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LOW REGULARITY WELL-POSEDNESS FOR THE PERIODIC KAWAHARA EQUATION

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Abstract. In this paper, we consider the well-posedness for the Cauchy problem of the Kawahara equation with low regularity data in the periodic case. We obtain the local well-posedness for $s \geq -3/2$ by a variant of the Fourier restriction norm method introduced by Bourgain. Moreover, these local solutions can be extended globally in time for $s \geq -1$ by the I-method. On the other hand, we prove ill-posedness for $s < -3/2$ in some sense. This is a sharp contrast to the results in the case of $\mathbb{R}$, where the critical exponent is equal to -2.

1. Introduction

We consider the well-posedness for the Cauchy problem of the Kawahara equation which is one of the fifth-order KdV-type equations:

$$\begin{cases} \partial_t u + \alpha \partial_x^5 u + \beta \partial_x^3 u + \gamma \partial_x (u^2) = 0, \quad (t,x) \in [0,T] \times \mathbb{T}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha, \gamma \neq 0$ and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Here the unknown function u is assumed to be real-valued or complex-valued in the case in which we deal with the local well-posedness (LWP for short) and to be real-valued when we consider the global well-posedness (GWP for short). By the renormalization of u, we may assume $\alpha = -1$, $\gamma = 1$, and $\beta = -1, 0$, or 1. We put $v = u - a$, where a is the integral mean value of initial data defined as $a := \fint_{\mathbb{T}} u_0(x)\,dx$. If u solves (1.1), then v satisfies the following equation:

$$\begin{cases} \partial_t v - \partial_x^5 v + \beta \partial_x^3 v + 2a\partial_x v + \partial_x (v^2) = 0, \quad (t,x) \in [0,T] \times \mathbb{T}, \\ v(0,x) = v_0(x), \quad x \in \mathbb{T}. \end{cases}$$

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Note that the Fourier coefficient $\mathcal{F}_x(v)(0)$ of zero mode vanishes. It suffices to consider the well-posedness for (1.1) under the mean-zero assumption $\int_{\mathbb{T}} u_0(x)\,dx = 0$ because the linear first-order term is harmless. This observation was used by Bourgain [2]. Without the mean-zero assumption, the data-to-solution map fails to be C^2 in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. So this assumption is crucial for some of the analysis that follows. From the above argument, we only consider the case $\dot{\mathbb{Z}} := \mathbb{Z} \setminus \{0\}$. The Kawahara equation models the capillary waves on a shallow layer and the magneto-sound propagation in plasma (see e.g. [17]). This equation has solitary waves with $\beta = 1$ and many conserved quantities. Our aim is to prove the well-posedness for (1.1) with low-regularity data given in the Sobolev space $\dot{H}^s(\mathbb{T})$. Here $\dot{H}^s(\mathbb{T})$ is defined by the norm

$$\|u\|_{\dot{H}^s(\mathbb{T})} := \|\langle k\rangle^s \mathcal{F}_x u\|_{l_k^2(\dot{\mathbb{Z}})}, \quad \text{where } \langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}.$$

We first use the Fourier restriction norm method to prove LWP for (1.1). This method was introduced by Bourgain [2]. Next, we extend local solutions to global-in-time ones by the I-method, which was exploited by Colliander, Keel, Staffilani, Takaoka, and Tao [7], [8]. The Kawahara equation with the periodic boundary condition does not have the Kato smoothing effect unlike the case of $\mathbb{R}$, though a weak version of the Strichartz estimate still holds in the periodic setting. It is possible to make a close investigation into the resonance of nonlinear interactions under the periodic boundary conditions.

The local well-posedness for the periodic KdV equation has been extensively studied. Bourgain [2] proved LWP in $\dot{H}^s$ for $s \geq 0$. Kenig, Ponce, and Vega [18] refined Bourgain's argument to show LWP in $\dot{H}^s$ for $s > -1/2$. Moreover, Colliander, Keel, Staffilani, Takaoka, and Tao [9] obtained LWP in the critical case $s = -1/2$. On the other hand, Christ, Colliander, and Tao [6] showed that the data-to-solution map fails to be uniformly continuous for $-2 < s < -1/2$. We now recall the local well-posedness results for the Kawahara equation. Hirayama [14] proved LWP in $\dot{H}^s$ for $s \geq -1$ in the periodic case, which was an adaptation of the argument to Kenig, Ponce, and Vega [18]. Moreover, he also showed that the Fourier restriction norm method cannot work for $s < -1$ as long as one is using the standard Bourgain space. On the other hand, there are many studies in the case of $\mathbb{R}$. Chen and Guo [4] proved a result for $s \geq -7/4$, using a modified Bourgain space $\bar{F}^s$ introduced in [12]. Following an idea of Bejenaru and Tao [1] and Kishimoto and Tsugawa [19], we improved the previous results to $s \geq -2$ in [15]. This result is optimal in the sense that the data-to-solution map fails to be continuous when $s < -2$. Earlier results can be found in [5], [11], and [21]. The main difficulty in obtaining LWP for the periodic equation is to recover no derivatives by the smoothing effects. So we need to make a more complex modification of the Bourgain space. Then we find a suitable function space reflecting the structure of the nonlinear term to obtain the following theorem.

Theorem 1.1. *Let $s \geq -3/2$. Then* (1.1) *is locally well-posed in $\dot{H}^s(\mathbb{T})$.*

On the other hand, we obtain the ill-posedness result in the following sense.

Theorem 1.2. *Let $s < -3/2$. Then, there is no $T > 0$ such that the flow map, $\dot{H}^s(\mathbb{T}) \ni u_0 \mapsto u(t) \in \dot{H}^s(\mathbb{T})$, can be C^3 for any $t \in (0, T]$.*

These theorems imply that the critical regularity is $s = -3/2$ in the above sense. Moreover, the local solutions obtained in Theorem 1.1 are shown to exist in an arbitrary time by the I-method. Colliander, Keel, Staffilani, Takaoka, and Tao [9] proved GWP for the periodic-case KdV equation when $s > -1/2$, which was improved to $s \geq -1/2$ in [10]. We now describe the global well-posedness results for the Kawahara equation in the non-periodic case. Note that it is difficult to apply the I-method to the Kawahara equation because this equation has fewer symmetries than the KdV equation. Chen and Guo [4] overcame this issue and used an argument similar to that of [9] to show GWP for $s \geq -7/4$. Recently, we have refined their argument and established GWP for $s \geq -38/21$ in [16]. We apply the argument presented for the non-periodic case to the periodic setting so that the following result is established.

Theorem 1.3. *Let $s \geq -1$. Then* (1.1) *is globally well-posed in $\dot{H}^s(\mathbb{T})$.*

Following Theorem 1.4 in [14], we cannot construct the local-in-time solution for $s < -1$ by using the Bourgain space. Hence this result is optimal as long as we use the standard Bourgain space.

We now use the scaling argument. For $\lambda \geq 1$,

$$u_\lambda(t,x) := \lambda^{-4} u(\lambda^{-5} t, \lambda^{-1} x), \quad u_{0,\lambda}(x) := \lambda^{-4} u_0(\lambda^{-1} x).$$

If u solves (1.1), u_λ satisfies the following rescaled Cauchy problem:

$$\begin{cases} \partial_t u_\lambda - \partial_x^5 u_\lambda + \ \lambda^{-2}\beta \partial_x^3 u_\lambda + \partial_x(u_\lambda^2) = 0, \quad (t,x) \in [0, \lambda^5 T] \times \mathbb{T}_\lambda, \\ u_\lambda(0,x) = u_{0,\lambda}(x), \quad x \in \mathbb{T}_\lambda, \end{cases} \tag{1.2}$$

where $\mathbb{T}_\lambda := \mathbb{R}/2\pi\lambda\mathbb{Z}$. $\widehat{\varphi}$ denotes the Fourier transform on $\mathbb{T}_\lambda$ of φ as follows:

$$\widehat{\varphi}(k) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi\lambda} e^{-ikx}\varphi(x)\,dx, \quad k \in \dot{\mathbb{Z}}_\lambda := \frac{1}{\lambda}\dot{\mathbb{Z}}.$$

Here the space $\dot{H}^s(\mathbb{T}_\lambda)$ is equipped with the norm

$$\|\varphi\|_{\dot{H}^s(\mathbb{T}_\lambda)} := \|\langle k\rangle^s \widehat{\varphi}\|_{l^2_k(\dot{\mathbb{Z}}_\lambda)}, \quad \text{where} \;\; \|f\|_{l^p_k(\dot{\mathbb{Z}}_\lambda)} := \Big(\frac{1}{\lambda}\sum_{k\in\dot{\mathbb{Z}}_\lambda} |f(k)|^p\Big)^{1/p},$$

for $1 \le p \le \infty$. A direct calculation shows that

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{T}_\lambda)} \le \lambda^{-7/2-s}\|u_0\|_{\dot{H}^s(\mathbb{T})} \text{ for } s < 0. \tag{1.3}$$

Therefore, we may assume smallness of initial data. So it suffices to solve (1.2) for sufficiently small data. We first summarize the local well-posedness theory. The main idea is how to define the function space to construct solutions. When s is small, especially negative, the Bourgain space plays an important role. The Bourgain space $X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)$ for $2\pi\lambda$-periodic is defined by the norm

