{F}^{-n+\frac{5}{4}}}.$$

Proof. By using the definition of $\bar{F}^s$, we derive $\|B(u,v)\|^2_{\bar{F}^{-n+\frac{5}{4}}} = I + II$, where

$$I = \sum_{k_1\geq 1}2^{(-2n+\frac{5}{2})k_1}\|\eta_{k_1}(\xi)\mathcal{F}[B(u,v)]\|^2_{X_{k_1}}, \quad II = \|P_{\leq 0}(B(u,v))\|^2_{\bar{X}_0}.$$

By decomposing u and v, we derive

$$\|\eta_{k_1}(\xi)\mathcal{F}[B(u,v)]\|_{X_{k_1}} \leq C \sum_{k_2,k_3\geq 0} \|\eta_{k_1}(\xi)\mathcal{F}[B(P_{k_2}u,P_{k_3}v)]\|_{X_{k_1}}. \tag{3.11}$$

From (2.2), the right-hand side of (3.11) is bounded by

$$C \sum_{k_2\geq 0,\, k_3\geq 0} \left\|(i+\tau+(-1)^n\xi^{2n+1})^{-1}\eta_{k_1}(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}P_{k_2}\partial_x u * \mathcal{F}P_{k_3}\partial_x v\right\|_{X_{k_1}}. \tag{3.12}$$

We can assume $k_2 \leq k_3$ due to the symmetry between k_2 and k_3. To derive

$$I \leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}},$$

it suffices to prove

$$\begin{aligned}&\sum_{k_1\geq 1} 2^{(-2n+\frac{5}{2})k_1}\\ &\times\Big[\sum_{k_2,k_3\geq 0}\|(i+\tau+(-1)^n\xi^{2n+1})^{-1}\eta_{k_1}(\xi)\frac{i\xi}{1+\xi^2}\mathcal{F}P_{k_2}\partial_x u*\mathcal{F}P_{k_3}\partial_x v\|_{X_{k_1}}\Big]^2\\ &\leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.\end{aligned} \tag{3.13}$$

When $k_{\max} \leq 20$, by using Lemma 3.1 and (2.5) of [15], we have that (3.12) is bounded by

$$\leq C \sum_{k_{\max}\leq 20} \|P_{k_2}u\|_{L_t^\infty L_x^2}\|P_{k_3}v\|_{L_t^\infty L_x^2}. \tag{3.14}$$

We have

$$\|P_k u\|^2_{L_t^\infty L_x^2} \leq C\|\mathcal{F}P_k u\|^2_{X_k} \leq \sum_{k\geq 1} 2^{k(-2n+\frac{5}{2})}\|\eta_k(\xi)\mathcal{F}u(\xi,\tau)\|^2_{X_k} \leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}$$

with $k \geq 1$ due to (2.11), which yields

$$\|P_k u\|_{L_t^\infty L_x^2} \leq C\|u\|_{\bar{F}^{-n+\frac{5}{4}}} \tag{3.15}$$

with $k \geq 1$ and

$$\|P_k u\|_{L_t^\infty L_x^2} \leq C\|P_k u\|_{\bar{X}_k} = C\|P_k u\|_{L_x^2 L_t^\infty} \leq C\|u\|_{\bar{F}^{-n+\frac{5}{4}}} \tag{3.16}$$

with $k = 0$ due to Minkowski's inequality and the definition of $\bar{F}^{-n+\frac{5}{4}}$. By using (3.14)–(3.16), we obtain

the left-hand side of (3.13)

$$\leq C\Big[\sum_{k_{\max}\leq 20}\|P_{k_2}u\|_{L_t^\infty L_x^2}\|P_{k_3}v\|_{L_t^\infty L_x^2}\Big]^2\leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.$$

When $k_{\max}\geq 20$ in (3.13), we consider the following four cases.

Case (1): When $|k_1-k_3|\leq 5$ and $k_2\leq k_1-10$.

