Differential and Integral Equations Volume 25, Numbers 11-12 (2012) , 993–1010

PERIODIC SOLUTIONS OF A SYSTEM OF COUPLED OSCILLATORS WITH ONE-SIDED SUPERLINEAR RETRACTION FORCES

Alessandro Fonda
Dipartimento di Matematica e Geoscienze, Università di Trieste
P.le Europa 1, I-34127 Trieste, Italy

Andrea Sfecci
SISSA - International School for Advanced Studies
Via Bonomea 265, I-34136 Trieste, Italy

(Submitted by: Reza Aftabizadeh)

Abstract. We generalize the phase-plane method approach introduced in [9] to the case of higher dimensions. To this aim, the phase-space is assumed to be decomposable as a product of planes, where the corresponding components of the solutions can be controlled by means of suitable plane curves. We then apply our general result to the periodic problem associated to a system of coupled oscillators, with retraction forces having a linear growth, or with one-sided superlinear nonlinearities.

1. Introduction

The aim of this paper is to extend to higher dimensions the existence results obtained in [9] for the periodic problem associated to some planar systems of ordinary differential equations.

We want to prove the existence of T-periodic solutions for a system like

$$u' = \mathcal{F}(t,u), \tag{1.1}$$

where $\mathcal{F} : \mathbb{R}\times\mathbb{R}^{2h} \to \mathbb{R}^{2h}$ is a continuous function which is T-periodic in its first variable. As a particular case, we have in mind a system of coupled oscillators of the type

$$\begin{cases} x_1'' + \phi_1(t,x_1) = e_1(t,x_1,\dots,x_h) \\ x_2'' + \phi_2(t,x_2) = e_2(t,x_1,\dots,x_h) \\ \quad\vdots \\ x_h'' + \phi_h(t,x_h) = e_h(t,x_1,\dots,x_h). \end{cases} \tag{1.2}$$

Accepted for publication: August 2012.
AMS Subject Classifications: 34C25, 34C15.

Some existence theorems for general systems of this type have been provided by the use of functional analytical methods, typically using bifurcation theory, or degree theory. We refer to [1, 10] and the references therein, for some classical results in this direction. Moreover, in the variational setting, when (1.1) is a Hamiltonian system, there is a large literature on this type of problem. See, e.g., [3, 4, 13, 15] and the references therein.

In [9], we studied the case $h = 1$, by the use of phase-plane methods. Such methods are frequently applied to planar systems, but very rarely used in higher dimensions, due to the difficulty in controlling the solutions in the phase-space. The aim of the present paper is to provide a setting where it is possible to have such control, at least when the coupling forces $e_i(t, \cdot)$ have a sublinear growth at infinity. On the other hand, we are able to deal with many different situations involving the growth of the functions $\phi_i(t, \cdot)$. As in [9], we can deal with functions having a linear growth, assuming either nonresonance at infinity, or a Landesman–Lazer type of situation, or even with one-sided superlinear nonlinearities. Notice that our result is not of perturbative type, like e.g. the ones in [5, 11, 14], and many others, in the sense that we do not require the functions $e_i(t, \cdot)$ to depend on a small parameter.

We will mainly concentrate on the situation of one-sided superlinear retraction forces, since, in our opinion, it has not yet been sufficiently studied in the literature, for higher-dimensional systems like (1.2). We recall that, in the case of the periodic problem for a second-order scalar equation, one-sided superlinear growth was first considered in the pioneering papers by Mawhin–Ward [12] and Fabry–Habets [7], while a particular higher-dimensional situation was studied by Arioli and Ruf in [2], by the use of a variational method.

Let us explain the main result of this paper, recalling first the approach we used in [9]. The idea there was inspired by [7], where a particular curve γ was constructed in order to control the solutions in the phase plane. We then used such a curve to have the necessary estimates in order to apply the Poincaré–Bohl fixed-point theorem to the Poincaré map associated to (1.1), thus obtaining the existence of a T-periodic solution. Passing to higher dimensions, one could try to generalize this approach, introducing some kind of manifolds in order to have the same type of control. This seems a very delicate problem, and it is not clear to us how such manifolds could be defined. As an alternative approach, we separate the phase space as the product of h planes, and on each of them we construct a curve γ^i, which controls the solutions in that particular plane. Assuming the coupling forces $e_i(t, \cdot)$ to have an appropriate sublinear growth at infinity, the behaviour of a large-amplitude solution $x(t) = (x_1(t), \dots, x_h(t))$ of (1.2) will be approximately the same as if the oscillators were uncoupled, so that each component $x_i(t)$ of the solution will be controlled by the corresponding curve γ^i.

Let us briefly describe how the paper is organized. In Section 2, we provide our general setting for a system in $\mathbb{R}^{2h}$, with $h \geq 1$. We state and prove our main existence result, thus generalizing what we have done in [9] in the case $h = 1$. In Section 3 we particularize our assumptions in view of the applications we have in mind. In Section 4, we deal with a system with nonlinearities having either linear growth, or one-sided superlinear growth. In Section 5, we show how our existence result applies for a system of coupled oscillators.

A few words about the notation to be used on each phase plane. We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^2$, and by $|\cdot|$ the corresponding norm. The open ball, centered at the origin, with radius $R > 0$, is $B^2_R = \{v \in \mathbb{R}^2 : |v| < R\}$, and by S^1 we denote the set $\{v \in \mathbb{R}^2 : |v| = 1\}$. The cone determined by two angles $\theta_1 < \theta_2$ is defined as

$$\Theta(\theta_1, \theta_2) = \{v \in \mathbb{R}^2 : v = \rho e^{i\theta},\ \rho \geq 0\,,\ \theta \in [\theta_1, \theta_2]\}.$$

(It will be sometimes convenient to use the complex notation for the points in $\mathbb{R}^2$.) The closed segment joining two points v_1 and v_2 is denoted by $[v_1, v_2]$. Finally, we use the standard notation $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2. The main result

In this section we are going to introduce a generalization of the main result obtained in [9] to a particular system in $\mathbb{R}^{2h}$ with $h \geq 1$.

Consider an open set $\mathcal{A}$, containing the origin, with the following shape: $\mathcal{A} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_h$, where, for $i \in \{1, \dots, h\}$, the set $\mathcal{U}_i \subset \mathbb{R}^2$ is open and bounded. In the following we will write any point $u \in \mathbb{R}^{2h}$ using the coordinates $(u_1, \dots, u_h)$, where $u_i \in \mathbb{R}^2$. In the same way, all the functions φ with image in $\mathbb{R}^{2h}$ will be written using the components $(\varphi_1, \dots, \varphi_h)$.