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} := \Big\|\langle k\rangle^s\langle \tau - p_\lambda(k)\rangle^b \widehat{u}\Big\|_{l^2_k(\dot{\mathbb{Z}}_\lambda;L^2_\tau(\mathbb{R}))},$$

where $p_\lambda(k) := k^5 + \beta\lambda^{-2}k^3$. Remark that the Bourgain space depends on the linear part of our target equation. One of the key estimates is the bilinear estimate in $X^{s,b}$ as follows:

$$\|\Lambda^{-1}\partial_x(uv)\|_{X^{s,b}} \le C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}, \tag{1.4}$$

where Λ^b is the Fourier multiplier defined as $\Lambda^b := \mathcal{F}^{-1}_{\tau,k}\langle\tau - p_\lambda(k)\rangle^b\mathcal{F}_{t,x}$ for $b \in \mathbb{R}$. From the bilinear estimate and some linear estimates, the standard argument of the Fourier restriction norm method works to obtain LWP. Hirayama [14] showed (1.4) for $s \ge -1$. On the other hand, he proved that this estimate fails for any $b \in \mathbb{R}$ when $s < -1$. So it is difficult to construct the local solutions by the iteration argument when $s < -1$. To avoid this difficulty, we modify the Bourgain space $X^{s,b}$ to control strong nonlinear interactions and establish the bilinear estimate at the critical regularity $s = -3/2$. An idea of a modification of $X^{s,b}$ was developed by Bejenaru and Tao [1]. They considered the quadratic Schrödinger equation with the nonlinearity u^2 and obtained LWP in the critical case $H^{-1}(\mathbb{R})$. Note that the way to modify the Bourgain space is determined by the balance between dispersion and nonlinearity. So there is no general framework for modifying $X^{s,b}$. This is one of the most difficult points in our study. Compared to the non-periodic case, fewer derivatives can be recovered by the smoothing effects in the periodic setting. So nonlinear interactions which can be ignored in the non-periodic case take effect. Therefore we need to make a more complex modification of $X^{s,b}$ to control three types of nonlinear interactions. We now mention how to modify $X^{s,b}$. From the counterexamples of (1.4) in the case $s < -1$, we find the regions in which strong nonlinear interactions appear. In these domains, we make a suitable modification of $X^{s,b}$ as follows:

$$\begin{aligned}\|u\|_{Z^s} &:= \|P_{D_1}u\|_{X^{s,3/4}} + \|P_{D_2}u\|_{X^{-3s-1,s+1}} \\ &\quad + \|P_{D_3}u\|_{X^{-s/2-1,s/2+1}} + \|u\|_{Y^s}, \quad \text{for} \quad -3/2 \le s \le -1,\end{aligned}$$

where P_Ω is the Fourier projection onto a set $\Omega \subset \mathbb{R} \times \dot{\mathbb{Z}}_\lambda$ and

$$\begin{aligned}D_1 &:= \big\{(\tau,k) \in \mathbb{R} \times \dot{\mathbb{Z}}_\lambda : |\tau - p_\lambda(k)| \le |k|^4/10 \text{ and } |k| \ge 1\big\}, \\ D_2 &:= \big\{(\tau,k) \in \mathbb{R} \times \dot{\mathbb{Z}}_\lambda : |k|^4/10 \le |\tau - p_\lambda(k)| \le |k|^5/10 \text{ and } |k| \ge 1\big\}, \\ D_3 &:= \big\{(\tau,k) \in \mathbb{R} \times \dot{\mathbb{Z}}_\lambda : |\tau - p_\lambda(k)| \ge |k|^5/10 \text{ or } \frac{1}{\lambda} \le |k| \le 1\big\}.\end{aligned}$$

Here $\|u\|_{Y^s} := \|\langle k\rangle^s \widehat{u}\|_{l^2_k L^1_\tau(\dot{\mathbb{Z}}_\lambda \times \mathbb{R})}$. From the definition, Y^s is continuously embedded into $C(\mathbb{R}; \dot{H}^s(\mathbb{T}_\lambda))$. Using the function space above, we obtain the following bilinear estimate, which is one of the main estimates in the present paper.

Proposition 1.4. *Let $-3/2 \le s < -1$. Then, the following estimate holds:*

$$\|\Lambda^{-1}\partial_x(uv)\|_{Z^s} \le C\|u\|_{Z^s}\|v\|_{Z^s}, \tag{1.5}$$

where the positive constant C is independent of λ.

Next, we extend the local solution obtained above globally in time. In the case where s is negative, we have no conservation laws. To avoid this difficulty, we apply the I-method exploited by Colliander, Keel, Staffilani, Takaoka, and Tao [7], [8]. The main idea is to use a modified energy defined for less regular functions, which is not conserved. If we control the growth of the modified energy in time, this enables us to iterate the local theory to continue the solution to any time T. We now mention the definition of the modified energy $E_I^{(2)}(u)$. The operator $I : H^s \to L^2$ is the Fourier multiplier defined as $I = \mathcal{F}_\xi^{-1} m(\xi)\mathcal{F}_x$. Here m is a smooth and monotone function satisfying

$$m(\xi) := \begin{cases} 1 & \text{for } |\xi| \le N \\ |\xi|^s N^{-s} & \text{for } |\xi| \ge 2N, \end{cases}$$

for $s < 0$ and $N \gg 1$. The modified energy $E_I^{(2)}(u)$ is defined as $E_I^{(2)}(u)(t) := \|Iu(t)\|_{L^2}^2$. In the I-method, the key estimate is the almost-conservation law, which implies the increment of the modified energy is sufficiently small for a short time interval and large N. Following the argument of [7], we obtain the almost-conservation law and show GWP for $s > -21/26$. However, the growth of the modified energy $E_I^{(2)}(u)$ in time cannot be controlled for $-1 \leq s \leq -21/26$. Then we add some correction terms to the original modified energy $E_I^{(2)}(u)$ to construct a new modified energy in order to remove some oscillations in this functional. This idea was developed by Colliander, Keel, Staffilani, Takaoka, and Tao [9]. They proved GWP of the KdV equation for $s > -3/4$ in the case of $\mathbb{R}$ and for $s > -1/2$ in the periodic case. Chen and Guo [4] establish the sharp upper bound of some multiplier to show GWP of the Kawahara equation for $s \geq -7/4$ in the case of $\mathbb{R}$. Following the argument of [4], we obtain the almost-conservation law for the modified energy $E_I^{(4)}(u)$ by adding two suitable correction terms to the original functional when $s \geq -1$. On the other hand, the difference between the almost-conserved quantities $E_I^{(4)}(u)$ and the first modified energy $E_I^{(2)}(u)$ can be controlled by $E_I^{(2)}(u)$ when the time is fixed. This estimate and the almost-conservation law imply well-posedness on any time interval. Remark that we do not expect to recover any derivatives by the smoothing effects in the periodic setting. So it is hard to control the strong nonlinear interaction of the almost-conservation law and construct the global-in-time solution when $s < -1$.

We use the following notation in this paper. $A \lesssim B$ means $A \leq CB$ for some positive constant C and $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$. $c+$ means $c + \varepsilon$, while $c-$ means $c - \varepsilon$, where $\varepsilon > 0$ is small enough. For a normed space $\mathcal{X}$ and a set Ω, $\|\cdot\|_{\mathcal{X}(\Omega)}$ denotes $\|f\|_{\mathcal{X}(\Omega)} := \|\chi_\Omega f\|_{\mathcal{X}}$, where χ_Ω is the characteristic function of Ω.

The rest of this paper is planned as follows. In Section 2, we give some preliminary lemmas. In Section 3 we prove the bilinear estimate (1.5), and we give the proof of LWP in Section 4. In Section 5, we show GWP by the I-method, following [4] and [9]. In Section 6, we give the proof of Theorem 1.2, which is based on Bourgain's work [3].

2. Preliminaries

In this section, we prepare the bilinear Strichartz estimate to show the main estimates. When we use the variables (τ, k), (τ_1, k_1), and (τ_2, k_2), we always assume the relation $(\tau, k) = (\tau_1, k_1) + (\tau_2, k_2)$. The bilinear estimate (1.5) can be established by the Hölder inequality, the Young inequality, and the following estimate.

Lemma 2.1. *If $b, b' \in \mathbb{R}$ satisfy $b + b' \geq 29/40$ and $b, b' > 9/40$, then we have*

$$\left\| P_{\{|k|\geq 1\}}(uv) \right\|_{L^2_{t,x}} \lesssim \|u\|_{X^{0,b}} \|v\|_{X^{0,b'}}, \tag{2.1}$$

$$\left\| u(P_{\{|k|\geq 1\}}v) \right\|_{X^{0,-b'}} \lesssim \|u\|_{X^{0,b}} \|v\|_{L^2_{t,x}}. \tag{2.2}$$

Proof. For a dyadic number $M \geq 1$, u_M denotes that the support of $\widehat{u}$ is restricted to the dyadic block $\{\langle \tau - p_\lambda(k) \rangle \sim M\}$. We use the triangle inequality and the Plancherel theorem to have

$$\begin{aligned} \left\| P_{\{|k|\geq 1\}}(uv) \right\|_{L^2_{t,x}} &\lesssim \sum_{M_1, M_2 \geq 1} \left\| P_{\{|k|\geq 1\}}(u_{M_1} v_{M_2}) \right\|_{L^2_{t,x}} \\ &\sim \sum_{M_1, M_2 \geq 1} \left\| \frac{1}{\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \widehat{u}_{M_1}(\tau_1, k_1) \widehat{v}_{M_2}(\tau_2, k_2) d\tau_1 \right\|_{l^2_k L^2_\tau(|k|\geq 1)}. \end{aligned}$$

Using the Schwarz inequality twice, the above is bounded by

$$\sum_{M_1, M_2 \geq 1} \sup_{(\tau,k)\in \mathbb{R}\times\mathbb{Z}_\lambda} \Big(\frac{1}{\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \chi_E(\tau, k, \tau_1, k_1) d\tau_1 \Big)^{\frac{1}{2}} \|u_{M_1}\|_{L^2_{t,x}} \|v_{M_2}\|_{L^2_{t,x}}, \tag{2.3}$$

where $E := \big\{ (\tau, k, \tau_1, k_1) \in (\mathbb{R} \times \mathbb{Z}_\lambda)^2 : |\tau_1 - p_\lambda(k_1)| \sim M_1,\ |\tau_2 - p_\lambda(k_2)| \sim M_2,\ |k| \geq 1 \big\}$. We now show the following estimate when $M_1 \geq M_2$:

$$\sup_{(\tau,k)\in \mathbb{R}\times\mathbb{Z}_\lambda} \frac{1}{\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \chi_E(\tau, k, \tau_1, k_1)\, d\tau_1 \lesssim M_1^{9/20} M_2. \tag{2.4}$$

The identity

$$\begin{aligned} \big(\tau - \frac{k^5}{16} - \beta\lambda^{-2}\frac{k^3}{4}\big) - (\tau_1 - p_\lambda(k_1)) - (\tau_2 - p_\lambda(k_2)) \\ = \frac{5}{16} k (k_1 - k_2)^2 \big\{ (k_1 - k_2)^2 + 2k^2 + \frac{12}{5}\beta\lambda^{-2} \big\} \end{aligned}$$

implies

$$(k_1 - k_2)^2 = \Big\{ \frac{L_0 + O(\max\{M_1, M_2\})}{|k|} + (k^2 + \frac{6}{5}\beta\lambda^{-2})^2 \Big\}^{1/2} - (k^2 + \frac{6}{5}\beta\lambda^{-2}),$$

where $L_0 := \frac{16}{5}\big|\tau - \frac{k^5}{16} - \beta\lambda^{-2}\frac{k^3}{4}\big|$. Now (τ, k) is fixed. Then the variation of k_1 is bounded by

$$\lambda\frac{\max\{M_1,M_2\}}{|k|}\Big\{\frac{L_0+O(\max\{M_1,M_2\})}{|k|}+(k^2+\frac{6}{5}\beta\lambda^{-2})^2\Big\}^{-1/2} \tag{2.5}$$
$$\times\Big[\Big\{\frac{L_0+O(\max\{M_1,M_2\})}{|k|}+(k^2+\frac{6}{5}\beta\lambda^{-2})^2\Big\}^{1/2}-(k^2+\frac{6}{5}\beta\lambda^{-2})\Big]^{-1/2}.$$