If $k_2=0$, by using (a) of Lemma 3.2 and the discrete Hölder inequality as well as (2.5) of [15], we have

the left-hand side of (3.13)

$$\leq C\sum_{k_1\geq 10}2^{(-2n+\frac{5}{2})k_1}\Big[\sum_{|k_1-k_3|\leq 5}2^{-nk_3}\|P_{\leq 0}u\|_{L_x^2L_t^\infty}\|\mathcal{F}P_{k_3}v\|_{X_{k_3}}\Big]^2$$
$$\leq C\big(\|P_{\leq 0}u\|_{L_x^2L_t^\infty}\big)^2\Big(\sum_{k_3\geq 1}2^{(-2n+\frac{5}{2})k_3}\|\mathcal{F}P_{k_3}v\|^2_{X_{k_3}}\Big)\leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.$$

If $k_2\geq 1$, by applying (b) of Lemma 3.2, (2.5) of [15] and

$$\frac{k^3}{2^{k(n+\frac{5}{4})}}\leq 1\quad (k\geq 10)$$

as well as the discrete Hölder inequality, we have

the left-hand side of (3.13)

$$\leq C\sum_{k_1\geq 1}2^{(-2n+\frac{5}{2})k_1}\Big[\sum_{|k_1-k_3|\leq 5, 1\leq k_2\leq k_1-10}k_1^3 2^{-2nk_3}\|\mathcal{F}P_{k_2}u\|_{X_{k_2}}\|\mathcal{F}P_{k_3}v\|_{X_{k_3}}\Big]^2$$
$$\leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.$$

Case (2): When $|k_1-k_3|\leq 5$ and $k_1-9\leq k_2\leq k_3$. By using Lemma 3.3 and the discrete Hölder inequality as well as (2.5) of [15], we have

the left-hand side of (3.13)

$$\leq C\sum_{k_1\geq 1}2^{(-2n+\frac{5}{2})k_1}\Big[\sum_{\substack{|k_1-k_3|\leq 5\\ k_1-9\leq k_2\leq k_3}}2^{(3-6n)k_2/2}\|\mathcal{F}P_{k_2}u\|_{X_{k_2}}\|\mathcal{F}P_{k_3}v\|_{X_{k_3}}\Big]^2$$
$$\leq C\Big[\sum_{k_2\geq 1}2^{(-2n+\frac{5}{2})k_2}\|\mathcal{F}P_{k_2}u\|^2_{X_{k_2}}\Big]\Big[\sum_{k_3\geq 1}2^{(-2n+\frac{5}{2})k_3}\|\mathcal{F}P_{k_3}v\|^2_{X_{k_3}}\Big]$$
$$\leq C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.$$

Case (3): When $|k_2 - k_3| \le 5$ and $k_1 = 0$. By using (a) of Lemma 3.4, we easily obtain

$$\text{the left-hand side of (3.13)} \le C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.$$

Case (4): When $|k_2 - k_3| \le 5$ and $1 \le k_1 \le k_2 - 5$. By using (b) of Lemma 3.4, (2.5) of [15], and the discrete Hölder inequality as well as $\frac{k^2}{2^k} \le C(k \ge 10), n \ge 2$ we have

$$\begin{aligned}
&\text{the left-hand side of (3.13)}\\
&\le C \sum_{1 \le k_1 \le k_2 - 5} 2^{(-2n+\frac{5}{2})k_1}\\
&\times \Big[\sum_{|k_2-k_3|\le 5} 2^{-2k_1}(2^{(-2n+\frac{5}{2})k_2} + k_2 2^{k_1/2}) 2^{(-2n+2)k} \|\mathcal{F}P_{k_2}u\|_{X_{k_2}} \|\mathcal{F}P_{k_3}v\|_{X_{k_3}} \Big]^2\\
&\le C \Big[\sum_{|k_2-k_3|\le 5} 2^{(-n+\frac{5}{4})k_2} \|\mathcal{F}P_{k_2}u\|_{X_{k_2}} 2^{(-n+\frac{5}{4})k_3} \|\mathcal{F}P_{k_3}v\|_{X_{k_3}} \Big]^2\\
&\le C \Big[\sum_{k_2 \ge 1} 2^{(-2n+\frac{5}{2})k_2} \|\mathcal{F}P_{k_2}u\|^2_{X_{k_2}} \Big] \Big[\sum_{k_3 \ge 1} 2^{(-2n+\frac{5}{2})k_3} \|\mathcal{F}P_{k_3}v\|^2_{X_{k_3}} \Big]\\
&\le C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\|v\|^2_{\bar{F}^{-n+\frac{5}{4}}}.
\end{aligned}$$

To prove Lemma 3.6, we also need to prove

$$\|P_{\le 0}(B(u,v))\|_{\bar{X}_0} \le C\|u\|_{\bar{F}^{-n+\frac{5}{4}}}\|v\|_{\bar{F}^{-n+\frac{5}{4}}}.$$