We start by stating the following result, reminiscent of the Poincaré–Bohl theorem, whose proof is a standard application of Brouwer degree theory (see, e.g., [6]).

Theorem 2.1. *Let $\varphi : \mathcal{A} \to \mathbb{R}^{2h}$ be a continuous function such that, for every $i \in \{1, \dots, h\}$, the following property holds:*

$$\varphi_i(u_1, \dots, u_h) \neq \mu u_i, \quad \text{for every } \mu > 1 \text{ and for every } u \in \overline{\mathcal{U}}_1 \times \cdots \times \overline{\mathcal{U}}_{i-1} \times \partial\mathcal{U}_i \times \overline{\mathcal{U}}_{i+1} \times \cdots \times \overline{\mathcal{U}}_h.$$

Then φ has a fixed point in $\overline{\mathcal{A}}$.

We consider the equation

$$u' = \mathcal{F}(t,u), \tag{2.1}$$

where $u = (u_1, \dots, u_h) \in \mathbb{R}^{2h}$ and $\mathcal{F} = (f_1, \dots, f_h)$, $f_i : \mathbb{R} \times \mathbb{R}^{2h} \to \mathbb{R}^2$ being continuous functions which are T-periodic in the first variable.

Let us recall the definition of clockwise rotating regular spiral in a plane, which can be found in [9].

Definition 2.2. *A* clockwise rotating regular spiral *is a continuous and injective curve $\gamma : [0,+\infty) \to \mathbb{R}^2$, satisfying the following properties:*

(1) *there exists an unlimited strictly increasing sequence*

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k < \sigma_{k+1} < \cdots$$

such that the restriction of γ to every closed interval $[\sigma_k, \sigma_{k+1}]$ is continuously differentiable, and such that

$$\langle J\dot\gamma(s), \gamma(s)\rangle > 0, \qquad \text{for every } s \in [\sigma_k, \sigma_{k+1}]\,; \tag{2.2}$$

(2) *the curve grows in norm to infinity:*

$$\lim_{s\to+\infty} |\gamma(s)| = +\infty\,; \tag{2.3}$$

(3) *the curve rotates clockwise infinitely many times:*

$$\int_0^{+\infty} \frac{\langle J\dot\gamma(s), \gamma(s)\rangle}{|\gamma(s)|^2}\, ds = +\infty. \tag{2.4}$$

A similar definition can be given for a counterclockwise rotating regular spiral, by changing the inequality in (2.2)*, and requiring the integral in* (2.4) *to be equal to $-\infty$.*

Without loss of generality, we will assume that the spiral has the following parametrization: $\gamma(s) = |\gamma(s)|(\cos s, -\sin s)$.

Now we want to introduce an admissibility condition for such spirals, extending the definition in [9] to the case of a higher-dimensional space.

Definition 2.3. *A clockwise rotating regular spiral γ is said to be i-admissible for system* (2.1)*, with $i \in \{1, \dots, h\}$, if, when restricted to any subinterval $[\sigma_k, \sigma_{k+1}]$, it satisfies*

$$\langle J\dot\gamma(s), f_i(t, u_1, \dots, u_{i-1}, \gamma(s), u_{i+1}, \dots, u_h)\rangle < 0, \tag{2.5}$$

for every $t \in [0,T]$, $s \in [\sigma_k, \sigma_{k+1}]$, and $u_j \in \mathbb{R}^2$ with $j \neq i$. (*The sequence $\{\sigma_k\}_k$ is the one introduced in Definition* 2.2.) *Moreover, given a subset U*

of $\mathbb{R}^2$*, the spiral is said to be* i-admissible in U *for system* (2.1) *if* (2.5) *is satisfied whenever* $\gamma(s) \in U$*.*

Starting from a spiral γ^i which is i-admissible for (2.1), it is convenient to define, for every $n \in \mathbb{N}$, the set Ω^i_n: it is the open region delimited by the Jordan curve Γ^i_n obtained by gluing together the piece of curve γ^i going from $\gamma^i(2\pi n)$ to $\gamma^i(2\pi(n+1))$, and the segment joining the two endpoints:

$$\Gamma^i_n = \big\{\gamma^i(s) : s \in [2\pi n, 2\pi(n+1)]\big\} \ \cup \ \big[\gamma^i(2\pi n), \gamma^i(2\pi(n+1))\big].$$

Notice moreover that, by the injectivity, one has

$$|\gamma^i(s)| < |\gamma^i(s+2\pi)| \qquad \text{for every } s > 0. \tag{2.6}$$

Let us now state our main result.

Theorem 2.4. *Suppose that the following assumptions hold, for every* $i \in \{1, \dots, h\}$*.*

(H1^i) *There exists a clockwise rotating regular spiral* $\gamma^i : [0, +\infty[\to \mathbb{R}^2$ *which is* i*-admissible for* (2.1).

(H2^i) *There exists* $R^i > 0$ *such that, for any solution* $u : [0,T] \to \mathbb{R}^{2h}$ *of* (2.1), *satisfying*

$$|u_i(t)| \geq R^i, \qquad \text{for every } t \in [0,T],$$

one has that, either $|u_i(T)| < |u_i(0)|$*, or*

$$\int_0^T \frac{\langle J u_i'(t), u_i(t)\rangle}{|u_i(t)|^2}\, dt \ \notin 2\pi\mathbb{N}.$$

(H3^i) *There exist* $C^i > 0$ *and* $\theta^i_1 < \theta^i_2$ *such that*

$$\langle J f_i(t,u), u_i\rangle \leq C^i(|u_i|^2 + 1),$$

for every $t \in [0,T]$ *and* $u \in \mathbb{R}^{2h}$ with $u_i \in \Theta(\theta^i_1, \theta^i_2)$.

Then, a T*-periodic solution of equation* (2.1) *exists.*

Proof. Take $R \geq \max\{1, R^1, \dots, R^h\}$ such that $\overline{\Omega}{}^i_0 \subseteq B^2_R$ for every i. Let m_1 be a positive integer such that $\overline{B}{}^2_R \subseteq \Omega^i_{m_1}$ for every i, and let $\bar{n}$ be an integer such that, for every i,

$$\bar{n} > \frac{(C^i+1)T}{\theta^i_2 - \theta^i_1}. \tag{2.7}$$

We can find a $R_1 > R$ such that $\overline{\Omega}{}^i_{m_1+\bar{n}+1} \subseteq B^2_{R_1}$ for every i. In the same way we can find an integer $m_2 > m_1 + \bar{n} + 1$ such that $\overline{B}{}^2_{R_1} \subseteq \Omega^i_{m_2}$ for every i, and a constant $R_2 > R_1$ such that $\overline{\Omega}{}^i_{m_2+\bar{n}+1} \subseteq B^2_{R_2}$ for every i.

