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ON THE EQUATION $\det \nabla\varphi = f$ PRESCRIBING $\varphi = 0$ ON THE BOUNDARY

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Abstract. We discuss the existence of a regular map φ satisfying

$$\begin{cases} \det \nabla\varphi = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain and f is a regular function satisfying $\int_\Omega f = 0$.

1. Introduction

In this article we discuss the existence of $\varphi : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{cases} \det \nabla\varphi = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain and $f : \overline{\Omega} \to \mathbb{R}$. Using the divergence theorem one immediately deduces that $\int_\Omega f = 0$ is a necessary condition to solve (1.1) (see Lemma 15 for a more precise statement). The main results of this article can be summarized as follows (see Theorems 2, 4, and 6 for a more general statement).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be the unit ball and $f \in C^1(\overline{\Omega})$ be such that $\int_\Omega f = 0$.*

(i) If $|f| > 0$ on $\partial\Omega$, then there exists $\varphi \in C^1(\Omega;\mathbb{R}^n) \cap C^{0,1/n}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1.1).

(ii) If f is of the form $\operatorname{dist}(\cdot,\partial\Omega)^{n-1}$ near $\partial\Omega$, then there exists $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ satisfying (1.1).

(iii) If $\operatorname{supp} f \subset \Omega$, then

$$\inf_{\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)} \| \det \nabla\varphi - f \|_{C^1(\overline{\Omega})} = 0.$$

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This article uses similar techniques as in [2] where the existence of a φ as regular as f (where $\int_\Omega f = \operatorname{meas}\Omega$) satisfying

$$\det \nabla\varphi = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id} \text{ on } \partial\Omega$$

was proved (see also [1] for a complete overview): roughly speaking we look for a solution φ of the form $\varphi = \phi \circ \psi$ where ψ is a diffeormophism and ψ a radial solution, i.e., a solution of the form $\alpha(x)x/|x|$ with α a function. However, contrary to the case where we prescribe the identity on the boundary, these techniques seem not to allow one to solve (1.1) in its full generality.

Finally, note that, contrary to the case $\varphi = \text{id}$ on the boundary, the integral condition $\int_\Omega f = 0$ is not sufficient for the existence of a solution C^k, $k \geq 1$, up to the boundary of (1.1). Indeed (cf. Lemma 17):

- f has to vanish identically on the boundary in order to have a solution C^1 up to the boundary.

- f has to be of the form $\operatorname{dist}(\cdot, \partial\Omega)^{n-1}h(\cdot)$ near $\partial\Omega$ for a continuous function h if we want the solution to be C^2 up to the boundary.

2. NOTATION

We gather here the main notation that will be used throughout the article. Let $\Omega, O \subset \mathbb{R}^n$ be bounded open sets.

- Balls in $\mathbb{R}^n$ are denoted by $B_\epsilon(x) := \{y \in \mathbb{R}^n : |y - x| < \epsilon\}$. When $x = 0$ we simply write B_ϵ instead of $B_\epsilon(0)$; similarly we write B instead of B_1.

- For $g \in C^0(\mathbb{R}^n)$ and $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ we let

$$\varphi^*(g) := g(\varphi) \det \nabla\varphi.$$

We will use several times the following elementary property: for $\varphi_1 \in C^1(\Omega; O)$ and $\varphi_2 \in C^1(O; \mathbb{R}^n)$ we have $(\varphi_2 \circ \varphi_1)^*(g) = \varphi_1^*(\varphi_2^*(g))$.

- The set of diffeormorphisms of class k, $k \geq 1$ an integer, is denoted by

$$\operatorname{Diff}^k(\overline{\Omega}; \overline{O}) := \{\varphi \in C^k(\overline{\Omega}; \overline{O}) : \varphi^{-1} \in C^k(\overline{O}; \overline{\Omega})\}.$$

We say that $\overline{\Omega}$ is C^k-diffeomorphic to $\overline{B}$ if there exists $\varphi \in \operatorname{Diff}^k(\overline{B}; \overline{\Omega})$.

- $C_0^k(\Omega)$ stands for the set of $C^k(\overline{\Omega})$ functions with compact support in Ω.

- For $k \geq 0$ an integer and g a function, $\nabla^k g$ stands for the set of derivatives of order k of g with the convention $\nabla^0 g = g$.

- $\operatorname{dist}(\cdot, \partial\Omega)$ is the usual distance function to the set $\partial\Omega$, i.e.,

$$\operatorname{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |y - x|.$$

3. Main results

In this section we state the three main results of this article, which will be proved in the next section. We start with a result giving a solution with interior regularity.

Theorem 2. *Let $n \geq 2$ and $k \geq 1$ be two integers and $\Omega \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B}$. Let also V be a neighborhood of $\partial\Omega$, $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{\Omega})$ be such that $\inf_{\mathbb{R}^n} |g| > 0$, $|f| > 0$ in $V \cap \Omega$, and $\int_\Omega f = 0$. Then there exists $\varphi \in C^k(\Omega; \mathbb{R}^n) \cap C^{0,1/n}(\overline{\Omega}; \mathbb{R}^n)$ such that*

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 3. (i) Recall that "$\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B}$" means that there exists $\psi \in \mathrm{Diff}^{k+1}(\overline{B}; \overline{\Omega})$.

(ii) The condition $\int_\Omega f = 0$ is always necessary (cf. Lemma 15).

(iii) In the particular case $|f| > 0$ on $\partial\Omega$, the obtained solution belongs to $\cap_{1 \leq p < n/(n-1)} W^{1,p}(\Omega, \mathbb{R}^n)$ but not to $W^{1,n/(n-1)}(\Omega, \mathbb{R}^n)$; moreover, it does not belong to $C^{0,\beta}(\overline{\Omega}; \mathbb{R}^n)$ for $1/n < \beta \leq 1$ (cf. Proposition 13 (i)).