Decomposing u and v yields

$$\|P_{\le 0}(B(P_{k_2}u, P_{k_3}v))\|_{\bar{X}_0} \le C \sum_{k_2 \ge 0, k_3 \ge 0} \|P_{\le 0}(B(P_{k_2}u, P_{k_3}v))\|_{\bar{X}_0}.$$

If $\max(k_2, k_3) \le 10$, by using (2.10), the second conclusion of Lemma 2.1, Lemma 3.1, and (3.15)–(3.16), we obtain

$$\begin{aligned}
&\|P_{\le 0}(B(P_{k_2}u, P_{k_3}v))\|_{\bar{X}_0} \le C\|\mathcal{F}P_{\le 0}(B(P_{k_2}u, P_{k_3}v))\|_{X_0}\\
&\le C\|(i + \tau + (-1)^n \xi^{2n+1})^{-1} \frac{i\xi}{1+\xi^2} \mathcal{F}P_{k_2}\partial_x u * \mathcal{F}P_{k_3}\partial_x v\|_{X_0}\\
&\le C \sum_{0 \le k_2,\, k_3 \le 10} \|P_{k_2}u\|_{L_t^\infty L_x^2} \|P_{k_3}v\|_{L_t^\infty L_x^2} \le C\|u\|_{\bar{F}^{-n+\frac{5}{4}}}\|v\|_{\bar{F}^{-n+\frac{5}{4}}}.
\end{aligned}$$

If $\max(k_2, k_3) \geq 10$, then $|k_2 - k_3| \leq 5$; by using Lemma 3.5, (2.5) of [15], and the discrete Hölder inequality, we obtain

$$\begin{aligned}
&\|P_{\leq 0}(B(u,v))\|_{\bar{X}_0} \leq C \sum_{k_2 \geq 0,\, k_3 \geq 0} \|P_{\leq 0}(B(P_{k_2}u, P_{k_3}v))\|_{\bar{X}_0} \\
&\leq C \sum_{k_2 \geq 0,\, k_3 \geq 0, |k_2 - k_3| \leq 5} 2^{(-n+\frac{5}{4})k_2} \|\mathcal{F}P_{k_2}u\|_{X_{k_2}} 2^{(-n+\frac{5}{4})k_3} \|\mathcal{F}P_{k_3}v\|_{X_{k_2}} \\
&\leq C \Big(\sum_{k_2 \geq 1} 2^{(-2n+\frac{5}{2})k_2} \|\mathcal{F}P_{k_2}u\|^2_{X_{k_2}} \Big)^{\frac{1}{2}} \Big(\sum_{k_3 \geq 1} 2^{(-2n+\frac{5}{2})k_3} \|\mathcal{F}P_{k_3}v\|^2_{X_{k_3}} \Big)^{\frac{1}{2}} \\
&\leq C \|u\|_{\bar{F}^{-n+\frac{5}{4}}} \|v\|_{\bar{F}^{-n+\frac{5}{4}}}.
\end{aligned}$$

Thus, the proof of Lemma 3.6 is completed.

Lemma 3.7. *Let*

$$D(u,v) = \psi\left(\tfrac{t}{4}\right) \int_0^t W(t-\tau)\partial_x\{(\psi(\tau)u(\tau))(\psi(\tau)v(\tau))\}\, d\tau$$

and $u, v \in \bar{F}^{-n+\frac{5}{4}}$. *Then*

$$\|D(u,v)\|_{\bar{F}^{-n+\frac{5}{4}}} \leq C\|u\|_{\bar{F}^{-n+\frac{5}{4}}} \|v\|_{\bar{F}^{-n+\frac{5}{4}}}.$$

Lemma 3.7 is proved similarly to Lemma 3.6.