Define, for any $r > 0$, $\mathcal{B}_r = (B_r^2)^h = \underbrace{B_r^2 \times \cdots \times B_r^2}_{h \text{ times}} \subset \mathbb{R}^{2h}$. Consider a sequence $(\mathcal{F}^n)_n = (f_1^n, \dots, f_h^n)_n$ of locally Lipschitz-continuous functions converging to $\mathcal{F}$ uniformly on $[0,T] \times \overline{\mathcal{B}}_{R_2}$. For any i, by (2.5), as long as, for some s, $\tilde{u} = (u_1, \dots, u_{i-1}, \gamma^i(s), u_{i+1}, \dots, u_h)$ belongs to $\overline{\mathcal{B}}_{R_2}$, then, for n large enough,

$$\langle J\dot{\gamma}^i(s), f_i^n(t, \tilde{u})\rangle < 0, \qquad \text{for every } t \in [0,T]\,; \tag{2.8}$$

moreover, by (H3^i), for n sufficiently large,

$$\frac{\langle J f_i^n(t,v), v_i\rangle}{|v_i|^2} \le C^i + 1, \tag{2.9}$$

for every $t \in [0,T]$ and $v \in \overline{\mathcal{B}}_{R_2}$ whose i-th component is such that $v_i \in \Theta(\theta_1^i, \theta_2^i) \cap (\overline{B}^2_{R_2} \setminus B^2_R)$.

The solution to the Cauchy problem associated to the equation

$$u' = \mathcal{F}^n(t,u) \tag{2.10}$$

is unique for every n, and, if u^n is a solution of (2.10) satisfying $|u_i^n(0)| \le R_1$ for every i, then, for sufficiently large n,

$$|u_i^n(t)| < R_2, \qquad \text{for every } t \in [0,T] \text{ and for every } i \tag{2.11}$$

(i.e., $u^n(t) \in \mathcal{B}_{R_2}$ for every $t \in [0,T]$). Indeed, for such n, assuming for the sake of contradiction that $\max\{|u_i^n(t)| : t \in [0,T],\, i = 1, \dots, h\} \ge R_2$ (i.e., $u^n(t) \notin \mathcal{B}_{R_2}$ for at least one $t \in [0,T]$), there exists an index j and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, such that

$$|u_j^n(t_1)| = R_1,\ |u_j^n(t_2)| = R_2,\ R_1 < |u_j^n(t)| < R_2 \text{ for every } t \in (t_1,t_2), \tag{2.12}$$

$$|u_i^n(t)| \le R_2 \text{ for every } t \in [0,t_2] \text{ and every } i \ne j. \tag{2.13}$$

Then, for t varying from t_1 to t_2, by (2.8) the component u_j^n of the solution would be driven by the curve γ^j to make at least $\bar{n}+1$ clockwise revolutions around the origin, thus crossing at least $\bar{n}$ times the cone $\Theta(\theta_1^j, \theta_2^j)$, in the clockwise sense. Writing the solution in polar coordinates

$$u_i(t) = \rho_i(t)(\cos(\vartheta_i(t)), \sin(\vartheta_i(t))\,), \tag{2.14}$$

from (2.9) we have that, if $\theta_1^j \le \vartheta_j^n(t) \le \theta_2^j$, then

$$-(\vartheta_j^n)'(t) = \frac{\langle J f_j^n(t, u^n(t))\,, u_j^n(t)\rangle}{|u_j^n(t)|^2} \le C^j + 1.$$

So, the time to cross the cone $\Theta(\theta_1^j,\theta_2^j)$ in the clockwise sense is at least $(\theta_2^j-\theta_1^j)/(C^j+1)$, and then, by (2.7), the time to cross it $\bar n$ times should be greater than T. Hence, $t_2-t_1>T$, which is impossible.

The Poincaré map associated to (2.10) is then well defined on $\overline{\mathcal{B}}_{R_1}$. Let us now see that Theorem 2.1 can be applied for every n large enough, up to a subsequence, taking as $\mathcal{A}$ the set $\mathcal{B}_{R_1}$, in order to find a periodic solution to the equation (2.10).

Assume for the sake of contradiction that, for every n large enough, there exist $\mu_n>1$ and $\bar u^n\in\partial\mathcal{B}_{R_1}$ with

$$\bar u^n=(\bar u_1^n,\dots,\bar u_{i_n}^n,\dots,\bar u_h^n)\in\overline{B}^2_{R_1}\times\cdots\times\partial B^2_{R_1}\times\cdots\times\overline{B}^2_{R_1} \tag{2.15}$$

for a suitable i_n, such that the solution u^n of (2.10) with $u^n(0)=\bar u^n$ satisfies $u_{i_n}^n(T)=\mu_n u_{i_n}^n(0)$. We claim that, for n large enough, it has to be

$$R<|u_{i_n}^n(t)|<R_2,\qquad\text{for every } t\in[0,T]. \tag{2.16}$$

Indeed, we already proved above that $\max\{|u_i^n(t)|:t\in[0,T],\ i=1,\dots,h\}<R_2$. Assume for the sake of contradiction that $\min\{|u_{i_n}^n(t)|:t\in[0,T]\}\le R$. Then, since $|u_{i_n}^n(T)|>R_1$, there would be $\hat t_1$ and $\hat t_2$ in $[0,T]$, with $\hat t_1<\hat t_2$, such that $|u_{i_n}^n(\hat t_1)|=R$, $|u_{i_n}^n(\hat t_2)|=R_1$, and $R<|u_{i_n}^n(t)|<R_1$, for every $t\in(\hat t_1,\hat t_2)$. Then, for t varying from $\hat t_1$ to $\hat t_2$, by (2.8) the component $u_{i_n}^n$ of the solution would be driven by the curve γ^{i_n} to make at least $\bar n+1$ clockwise revolutions around the origin, thus crossing at least $\bar n$ times the cone $\Theta(\theta_1^{i_n},\theta_2^{i_n})$, in the clockwise sense. Arguing as above, we see that $\hat t_2-\hat t_1>T$, which is impossible. By (2.16), necessarily it has to be $1<\mu_n<\frac{R_2}{R_1}$, so, up to subsequences, we can assume that

$$i_n\equiv\iota,\qquad \mu_n\to\bar\mu\in[1,\tfrac{R_2}{R_1}],\quad\text{and}\quad \bar u^n\to\bar u\in\partial\mathcal{B}_{R_1}.$$