(iv) With a more involved proof we can show that for every $\epsilon > 0$ there exists φ_ϵ satisfying all the conclusions of the theorem with in addition $\varphi_\epsilon(\overline{\Omega}) \subset B_\epsilon$. This can be seen as an optimal control of the size of the image of the solution.

Our second result gives a sufficient condition (and somehow partially necessary; cf. the following remark) to have a solution regular up to the boundary.

Theorem 4. *Let $n \geq 2$ be an integer, $\Omega \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega}$ is C^3-diffeomorphic to $\overline{B}$, $g \in C^1(\mathbb{R}^n)$ and $f \in C^1(\overline{\Omega})$ be such that $\inf_{\mathbb{R}^n} |g| > 0$, $\int_B f = 0$, and*

$$f(x) = \mathrm{dist}(x, \partial\Omega)^m c(x) \quad \text{near } \partial\Omega,$$

where $m \geq n - 1$ and where $c \in C^1(\overline{\Omega})$ satisfies $|c| > 0$ on $\partial\Omega$. Then there exists $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Remark 5. (i) The condition $\int_\Omega f = 0$ is always necessary (cf. Lemma 15) and Remark 3 (iv) is also verified.

(ii) Surprisingly enough, even if g, f, c, and Ω are smooth, the proof does not give, in general, $\varphi \in C^2(\overline{\Omega}; \mathbb{R}^n)$. For example, taking $n = 2$, $\Omega = B$,

$g \equiv 1$, and $f = (1 - |x|)|x|$ near the boundary, the solution constructed in the proof is not C^2 up to the boundary (cf. Remark 14 (i)).

(iii) However, with exactly the same proof (using Proposition 13 (iii)), one can show the following: let $\Omega = B$, $k \geq 2$ be an integer, $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{B})$ be such that $\inf_{\mathbb{R}^n} |g| > 0$,

$$\int_B f = 0, \quad \text{and} \quad f(x) = (1 - |x|)^{ns-1} \frac{1}{|x|^{n-1}} \quad \text{near } \partial B,$$

where $s \geq 1$ is an integer; then there exists $\varphi \in C^k(\overline{B}; \mathbb{R}^n)$ satisfying (3.1).

(iv) Using Proposition 17, f has to be of the form

$$f(x) = \operatorname{dist}(x; \partial\Omega)^{n-1} h(x)$$

near the the boundary where h is continuous if we want the solution to be $C^2(\overline{\Omega}; \mathbb{R}^n)$. This gives a condition of the behavior of f near the boundary: indeed, in order to have a solution of (3.1) C^2 up to the boundary, f has to vanish on $\partial\Omega$ at least as fast as $\operatorname{dist}(\cdot; \partial\Omega)^{n-1}$.

Our last theorem is an approximation result.

Theorem 6. *Let $n \geq 2$ and $k \geq 0$ be two integers, and let $\Omega \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B}$. Let also $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{\Omega})$ be such that $\inf_{\mathbb{R}^n} |g| > 0$,*

$$\nabla^s f = 0 \quad \text{on } \partial\Omega, \text{ for every } 0 \leq s \leq k$$

and $\int_B f = 0$. Then

$$\inf_{\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)} \|\varphi^*(g) - f\|_{C^k(\overline{\Omega})} = 0.$$

Remark 7. The previous theorem raises the following interesting open question: given $f \in C_0^k(\Omega)$ with $\int_\Omega f = 0$, does there exist $\varphi \in C_0^k(\Omega; \mathbb{R}^n)$ such that

$$\det \nabla\varphi = f \quad \text{in } \Omega?$$

Note that the above result says that we can always uniformly approximate any such f by the Jacobian of smooth maps with compact support.

4. Proof of the main results

In this section we prove the three results stated in the last section. We start by proving an elementary lemma which will allow us to assume $g \equiv 1$.

Lemma 8. *Let $n \geq 2$ and $k \geq 1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $g \in C^k(\mathbb{R}^n)$ with $\inf_{\mathbb{R}^n} |g| > 0$, $f \in C^k(\overline{\Omega})$, and $\psi \in C^k(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\begin{cases} \psi^*(1) = f & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists $\varphi \in C^k(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 9. With exactly the same proof we show that if $\psi \in C^k(\Omega; \mathbb{R}^n) \cap C^{0,1/n}(\overline{\Omega}; \mathbb{R}^n)$, then $\varphi \in C^k(\Omega; \mathbb{R}^n) \cap C^{0,1/n}(\overline{\Omega}; \mathbb{R}^n)$.

Proof. Since $\inf_{\mathbb{R}^n} |g| > 0$, we can define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$\int_0^{h(x)} g(x_1, \ldots, x_{n-1}, t)\, dt = x_n.$$

Note that $h \in C^k(\mathbb{R}^n)$. Then ϕ defined by $\phi(x) = (x_1, \ldots, x_{n-1}, h(x))$ belongs to $\mathrm{Diff}^k(\mathbb{R}^n; \phi(\mathbb{R}^n))$ and satisfies

$$\phi^*(g) = 1 \text{ in } \mathbb{R}^n \quad \text{and} \quad \phi(0) = 0.$$

It is then easily checked that $\varphi = \phi \circ \psi$ has all the wished-for properties. □

We now prove the first main result.