Lemma 3.8. *Let*

$$H(u,v) = \psi\left(\tfrac{t}{4}\right) \int_0^t W(t-\tau)\partial_x(1-\partial_x^2)^{-1}\{(\psi(\tau)u(\tau))(\psi(\tau)v(\tau))\}\, d\tau$$

and $u, v \in \bar{F}^{-n+\frac{5}{4}}$. *Then*

$$\|H(u,v)\|_{\bar{F}^{-n+\frac{5}{4}}} \leq C\|u\|_{\bar{F}^{-n+\frac{5}{4}}} \|v\|_{\bar{F}^{-n+\frac{5}{4}}}.$$

Lemma 3.8 is proved similarly to Lemma 3.6.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Noticing that (1.1)–(1.2) are formally equivalent to (1.3), we define the mapping G as follows:

$$\begin{aligned}
G(u) = \psi(\tfrac{t}{4})\Big[&W(t)u_0 \\
&- \int_0^t W(t-\tau)\psi(\tau)^2 \left(\tfrac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x[u^2 + \tfrac{1}{2}(\partial_x u)^2]\right) d\tau\Big].
\end{aligned}$$

Let

$$B(0,r)=\{u\in \bar{F}^{-n+\frac{5}{4}}:\|u\|_{\bar{F}^{-n+\frac{5}{4}}}\le 2Cr\},$$

where $r=\|u_0\|_{H^{-n+\frac{5}{4}}}\le\epsilon\ll 1$. Thus, by using the first conclusion of Lemma 2.1, Lemma 3.6, and Lemma 3.7 as well as Lemma 3.8, we obtain

$$\begin{aligned}&\|G(u)\|_{\bar{F}^{-n+\frac{5}{4}}}\le\|\psi(\tfrac{t}{4})W(t)u_0\|_{\bar{F}^{-n+\frac{5}{4}}}\\&+\left\|\psi(\tfrac{t}{4})\int_0^t W(t-\tau)\psi(\tau)^2(\tfrac{1}{2}\partial_x(u^2)+(1-\partial_x^2)^{-1}\partial_x[u^2+\tfrac{1}{2}(\partial_x u)^2])\,d\tau\right\|_{\bar{F}^{-n+\frac{5}{4}}}\\&\le C\|u_0\|_{H^{-n+\frac{5}{4}}}+C\|u\|^2_{\bar{F}^{-n+\frac{5}{4}}}\le Cr+C(2Cr)^2\le 2Cr\end{aligned}$$

since $\|u_0\|_{H^{-n+\frac{5}{4}}}\ll 1$. Similarly,

$$\begin{aligned}\|G(u)-G(v)\|_{\bar{F}^{-n+\frac{5}{4}}}&\le C\|u-v\|_{\bar{F}^{-n+\frac{5}{4}}}\|u+v\|_{\bar{F}^{-n+\frac{5}{4}}}\\&\le 4C^2r\|u-v\|_{\bar{F}^{-n+\frac{5}{4}}}\le\frac{1}{2}\|u-v\|_{\bar{F}^{-n+\frac{5}{4}}}.\end{aligned}$$

Then the mapping G is the contraction mapping from $B(0,r)$ into itself. From the fixed-point theorem, we know that there exists a u such that $G(u)=u$. The rest of Theorem 1.1 follows from a standard argument.

5. Proof of Theorem 1.2

We prove Theorem 5.1 before proving Theorem 1.2.

Theorem 5.1. *Let $s<-n+\frac{5}{4}$, $n\ge 2$, $n\in N$, and T be a positive real number. Then there does not exist a space Y_T continuously embedded in $C([0,T];\dot{H}^s(\mathbf{R}))$ such that*

$$\|W(t)u_0\|_{Y_T}\le C\|u_0\|_{\dot{H}^s},\quad\forall u_0\in\dot{H}^s(\mathbf{R}),\tag{5.1}$$

$$\left\|\int_0^{\mathrm{t}}\mathrm{W}(\mathrm{t}-\tau)\partial_x f(u,\partial_x u))d\tau\right\|_{Y_T}\le C\|u\|^2_{Y_T},\quad\forall u\in Y_T,\tag{5.2}$$

where $f(u,\partial_x u)=\frac{1}{2}u^2+(1-\partial_x^2)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right].$

Proof. Assume that there exists a space Y_T such that (5.1) and (5.2) hold. Let $u=W(t)u_0$ in (5.2); by using (5.1) and the fact that Y_T is continuously embedded in $C([0,T];\dot{H}^s(\mathbf{R}))$, for any t $\in$ [0, T], we have, $\forall u_0\in\dot{H}^s(\mathbf{R})$,

$$\left\|\int_0^t W(t-\tau)\partial_x\left[f(W(\tau)u_0,\partial_x W(\tau)u_0)\right]d\tau\right\|_{\dot{H}^s}\le C\|u_0\|^2_{\dot{H}^s}.\tag{5.3}$$