Note that for $|k| \geq 1$

$$\Big[\Big\{\frac{L_0+O(\max\{M_1,M_2\})}{|k|}+(k^2+\frac{6}{5}\beta\lambda^{-2})^2\Big\}^{1/2}-(k^2+\frac{6}{5}\beta\lambda^{-2})\Big]^{-1/2}$$
$$\gtrsim |k|^{-3/2}O(\max\{M_1,M_2\})^{1/4}. \tag{2.6}$$

We apply (2.6) and the Young inequality to (2.5) so that the variation of k_1 is at most $\lambda|k|^{3/2-5/2p}\max\{M_1,M_2\}^{3/4-1/2p}$ for $1 < p < \infty$. The above is equal to $\lambda\max\{M_1,M_2\}^{9/20}$ with $p = 5/3$. If we also fix k_1; τ_1 is restricted to the interval of measure $O(\max\{M_1,M_2\})$. Therefore we obtain (2.4) when $M_1 \geq M_2$. Substituting (2.4) into (2.3), we have

$$\begin{aligned}\|P_{\{|k|\geq 1\}}uv\|_{L^2_{t,x}} &\lesssim \sum_{M_1,M_2\geq 1} M_1^{9/40}M_2^{1/2}\|u_{M_1}\|_{L^2_{t,x}}\|v_{M_2}\|_{L^2_{t,x}}\\ &= \sum_{N,M_2\geq 1} M_2^{29/40}N^{9/40}\|u_{NM_2}\|_{L^2_{t,x}}\|v_{M_2}\|_{L^2_{t,x}}\\ &\lesssim \sum_{N\geq 1}\sum_{M_2\geq 1} N^{9/40-b}M_2^{29/40-(b+b')}(NM_2)^b\|u_{NM_2}\|_{L^2_{t,x}}M_2^{b'}\|v_{M_2}\|_{L^2_{t,x}}.\end{aligned}$$

Applying the Schwarz inequality in M_2 and summing over N, we obtain the desired estimate.

On the other hand, we immediately obtain (2.2) from the duality argument. □

We put a one-parameter semigroup $U_\lambda(t)$ as follows:

$$U_\lambda(t) := \mathcal{F}_k^{-1}\exp(ip_\lambda(k)t)\mathcal{F}_x.$$

For any time interval I, we define the restricted space $Z^s(I)$ by the norm

$$\|u\|_{Z^s(I)} := \inf\big\{\|v\|_{Z^s} : u(t) = v(t) \text{ on } t \in I\big\}.$$

From the definition, $Z^s([0,T])$ has the following property:

$$X^{s,3/4}([0,T]) \hookrightarrow Z^s([0,T]) \hookrightarrow C([0,T];\dot{H}^s(\mathbb{T}_\lambda)).$$

The above property implies the following linear estimates.

Proposition 2.2. *Let $s \in \mathbb{R}$, $T > 0$, and $\lambda \geq 1$. Then, we have*

$$\|U_\lambda(t)u_0\|_{Z^s([0,T])} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{T})}.$$

Proposition 2.3. *Let $s \in \mathbb{R}$, $T > 0$, and $\lambda \geq 1$. If the bilinear estimate* (1.5) *holds, then we have*

$$\Big\|\int_0^t U_\lambda(t-t')F(t')\,dt'\Big\|_{Z^s([0,T])} \lesssim \|u\|_{Z^s([0,T])}\|v\|_{Z^s([0,T])}.$$

For the proofs of these propositions, see [1].

3. Proof of the bilinear estimate

In this section, we give a proof of the bilinear estimate (1.5). For simplicity, we introduce the Fourier multiplier $J^\sigma := \mathcal{F}_k^{-1}\langle k\rangle^\sigma \mathcal{F}_x$ for $\sigma \in \mathbb{R}$. Proposition 1.4 can be established by Hölder's and Young's inequalities and Lemma 2.1.

Proof of Proposition 1.4. We prove the following two estimates to obtain (1.5):

$$\|\Lambda^{-1}\partial_x(uv)\|_{X_w^s} \lesssim \|u\|_{Z^s}\|v\|_{Z^s}, \tag{3.1}$$

$$\|\Lambda^{-1}\partial_x(uv)\|_{Y^s} \lesssim \|u\|_{Z^s}\|v\|_{Z^s}, \tag{3.2}$$

where $\|\cdot\|_{X_w^s}$ is the norm removing $\|\cdot\|_{Y^s}$ from $\|\cdot\|_{Z^s}$. Firstly, we divide $(\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2$ into six parts as follows:

$$\Omega_0 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 : |k|,|k_1| \lesssim 1\big\},$$
$$\Omega_1 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 \setminus \Omega_0 : |k_1| \sim |k-k_1| \gg |k| \geq 1\big\},$$
$$\Omega_2 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 \setminus \Omega_0 : |k_1| \sim |k-k_1| \gg |k| \text{ and } 1 \geq |k| \geq 1/\lambda\big\},$$
$$\Omega_3 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 \setminus \Omega_0 : |k| \sim |k-k_1| \gg |k_1| \geq 1\big\},$$
$$\Omega_4 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 \setminus \Omega_0 : |k| \sim |k-k_1| \gg |k_1| \text{ and } 1 \geq |k_1| \geq 1/\lambda\big\},$$
$$\Omega_5 := \big\{(\tau,k,\tau_1,k_1) \in (\mathbb{R}\times\dot{\mathbb{Z}}_\lambda)^2 \setminus \Omega_0 : |k| \sim |k_1| \sim |k-k_1| \geq 1\big\}.$$

We note that the cases Ω_3 and Ω_4 are also true with k_1 and $k-k_1$ exchanged because of symmetry. Recall that Z^s has the following properties:

$$\|u\|_{X^{s,1/4}} \lesssim \|u\|_{Z^s} \lesssim \|u\|_{X^{s,3/4}} \text{ and } \|u\|_{X^{s,1/2}(D_1\cup D_2)} \lesssim \|u\|_{Z^s(D_1\cup D_2)}.$$

Estimate in Ω_0. From the property of Z^s, we only estimate the norm $X^{s,3/4}$ of $\Lambda^{-1}\partial_x(uv)$. From $|k|,|k_1|,|k-k_1| \lesssim 1$, we use the Hölder inequality and the Young inequality to have

$$\||k|\langle\tau - p_\lambda(k)\rangle^{-1/4}\widehat{u} * \widehat{v}\|_{l^2_k L^2_\tau} \lesssim \||k|\|_{l^2_k}\|\widehat{u} * \widehat{v}\|_{l^\infty_k L^2_\tau} \lesssim \|\widehat{u}\|_{l^2_k L^2_\tau}\|\widehat{v}\|_{l^2_k L^1_\tau},$$

which is an appropriate bound.

Here we put $L_{\max} = \max\{|\tau - p_\lambda(k)|, |\tau_1 - p_\lambda(k_1)|, |(\tau-\tau_1) - p_\lambda(k-k_1)|\}$. In the remainder case, we often use the algebraic relation as follows:

$$\begin{aligned} L_{\max} \geq & \frac{1}{3}\Big|(\tau - p_\lambda(k)) - (\tau_1 - p_\lambda(k_1)) - \big\{(\tau-\tau_1) - p_\lambda(k-k_1)\big\}\Big| \\ \geq & \frac{5}{6}\Big|kk_1(k-k_1)\big\{k^2 + k_1^2 + (k-k_1)^2 + \frac{6}{5}\beta\lambda^{-2}\big\}\Big|. \end{aligned} \tag{3.3}$$

(I) We prove the estimate for Ω_1. We first decompose Ω_1 into three parts as follows:

$$\begin{aligned} \Omega_{11} &:= \big\{(\tau, k, \tau_1, k_1) \in \Omega_1 : L_{\max} = |\tau - p_\lambda(k)|\big\}, \\ \Omega_{12} &:= \big\{(\tau, k, \tau_1, k_1) \in \Omega_1 : L_{\max} = |\tau_1 - p_\lambda(k_1)|\big\}, \\ \Omega_{13} &:= \big\{(\tau, k, \tau_1, k_1) \in \Omega_1 : L_{\max} = |(\tau-\tau_1) - p_\lambda(k-k_1)|\big\}. \end{aligned}$$

The case Ω_{13} is identical to the case Ω_{12}, so we omit this case. Note that $L_{\max} \gtrsim |kk_1^4|$ in Ω_1 from (3.3).

(Ia) In Ω_{11}, $\widehat{u}*\widehat{v}$ is supported on D_3 from the definition. We use Lemma 2.1 with $b' = 1/4$ and $b = 1/2$ to obtain

$$\|\langle k\rangle^{-s/2}\langle\tau - p_\lambda(k)\rangle^{s/2}\widehat{u} * \widehat{v}\|_{l^2_k L^2_\tau} \lesssim \|J^s u J^s v\|_{L^2_{t,x}} \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s,1/4}},$$

which implies the desired estimate except for the case where $\widehat{u}$ and $\widehat{v}$ are restricted to D_3. Next we consider the case that both $\widehat{u}$ and $\widehat{v}$ are supported on D_3. We use Hölder's inequality and Young's inequality to have

$$\begin{aligned} &\|\langle k\rangle^{-\frac{s}{2}}\langle\tau - p_\lambda(k)\rangle^{\frac{s}{2}}\widehat{u} * \widehat{v}\|_{l^2_k L^2_\tau} \lesssim \|\widehat{u} * (\langle k\rangle^{2s}\widehat{v})\|_{l^2_k L^2_\tau} \\ &\lesssim \|(\langle k\rangle^{-\frac{s}{2}-1}\langle\tau - p_\lambda(k)\rangle^{\frac{s}{2}+1}\widehat{u}) * (\langle k\rangle^{-4}\widehat{v})\|_{l^2_k L^2_\tau} \lesssim \|u\|_{X^{-\frac{s}{2}-1,\frac{s}{2}+1}}\|\langle k\rangle^{-4}\widehat{v}\|_{l^1_k L^1_\tau}, \end{aligned}$$

which shows the required estimate since $\|\langle k\rangle^{-4}\widehat{v}\|_{l^1_k L^1_\tau} \lesssim \|v\|_{Y^s}$ by the Schwarz inequality. Moreover, we estimate the norm Y^s of $\Lambda^{-1}\partial_x(uv)$. Following $|\tau - p_\lambda(k)| \gtrsim |kk_1^4|$, we use the Hölder inequality and the Young inequality to obtain

$$\|\langle k\rangle^{s+1}\langle\tau - p_\lambda(k)\rangle^{-1}\widehat{u} * \widehat{v}\|_{l^2_k L^1_\tau} \lesssim \|(\langle k\rangle^{-2}\widehat{u}) * (\langle k\rangle^{-2}\widehat{v})\|_{l^\infty_k L^1_\tau} \lesssim \|u\|_{Y^s}\|v\|_{Y^s}.$$