Proof of Theorem 2. Step 1 (simplification). Using Lemma 8, we can assume with no loss of generality that $g \equiv 1$. We also claim that we can assume that $\Omega = B$. Indeed, by hypothesis there exists $\psi \in \mathrm{Diff}^{k+1}(\overline{B}; \overline{\Omega})$. Note that $\widetilde{f} := \psi^*(f)$ belongs to $C^k(\overline{B})$ and satisfies

$$\int_B \widetilde{f} = 0 \quad \text{and} \quad |\widetilde{f}| > 0 \text{ in } B \setminus B_{1-\epsilon},$$

for some $\epsilon > 0$ small enough. If $\phi \in C^k(B; \mathbb{R}^n) \cap C^{0,1/n}(\overline{B}; \mathbb{R}^n)$ solves

$$\phi^*(1) = \widetilde{f} \text{ in } B \quad \text{and} \quad \phi = 0 \text{ on } \partial B,$$

then it is easily seen that $\varphi = \phi \circ \psi^{-1} \in C^k(\Omega; \mathbb{R}^n) \cap C^{0,1/n}(\overline{\Omega}; \mathbb{R}^n)$ and solves

$$\varphi^*(1) = f \text{ in } \Omega \quad \text{and} \quad \varphi = 0 \text{ on } \partial\Omega,$$

which proves the assertion. We also claim that we can assume $f < 0$ in $B \setminus B_{1-\epsilon}$. Indeed, if $f > 0$ in $B \setminus B_{1-\epsilon}$ and if φ satisfies

$$\varphi^*(1) = -f \text{ in } B \quad \text{and} \quad \varphi = 0 \text{ on } \partial B,$$

then $\psi = (-\varphi^1, \varphi^2, \ldots, \varphi^n)$ satisfies

$$\psi^*(1) = f \text{ in } B \quad \text{and} \quad \psi = 0 \text{ on } \partial B.$$

Step 2. Applying Lemma 10 there exists $\varphi_1 \in \operatorname{Diff}^k(\overline{B}; \overline{B})$ such that $\operatorname{supp}(\varphi_1 - \mathrm{id}) \subset B$, $\varphi_1^*(f) \in C^k(\overline{B})$, $\varphi_1^*(f) \equiv 1$ in a neighborhood of 0, and

$$\int_0^r s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds > 0 \quad \text{for every } r \in (0,1) \text{ and every } x \neq 0$$

$$\int_0^1 s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds = 0 \quad \text{for every } x \neq 0.$$

Step 3. Let $\alpha : \overline{B} \to \mathbb{R}$ be defined by $\alpha(0) = 0$ and, for $x \in \overline{B} \setminus \{0\}$,

$$\alpha(x) = \Big(n \int_0^{|x|} s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds\Big)^{1/n}$$

and $\varphi_2 : \overline{B} \to \mathbb{R}^n$ be defined by $\varphi_2(x) = \alpha(x)\frac{x}{|x|}$. Using Lemma 13 we have $\varphi_2 \in C^k(B; \mathbb{R}^n) \cap C^{0,1/n}(\overline{B}; \mathbb{R}^n)$ and

$$\begin{cases} \varphi_2^*(1) = \varphi_1^*(f) & \text{in } B \\ \varphi_2 = 0 & \text{on } \partial B. \end{cases}$$

Then $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$ belongs to $C^k(B; \mathbb{R}^n) \cap C^{0,1/n}(\overline{B}; \mathbb{R}^n)$ and satisfies

$$\begin{cases} \varphi^*(1) = f & \text{in } B \\ \varphi = 0 & \text{on } \partial B, \end{cases}$$

which concludes the proof. □

We now prove the second main result.

Proof of Theorem 4. Step 1. Reasoning exactly as in Steps 1 and 2 of the previous proof (using in particular Lemma 18) we can assume $\Omega = B$, $g \equiv 1$, $f = (1 - |x|)^m \widetilde{c}(x)$ with $\widetilde{c} \in C^1(\overline{B})$ such that $\widetilde{c} < 0$ on $\partial\Omega$, and the existence of $\varphi_1 \in \operatorname{Diff}^1(\overline{B}; \overline{B})$ such that $\operatorname{supp}(\varphi_1 - \mathrm{id}) \subset B$, $\varphi_1^*(f) \in C^1(\overline{B})$, $\varphi_1^*(f) \equiv 1$ in a neighborhood of 0, and

$$\int_0^r s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds > 0 \quad \text{for every } r \in (0,1) \text{ and every } x \neq 0$$

$$\int_0^1 s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds = 0 \quad \text{for every } x \neq 0.$$

Step 2. Let $\alpha : \overline{B} \to \mathbb{R}$ be defined by $\alpha(0) = 0$ and, for $x \in \overline{B} \setminus \{0\}$,

$$\alpha(x) = \Big(n \int_0^{|x|} s^{n-1} \varphi_1^*(f)(s\frac{x}{|x|})\, ds\Big)^{1/n},$$

and let $\varphi_2 : \overline{B} \to \mathbb{R}^n$ be defined by $\varphi_2(x) = \alpha(x)\frac{x}{|x|}$. Since $\varphi_1^*(f) = f = (1-|x|)^m \widetilde{c}(x)$ near ∂B we get, using Lemma 13 (ii), that φ_2 belongs to $C^1(\overline{B};\mathbb{R}^n)$ and satisfies

$$\begin{cases} \varphi_2^*(1) = \varphi_1^*(f) & \text{in } B \\ \varphi_2 = 0 & \text{on } \partial B. \end{cases}$$

Then $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$ has all the desired properties, which concludes the proof. □

Finally, we prove the last main result.

Proof of Theorem 6. Step 1. As in Step 1 of the proof of Theorem 2, we can assume $g \equiv 1$ and $\Omega = B$. Using that $\nabla^s f = 0$ on ∂B for every $0 \le s \le k$, we deduce that

$$\inf_{h \in C_0^\infty(B)} \|h - f\|_{C^k(\overline{B})} = 0,$$

and hence we can also assume that $f \in C_0^\infty(B)$. Then, in particular, there exists $\sigma > 0$ small enough such that $f \equiv 0$ in $\overline{B} \setminus B_{1-3\sigma}$. For every $\delta > 0$ small enough let $f_\delta \in C_0^\infty(B)$ be such that

$$f_\delta(x) = \begin{cases} f & \text{in } (\overline{B} \setminus B_{1-\sigma}) \cup B_{1-3\sigma} \\ -\delta \frac{e^{-1/(1-\sigma-|x|)^n}}{|x|^{n-1}(1-\sigma-|x|)^{n+1}} & \text{in } B_{1-\sigma} \setminus B_{1-2\sigma} \end{cases}$$

$$\int_{B \setminus B_{1-3\sigma}} f_\delta = 0 \quad \text{and} \quad \lim_{\delta \to 0} \|f_\delta - f\|_{C^k(\overline{B})} = 0.$$

Using the last equation the theorem will be proved if we show, for every $\delta > 0$ small, the existence of a $\varphi_\delta \in C_0^\infty(B;\mathbb{R}^n)$ such that $\varphi_\delta^*(1) = f_\delta$. This will be done in Step 2.