We consider the initial data as follows:

$$u_0(x) = r^{-1/2}N^{-s}\Big\{e^{-iNx}\Big(\int_0^r e^{ix\xi}\,d\xi\Big) + e^{iNx}\Big(\int_r^{2r} e^{ix\xi}\,d\xi\Big)\Big\},$$

which can be seen in [30], where $r = N^{-\frac{2}{4n-7}}$ and $N \gg 1$. Thus we have

$$\mathcal{F}_x u_0(\xi) = Cr^{-1/2}N^{-s}\left\{\chi_{[-N,-N+r]}(\xi) + \chi_{[N+r,N+2r]}(\xi)\right\},$$

where χ_I denotes the characteristic function of a set $I \subset \mathbf{R}$. It is easy to check that $\|u_0\|_{\dot{H}^s} \sim 1$. Let $I_1 = [-N,-N+r]$ and $I_2 = [N+r, N+2r]$. We define

$$F(x,t) = \int_0^t W(t-\tau)\partial_x\left[f(W(\tau)u_0, \partial_x W(\tau)u_0)\right]d\tau.$$

Thus,

$$\begin{aligned}\mathcal{F}_x F(\xi,t) = C&\frac{i\xi}{2(1+\xi^2)}e^{it(-1)^{n+1}\xi^{2n+1}}\\ &\times\int_{\mathbf{R}}(3+\xi^2-\xi_1\xi_2)\mathcal{F}_x u_0(\xi_1)\mathcal{F}_x u_0(\xi_2)\frac{e^{itQ}-1}{Q}\,d\xi_1;\end{aligned} \tag{5.4}$$

in the above process, we use

$$\xi = \xi_1 + \xi_2,\ Q = (-1)^{n+1}[\phi(\xi_1) + \phi(\xi_2) - \phi(\xi)],$$
$$\left|\xi_1^{2n+1} + \xi_2^{2n+1} - \xi^{2n+1}\right| \sim \min\{|\xi|,|\xi_1|,|\xi_2|\}\max\{|\xi|,|\xi_1|,|\xi_2|\}^{2n}.$$

Take $t = N^{-2n+\frac{2}{4n-7}} \leq T$ for sufficiently large N and $n \geq 2$, which yields $\left|e^{itQ}-1\right| \sim$ Constant; thus, we have

$$\begin{aligned}C \sim \|F\|_{\dot{H}^s} &\geq \Big(\int_{-\frac{r}{2}}^{\frac{r}{2}}|\xi|^{2s}|\mathcal{F}_x F(t,\xi)|^2\,d\xi\Big)^{1/2}\\ &\geq Cr^{-1}N^{-2s}r^{3/2}\frac{1}{1+r^2}r^sN^{-2n+2} \geq CN^{-2s\frac{4n-6}{4n-7}-\frac{8n^2-22n+15}{4n-7}}.\end{aligned} \tag{5.5}$$

Since $s < -n + \frac{5}{4}$, $n \geq 2$, and $N \gg 1$, we have $N^{-2s\frac{4n-6}{4n-7}-\frac{8n^2-22n+15}{4n-7}} \longrightarrow \infty$ as $N \longrightarrow \infty$. This contradicts (5.5). Thus we complete the proof of Theorem 5.1.

Now we prove Theorem 1.2. Since

$$u(x,t,u_0) = W(t)u_0 - \int_0^t W(t-\tau)\partial_x(f(u,\partial_x u)(\cdot,\tau,u_0))\,d\tau,$$

assume that the flow-map is C^2. Since $u(x,t,0)=0$, we have

$$u_1(x,t) = \frac{\partial u}{\partial u_0}(x,t,0)[h] = W(t)h,$$
$$u_2(x,t) = \frac{\partial^2 u}{\partial^2 u_0}(x,t,0)[h,h] = -2\int_0^t W(t-\tau)\partial_x \left[f(W(t)h, \partial_x W(t)h)\right] d\tau.$$

Since the flow map is C^2, we have

$$\|u_2(t)\|_{\dot{H}^s} \le C\|h\|_{\dot{H}^s}^2, \ \forall h \in \dot{H}^s(\mathbf{R}). \tag{5.6}$$

From Theorem 5.1, we know that (5.6) is untrue. Thus, we complete the proof of Theorem 1.2.

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