(Ib) We show the estimate for Ω_{12}. Consider three subregions

$$\begin{aligned} \Omega_{12a} &:= \big\{(\tau, k, \tau_1, k_1) \in \Omega_{12} : |\tau_1 - p_\lambda(k_1)| \sim |kk_1^4|\big\}, \\ \Omega_{12b} &:= \big\{(\tau, k, \tau_1, k_1) \in \Omega_{12} : |k_1|^5 \gtrsim |\tau_1 - p_\lambda(k_1)| \gg |kk_1^4|\big\}, \end{aligned}$$

$$\Omega_{12c} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_{12} : |\tau_1 - p_\lambda(k_1)| \gtrsim |k_1|^5\big\}.$$

In Ω_{12a}, $\widehat{u}$ is restricted to D_2. Then we use Lemma 2.1 with $b'=1/4$ and $b=1/2$ to obtain

$$\begin{aligned}\|\langle k\rangle^s\langle\tau-p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2} &\lesssim \|(J^{-3s-1}\Lambda^{s+1}u)(J^{-s-3}v)\|_{X^{0,-1/4}}\\ &\lesssim \|u\|_{X^{-3s-1,s+1}}\|v\|_{X^{s,1/2}}.\end{aligned}$$

In Ω_{12b} and Ω_{12c}, either $|\tau_1-p_\lambda(k_1)|\sim|\tau-p_\lambda(k)|$ or $|\tau_1-p_\lambda(k_1)|\sim|(\tau-\tau_1)-p_\lambda(k-k_1)|$ happens. The former case is almost identical to the case (Ia). So we consider only the latter case.

We prove the estimate for Ω_{12b}. In this case, we may assume that both $\widehat{u}$ and $\widehat{v}$ are supported on D_2 and $|\tau-p_\lambda(k)|\lesssim|k_1|^5$. From $\langle k_1\rangle^{3s+1}\langle\tau_1-p_\lambda(k_1)\rangle^{-s-1}\lesssim\langle k_1\rangle^{-2s-4}$, we use the Hölder inequality and the Young inequality to have

$$\begin{aligned}&\|\langle k\rangle^{s+1}\langle\tau-p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim\|\langle k\rangle^{s+3/2}\langle\tau-p_\lambda(k)\rangle^{1/4}\widehat{u}*\widehat{v}\|_{l_k^\infty L_\tau^\infty}\\ &\quad\lesssim\|(\langle k\rangle^{s+11/4}\widehat{u})*\widehat{v}\|_{l_k^\infty L_\tau^\infty}\lesssim\|\langle k\rangle^{-3s-21/4}\|_{l_k^\infty}\|u\|_{X^{-3s-1,s+1}}\|v\|_{X^{-3s-1,s+1}},\end{aligned}$$

which is an appropriate bound. From an argument similar to that above, we obtain the desired estimate for Ω_{12c}.

Estimate for Ω_2.

(II) We divide Ω_2 into three parts as follows:

$$\begin{aligned}\Omega_{21}&:=\big\{(\tau,k,\tau_1,k_1)\in\Omega_2 : L_{\max}=|\tau-p_\lambda(k)|\big\},\\ \Omega_{22}&:=\big\{(\tau,k,\tau_1,k_1)\in\Omega_2 : L_{\max}=|\tau_1-p_\lambda(k_1)|\big\},\\ \Omega_{23}&:=\big\{(\tau,k,\tau_1,k_1)\in\Omega_2 : L_{\max}=|\tau_2-p_\lambda(k_2)|\big\}.\end{aligned}$$

We omit the estimate for Ω_{23} because this case is identical to Ω_{22}. Note that $\widehat{u}*\widehat{v}$ is supported on D_3 in Ω_2. When $|k|\lesssim|k_1|^{-4}$, Hölder's and Young's inequalities show

$$\||k|\langle\tau-p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim\|(\langle k\rangle^{-3}\widehat{u})*(\langle k\rangle^{-3}\widehat{v})\|_{l_k^\infty L_\tau^2}\lesssim\|u\|_{X^{-3,0}}\|v\|_{Y^{-3}},$$

which implies the desired estimate. So we deal only with the case $|k_1|^{-4}\lesssim|k|\le1$.

(IIa) We prove the estimate for Ω_{21}. We use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned}&\||k|\langle\tau-p_\lambda(k)\rangle^{s/2}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim\||k|^{1+s/2}(\langle k\rangle^s\widehat{u})*(\langle k\rangle^s\widehat{v})\|_{l_k^2L_\tau^2}\\ &\quad\lesssim\|(\langle k\rangle^s\widehat{u})*(\langle k\rangle^s\widehat{v})\|_{l_k^\infty L_\tau^2}\lesssim\|u\|_{X^{s,0}}\|v\|_{Y^s}.\end{aligned}$$

Next we estimate the Y^s norm of $\Lambda^{-1}\partial_x(uv)$. Combining Hölder's and Young's inequalities, we have

$$\begin{aligned}\||k|\langle\tau-p_\lambda(k)\rangle^{-1}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^1} \lesssim&\|(\langle k\rangle^{-2}\widehat{u})*(\langle k\rangle^{-2}\widehat{v})\|_{l_k^\infty L_\tau^1}\\ \lesssim&\|\langle k\rangle^{-2}\widehat{u}\|_{l_k^2L_\tau^1}\|\langle k\rangle^{-2}\widehat{v}\|_{l_k^2L_\tau^1},\end{aligned}$$

which is an appropriate bound.

(IIb) We consider the estimate for Ω_{22}. Following $\|u\|_{Y^s}\lesssim\|u\|_{X^{s,1/2+}}$, it suffices to show

$$\||k|\langle\tau-p_\lambda(k)\rangle^{-1/2+}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim\|u\|_{Z^s}\|v\|_{Z^s} \tag{3.4}$$

in Ω_{22}. We consider three subregions as follows:

$$\begin{aligned}\Omega_{22a}&:=\{(\tau,k,\tau_1,k_1)\in\Omega_{22}:|kk_1|^4\lesssim|\tau_1-p_\lambda(k_1)|\lesssim|k_1|^4\},\\ \Omega_{22b}&:=\{(\tau,k,\tau_1,k_1)\in\Omega_{22}:|k_1^4|\lesssim|\tau_1-p_\lambda(k_1)|\lesssim|k_1|^5\},\\ \Omega_{22c}&:=\{(\tau,k,\tau_1,k_1)\in\Omega_{22}:|k_1|^5\lesssim|\tau_1-p_\lambda(k_1)|\}.\end{aligned}$$

In Ω_{22a}, $\widehat{u}$ is restricted to D_1. We use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned}&\||k|\langle\tau-p_\lambda(k)\rangle^{-\frac{1}{2}+}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim\||k|^{\frac{1}{4}}(\langle k\rangle^s\langle\tau-p_\lambda(k)\rangle^{\frac{3}{4}}\widehat{u})*(\langle k\rangle^{-s-3}\widehat{v})\|_{l_k^2L_\tau^2}\\ &\lesssim\|(\langle k\rangle^s\langle\tau-p_\lambda(k)\rangle^{3/4}\widehat{u})*(\langle k\rangle^s\widehat{v})\|_{l_k^\infty L_\tau^2}\lesssim\|u\|_{X^{s,3/4}}\|v\|_{Y^s}.\end{aligned}$$

We consider the estimate for Ω_{22b} and Ω_{22c}. From the estimate for Ω_{12}, we immediately obtain (3.4) in the case $|\tau-p_\lambda(k)|\sim|\tau_1-p_\lambda(k_1)|$. So we deal only with the case $|\tau_1-p_\lambda(k_1)|\sim|(\tau-\tau_1)-p_\lambda(k-k_1)|$. In Ω_{22b}, both $\widehat{u}$ and $\widehat{v}$ are restricted to D_2. We use Lemma 2.1 with $b'=1/2-$ and $b=s/2+1$ to have

$$\begin{aligned}\||k|\langle\tau-p_\lambda(k)\rangle^{-1/2+}\widehat{u}*\widehat{v}\|_{l_k^2L_\tau^2}\lesssim&\|(J^{-3s-1}\Lambda^{s+1}u)(J^{-2s-4}v)\|_{X^{0,-1/2+}}\\ \lesssim&\|u\|_{X^{-3s-1,s+1}}\|v\|_{X^{-2s-4,s/2+1}},\end{aligned}$$

which shows the desired estimate since $\|v\|_{X^{-2s-4,s/2+1}}\lesssim\|v\|_{X^{-3s-1,s+1}}$ in D_2 for $s\geq-3/2$.

The case Ω_{22c} is almost identical to the above case.

Estimate for Ω_3.

(III) From the algebraic relation (3.3), $L_{\max}\gtrsim|k_1k^4|$. We decompose Ω_3 into three parts as follows:

$$\begin{aligned}\Omega_{31}&:=\{(\tau,k,\tau_1,k_1)\in\Omega_3:L_{\max}=|\tau_1-p_\lambda(k_1)|\},\\ \Omega_{32}&:=\{(\tau,k,\tau_1,k_1)\in\Omega_3:L_{\max}=|\tau-p_\lambda(k)|\},\end{aligned}$$

$$\Omega_{33} := \big\{(\tau,k,\tau_1,k_1) \in \Omega_3 : L_{\max} = |\tau_2 - p_\lambda(k_2)|\big\}.$$

(IIIa) Firstly, we consider the case in which $\widehat{u}*\widehat{v}$ is supported on D_3. In this case, either $|\tau - p_\lambda(k)| \sim |\tau_1 - p_\lambda(k_1)| \gtrsim |k|^5$ or $|\tau - p_\lambda(k)| \sim |\tau_2 - p_\lambda(k_2)| \gtrsim |k|^5$ holds. In the former case, $\widehat{u}$ is supported on D_3. We use the Young inequality to obtain

$$\begin{aligned}\|\langle k\rangle^{-s/2}\langle \tau - p_\lambda(k)\rangle^{s/2}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} &\lesssim \|(J^{-s/2-1}\Lambda^{s/2+1}u)(J^{-4}v)\|_{L^2_{t,x}} \\ &\lesssim \|u\|_{X^{-s/2-1,s/2+1}}\|\langle k\rangle^{-4}v\|_{l^1_k L^1_\tau},\end{aligned}$$

which is bounded by $\|u\|_{X^{-s/2-1,s/2+1}}\|v\|_{Y^s}$ from the Schwarz inequality. The latter case is almost identical to the above case.