Step 2. Let $\delta > 0$ be small enough. Note that

$$\int_{B_{1-\sigma}} f_\delta = 0 \quad \text{and} \quad f_\delta < 0 \text{ in } B_{1-\sigma} \setminus B_{1-2\sigma}.$$

Hence, using Lemma 10 (applied to $B_{1-\sigma}$) there exists $\varphi_1 \in \text{Diff}^\infty(\overline{B};\overline{B})$ such that $\text{supp}(\varphi_1 - \text{id}) \subset B_{1-\sigma}$, $\varphi_1^*(f_\delta) \in C^\infty(\overline{B})$, $\varphi_1^*(f_\delta) \equiv 1$ in a neighborhood of 0, and

$$\int_0^r s^{n-1} \varphi_1^*(f_\delta)(s\frac{x}{|x|})\, ds > 0 \quad \text{for every } r \in (0, 1-\sigma) \text{ and every } x \neq 0$$

$$\int_0^{1-\sigma} s^{n-1} \varphi_1^*(f_\delta)(s\frac{x}{|x|})\, ds = 0 \quad \text{for every } x \neq 0.$$

Define $\alpha : \overline{B} \to \mathbb{R}$ by $\alpha(0) = 0$ and, for $x \in \overline{B} \setminus \{0\}$,

$$\alpha(x) = \Big(n \int_0^{|x|} s^{n-1} \varphi_1^*(f_\delta)(s\frac{x}{|x|})\, ds\Big)^{1/n}$$

and $\varphi_2 : \overline{B} \to \mathbb{R}^n$ by $\varphi_2(x) = \alpha(x)\frac{x}{|x|}$. Since $\varphi_1^*(f_\delta) = f_\delta$ near $\partial B_{1-\sigma}$, we get, using Lemma 13 (iv), that $\varphi_2 \in C_0^\infty(B;\mathbb{R}^n)$ and satisfies $\varphi_2^*(1) = \varphi_1^*(f_\delta)$ in B. Finally, $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$ belongs to $C_0^\infty(B;\mathbb{R}^n)$ and satisfies $\varphi^*(1) = f_\delta$ in B, which concludes the proof. □

5. Positive radial integration

In this section we show how to modify the mass distribution of some $f \in C^k(\overline{B})$ with $\int_B f = 0$ in order to have strictly positive integrals on every radius starting from 0 and going to points in B and vanishing integrals on every radius starting from 0 and going to points on ∂B.

Lemma 10. *Let $n \geq 2$ and $k \geq 1$ be two integers, $\epsilon > 0$, and $f \in C^k(\overline{B})$ be such that*

$$\int_B f = 0 \quad \textit{and} \quad f < 0 \textit{ in } B \setminus B_{1-\epsilon}.$$

Then there exists $\varphi \in \operatorname{Diff}^k(\overline{B};\overline{B})$ such that $\operatorname{supp}(\varphi - \mathrm{id}) \subset B$, $\varphi^*(f) \in C^k(\overline{B})$, $\varphi^*(f) \equiv 1$ in a neighborhood of 0, and

$$\int_0^r s^{n-1}\varphi^*(f)(s\frac{x}{|x|})\, ds > 0 \quad \textit{for every } r \in (0,1) \textit{ and every } x \neq 0$$

$$\int_0^1 s^{n-1}\varphi^*(f)(s\frac{x}{|x|})\, ds = 0 \quad \textit{for every } x \neq 0.$$

Proof. Step 1. Note that $\int_{B_{1-\epsilon}} f > 0$. Hence, applying Lemma 11.21 of [1] to f and $B_{1-\epsilon}$, there exists $\varphi_1 \in \operatorname{Diff}^\infty(\overline{B};\overline{B})$ such that $\operatorname{supp}(\varphi_1 - \mathrm{id}) \subset B_{1-\epsilon}$ and such that, writing $f_1 = \varphi_1^*(f)$, we have $f_1(0) > 0$ and

$$\int_0^r s^{n-1} f_1(s\frac{x}{|x|})\, ds > 0, \quad \text{for every } x \neq 0 \text{ and every } r \in (0, 1-\epsilon].$$

Note that $\int_B f_1 = 0$.

Step 2. Choose $\eta > 0$ small enough so that $f_1 > 0$ in $\overline{B}_\eta$ and

$$\int_\eta^r s^{n-1} f_1(s\frac{x}{|x|})\, ds > 0, \quad \text{for every } x \neq 0 \text{ and every } r \in (\eta, 1-\epsilon].$$

Let $f_2 \in C^k(\overline{B})$ be such that $\operatorname{supp}(f_2 - f_1) \subset B_\eta$, $f_2 > 0$ in $\overline{B}_\eta$, $f_2 \equiv 1$ in a neighborhood of 0, and

$$\int_{B_\eta} f_2 = \int_{B_\eta} f_1.$$

Using Theorem 11 there exists $\varphi_2 \in \operatorname{Diff}^k(\overline{B}; \overline{B})$ such that

$$\varphi_2^*(f_1) = f_2 \text{ in } B \quad \text{and} \quad \operatorname{supp}(\varphi_2 - \mathrm{id}) \subset B_\eta.$$

In particular, we have $f_2 = f_1 = f < 0$ in $B \setminus B_{1-\epsilon}$ and

$$\int_0^r s^{n-1} f_2(s\frac{x}{|x|})\, ds > 0, \quad \text{for every } x \neq 0 \text{ and every } r \in (0, 1-\epsilon]. \tag{5.1}$$

Note that $\int_B f_2 = 0$.