Moreover, since $(\mathcal{F}^n)_n$ converges to $\mathcal{F}$ uniformly in $[0,T]\times\overline{\mathcal{B}}_{R_2}$, there is a constant $M>0$ such that

$$|\mathcal{F}^n(t,u)|\le M,\qquad\text{for every } n\in\mathbb{N},\ t\in[0,T],\ \text{and } u\in\overline{\mathcal{B}}_{R_2}.$$

By (2.11), $u^n(t)\in\mathcal{B}_{R_2}$ for every $t\in[0,T]$, so $(u^n)_n$ is bounded in $C^1([0,T])$ and, by the Ascoli–Arzelà theorem, there is a continuous function $u:[0,T]\to\mathbb{R}^{2h}$ such that, up to a subsequence, $u^n\to u$ uniformly. Passing to the limit in

$$u^n(t)=\bar u^n+\int_0^t\mathcal{F}^n(\tau,u^n(\tau))\,d\tau,$$

we obtain

$$u(t)=\bar u+\int_0^t\mathcal{F}(\tau,u(\tau))\,d\tau,$$

so that u is a solution to the equation (2.1) with initial value $u(0) = \bar{u} \in \partial\mathcal{B}_{R_1}$. By (2.16),

$$R \leq |u_\iota(t)| \leq R_2, \qquad \text{for every } t \in [0, T], \tag{2.17}$$

and $u_\iota(T) = \bar{\mu} u_\iota(0)$. Hence, $|u_\iota(T)| \geq |u_\iota(0)|$ and, using polar coordinates as in (2.14), there is an integer k such that

$$\vartheta_\iota(T) = \vartheta_\iota(0) - 2\pi k.$$

By the angular velocity formula,

$$-(\vartheta_\iota)'(t) = \frac{\langle J u_\iota'(t), u_\iota(t)\rangle}{|u_\iota(t)|^2} = \frac{\langle J f_\iota(t, u(t)), u_\iota(t)\rangle}{|u_\iota(t)|^2};$$

as a consequence of (H2$^\iota$) we have that $k \leq -1$. Taking into account (2.17) and the fact that $\overline{\Omega}_0^\iota \subseteq B_R^2$, let $\bar{m} \in \mathbb{Z}$ be such that

$$|\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar{m} - 1))| < |u_\iota(0)| \leq |\gamma^\iota(-\vartheta_\iota(0) + 2\pi\bar{m})|.$$

(Recall that γ^ι is parametrized in clockwise polar coordinates.) Then, by the admissibility of the curve γ^ι and (2.17), since B_R^2 contains $\overline{\Omega}_0^\iota$, it has to be that

$$|u_\iota(t)| < |\gamma^\iota(-\vartheta_\iota(t) + 2\pi\bar{m})|, \qquad \text{for every } t \in]0, T].$$

So, using (2.6),

$$\begin{aligned} |u_\iota(T)| &< |\gamma^\iota(-\vartheta_\iota(T) + 2\pi\bar{m})| = |\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar{m} + k))| \\ &\leq |\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar{m} - 1))| < |u_\iota(0)|, \end{aligned}$$

and we get a contradiction with the fact that $|u_\iota(T)| \geq |u_\iota(0)|$.

So, up to a subsequence, for every $\bar{u}^n \in \partial\mathcal{B}_{R_1}$ (associated, as in (2.15), with an index i_n such that $|\bar{u}^n_{i_n}| = R_1$), the solution u^n of (2.10) with $u^n(0) = \bar{u}^n$ is such that $u^n_{i_n}(T) \neq \mu\bar{u}^n_{i_n}$, for every $\mu > 1$. We can then apply Theorem 2.1 to find a T-periodic solution $v^n(t)$ of (2.10), for n large enough, up to a subsequence, starting from a point $\bar{v}^n \in \overline{\mathcal{B}}_{R_1}$. Using the Ascoli–Arzelà theorem again, we find that, up to a subsequence, $(v^n)_n$ converges to a T-periodic solution of equation (2.1). □

3. Some applicative conditions

In this section we introduce three other hypotheses which are useful to obtain (H1^i), (H2^i), and (H3^i). These conditions are simply the generalizations to the $2h$-dimensional case of the conditions (H4), (H5), and (H6) introduced in [9].

(H4^i) There exist $R > 0$ and $\eta > 0$ such that, for every $v \in \mathbb{R}^{2h}$,

$$|v_i| \geq R \quad \Rightarrow \quad \langle Jf_i(t,v), v_i\rangle \geq \eta |v_i|^2, \quad \text{for every } t \in [0,T].$$

(H5^i) There exists a continuous function $\chi : [0,+\infty) \to (0,+\infty)$ such that

$$\langle f_i(t,v), v_i\rangle \leq \chi(|v_i|), \quad \text{for every } t \in [0,T] \text{ and } v \in \mathbb{R}^{2h}, \tag{3.1}$$

and

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty.$$

(H6^i) There exist some values $w_1, \dots, w_m \in S^1$ and two positive functions

$$\psi_1, \psi_2 : S^1 \setminus \{w_1, \dots, w_m\} \to (0, +\infty],$$

not identically equal to $+\infty$, with the following properties:
(i) in each open arc of the domain these functions are either continuous and bounded with all values in $\mathbb{R}$, or identically equal to $+\infty$;
(ii) one has

$$\psi_1(w) \leq \liminf_{\alpha\to+\infty} \Big\langle \frac{Jf_i(t,u_1,\dots,u_{i-1},\alpha w,u_{i+1},\dots,u_h)}{\alpha}\ ,\ w \Big\rangle \tag{3.2}$$
$$\leq \limsup_{\alpha\to+\infty} \Big\langle \frac{Jf_i(t,u_1,\dots,u_{i-1},\alpha w,u_{i+1},\dots,u_h)}{\alpha}\ ,\ w \Big\rangle \leq \psi_2(w),$$

uniformly for $t \in [0,T]$, $(u_1,\dots,u_{i-1},u_{i+1},\dots,u_h) \in \mathbb{R}^{2h-2}$, and w in any compact subset of $S^1 \setminus \{w_1,\dots,w_n\}$;
(iii) moreover,

$$\Big[\int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}\, , \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})} \Big] \ \cap\ \Big\{ \frac{T}{N} : N \in \mathbb{N}_0 \Big\} = \emptyset, \tag{3.3}$$

where $\mathbb{N}_0$ denotes the set of positive integers.