Secondly, we deal with the case in which $\widehat{v}$ is supported on D_3. From (3.3), $|\tau_2 - p_\lambda(k_2)| \sim |\tau - p_\lambda(k)| \gtrsim |k|^5$ or $|\tau_2 - p_\lambda(k_2)| \sim |\tau_1 - p_\lambda(k_1)| \gtrsim |k|^5$ holds. In the former case, we have already proven (1.5). So we consider the latter case. We may assume that $\widehat{u}$ is restricted to D_3 and $|\tau - p_\lambda(k)| \lesssim |k|^5$. We use the Hölder inequality and the Young inequality to have

$$\begin{aligned}&\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} \lesssim \|\langle k\rangle^{s+3/2}\langle \tau - p_\lambda(k)\rangle^{1/4}\widehat{u}*\widehat{v}\|_{l^\infty_k L^\infty_\tau} \\ &\lesssim \|\langle k\rangle^{-3s-21/4}\|_{l^\infty_k}\|u\|_{X^{-s/2-1,s/2+1}}\|v\|_{X^{-s/2-1,s/2+1}},\end{aligned}$$

which shows the required estimate. Therefore we deal only with the case in which both $\widehat{u}*\widehat{v}$ and $\widehat{v}$ are supported on $D_1 \cup D_2$.

(IIIb) We estimate (1.5) for Ω_{31}. In Ω_{31}, $\widehat{u}$ is supported on D_3 from $L_{\max} \gtrsim |k_1 k^4|$. We use Lemma 2.1 with $b' = 1/4$ and $b = 1/2$ to obtain

$$\begin{aligned}\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} &\lesssim \|(J^{-s/2-1}\Lambda^{s/2+1}u)(J^{-s-3}v)\|_{X^{0,-1/4}} \\ &\lesssim \|u\|_{X^{-s/2-1,s/2+1}}\|v\|_{X^{s,1/2}},\end{aligned}$$

which is an appropriate bound.

(IIIc) We consider the estimate for Ω_{32}. From (3.3), $\widehat{u}*\widehat{v}$ is supported on D_2. We use Lemma 2.1 with $b = 1/4$ and $b' = 1/2$ to have

$$\|\langle k\rangle^{-3s}\langle \tau - p_\lambda(k)\rangle^{s}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} \lesssim \|(J^s u)(J^s v)\|_{L^2_{t,x}} \lesssim \|u\|_{X^{s,1/4}}\|v\|_{X^{s,1/2}},$$

which is an appropriate bound.

Next, we estimate the Y^s norm of $\Lambda^{-1}\partial_x(uv)$. The Young inequality shows

$$\begin{aligned}\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1}\widehat{u}*\widehat{v}\|_{l^2_k L^1_\tau} &\lesssim \|(\langle k\rangle^{-4}\widehat{u})*(\langle k\rangle^{s}\widehat{v})\|_{l^2_k L^1_\tau} \\ &\lesssim \|\langle k\rangle^{-4}\widehat{u}\|_{l^1_k L^2_\tau}\|v\|_{Y^s},\end{aligned}$$

which implies the desired estimate from the Schwarz inequality.

(IIId) We consider the estimate for Ω_{33}. From (3.3), we may assume that $\widehat{v}$ is supported on D_2 and $|\tau - p_\lambda(k)| \lesssim |k|^5$. In the case $\widehat{u}$ is supported on $D_1 \cup D_2$, we use Lemma 2.1 with $b' = 1/4$ and $b = 1/2$ to obtain

$$\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2} \lesssim \|(J^{-s-3}u)(J^{-3s-1}\Lambda^{s+1}v)\|_{X^{0,-1/4}}$$
$$\lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{-3s-1,s+1}}.$$

On the other hand, we consider the case in which $\widehat{u}$ is supported on D_3. Then we use Lemma 2.1 with $b' = -s/2 - 1/4$ and $b = s/2 + 1$ to have

$$\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-\frac{1}{4}}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2} \lesssim \|\langle k\rangle^{-\frac{3s}{2}-\frac{3}{2}}\langle \tau - p_\lambda(k)\rangle^{\frac{s}{2}+\frac{1}{4}}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2}$$
$$\lesssim \|(J^{-\frac{7s}{2}-\frac{11}{2}}u)(J^{-3s-1}\Lambda^{s+1}v)\|_{X^{0,\frac{s}{2}+\frac{1}{2}}} \lesssim \|u\|_{X^{-\frac{7s}{2}-\frac{11}{2},\frac{s}{2}+1}}\|v\|_{X^{-3s-1,s+1}},$$

which shows the desired estimate since $-\frac{7s}{2} - \frac{11}{2} \leq -\frac{s}{2} - 1$ for $s \geq -\frac{3}{2}$.

Estimate for Ω_4.

(VI) In Ω_4, $\widehat{u}$ is restricted to D_3. We divide Ω_4 into three parts as follows:

$$\Omega_{41} := \{(\tau, k, \tau_1, k_1) \in \Omega_4 : L_{\max} = |\tau_1 - p_\lambda(k_1)|\},$$
$$\Omega_{42} := \{(\tau, k, \tau_1, k_1) \in \Omega_4 : L_{\max} = |\tau - p_\lambda(k)|\},$$
$$\Omega_{43} := \{(\tau, k, \tau_1, k_1) \in \Omega_4 : L_{\max} = |\tau_2 - p_\lambda(k_2)|\}.$$

When $|k_1| \lesssim |k|^{-4}$, we easily obtain the desired estimate combining Hölder's and Young's inequalities. So we deal only with the case $|k|^{-4} \lesssim |k_1| \leq 1$.

(VIa) In Ω_{41}, we use the Hölder inequality and the Young inequality to obtain

$$\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2} \lesssim \|(|k|^{-1/4}\langle \tau - p_\lambda(k)\rangle^{1/4}\widehat{u}) * (\langle k\rangle^s\widehat{v})\|_{l_k^2 L_\tau^2}$$
$$\lesssim \||k|^{-1/4}\langle \tau - p_\lambda(k)\rangle^{1/4}\widehat{u}\|_{l_k^1 L_\tau^2}\|v\|_{Y^s} \lesssim \|u\|_{X^{0,1/4}}\|v\|_{Y^s}.$$

(VIb) In Ω_{42}, we use Young's inequality to have

$$\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2} \lesssim \|(|k|^{-1/4}\widehat{u}) * (\langle k\rangle^s\widehat{v})\|_{l_k^2 L_\tau^2}$$
$$\lesssim \||k|^{-1/4}\widehat{u}\|_{l_k^1 L_\tau^2}\|v\|_{Y^s},$$

which is an appropriate bound from Schwarz's inequality.

(VIc) From $\langle k_2\rangle^{-s}\langle \tau_2 - p_\lambda(k_2)\rangle^{-1/4} \lesssim |k_1|^{-1/4}\langle k_2\rangle^{-s-1}$ in Ω_{43}, we use the Hölder inequality and the Young inequality to have

$$\|\langle k\rangle^{s+1}\langle \tau - p_\lambda(k)\rangle^{-1/4}\widehat{u} * \widehat{v}\|_{l_k^2 L_\tau^2} \lesssim \|(|k|^{-1/4}\widehat{u}) * (\langle k\rangle^s\langle \tau - p_\lambda(k)\rangle^{1/4}\widehat{v})\|_{l_k^2 L_\tau^2}$$
$$\lesssim \||k|^{-1/4}\widehat{u}\|_{l_k^1 L_\tau^1}\|v\|_{X^{s,1/4}} \lesssim \|u\|_{Y^s}\|v\|_{X^{s,1/4}}.$$

Estimate for Ω_5. We decompose Ω_5 into three parts as follows:

$$\Omega_{51} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_5 : L_{\max} = |\tau - p_\lambda(k)|\big\},$$
$$\Omega_{52} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_5 : L_{\max} = |\tau_1 - p_\lambda(k_1)|\big\},$$
$$\Omega_{53} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_5 : L_{\max} = |\tau_2 - p_\lambda(k_2)|\big\}.$$

(Va) In Ω_{51}, $\widehat{u}*\widehat{v}$ is supported on D_3. We divide this region into

$$\Omega_{51a} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_{51} : |\tau - p_\lambda(k)| \sim |k^5|\big\},$$
$$\Omega_{51b} := \Omega_{51}\setminus\Omega_{51a}.$$

In Ω_{51a}, both $\widehat{u}$ and $\widehat{v}$ are supported on $D_1\cup D_2$ from (3.3). We use Lemma 2.1 with $b'=1/4$ and $b=1/2$ to have

$$\|\langle k\rangle^{-s/2}\langle\tau - p_\lambda(k)\rangle^{s/2}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} \lesssim \|(J^s u)(J^s v)\|_{L^2_{t,x}} \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s,1/4}}.$$

In Ω_{51b}, either $|\tau - p_\lambda(k)|\sim|\tau_1 - p_\lambda(k_1)|$ or $|\tau - p_\lambda(k)|\sim|\tau_2 - p_\lambda(k_2)|$ holds. Following an argument similar to that in the case Ω_{11}, we obtain the desired estimate in Ω_{51b}.

(Vb) We consider the estimate for Ω_{52}. From (3.3), $\widehat{u}$ is supported on D_3. We divide Ω_{52} into

$$\Omega_{52a} := \big\{(\tau,k,\tau_1,k_1)\in\Omega_{52} : |\tau_1 - p_\lambda(k_1)| \sim |k_1^5|\big\},$$
$$\Omega_{52b} := \Omega_{52}\setminus\Omega_{52a}.$$

In Ω_{52a}, $\widehat{u}*\widehat{v}$ and $\widehat{v}$ are supported on $D_1\cup D_2$ under this assumption. Then we use Lemma 2.1 with $b'=1/4$ and $b=1/2$ to have

$$\begin{aligned}\|\langle k\rangle^{s+1}\langle\tau - p_\lambda(k)\rangle^{-1/4}\widehat{u}*\widehat{v}\|_{l^2_k L^2_\tau} &\lesssim \|(J^{-s/2-1}\Lambda^{s/2+1}u)(J^{-s-3}v)\|_{X^{0,-1/4}}\\ &\lesssim \|u\|_{X^{-s/2-1,s/2+1}}\|v\|_{X^{s,1/2}},\end{aligned}$$

which is an appropriate bound.

In Ω_{52b}, either $|\tau_1 - p_\lambda(k_1)|\sim|\tau - p_\lambda(k)|$ or $|\tau_1 - p_\lambda(k_1)|\sim|\tau_2 - p_\lambda(k_2)|$ holds. These cases are almost identical to the case (IIIa).

In the same manner as above, we obtain the desired estimate in Ω_{53} by symmetry. □

4. Proof of the local well-posedness

In this section, we give the proof of Theorem 1.1 by the iteration method. Here we put $U(t) := \mathcal{F}_k^{-1}\exp(ip(k)t)\mathcal{F}_x$ and $p(k) := k^5 + \beta k^3$. We obtain the local well-posedness result in the following sense.