Step 3. We claim the existence of $f_3 \in C^k(\overline{B})$ such that $\operatorname{supp}(f_3 - f_2) \subset B \setminus \overline{B}_{1-\epsilon}$, $f_3 < 0$ in $B \setminus B_{1-\epsilon}$ and

$$\int_0^1 s^{n-1} f_3(s\frac{x}{|x|})\, ds = 0, \quad \text{for every } x \neq 0. \tag{5.2}$$

Using (5.1), there exists $\delta > 0$ small enough so that, for every $x \neq 0$,

$$\int_0^{1-\epsilon+2\delta} s^{n-1} f_2(s\frac{x}{|x|})\, ds + \int_{1-2\delta}^1 s^{n-1} f_2(s\frac{x}{|x|})\, ds > 0. \tag{5.3}$$

Let $\rho \in C^\infty([0,1];[0,1])$ be such that

$$\rho = \begin{cases} 1 & \text{in } [1-\epsilon+2\delta, 1-2\delta] \\ 0 & \text{in } [0, 1-\epsilon+\delta] \cup [1-\delta, 1]. \end{cases}$$

Define $h \in C^k(\mathbb{R}^n \setminus \{0\})$ by

$$h(x) = \frac{-\int_{1-\epsilon}^1 s^{n-1}(1-\rho(s)) f_2(s\frac{x}{|x|})\, ds - \int_0^{1-\epsilon} s^{n-1} f_2(s\frac{x}{|x|})\, ds}{\int_{1-\epsilon}^1 s^{n-1}\rho(s)\, ds}.$$

Using (5.3), the definition of ρ, and the fact that $f_2 < 0$ in $B \setminus B_{1-\epsilon}$, we get $h < 0$. Note also that $h(\lambda x) = h(x)$ for every $x \neq 0$ and every $\lambda > 0$. It is easily seen that f_3 defined by $f_3(x) = (1 - \rho(|x|)) f_2(x) + \rho(|x|) h(x)$ has all the required properties.

Step 4. Integrating (5.2) over the unit sphere we get that $\int_B f_3 = 0$, and therefore, recalling that $\int_B f_2 = 0$ and that $\operatorname{supp}(f_2 - f_3) \subset B \setminus B_{1-\epsilon}$,

$$\int_{B_{1-\epsilon}} f_2 = \int_{B_{1-\epsilon}} f_3.$$

Hence, recalling also that $f_2, f_3 < 0$ in $B \setminus B_{1-\epsilon}$, we can apply Theorem 11 and find $\varphi_3 \in \mathrm{Diff}^k(\overline{B};\overline{B})$ such that

$$\varphi_3^*(f_2) = f_3 \text{ in } B \quad \text{and} \quad \mathrm{supp}(\varphi_3 - \mathrm{id}) \subset B \setminus \overline{B}_{1-\epsilon}.$$

Step 5. Note that $f_3 \in C^k(\overline{B})$ satisfies $f_3 \equiv 1$ in a neighborhood of 0 and

$$\int_0^r s^{n-1} f_3(s\frac{x}{|x|})\, ds > 0 \quad \text{for every } r \in (0,1) \text{ and every } x \neq 0$$

$$\int_0^1 s^{n-1} f_3(s\frac{x}{|x|})\, ds = 0 \quad \text{for every } x \neq 0.$$

Let $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$. Noticing that $\varphi \in \mathrm{Diff}^k(\overline{B};\overline{B})$ and satisfies $\mathrm{supp}(\varphi - \mathrm{id}) \subset B$ and $\varphi^*(f) = f_3$, the proof is finished. □

In the previous proof we have used a classical result about the Jacobian (see [1] Theorem 10.11 or [3] for a proof).

Theorem 11. *Let $k \geq 1$ be an integer, Ω be a bounded connected open set in $\mathbb{R}^n$, and $f, g \in C^k(\overline{\Omega})$ be such that*

$$f \cdot g > 0 \text{ in } \Omega, \quad \int_\Omega f = \int_\Omega g, \quad \text{and} \quad \mathrm{supp}(f - g) \subset \Omega.$$

Then there exists $\varphi \in \mathrm{Diff}^k(\overline{\Omega};\overline{\Omega})$ such that

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \mathrm{supp}(\varphi - \mathrm{id}) \subset \Omega. \end{cases}$$

6. Radial solutions

In this section, we investigate some properties of radial solutions, i.e. solutions of the form $\alpha(x)x/|x|$ where α is a function. We start with an elementary lemma.

Lemma 12. *Let $n \geq 2$ be an integer and $f \in C^{0,1}(\overline{B})$ be such that*

$$\int_0^r s^{n-1} f(s\frac{x}{|x|})\, ds \geq 0, \quad \text{for every } r \in (0,1] \text{ and every } x \neq 0.$$

Let $\alpha : \overline{B} \to \mathbb{R}$ be defined by $\alpha(0) = 0$ and, for $x \in \overline{B} \setminus \{0\}$,

$$\alpha(x) = \Big(n \int_0^{|x|} s^{n-1} f(s\frac{x}{|x|})\, ds\Big)^{1/n}.$$

Then $\alpha \in C^{0,1/n}(\overline{B})$.