Notice that, in (3.3), we use the convention that $\frac{1}{+\infty} = 0$, and implicitly assume that the integrals have finite values. In [9], the following two propositions have been proved.

Proposition 3.1. *If* (H4^i) *and* (H5^i) *hold, then* (H1^i) *is satisfied.*

Proposition 3.2. *If* (H4^i) *and* (H6^i) *hold, then* (H2^i) *and* (H3^i) *are satisfied.*

We will see below how condition (H5^i) can be weakened when dealing with different regions in the phase plane.

4. Applications

In this section we are going to prove the existence of a T-periodic solution to the following system:

$$\begin{cases} Ju_1' = g_1(t,u_1) + r_1(t,u_1,\dots,u_h) \\ Ju_2' = g_2(t,u_2) + r_2(t,u_1,\dots,u_h) \\ \quad\vdots \\ Ju_h' = g_h(t,u_h) + r_h(t,u_1,\dots,u_h). \end{cases} \tag{4.1}$$

We assume that, for every $i \in \{1,\dots,h\}$, writing $u_i = (x_i, y_i)$, the i-th equation of the system has the following form:

$$\begin{cases} -y_i' = g_{i,1}(t,x_i) + r_{i,1}(t,x_1,y_1,\dots,x_h,y_h) \\ x_i' = g_{i,2}(t,y_i) + r_{i,2}(t,x_1,y_1,\dots,x_h,y_h), \end{cases} \tag{4.2}$$

where the functions $r_{i,j} : \mathbb{R}\times\mathbb{R}^{2h} \to \mathbb{R}$ are continuous, and T-periodic in their first variable. Moreover, we assume that there exist functions $p_{i,j} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|r_{i,j}(t,u_1,\dots,u_h)| \le p_{i,j}(u_i), \quad \text{with} \quad \lim_{|u_i|\to+\infty} \frac{p_{i,j}(u_i)}{|u_i|} = 0.$$

The functions $g_{i,j} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and T-periodic in their first variable. We assume that, for $j,k \in \{1,2\}$ and $i \in \{1,\dots,h\}$, there are some constants $\mu^i_{j,k}, \nu^i_{j,k} \in (0,+\infty]$ such that

$$\mu^i_{j,1} \;\le\; \liminf_{\xi\to+\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\le\; \limsup_{\xi\to+\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\le\; \mu^i_{j,2}, \tag{4.3}$$

$$\nu^i_{j,1} \;\le\; \liminf_{\xi\to-\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\le\; \limsup_{\xi\to-\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\le\; \nu^i_{j,2}. \tag{4.4}$$

With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau^i_k = \frac{\pi}{2}\Big(\frac{1}{\sqrt{\mu^i_{1,k}\,\mu^i_{2,k}}} + \frac{1}{\sqrt{\nu^i_{1,k}\,\mu^i_{2,k}}} + \frac{1}{\sqrt{\nu^i_{1,k}\,\nu^i_{2,k}}} + \frac{1}{\sqrt{\mu^i_{1,k}\,\nu^i_{2,k}}}\Big), \tag{4.5}$$

for $i \in \{1,\dots,h\}$ and $k \in \{1,2\}$.

Theorem 4.1. *Assume that all the constants in* (4.3) *and* (4.4) *are finite, and*

$$[\tau^i_2, \tau^i_1] \cap \Big\{ \tfrac{T}{N} : N \in \mathbb{N}_0 \Big\} = \emptyset, \qquad \textit{for every } i \in \{1,\dots,h\}. \tag{4.6}$$

Then, system (4.1) *has a T-periodic solution. The same is true if, for one or more indices i, one of the constants $\mu^i_{1,2}$, $\nu^i_{1,2}$, $\mu^i_{2,2}$, and $\nu^i_{2,2}$ is equal to $+\infty$, the three others being finite.*

The procedure to verify that (H1^i), (H2^i), and (H3^i) hold for every i is independent of the index i, so in the following we will consider the case $i = 1$ and, to simplify the notation, we will often write $u_1 = (x, y)$ instead of (x_1, y_1), and $\lambda = (x_2, y_2, \dots, x_h, y_h)$. Hence we have

$$Jf_1(t, x, y, \lambda) = g_1(t, x, y) + r_1(t, x, y, \lambda).$$

In the proof we will need the following refinement of Proposition 3.1, where we focus our attention on the admissibility of a regular spiral on a particular subset U of $\mathbb{R}^2$.

Lemma 4.2. *Given a positive constant η, a point $P_0 \in \mathbb{R}^2$, and a continuous function $\chi : [0, +\infty) \to (0, +\infty)$ such that*

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty,$$

it is possible to build a clockwise rotating regular spiral $\tilde{\gamma}$, passing through P_0, such that $s \mapsto |\tilde{\gamma}(s)|$ is strictly increasing, which is 1-admissible for system (4.1) *in any set U where*

(1) $\langle Jf_1(t, x, y, \lambda), (x, y)\rangle \geq \eta(x^2 + y^2)$,
(2) $\langle f_1(t, x, y, \lambda), (x, y)\rangle \leq \chi(\sqrt{x^2 + y^2})$,

for every $t \in [0, T]$, $(x, y) \in U$, and $\lambda \in \mathbb{R}^{2h-2}$.