Proposition 4.1. *Let* $-3/2 \leq s \leq -1$ *and* $r > 1$. *For any* $u_0 \in B_r(\dot{H}^s)$, *there exist* $T = T(r) > 0$ *and a unique solution* $u \in Z^s([0,T])$ *satisfying the following integral form for* (1.1):

$$u(t) = U(t)u_0 - \int_0^t U(t-s)\partial_x (u(s))^2 \, ds. \tag{4.1}$$

Moreover, the data-to-solution map, $B_r(\dot{H}^s) \ni u_0 \mapsto u \in Z^s([0,T])$, *is locally Lipschitz continuous.*

Proof. We first prove the existence of the solution by a fixed-point argument. Here λ is a sufficiently large number determined later. For any $u_0 \in B_r(\dot{H}^s)$, from (1.3), $\|u_{0,\lambda}\|_{\dot{H}^s} \leq \lambda^{-2} r$ when $-3/2 \leq s < 0$. Therefore, we prove, for any $u_{0,\lambda} \in B_{\lambda^{-2}r}(\dot{H}^s)$, there exists $u_\lambda \in Z^s([0,1])$ satisfying

$$M[u_\lambda](t) = u_\lambda(t), \; M[u_\lambda](t) = U_\lambda(t)u_{0,\lambda} - \int_0^t U_\lambda(t-s)\partial_x (u_\lambda(s))^2 \, ds. \tag{4.2}$$

Following Propositions 1.4 and 2.3, we obtain the bilinear estimate as follows:

$$\Big\| \int_0^t U_\lambda(t-s)\partial_x (u_\lambda(s) v_\lambda(s)) \, ds \Big\|_{Z^s([0,1])} \leq C_1 \|u_\lambda\|_{Z^s([0,1])} \|v_\lambda\|_{Z^s([0,1])}, \tag{4.3}$$

for some constant $C_1 > 0$. From Proposition 2.2 and (4.3), we have

$$\|M[u_\lambda]\|_{Z^s([0,1])} \leq C_1 \big(\|u_{0,\lambda}\|_{\dot{H}^s} + \|u_\lambda\|^2_{Z^s([0,1])} \big).$$

Here we choose $\lambda^2 \geq 8C_1^2 r$ so that M is a map from $B_{2C_1\lambda^{-2}r}(Z^s([0,1]))$ to itself. In the same manner as above, we obtain

$$\begin{aligned} \|M[u_\lambda] - M[v_\lambda]\|_{Z^s([0,1])} &\leq C_1 \|u_\lambda + v_\lambda\|_{Z^s([0,1])} \|u_\lambda - v_\lambda\|_{Z^s([0,1])} \\ &\leq 4\lambda^{-2} C_1^2 r \|u_\lambda - v_\lambda\|_{Z^s([0,1])} \leq \frac{1}{2} \|u_\lambda - v_\lambda\|_{Z^s([0,1])}, \end{aligned}$$

which implies that M is a contraction map on $B_{2C_1\lambda^{-2}r}(Z^s([0,1]))$. From the fixed-point argument, we construct the solution to (4.2) on $[0,1]$. Here we put $u(t,x) := \lambda^4 u_\lambda(\lambda^5 t, \lambda x)$. Then u solves (4.1) on $[0,T]$, where the lifetime T satisfies $T \sim \lambda^{-5} \sim r^{-5/2}$. Moreover, following the standard argument, we show that the data-to-solution map is locally Lipschitz continuous.

Moreover, uniqueness can be extended to the whole of $Z^s([0,T])$. This proof is based on Muramatu and Taoka's work [20]. For the details, see [15]. □

5. PROOF OF THE GLOBAL WELL-POSEDNESS

In this section, we extend the local solution obtained above globally in time by the I-method. When $s \geq -1$, Hirayama [14] obtained LWP for (1.1) in the function space $W^s([0,T])$ equipped with the norm

$$\|u\|_{W^s([0,T])} := \|u\|_{X^{s,1/2}([0,T])} + \|u\|_{Y^s([0,T])}.$$

These local-in-time solutions are shown to exist on an arbitrary time interval for $0 > s \geq -1$. Note that $s = -1$ is optimal in the sense that the bilinear estimate in the standard Bourgain space fails for $s < -1$. The proof is an adaptation of the argument presented for the periodic KdV equation in [9]. Remark that we encounter difficulty in that the Kawahara equation has fewer symmetries than the KdV equation. Before modified energies are introduced, we prepare some notation. An l multiplier is a function $M : \mathbb{R}^l \to \mathbb{C}$. We say an l multiplier M is symmetric if $M(k_1, k_2, \dots, k_l) = M(k_{\sigma(1)}, k_{\sigma(2)}, \dots, k_{\sigma(l)})$ for all $\sigma \in S_l$, the group of all permutations on l objects. The symmetrization of an l multiplier M is defined by

$$[M]_{sym}(k_1, k_2, \dots, k_l) := \frac{1}{l!} \sum_{\sigma \in S_l} M(k_{\sigma(1)}, k_{\sigma(2)}, \dots, k_{\sigma(l)}).$$

We define an l-linear functional associated to the function M acting on l functions $u_1, u_2, \dots, u_l$:

$$\Lambda_l(M; u_1, u_2, \dots, u_l) := \int_{k_1+k_2+\cdots+k_l=0} M(k_1, k_2, \dots, k_l) \prod_{i=1}^{l} \widehat{u}_i(k_i).$$

$\Lambda_l(M; u, \dots, u)$ is simply written as $\Lambda_l(M)$. We recall the original modified energy $E_I^{(2)}(u)(t) = \|Iu(t)\|_{L^2}^2$. We use this functional to obtain GWP for $-21/26 < s < 0$ but not $-1 \leq s \leq -21/26$. Then we construct new modified energies by adding some correction terms to $E_I^{(2)}(u)$, following the argument in [9]. Using u real-valued and m even, we use the Plancherel theorem to have

$$E_I^{(2)}(u)(t) = \Lambda_2(m(k_1)m(k_2))(t).$$

Here a_l and b_l denote $a_l = i\sum_{j=1}^{l} k_j^5$ and $b_l = i\sum_{j=1}^{l} k_j^3$. We compute the time derivative of the modified energy $E_I^{(2)}(u)$. Now we use the Leibniz rule with respect to the time derivative and substitute the equation

$$\partial_t \widehat{u}(k) = ik^5 \widehat{u}(k) + i\lambda^{-2}\beta k^3 \widehat{u}(k) - ik \sum_{k_1 \in \dot{\mathbb{Z}}_\lambda} \widehat{u}(k_1)\widehat{u}(k - k_1)$$

to $E_I^{(2)}(u)$ to have

$$\begin{aligned}\frac{d}{dt}E_I^{(2)}(u)(t) =&\Lambda_2((a_2+\lambda^{-2}\beta b_2)m(k_1)m(k_2))(t)\\ &-2i\Lambda_3([(k_2+k_3)m(k_1)m(k_2+k_3)]_{sym})(t).\end{aligned}$$

Here the first term vanishes because $a_2=0$ and $b_2=0$. Therefore the time derivative of $E_I^{(2)}(u)$ has the cubic form as follows:

$$\frac{d}{dt}E_I^{(2)}(u)(t)=\Lambda_3(M_3)(t),\quad M_3(k_1,k_2,k_3)=-2i[m(k_1)m(k_{23})k_{23}]_{sym},$$

where $k_{ij}=k_i+k_j$ for $i\neq j$. We add a correction term $\Lambda_3(\sigma_3)$ to the modified energy $E_I^{(2)}(u)$ to construct a new modified energy $E_I^{(3)}(u)$, namely,

$$E_I^{(3)}(u)(t)=E_I^{(2)}(u)(t)+\Lambda_3(\sigma_3)(t),$$

where the symmetric function σ_3 is determined later. Similarly, the time derivative of $E_I^{(3)}(u)$ is expressed by

$$\begin{aligned}\frac{d}{dt}E_I^{(3)}(u)(t) =&\Lambda_3(M_3)(t)+\Lambda_3((a_3+\lambda^{-2}\beta b_3)\sigma_3)(t)\\ &-3i\Lambda_4([\sigma_3(k_1,k_2,k_{34})k_{34}]_{sym})(t).\end{aligned}$$

Here we choose $\sigma_3=-M_3/(a_3+\lambda^{-2}\beta b_3)$ to cancel the cubic terms. Then,

$$\frac{d}{dt}E_I^{(3)}(u)(t)=\Lambda_4(M_4)(t),\ M_4(k_1,k_2,k_3,k_4):=-3i\Lambda_4([\sigma_3(k_1,k_2,k_{34})k_{34}]_{sym}).$$

In the same manner, we define the third modified energy as follows:

$$E_I^{(4)}(u)(t):=E_I^{(3)}(u)(t)+\Lambda_4(\sigma_4)(t),\ \sigma_4:=-M_4/(a_4+\lambda^{-2}\beta b_4).$$

Then we have

$$\begin{aligned}\frac{d}{dt}E_I^{(4)}(u)(t) :=&\Lambda_5(M_5)(t),\\ M_5(k_1,k_2,k_3,k_4,k_5):=&-4i[\sigma_4(k_1,k_2,k_3,k_{45})k_{45}]_{sym}.\end{aligned}$$

Chen and Guo [4] obtained the upper bound of M_4 as follows.

Lemma 5.1. *Let $|k_1|\geq|k_2|\geq|k_3|\geq|k_4|$. Then we have*

$$|M_4(k_1,k_2,k_3,k_4)|\lesssim\frac{|a_4+\beta\lambda^{-2}b_4|m(k_4^*)}{(N+|k_1|)^2(N+|k_2|)^2(N+|k_3|)^3(N+|k_4|)},\tag{5.1}$$

where $k_4^*:=\min\{|k_l|,|k_{ij}|\}$.

To establish this upper bound for the Kawahara equation is difficult because this equation has fewer symmetries than the KdV equation. Combining the bilinear Strichartz estimate (2.1) and this upper bound (5.1), we establish the following almost-conservation law, which controls the increment of the modified energy $E_I^{(4)}(u)$ in time.

Proposition 5.2. *Let $0 > s \geq -1$ and $N \gg 1$. Then there exists $C_1 > 0$ such that*

$$\left|E_I^{(4)}(u)(t) - E_I^{(4)}(u)(t_0)\right| \leq C_1 N^{5s} \|Iu\|^5_{W^0([t_0-1,t_0+1])}, \tag{5.2}$$

for any $t_0 \in \mathbb{R}$ and $t \in [t_0 - 1, t_0 + 1]$.