Proof. It is enough to note that $\alpha^n \in C^{0,1}(\overline{B})$. $\square$

Proposition 13. *Let $n \geq 2$ and $k \geq 1$ be two integers and let $f \in C^k(\overline{B})$ be such that $f \equiv 1$ in a neighborhood of 0 and*

$$\int_0^r s^{n-1} f(s\frac{x}{|x|})\, ds > 0, \quad \text{for every } r \in (0,1) \text{ and every } x \neq 0$$

$$\int_0^1 s^{n-1} f(s\frac{x}{|x|})\, ds = 0, \quad \text{for every } x \neq 0.$$

Let $\alpha : \overline{B} \to \mathbb{R}$ be defined by $\alpha(0) = 0$ and, for $x \in \overline{B} \setminus \{0\}$,

$$\alpha(x) = \Big(n \int_0^{|x|} s^{n-1} f(s\frac{x}{|x|})\, ds\Big)^{1/n},$$

and let $\varphi : \overline{B} \to \mathbb{R}^n$ be defined by $\varphi(x) = \alpha(x)\frac{x}{|x|}$. Then $\varphi \in C^k(B;\mathbb{R}^n) \cap C^{0,1/n}(\overline{B};\mathbb{R}^n)$ and

$$\begin{cases} \varphi^*(1) = f & \text{in } B \\ \varphi = 0 & \text{on } \partial B. \end{cases}$$

The following extra assertions are also satisfied:

(i) If $f < 0$ on ∂B, then $\varphi \in \cap_{1\leq p<n/(n-1)} W^{1,p}(B;\mathbb{R}^n)$, $\varphi \notin W^{1,n/(n-1)}(B;\mathbb{R}^n)$, and $\varphi \notin C^{0,\beta}(\overline{B};\mathbb{R}^n)$ for every $1/n < \beta \leq 1$.

(ii) If $f(x) = c(x)(1-|x|)^m$ near ∂B, where $m \geq n-1$ and where $c \in C^1(\overline{B})$ is such that $c < 0$ on ∂B, then $\varphi \in C^1(\overline{B};\mathbb{R}^n)$.

(iii) If $f(x) = -\frac{(1-|x|)^{ns-1}}{|x|^{n-1}}$ near ∂B, where $s \geq 1$ is an integer, then $\varphi \in C^k(\overline{B};\mathbb{R}^n)$.

(iv) If $f \in C^\infty(\overline{B})$ and

$$f(x) = -\frac{e^{-1/(1-|x|)^n}}{|x|^{n-1}(1-|x|)^{n+1}} \quad \text{near } \partial B,$$

then $\varphi \in C^\infty(\overline{B};\mathbb{R}^n)$ and satisfies

$$\nabla^l \varphi = 0 \quad \text{on } \partial B \text{ for every integer } l \geq 0.$$

In particular, φ extended by 0 outside B belongs to $C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Remark 14. (i) The conclusion concerning the regularity of the third extra statement is no longer true in general if $f(x)$ is only assumed to be of the form $c(x)(1-|x|)^{ns-1}$ near the boundary for some $c \in C^\infty(\overline{B})$ with $c < 0$ on ∂B. Indeed, taking $n = 2$, $s = 1$, and $c(x) = -|x|$ near ∂B, a simple calculation gives that $\varphi \notin C^2(\overline{B};\mathbb{R}^n)$.

(ii) In the third extra statement, if f is of the form $f(x) = -(1-|x|)^m/|x|^{n-1}$ near ∂B, then $\varphi(x) = (n/(k+1))^{1/n}(1-|x|)^{(m+1)/n}x/|x|$ near ∂B. This shows in particular that, if we want the solution to be smooth near the boundary, m has to be of the form $ns-1$ with $s \geq 1$ an integer.

Proof. Step 1. Using the implicit function theorem, the fact that $f \equiv 1$ in a neighborhood of 0, and the previous lemma, we easily obtain that $\varphi \in C^k(B;\mathbb{R}^n) \cap C^{0,1/n}(\overline{B};\mathbb{R}^n)$ and $\varphi = 0$ on ∂B. Finally (cf. Step 2.2 in the proof of Lemma 11.10 in [1]) we get $\varphi^*(1) = f$ in B, which proves the main assertion.

Step 2. Let us show the first extra assertion. Since $f < 0$ on ∂B, there exists $\epsilon > 0$ small enough such that

$$-1/\epsilon < f < -\epsilon \quad \text{in } \overline{B} \setminus B_{1-\epsilon}. \tag{6.1}$$

Step 2.1. We start by showing that $\varphi \in \cap_{1 \leq p < n/(n-1)} W^{1,p}(B;\mathbb{R}^n)$. Since $\varphi \in C^k(B;\mathbb{R}^n)$, it is enough to prove that $\varphi \in W^{1,p}(B \setminus B_{1-\epsilon};\mathbb{R}^n)$ for every $1 \leq p < n/(n-1)$, which is equivalent to proving that $\alpha \in W^{1,p}(B \setminus B_{1-\epsilon})$ for every $1 \leq p < n/(n-1)$. Note that

$$\alpha(x) = \Big(n\int_0^{|x|} s^{n-1} f(s\frac{x}{|x|})\,ds\Big)^{1/n} = \Big(-n\int_{|x|}^1 s^{n-1} f(s\frac{x}{|x|})\,ds\Big)^{1/n}.$$

Fix $1 \leq i \leq n$. Differentiating the previous equation with respect to x_i, we get that, for every $x \in B$, $x \neq 0$,

$$\alpha_{x_i}(x) = \frac{n|x|^{n-1}f(x)\frac{x_i}{|x|} - n\sum_{j=1}^n \int_{|x|}^1 s^n f_{x_j}(s\frac{x}{|x|})\Big(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2}\Big)\,ds}{\Big(-n\int_{|x|}^1 s^{n-1} f(s\frac{x}{|x|})\,ds\Big)^{(n-1)/n}}, \tag{6.2}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Using (6.1) we get that

$$\begin{aligned}\int_{B\setminus B_{1-\epsilon}} |\alpha_{x_i}(x)|^p\,dx &\leq K\int_{B\setminus B_{1-\epsilon}} \frac{1}{\Big(-n\int_{|x|}^1 s^{n-1}f(s\frac{x}{|x|})\,ds\Big)^{p(n-1)/n}}\,dx \\ &\leq \frac{K}{\epsilon^{p(n-1)/n}}\int_{B\setminus B_{1-\epsilon}} \frac{1}{(1-|x|^n)^{p(n-1)/n}}\,dx \\ &\leq \frac{K}{\epsilon^{p(n-1)/n}}\int_{B\setminus B_{1-\epsilon}} \frac{1}{(1-|x|)^{p(n-1)/n}}\,dx.\end{aligned}$$

Since the previous integral is finite when $1 \leq p < n/(n-1)$, we have proved that $\varphi \in \cap_{1 \leq p < n/(n-1)} W^{1,p}(B;\mathbb{R}^n)$.