Proof. Let $P_0 = r_0 e^{-is_0}$. We define the curve $\tilde{\gamma} : [0, +\infty) \to \mathbb{R}^2$ as

$$\tilde{\gamma}(s) = r(s)(\cos s, -\sin s),$$

where $r(s)$ is the solution of the Cauchy problem

$$\dot{r} = \frac{2}{\eta}\,\frac{\chi(r)}{r}, \qquad r(s_0) = r_0.$$

Notice that $\gamma(s_0) = P_0$. Since this curve is smooth, the sequence $(\sigma_k)_k$, in this case, is arbitrary. Clearly, (2.2) and (2.4) hold, since s is the angle in clockwise polar coordinates. We see that $r(s)$ is strictly increasing, and remains bounded for s bounded. Moreover, $r(s) \to +\infty$ for $s \to +\infty$, so that condition (2.3) is satisfied as well. We compute:

$$\langle J\dot{\tilde{\gamma}}(s), f_1(t, \tilde{\gamma}(s), \lambda)\rangle = \frac{\dot{r}(s)}{r(s)}\,\langle J\tilde{\gamma}(s), f_1(t, \tilde{\gamma}(s), \lambda)\rangle + \langle \tilde{\gamma}(s), f_1(t, \tilde{\gamma}(s), \lambda)\rangle.$$

Using the assumptions, we have that, for $\tilde{\gamma}(s) \in U$,

$$\langle J\dot{\tilde{\gamma}}(s), f_1(t, \tilde{\gamma}(s), \lambda)\rangle \leq -\eta\, \dot{r}(s) r(s) + \chi(r(s)) < 0,$$

for every $t \in [0, T]$ and $\lambda \in \mathbb{R}^{2h-2}$, thus completing the proof. □

Proof of Theorem 4.1. It is easy to see that the functions ψ_1 and ψ_2 which are involved in ($\mathrm{H6}^1$) are

$$\psi_1(e^{i\theta}) = \begin{cases} \mu^1_{1,1} \cos^2\theta + \mu^1_{2,1} \sin^2\theta, & \text{if } \theta \in (0, \frac{\pi}{2}), \\ \nu^1_{1,1} \cos^2\theta + \mu^1_{2,1} \sin^2\theta, & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ \nu^1_{1,1} \cos^2\theta + \nu^1_{2,1} \sin^2\theta, & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ \mu^1_{1,1} \cos^2\theta + \nu^1_{2,1} \sin^2\theta, & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi), \end{cases}$$

and

$$\psi_2(e^{i\theta}) = \begin{cases} \mu^1_{1,2} \cos^2\theta + \mu^1_{2,2} \sin^2\theta, & \text{if } \theta \in (0, \frac{\pi}{2}), \\ \nu^1_{1,2} \cos^2\theta + \mu^1_{2,2} \sin^2\theta, & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ \nu^1_{1,2} \cos^2\theta + \nu^1_{2,2} \sin^2\theta, & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ \mu^1_{1,2} \cos^2\theta + \nu^1_{2,2} \sin^2\theta, & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Since all the constants in (4.3) and (4.4) are strictly positive, ($\mathrm{H4}^1$) holds. Solving the integral, it is easy to see that also ($\mathrm{H6}^1$) holds (see [8] for computations). So, by Proposition 3.2, conditions ($\mathrm{H2}^1$) and ($\mathrm{H3}^1$) hold.

If all the constants $\mu^1_{1,2}$, $\nu^1_{1,2}$, $\mu^1_{2,2}$, and $\nu^1_{2,2}$ are finite, the nonlinearity has an at-most linear growth, so ($\mathrm{H5}^1$) holds with $\chi(r) = ar^2 + b$ for some suitable constants a and b. By Proposition 3.1, condition ($\mathrm{H1}^1$) is satisfied, and the proof is completed in this case, by Theorem 2.4.

We now consider the case in which one of these constants is equal to $+\infty$. For example, we assume $\mu^1_{1,2} = +\infty$.

In order to build an admissible spiral γ^1 in this case, we will glue together pieces of curves belonging to some regions of the plane. By construction the curve will pass through some points P_α, whose distance from the origin gradually increases, giving to it the shape of a regular spiral. In what follows, we will sometimes use Lemma 4.2, whose condition 1 is satisfied by ($\mathrm{H4}^1$), so that we will only need to find a suitable function χ in order to apply it.

Let $\epsilon > 0$ be fixed, in such a way that $\epsilon < \frac{1}{8} \min\left\{\mu^1_{1,1}, \nu^1_{1,1}, \mu^1_{2,1}, \nu^1_{2,1}\right\}$. Then, there exists $R > 0$ such that, for every (x, y) for which $|x| \geq R$ and $|y| \geq R$,

$$|r_{i,j}(t, x, y, \lambda)| \leq p_{1,j}(x, y) \leq \epsilon(\,|x| + |y|\,), \qquad j = 1, 2.$$

We can assume $R > 0$ large enough to have

$$x \geq R \qquad \Rightarrow \qquad 0 < \mu^1_{1,1}\, x \leq g_{1,1}(t, x),$$

$$
\begin{aligned}
x \le -R \quad &\Rightarrow \quad \nu^1_{1,2}\, x \le g_{1,1}(t,x) \le \nu^1_{1,1}\, x < 0,\\
y \ge R \quad &\Rightarrow \quad 0 < \mu^1_{2,1}\, y \le g_{1,2}(t,y) \le \mu^1_{2,2}\, y,\\
y \le -R \quad &\Rightarrow \quad \nu^1_{2,2}\, y \le g_{1,2}(t,y) \le \nu^1_{2,1}\, y < 0,
\end{aligned}
$$

slightly modifying these constants, if necessary, without affecting (4.6).

Moreover, we have the existence of a constant $C > 0$ such that

$$
\begin{aligned}
|x| \le R \quad &\Rightarrow \quad |g_{1,1}(t,x)| \le C,\\
|y| \le R \quad &\Rightarrow \quad |g_{1,2}(t,y)| \le C.
\end{aligned}
$$

We consider five different regions in the phase plane (see Figure 1):

$$
\begin{aligned}
W &= (-\infty, R] \times \mathbb{R}, \quad NE = [R,+\infty) \times [R,+\infty),\\
E &= [R,+\infty) \times [-R,R],\\
ESE &= [R,+\infty) \times (-\infty,-R] \cap \{(x,y) : x \ge -y\},\\
SSE &= [R,+\infty) \times (-\infty,-R] \cap \{(x,y) : x \le -y\}.
\end{aligned}
$$

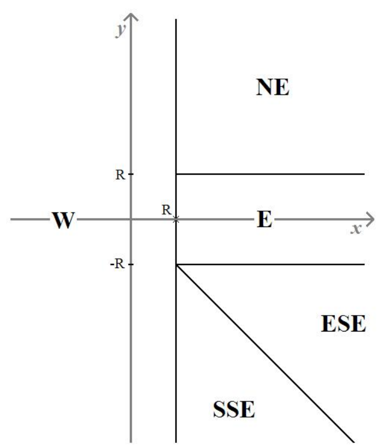

FIGURE 1. The regions in the phase plane.

The regular spiral γ^1 will be constructed by gluing together pieces of curves belonging to each of these regions.