Proof. We may assume $t_0 = 0$ and $\widehat{u}$ is non-negative. Since

$$|E_I^{(4)}(u)(t) - E_I^{(4)}(u)(0)| \lesssim \int_{-1}^{1} \Lambda(M_5)(t)\, dt,$$

for any $t \in [-1, 1]$, it suffices to show that

$$\int_{-1}^{1} \Lambda_5\Big(\frac{M_5(k_1,k_2,k_3,k_4,k_5)}{m(k_1)m(k_2)m(k_3)m(k_4)m(k_5)}\Big)(t)\, dt \lesssim N^{5s}\|u\|^5_{W^0([-1,1])}. \tag{5.3}$$

We suppose that $|k_1| \geq |k_2| \geq |k_3| \geq |k_4| \geq |k_5|$ without loss of generality. M_5 vanishes when $|k_i| \ll N$ for any $i = 1, 2, 3, 4, 5$. So we can assume $|k_1| \sim |k_2| \gtrsim N$. From the definition of M_5, we have

$$|M_5(k_1,k_2,k_3,k_4,k_5)| \lesssim |\sigma_4(k_3,k_4,k_5,k_{12})k_{12}|.$$

From the fact that $k_3 + k_4 + k_5 + k_{12} = 0$, we consider only two cases as follows:

$$D_1 := \big\{(\vec{\tau},\vec{k}) \in \mathbb{R}^5 \times \dot{\mathbb{Z}}^5_\lambda : |k_3| \sim |k_{12}| \gtrsim |k_4| \geq |k_5| \text{ and } |k_3| \sim |k_{12}| \gtrsim N\big\},$$
$$D_2 := \big\{(\vec{\tau},\vec{k}) \in \mathbb{R}^5 \times \dot{\mathbb{Z}}^5_\lambda : |k_3| \sim |k_4| \gg \max\{|k_{12}|, |k_5|\} \text{ and } |k_3| \sim |k_4| \gtrsim N\big\},$$

where $\vec{\tau} := (\tau_1, \tau_2, \ldots, \tau_5)$ and $\vec{k} := (k_1, k_2, \ldots, k_5)$.

(I) Firstly, we prove (5.3) in D_1. From (5.1), we easily obtain the upper bound of M_5 as follows:

$$|M_5(k_1,k_2,k_3,k_4,k_5)| \lesssim \frac{|k_{12}|}{(N+|k_3|)^4(N+|k_4|)^3(N+|k_5|)^1}.$$

From $|k_{12}| \sim |k_3| \gtrsim N$, we substitute this estimate into (5.3) and use the dyadic decompositions to have

(L. H. S. of (5.3))

$$\lesssim N^{5s} \int_{-1}^{1} \Lambda_5 \big(|k_{12}| \langle k_1 \rangle^{-s} \langle k_2 \rangle^{-s} \langle k_3 \rangle^{-s-4} \langle k_4 \rangle^{-s-3} \langle k_5 \rangle^{-s-1} \big)(t)\, dt$$

$$\lesssim N^{5s} \sum_{N_1} \sum_{N_2 \sim N_1} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} \sum_{N_5 \leq N_4}$$

$$\times N_1^{-s} N_2^{-s} N_3^{-s-4} \langle N_4 \rangle^{-s-3} \langle N_5 \rangle^{-s-1} \Big\| \prod_{i=1}^{5} u_{N_i} \Big\|_{L_x^1 L^1_{t\in[-1,1]}},$$

where $u_{N_i} := P_{\{|k_i| \sim N_i\}} u$ for dyadic numbers N_i with $i = 1, 2, 3, 4, 5$. From the Schwarz inequality, (5.3) is reduced to two estimates as follows:

$$N_1^{-s} N_2^{-s} \| |\partial_x| u_{N_1} u_{N_2} \|_{X^{s,-1/2}} \lesssim N_1^{-s-1} N_2^{-s-1} \|u_{N_1}\|_{W^0} \|u_{N_2}\|_{W^0}, \tag{5.4}$$

$$N_3^{-s-4} \langle N_4 \rangle^{-s-3} \langle N_5 \rangle^{-s-1} \| \prod_{i=3}^{5} u_{N_i} \|_{X^{-s,1/2}} \lesssim N_3^{-2s-2} \langle N_4 \rangle^{-s-2} \prod_{i=3}^{5} \|u_{N_i}\|_{W^0}. \tag{5.5}$$

If these estimates hold, the left-hand side of (5.3) is bounded by

$$N^{5s} \sum_{N_1} \sum_{N_2 \sim N_1} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} \sum_{N_5 \leq N_4} N_1^{-s-1} N_2^{-s-1} N_3^{-2s-2} \langle N_4 \rangle^{-s-2}$$

$$\times \|u_{N_1}\|_{W^0([-1,1])} \|u_{N_2}\|_{W^0([-1,1])} \|u\|^3_{W^0([-1,1])}$$

$$\lesssim N^{5s} \sum_{N_1} \sum_{N_2 \sim N_1} N_1^{-4s-4} \|u_{N_1}\|_{W^0([-1,1])} \|u_{N_2}\|_{W^0([-1,1])} \|u\|^3_{W^0([-1,1])},$$

which shows the desired estimate for $-1 \leq s < 0$.

The bilinear estimate (5.4) has been already proven by Hirayama (see Theorem 1.3 in [14]), so we prove only the trilinear estimate (5.5). From the Plancherel theorem, we have the identity

$$\|u_{N_3} u_{N_4} u_{N_5}\|_{X^{-s,1/2}} = \Big\| \langle k \rangle^{-s} \langle \tau - p_\lambda(k) \rangle^{1/2} \prod_{i=3}^{5} \widehat{u}_{N_i}(\tau_i, k_i) \Big\|_{l_k^2 L_\tau^2},$$

where $k = k_3 + k_4 + k_5$ and $\tau = \tau_3 + \tau_4 + \tau_5$. From the definition, $|k| \sim |k_{12}| \sim N_3$ in this case.

(Ia) We first consider the case $\langle \tau - p_\lambda(k) \rangle \lesssim \langle \tau_i - p_\lambda(k_i) \rangle$ for some $i = 3, 4, 5$. By symmetry, we may assume $\langle \tau - p_\lambda(k) \rangle \lesssim \langle \tau_3 - p_\lambda(k_3) \rangle$. It suffices show that

$$N_3^{-2s-4} \langle N_4 \rangle^{-s-3} \langle N_5 \rangle^{-s-1} \|u_{N_3} u_{N_4} u_{N_5}\|_{L^2_{t,x}}$$
$$\lesssim N_3^{-2s-4} \langle N_4 \rangle^{-s-2} \langle N_5 \rangle^{-s-1} \|u_{N_3}\|_{X^{0,0}} \|u_{N_4}\|_{Y^0} \|u_{N_5}\|_{Y^0}. \tag{5.6}$$

Hölder's and Young's inequalities imply

$$\|\prod_{i=3}^{5}\widehat{u}_{N_i}(\tau_i,k_i)\|_{l_k^2L_\tau^2} \lesssim \|\widehat{u}_{N_3}\|_{l_k^2L_\tau^2}\|\widehat{u}_{N_4}\|_{l_k^1L_\tau^1}\|\widehat{u}_{N_5}\|_{l_k^1L_\tau^1}$$
$$\lesssim N_4^{1/2}N_5^{1/2}\|u_{N_3}\|_{X^{0,0}}\|u_{N_4}\|_{Y^0}\|u_{N_5}\|_{Y^0}.$$

We insert this into the left-hand side of (5.6) to obtain the required estimate.

(Ib) Next, we consider the case $\langle \tau - p_\lambda(k)\rangle \gg \langle \tau_i - p_\lambda(k_i)\rangle$ for all $i = 3,4,5$. In this case, we use the algebraic relation to have

$$|\tau - p_\lambda(k)| \sim |-p_\lambda(k) + p_\lambda(k_1) + p_\lambda(k_2) + p_\lambda(k_3)| \lesssim |k_3|^4|k_4|. \tag{5.7}$$

We use (5.7) and the Hölder inequality to obtain

$$N_3^{-2s-4}\langle N_4\rangle^{-s-3}\langle N_5\rangle^{-s-1}\|u_{N_3}u_{N_4}u_{N_5}\|_{X^{0,1/2}}$$
$$\lesssim N_3^{-2s-2}\langle N_4\rangle^{-s-5/2}\langle N_5\rangle^{-s-1}\|u_{N_3}u_{N_4}u_{N_5}\|_{L^2_{t,x}}$$
$$\lesssim N_3^{-2s-2}\langle N_4\rangle^{-s-5/2}\langle N_5\rangle^{-s-1}\|u_{N_3}u_{N_4}\|_{L^2_{t,x}}\|u_{N_5}\|_{L^\infty_{t,x}}.$$

When $|k_{34}| \geq 1$, from (2.1) and the Sobolev inequality, the right-hand side is bounded by

$$N_3^{-2s-2}\langle N_4\rangle^{-s-5/2}\langle N_5\rangle^{-s-1/2}\|u_{N_3}\|_{X^{0,3/8}}\|u_{N_4}\|_{X^{0,3/8}}\|u_{N_5}\|_{Y^0},$$

which shows the required estimate. On the other hand, we deal with the case $|k_{34}| \leq 1$. Combining the Hölder inequality and the Young inequality, we have

$$\|u_{N_3}u_{N_4}\|_{L^2_{t,x}}\|u_{N_5}\|_{L^\infty_{t,x}} \lesssim N_5^{1/2}\|\widehat{u}_{N_3} * \widehat{u}_{N_4}\|_{l_k^\infty L_\tau^2}\|u_{N_5}\|_{Y^0}$$
$$\lesssim N_5^{1/2}\|u_{N_3}\|_{L^2_{t,x}}\|u_{N_4}\|_{Y^0}\|u_{N_5}\|_{Y^0}.$$

From this, we immediately obtain the desired estimate.

(II) Secondly, we prove (5.3) in D_2. In this case, we have the upper bound of M_5 as follows:

$$|M_5(k_1,k_2,k_3,k_4,k_5)| \lesssim \frac{|k_{12}|}{(N+|k_3|)^2(N+|k_4|)^2(N+|k_{12}|)^2(N+|k_5|)^2}.$$

In the same manner as above, (5.3) is reduced to (5.4) and

$$N_3^{-s-2}N_4^{-s-2}\langle N_5\rangle^{-s-2}\|\prod_{i=3}^{5}\widehat{u}_{N_i}(\tau_i,k_i)\|_{X^{-s-2,1/2}} \tag{5.8}$$

$$\lesssim N_3^{-2s-2}\langle N_5\rangle^{-s-3/2}\prod_{i=3}^{5}\|u_{N_i}\|_{W^0}.$$

We now show the trilinear estimate (5.8).