Step 2.2. We now show that $\varphi \notin W^{1,n/(n-1)}(B;\mathbb{R}^n)$. We start by choosing $0 < \delta < \epsilon$ small enough and choosing C_i a cone of axis $\mathbb{R}_+ e_i$ having vertex zero and of aperture small enough so that, for every $x \in C_i \cap (B \setminus B_{1-\delta})$,

$$\Big| n|x|^{n-1} f(x)\frac{x_i}{|x|} - n\sum_{j=1}^{n}\int_{|x|}^{1} s^n f_{x_j}(s\frac{x}{|x|})\Big(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2}\Big)\,ds\Big| \geq m > 0.$$

Combining the previous equation with (6.1) we obtain

$$\begin{aligned}&\int_{C_i\cap(B\setminus B_{1-\delta})} |\alpha_{x_i}(x)|^{n/(n-1)}\,dx\\ \geq&(m)^{n/(n-1)}\int_{C_i\cap(B\setminus B_{1-\delta})}\frac{1}{\left(-n\int_{|x|}^{1} s^{n-1} f(s\frac{x}{|x|})\,ds\right)}\,dx\\ \geq&(m)^{n/(n-1)}\epsilon\int_{C_i\cap(B\setminus B_{1-\delta})}\frac{1}{1-|x|^n}\,dx = \infty.\end{aligned}$$

Thus, $\varphi \notin W^{1,n/(n-1)}(B;\mathbb{R}^n)$.

Step 2.3. We finally prove that $\varphi \notin C^{0,\beta}(\overline{B};\mathbb{R}^n)$ for every $1/n < \beta \leq 1$. Using (6.1), we deduce that, for $x \in \overline{B} \setminus B_{1-\epsilon}$,

$$\alpha(x) \geq \Big(\int_{|x|}^{1} s^{n-1}\epsilon\,ds\Big)^{1/n} \geq (\epsilon(1-|x|)/n)^{1/n}.$$

Hence, for $|x| = 1$, we deduce that, for $1-\epsilon < \lambda < 1$,

$$\frac{|\alpha(x)-\alpha(\lambda x)|}{|x-\lambda x|^{\beta}} = \frac{|\alpha(\lambda x)|}{|1-\lambda|^{\beta}} \geq \frac{(\epsilon(1-\lambda)/n)^{1/n}}{|1-\lambda|^{\beta}}.$$

Letting λ go to 1, we deduce that $\alpha \notin C^{0,\beta}(\overline{B})$ for $1/n < \beta \leq 1$, which proves the claim.

Step 3. We prove the second extra assertion. Using (6.2), an elementary calculation gives that, for every $\overline{x} \in \partial B$,

$$\lim_{x\to\overline{x}, x\in B} \alpha_{x_i}(x) = \begin{cases} -n\overline{x}_i(-c(\overline{x}))^{1/n} & \text{if } m = n-1\\ 0 & \text{if } m > n-1.\end{cases}$$

This shows that $\varphi \in C^1(\overline{B};\mathbb{R}^n)$.

Step 4. The third and fourth extra statement are elementary since, near the boundary, a direct calculation gives $\varphi(x) = (1-|x|)^s x/|x|$ in the first case and $\varphi(x) = e^{-1/(1-|x|)}x/|x|$ in the second case. □

7. Necessary conditions

In this last section we give several necessary conditions to solve our problem, namely

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

We start with a lemma asserting that the total mass of f always has to be zero in order to solve the previous problem.

Lemma 15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that* $\operatorname{meas}(\partial\Omega) = 0$, $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\overline{\Omega})$, *and* $\varphi \in C^1(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}; \mathbb{R}^n)$ *be such that* $|g| > 0$ *in* $\mathbb{R}^n$ *and*

$$\begin{cases} \varphi^*(g) = f & \textit{in } \Omega \\ \varphi = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then

$$\int_\Omega f = 0.$$

Remark 16. If we moreover assume $\varphi \in W^{1,n}(\Omega; \mathbb{R}^n)$, then the previous result can be proved using smooth approximations of φ and the divergence theorem.

Proof. We extend φ by 0 outside of Ω. Note that this extension (still written φ) satisfies $\varphi \in C^0(\mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \partial\Omega; \mathbb{R}^n)$. Since $\operatorname{meas}(\partial\Omega) = 0$, φ is differentiable almost everywhere. Note that $\det \nabla\varphi \in L^1_{loc}(\mathbb{R}^n)$. Since φ is locally Lipschitz in $\mathbb{R}^n \setminus \partial\Omega$ and $\varphi = 0$ on $\partial\Omega$, we deduce that

$$\operatorname{meas}(\varphi(A)) = 0, \quad \text{for every } A \subset \mathbb{R}^n \text{ such that } \operatorname{meas}(A) = 0.$$

Hence, we can apply Theorem 5.27 of [4] (with $D = \mathbb{R}^n$, $G = \Omega$, and $v = g$) and get

$$\int_\Omega g(\varphi) \det \nabla\varphi = \int_{\mathbb{R}^n} g(y) \deg(\varphi, \Omega, y)\, dy,$$

where deg is the usual topological degree (see for example [4]). Since $\varphi = 0$ on $\partial\Omega$, we get, by a well-known property of the degree, that $\deg(\varphi, \Omega, y) = \deg(0, \Omega, y) = 0$, for every $y \neq 0$. Hence

$$\int_\Omega f = \int_\Omega g(\varphi) \det \nabla\varphi = 0,$$

which concludes the proof. □

The next proposition gives conditions on f if we want the solution to be regular up to the boundary.