Region W. We note that, in this region,

$$
\begin{aligned}
|g_{1,2}(t,y)| &\le C + \max\{\mu^1_{2,1}, \mu^1_{2,2}, \nu^1_{2,1}, \nu^1_{2,2}\}|y|,\\
|g_{1,1}(t,x)| &\le C + \max\{\nu^1_{1,2}, \nu^1_{1,1}\}|x|,
\end{aligned}
$$

giving us, for every $(x,y) \in W$,

$$\begin{aligned}
&\langle f_1(t,x,y,\lambda)(x,y)\rangle \\
&= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y \\
&\le C_1\,(x^2+y^2) + C_2,
\end{aligned}$$

for some suitable constants C_1 and C_2. Fix a point $P_0 = (R, y_0)$ with $y_0 < -R$. By Lemma 4.2, taking $U = W$, we can build the spiral $\tilde\gamma$ which passes through $P_0 = \tilde\gamma(s_0)$. There exists $s_1 > s_0$ such that $\tilde\gamma([s_0,s_1]) \subset W$ and $\tilde\gamma(s_1) = P_1 = (R, y_1)$ with $y_1 > R$. The spiral γ_1 in W consists of the branch of $\tilde\gamma$ which goes from P_0 to P_1, and it is abmissible in W by construction.

Region NE. We have, for every $(x,y) \in NE$,

$$\begin{aligned}
&\langle f_1(t,x,y,\lambda)(x,y)\rangle \\
&= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y \\
&\le \mu^1_{2,2}\,xy + 0 + \epsilon(x+y)^2 \le M\,(x^2+y^2)
\end{aligned}$$

for a suitable constant M. Similarly to what has been done in the region W, applying Lemma 4.2 with $U = NE$, we can construct γ^1 going from P_1 to a point $P_2 = (x_2, R)$ with $x_2 > R$.

Region E. In this region, we construct the spiral γ^1 as a line $y = -mx$ where $0 < m < 1$ is sufficiently small. We recall that, here,

$$|g_{1,2}(t,y)| \le C,\ g_{1,1}(t,x) \ge \mu^1_{1,1}x,\ |r_{1,i}(t,x,y,\lambda)| \le \epsilon(|x|+|y|) \le 2\epsilon x,\ i=1,2.$$

Hence, we have

$$\begin{aligned}
&\langle J\dot\gamma^1(s) f_1(t,\gamma^1(s),\lambda)\rangle = \langle (m,1)(x',y')\rangle \\
&= m(g_{1,2}(t,y) + r_{1,2}(t,x,y,\lambda)) - (g_{1,1}(t,x) + r_{1,1}(t,x,y,\lambda)) \\
&\le mC + 2m\epsilon x - \mu^1_{1,1}x + 2\epsilon x \\
&\le mC - (\mu^1_{1,1} - 4\epsilon)x \le mC - \tfrac{1}{2}\mu^1_{1,1}R,
\end{aligned}$$

which is negative choosing $m < \mu^1_{1,1}R/2C$. In this way we build a branch of the spiral γ^1 which goes from P_2 to a point $P_3 = (x_3, -R)$, with $x_3 > x_2 > R$.

Region ESE. In this region the spiral γ^1 simply coincides with the line $y = -2(x - x_3) - R$. Let $P_4 = (x_4, -x_4)$ be the intersection between this line and the line $y = -x$. We recall that, here,

$$g_{1,2}(t,y) \le \nu^1_{2,1}\,y < 0, \quad g_{1,1}(t,x) \ge \mu^1_{1,1}\,x,$$

$$|r_{1,i}(t,x,y,\lambda)| \le \epsilon(|x|+|y|) \le 2\epsilon\,x, \quad i = 1,2.$$

So we have

$$\begin{aligned}&\langle J\dot\gamma^1(s) f_1(t,\gamma^1(s),\lambda)\rangle = \langle (2,1)(x',y')\rangle\\&= 2(g_{1,2}(t,y)+r_{1,2}(t,x,y,\lambda)) - (g_{1,1}(t,x)+r_{1,1}(t,x,y,\lambda))\\&\le 2(0+2\epsilon\, x) - (\mu^1_{1,1}\, x - 2\epsilon\, x) = -(\mu^1_{1,1}-6\epsilon)\, x < 0.\end{aligned}$$

Region SSE. In this region the following inequalities hold for $i = 1,2$:

$$g_{1,2}(t,y) \le \nu^1_{2,1} y < 0, \quad g_{1,1}(t,x) \ge \mu^1_{1,1} x, \quad |r_{1,i}(t,x,y,\lambda)| \le \epsilon(x-y) \le -2\epsilon y.$$

At first, we note that for a solution of (4.1), with the above notation, as long as $(x(t), y(t))$ belongs to this region, we have that

$$x'(t) = g_{1,2}(t,y(t)) + r_{1,2}(t,x(t),y(t),\lambda) \le (\nu^1_{2,1} - 2\epsilon)\, y(t) < 0. \tag{4.7}$$

We have to build the spiral γ^1 starting from the point $P_4 = (x_4, -x_4)$. Call SSE_{good} the region $SSE \cap \{x \le x_4\}$ and SSE_{bad} the region $SSE \cap \{x > x_4\}$ (see Figure 2).

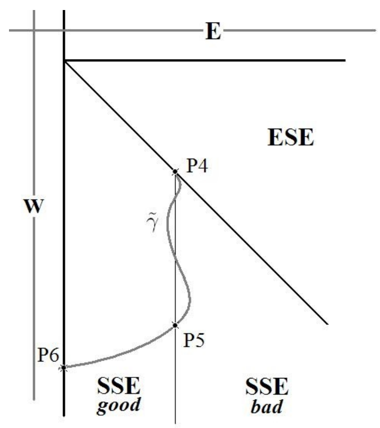

FIGURE 2. The construction of the curve in the region SSE.

Note that, for every $(x,y) \in SSE_{good}$,

$$\begin{aligned}&\langle f_1(t,x,y,\lambda)(x,y)\rangle\\&= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y\\&\le 0 - M_1 y + \epsilon(x-y)^2 \le M_2\,(x^2+y^2) + M_1,\end{aligned}$$

for some suitable constants M_1 and M_2. Setting $U = SSE_{good}$, it is possible to apply Lemma 4.2 to obtain a spiral $\tilde\gamma$ which links P_4 to a point $P_6 = (R, y_6)$, with $y_6 < -R$, passing through SSE. By construction, this spiral is 1-admissible only in SSE_{good}. Nothing tells us that $\tilde{\gamma}$ does not enter in the region SSE_{bad}, but there exists a point $P_5 = (x_5, y_5)$ with $x_5 = x_4$ and $y_5 \leq y_4$ (possibly $P_5 = P_4$) on the curve $\tilde{\gamma}$ after which $\tilde{\gamma}$ is contained in SSE_{good}. Using (4.7), we choose the spiral γ^1 to be made of the vertical line linking P_4 and P_5 and of the branch of $\tilde{\gamma}$ linking P_5 and P_6.