(IIa) We first consider the case $|\tau-p_\lambda(k)|\lesssim|\tau_3-p_\lambda(k_3)|$. We use Hölder's inequality and Young's inequality to have

$$\begin{aligned}\|\langle k\rangle^{-s-2}\prod_{i=3}^{5}\widehat{u}_{N_i}\|_{l_k^2L_\tau^2}&\lesssim\|\prod_{i=3}^{5}\widehat{u}_{N_i}\|_{l_k^\infty L_\tau^2}\\ \lesssim\|\widehat{u}_{N_3}\|_{l_k^2L_\tau^2}\|\widehat{u}_{N_4}\|_{l_k^2L_\tau^1}\|\widehat{u}_{N_5}\|_{l_k^1L_\tau^1}&\lesssim N_5^{1/2}\|u_{N_3}\|_{L^2_{t,x}}\|u_{N_4}\|_{Y^0}\|u_{N_5}\|_{Y^0},\end{aligned}$$

which implies the desired estimate.

(IIb) Next, we consider the case $|\tau-p_\lambda(k)|\gg|\tau_i-p_\lambda(k_i)|$ for all $i=3,4,5$. In this case, the algebraic relation implies

$$|\tau-p_\lambda(k)|\lesssim\max\{|k|,|k_5|\}|k_3|^4.$$

We use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned}(\text{L. H. S. of (5.8)})&\lesssim N_3^{-2s-2}\langle N_5\rangle^{-s-3/2}\|\prod_{i=3}^{5}\widehat{u}_{N_i}\|_{l_k^\infty L_\tau^2}\\ &\lesssim N_3^{-2s-2}\langle N_5\rangle^{-s-3/2}\|u_{N_3}u_{N_4}\|_{L^2_{t,x}}\|u_{N_5}\|_{Y^0},\end{aligned}$$

which is an appropriate bound from the above argument. □

Next, we estimate the difference between the almost-conserved quantity $E_I^{(4)}(u)$ and the first modified energy $E_I^{(2)}(u)$ when the time is fixed. We call this estimate the fixed-time difference.

Proposition 5.3. *Let $0>s\geq-1$ and $N\gg1$. Then there exists $C_2>0$ such that*

$$|E_I^{(4)}(u)(t_0)-E_I^{(2)}(u)(t_0)|\leq C_2(\|Iu(t_0)\|^3_{L^2_x}+\|Iu(t_0)\|^4_{L^2_x}),\tag{5.9}$$

for any $t_0\in\mathbb{R}$.

Proof. From the definition of the modified energies, it suffices to show that

$$|\Lambda_3(\sigma_3)(t_0)|\lesssim\|Iu(t_0)\|^3_{L^2},\quad|\Lambda_4(\sigma_4)(t_0)|\lesssim\|Iu(t_0)\|^4_{L^2}.$$

These estimates are reduced to the following estimates:

$$\Big|\Lambda_3\Big(\frac{M_3(k_1,k_2,k_3)}{(a_3+\beta\lambda^{-2}b_3)m(k_1)m(k_2)m(k_3)}(t_0)\Big)\Big|\lesssim\|u(t_0)\|^3_{L^2},\tag{5.10}$$

$$\Big|\Lambda_4\Big(\frac{M_4(k_1,k_2,k_3,k_4)}{(a_4+\beta\lambda^{-2}b_4)m(k_1)m(k_2)m(k_3)m(k_4)}(t_0)\Big)\Big| \lesssim \|u(t_0)\|_{L^2}^4. \tag{5.11}$$

Firstly, we prove (5.10) when $|k_1| \geq |k_2| \geq |k_3|$. Following the mean-value theorem, we easily obtain the upper bound of M_3 as follows:

$$|M_3(k_1,k_2,k_3)| \lesssim |k_3| m(k_3)^2.$$

If $|k_i| \ll N$ for all $i = 1,2,3$, then M_3 vanishes. So we consider only the case $|k_1| \sim |k_2| \gtrsim N$. The algebraic relation shows $|a_3 + \beta\lambda^{-2}b_3| \sim |k_1|^4|k_3|$ Following these, we use the Hölder inequality and the Sobolev inequality to have

$$\begin{aligned}(\text{L. H. S. of (5.10)}) &\lesssim N^{2s}\int |\langle\partial_x\rangle^{-2-s}u(t_0)|^2|Iu(t_0)|\,dx \\ &\lesssim N^{2s}\|\langle k\rangle^{-2-s}u(t_0)\|_{L^4}^2\|Iu(t_0)\|_{L^2} \lesssim N^{2s}\|u(t_0)\|_{L^2}^2\|m\widehat{u}(t_0)\|_{L^2_\xi},\end{aligned}$$

which is bounded by $N^{2s}\|u(t_0)\|_{L^2}^3$ from the definition of m.

Secondly, we prove (5.11) when $|k_1| \geq |k_2| \geq |k_3| \geq |k_4|$. From (5.1) and Sobolev's inequality, the left-hand side of (5.11) is bounded by

$$\begin{aligned}\Big|\Lambda_4\Big(\frac{1}{\prod_{i=1}^4(N+|k_i|)^2 m(k_i)}\Big)(t_0)\Big| &\lesssim N^{4s}\int |\langle\partial_x\rangle^{-s-2}u(t_0)|^4\,dx \\ &\lesssim N^{4s}\|\langle\partial_x\rangle^{-s-2}u(t_0)\|_{L^4}^4 \lesssim N^{4s}\|u(t_0)\|_{L^2}^4.\end{aligned}$$

□

Propositions 5.2 and 5.3 imply that we can find a constant $C_3 > 0$ such that

$$\sup_{-N^{-5s}\leq t\leq N^{-5s}} \|Iu(t)\|_{L^2} \leq C_3\|Iu(0)\|_{L^2}. \tag{5.12}$$

For the details of the proof, see [9]. A direct calculation shows that

$$\|Iu_\lambda(0,\cdot)\|_{L^2} \leq C_0\lambda^{-s-7/2}N^{-s}\|u_0\|_{\dot{H}^s} \tag{5.13}$$

for some constant $C_0 > 0$. Here we take $\lambda \geq 1$ satisfying the following condition:

$$\lambda^{-s-7/2}N^{-s} = \varepsilon_0 \ll 1.$$

Then we combine (5.12) and (5.13) to have

$$\begin{aligned}\sup_{-T\leq t\leq T}\|u(t)\|_{\dot{H}^s} \leq& \lambda^{7/2}\sup_{-\lambda^5T\leq t\leq\lambda^5T}\|Iu_\lambda(t)\|_{L^2} \\ \leq& C_3\lambda^{7/2}\|Iu_\lambda(0)\|_{L^2} \leq \varepsilon_0C_1C_3\lambda^{-s}N^{-s}\|u_0\|_{\dot{H}^s},\end{aligned}$$

when $\lambda^5 T \le N^{-5s}$. Therefore we have the following upper bound of the growth order of $\dot{H}^s$:

$$\sup_{-T\le t\le T} \|u(t)\|_{\dot{H}^s} \le CT^{7/5(2s+5)}\|u_0\|_{\dot{H}^s}, \quad \text{for } -1 \le s < 0.$$

6. Proof of the ill-posedness

In this section, we give the proof Theorem 1.2, which is based on [3]. From the argument introduced to [13], it suffices to show that we seek initial data such that, for $|t|$ bounded,

$$\|A_3(u_0)(t)\|_{\dot{H}^s} \lesssim \|u_0\|^3_{\dot{H}^s} \tag{6.1}$$

fails when $s < -3/2$. Here $A_3(u_0)$ is the cubic term of the Taylor expansion of the flow map as follows:

$$A_3(u_0)(t) = 2\int_0^t U(t-s)\partial_x(u_1(s)A_2(u_0)(s))\,ds, \tag{6.2}$$

where $u_1(t) = U(t)u_0$ and

$$A_2(u_0)(t) = \int_0^t U(t-s)\partial_x(u_1(s)^2)\,ds,$$

which is the quadratic term of the Taylor expansion of the flow map. We put a sequence of initial data $\{\phi_N\}_{N=1}^\infty \in H^\infty$ as follows:

$$\widehat{\phi}_N(k) = N^{-s}(\chi_N(k) + \chi_{-N}(k)). \tag{6.3}$$

Clearly $\|\phi\|_{\dot{H}^s} \sim 1$. A simple computation shows that

$$\widehat{A}_2(u_0)(t) = \sum_{k_1\neq 0, k\neq k_1} k\,\frac{e^{ip(k)t} - e^{ip(k_1)t + ip(k-k_1)t}}{q_0(k_1, k-k_1)}\widehat{u}_0(k_1)\widehat{u}_0(k-k_1),$$

where

$$q_0(k_1, k-k_1) := \frac{5}{2}kk_1(k-k_1)\big\{k^2 + k_1^2 + (k-k_1)^2 + \frac{6}{5}\beta\big\}.$$

Substituting this into (6.2), we use the Fourier inversion formula to have

$$\begin{aligned} &A_3(u_0)(t)\\ &= 2\sum_{k_1\neq 0}\sum_{k_2\neq 0}\sum_{k_3\neq 0} e^{i(k_1+k_2+k_3)x + ip(k_1+k_2+k_3)t}\Big(-\frac{1-e^{-iq_1 t}}{q_1} + \frac{1-e^{-iq_2 t}}{q_2}\Big)\\ &\quad\times \frac{(k_1+k_2+k_3)(k_2+k_3)}{q_0(k_2,k_3)}\widehat{u}_0(k_1)\widehat{u}_0(k_2)\widehat{u}_0(k_3), \end{aligned} \tag{6.4}$$

where

$$q_1 := \frac{5}{2}(k_1+k_2)(k_1+k_3)(k_2+k_3)\big\{(k_1+k_2)^2+(k_1+k_3)^2+(k_2+k_3)^2+\frac{6}{5}\beta\big\},$$
$$q_2 := \frac{5}{2}k_1(k_2+k_3)(k_1+k_2+k_3)\big\{k_1^2+(k_2+k_3)^2+(k_1+k_2+k_3)^2+\frac{6}{5}\beta\big\}.$$

Note that q_2 does not vanish, but q_1 vanishes when $k_1=-N$ and $k_2=k_3=N$. In this case, inserting (6.3) into (6.4), we obtain

$$|\widehat{A}_3(\phi_N)(t)| \gtrsim C_1|t|N^{-3s-4}|k|\chi_N(k) - C_2N^{-3s-8}\chi_N(k)$$

for some constants $C_1>0$ and $C_2\geq 0$. So there exists $C_3>0$ such that

$$\|A_3(\phi_N)(t)\|_{\dot{H}^s} \gtrsim C_3N^{-2s-3}$$

for $|t|$ bounded. From $\|\phi_N\|_{\dot{H}^s}\sim 1$, (6.1) fails for $s<-3/2$.

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