Proposition 17. *Let $n \geq 2$ be an integer.*

(i) Let Ω be a bounded open C^1 set and $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ be such that $\varphi = 0$ on $\partial\Omega$. Then

$$\det \nabla\varphi = 0 \quad \text{on } \partial\Omega.$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded open C^2 set and $\varphi \in C^2(\overline{\Omega};\mathbb{R}^n)$ be such that $\varphi = 0$ on $\partial\Omega$. Then, near $\partial\Omega$,

$$\det \nabla\varphi(x) = \operatorname{dist}(x,\partial\Omega)^{n-1} h(x)$$

with $h \in C^1(\Omega) \cap C^0(\overline{\Omega})$.

Proof. Part 1. We prove the first assertion. Since Ω is C^1 there exist $U \subset V$ a neighborhood of $\overline{x}$ and $\phi \in \operatorname{Diff}^1(U;\phi(U))$ such that

$$\phi(U\cap\Omega) \subset \{x \in \mathbb{R}^n : x_n > 0\}, \quad \phi(U\cap\partial\Omega) \subset \{x \in \mathbb{R}^n : x_n = 0\}, \quad \phi(\overline{x}) = 0.$$

In particular, there exists $\epsilon > 0$ small enough so that $\varphi\circ\phi^{-1} \in C^1(\overline{B}_\epsilon \cap \{y_n \geq 0\};\mathbb{R}^n)$. Note also that $\varphi \circ \phi^{-1} = 0$ on $\overline{B}_\epsilon \cap \{y_n = 0\}$. Since $\varphi \circ \phi^{-1}$ is C^1 and vanishes on $y_n = 0$ we immediately deduce that

$$\det \nabla(\varphi \circ \phi^{-1}) = 0 \quad \text{on } \overline{B}_\epsilon \cap \{y_n = 0\}.$$

This directly implies that $\det \nabla\varphi(\overline{x}) = 0$, and hence the first assertion is proved.

Part 2. We prove the second assertion.

Step 1. Let $\overline{x} \in \partial\Omega$. By hypothesis there exist U a neighborhood of $\overline{x}$ and $\phi \in \operatorname{Diff}^2(U;\phi(U))$ such that

$$\phi(U\cap\Omega) \subset \{x \in \mathbb{R}^n : x_n > 0\}, \quad \phi(U\cap\partial\Omega) \subset \{x \in \mathbb{R}^n : x_n = 0\}, \quad \phi(\overline{x}) = 0.$$

In particular, there exists $\epsilon > 0$ small enough so that $\varphi\circ\phi^{-1} \in C^2(\overline{B}_\epsilon \cap \{y_n \geq 0\};\mathbb{R}^n)$. Note also that $\varphi \circ \phi^{-1} = 0$ on $\overline{B}_\epsilon \cap \{y_n = 0\}$. We claim that, for every $y \in \overline{B}_\epsilon \cap \{y_n \geq 0\}$, we have

$$\det \nabla(\varphi \circ \phi^{-1})(y) = (y_n)^{n-1} w(y),$$

where $w \in C^1(\overline{B}_\epsilon \cap \{y_n > 0\}) \cap C^0(\overline{B}_\epsilon \cap \{y_n \geq 0\})$. Since $\varphi \circ \phi^{-1} = 0$ on $y_n = 0$ and since $\varphi \circ \phi^{-1} \in C^2$, we deduce that, for every $1 \leq s \leq n$ and every $1 \leq l \leq n-1$,

$$\frac{\partial(\varphi \circ \phi^{-1})^s)}{\partial y_l}(y) = y_n w_{sl}(y),$$

where $w_{sl} \in C^1(\overline{B}_\epsilon \cap \{y_n > 0\}) \cap C^0(\overline{B}_\epsilon \cap \{y_n \geq 0\})$. Recalling that

$$\det \nabla(\varphi \circ \phi^{-1}) = \sum_{\sigma\in\operatorname{Sym}(n)} \operatorname{sign}(\sigma) \prod_{i=1}^{n} \frac{\partial(\varphi \circ \phi^{-1})^i)}{\partial y_{\sigma(i)}},$$

we immediately get the assertion.

Step 2. Using Lemma 18 (with $\Omega_1 = \Omega$ and $\Omega_2 = \{x_n > 0\}$), we immediately get the existence of $g \in C^1(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\phi^n(x) = g(x)\,\mathrm{dist}(x, \partial\Omega), \quad \text{for } x \in \overline{\Omega} \text{ near } \partial\Omega.$$

Step 3 (conclusion). Let $x \in \overline{\Omega}$ near $\partial\Omega$. Using Steps 1 and 2 we obtain

$$\begin{aligned}\det \nabla\varphi(x) &= \det \nabla[(\varphi \circ \phi^{-1}) \circ \phi](x) = \det \nabla(\varphi \circ \phi^{-1})(\phi(x)) \det \nabla\phi(x)\\ &= (\phi^n(x))^{n-1} w(\varphi(x)) \det \nabla\phi(x)\\ &= \mathrm{dist}(x, \partial\Omega)^{n-1} g(x)^{n-1} w(\phi(x)) \det \nabla\phi(x),\end{aligned}$$

whence the theorem, letting $h = g^{n-1} w(\phi) \det \nabla\phi$. □

In the previous proof we have used the following elementary lemma which we do not prove.

Lemma 18. *Let $k \geq 2$ be an integer and $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be two bounded open C^k sets. Let $\overline{x} \in \partial\Omega_1$, V be a neighborhood of $\overline{x}$, and $\phi \in \mathrm{Diff}^k(V; \varphi(V))$ be such that $\phi(\partial\Omega_1 \cap V) \subset \partial\Omega_2$. Then there exist $U \subset V$ a neighborhood of $\overline{x}$ and $g \in C^{k-1}(U \setminus \partial\Omega_1) \cap C^{k-2}(U)$ such that*

$$\mathrm{dist}(\phi(x), \partial\Omega_2) = g(x)\,\mathrm{dist}(x, \partial\Omega_1), \quad \text{for every } x \in U$$

and $g \geq m > 0$ in U.

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