With such a procedure we have constructed the first lap of the spiral γ^1. In the same way we can obtain the other ones. Such a spiral is 1-admissible in the whole plane by construction. So, (H1^1) holds, and the proof is completed in this case, too. □

5. An example: coupled oscillators

As a particular case of (4.1), we have the following system of coupled oscillators:

$$\begin{cases} x_1'' + \phi_1(t, x_1) = e_1(t, x_1, \dots, x_h) \\ x_2'' + \phi_2(t, x_2) = e_2(t, x_1, \dots, x_h) \\ \quad\vdots \\ x_h'' + \phi_h(t, x_h) = e_h(t, x_1, \dots, x_h). \end{cases} \tag{5.1}$$

Here we assume that, for every i,

$$|e_i(t, x_1, \dots, x_h)| \leq p_i(x_i), \quad \text{with} \quad \lim_{|x_i| \to +\infty} \frac{p_i(x_i)}{|x_i|} = 0,$$

and that the function ϕ_i satisfies

$$\mu_1^i \leq \liminf_{\xi \to +\infty} \frac{\phi_i(t, \xi)}{\xi} \leq \limsup_{\xi \to +\infty} \frac{\phi_i(t, \xi)}{\xi} \leq \mu_2^i, \tag{5.2}$$

$$\nu_1^i \leq \liminf_{\xi \to -\infty} \frac{\phi_i(t, \xi)}{\xi} \leq \limsup_{\xi \to -\infty} \frac{\phi_i(t, \xi)}{\xi} \leq \nu_2^i, \tag{5.3}$$

for some suitable constants in $(0, +\infty]$. With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau_1^i = \frac{\pi}{\sqrt{\mu_1^i}} + \frac{\pi}{\sqrt{\nu_1^i}}, \qquad \tau_2^i = \frac{\pi}{\sqrt{\mu_2^i}} + \frac{\pi}{\sqrt{\nu_2^i}}, \tag{5.4}$$

for $i \in \{1, \dots, h\}$. As an immediate consequence of Theorem 2.4, we have the following.

Corollary 5.1. *Assume that all the constants in* (5.2) *and* (5.3) *are finite, and*

$$[\tau_2^i, \tau_1^i] \cap \left\{ \tfrac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset, \qquad \textit{for every } i \in \{1, \dots, h\}. \tag{5.5}$$

Then, system (5.1) *has a T-periodic solution. The same is true if, for one or more indices i, one of the constants μ_2^i and ν_2^i is equal to $+\infty$, the other being finite.*

Let us show an example of a situation which permits us to apply the previous corollary. We will use the following notation: for every $\xi \in \mathbb{R}$ we write $\xi^+ = \max\{\xi, 0\}$ and $\xi^- = \max\{-\xi, 0\}$, and for every $x = (x_1, x_2, \dots, x_h) \in \mathbb{R}^h$ we write $x^+ = (x_1^+, x_2^+, \dots, x_h^+)$, $x^- = (x_1^-, x_2^-, \dots, x_h^-)$, and $\exp(x) = (e^{x_1}, e^{x_2}, \dots, e^{x_h})$. Consider the equation in $\mathbb{R}^h$

$$x'' - B(t)\arctan(\|x\|)\, x^- + \exp(x^+) = a(t), \tag{5.6}$$

where $\|x\|$ is a norm in $\mathbb{R}^h$, $a : \mathbb{R} \to \mathbb{R}^h$ is a T-periodic continuous function, and $B(t) = \operatorname{diag}(b_1(t), \dots, b_h(t))$ is a diagonal matrix where each $b_i : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. We assume that there exist some positive integers N_i, and a constant $\delta > 0$, such that

$$\tfrac{1}{2\pi}(\lambda_{N_i} + \delta) < b_i(t) < \tfrac{1}{2\pi}(\lambda_{N_i+1} - \delta),$$

where $\lambda_k = (2\pi k/T)^2$ is the k-th eigenvalue for the T-periodic problem. We can see that this is a particular case of system (5.1), with

$$\begin{aligned} \phi_i(t, x_i) &= -\frac{\pi}{2} b_i(t)\, x_i^- + \exp(x_i^+), \\ e_i(t, x_1, \dots, x_h) &= \left(\arctan(\|x\|) - \tfrac{\pi}{2}\right) b_i(t)\, x_i^- + a_i(t), \end{aligned}$$

and we find the values

$$\mu_2^i = +\infty, \quad \nu_1^i = \tfrac{1}{4}(\lambda_{N_i} + \delta), \quad \text{and} \quad \nu_2^i = \tfrac{1}{4}(\lambda_{N_i+1} - \delta),$$

for every i. Thus, choosing μ_1^i large enough, we have

$$\frac{T}{N_i + 1} < \tau_2^i = \frac{2\pi}{\sqrt{\lambda_{N_i+1} - \delta}} < \tau_1^i = \frac{2\pi}{\sqrt{\lambda_{N_i} + \delta}} + \frac{\pi}{\sqrt{\mu_1^i}} < \frac{T}{N_i},$$

for every i. Corollary 5.1 can thus be applied, so that equation (5.6) has a T-periodic solution.

Remark 5.2. We have focused our attention on one particular situation where our conditions ($H1^i$), ($H2^i$) and ($H3^i$) hold, for every i. As we have shown in [9], many other different cases can be treated with the same approach, like, e.g., nonlinearities controlled by positively homogeneous Hamiltonian functions, Landesman–Lazer situations at resonance, and nonlinearities with a singularity. Our theorem permits us to mix together all these situations. For example, one could think about a system in $\mathbb{R}^6$ with a one-sided superlinearity in the first couple of variables, a resonance case with a

Landesman–Lazer condition in the second one, and a singularity in the last one.

Remark 5.3. One can extend our results to the case when the phase space $\mathbb{R}^{2h}$ is replaced by a space of the type $\mathbb{R}^{2h+k}$, for some $k \geq 1$, introducing some hypotheses on the last k coordinates. For example, one could think of some kind of dissipative situation, so that a variant of Theorem 2.1 will be applicable. For briefness, we will not enter into details here.

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