

WELL POSEDNESS FOR HIROTA-SATSUMA'S EQUATION

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Abstract. We are interested in the initial-value problem associated to the Hirota-Satsuma equation in the real line

$$u_t + u_x - 2uu_t + 2u_x \int_x^\infty u_t dx' - u_{txx} = 0, \quad x \in \mathbb{R},$$

where u is a real-valued function. This equation models the unidirectional propagation of shallow water waves as the well-known Korteweg-de Vries and Benjamin-Bona-Mahony equations. Here we show local well posedness for initial data in the space

$$\Omega_s = \{\phi \in H^s(\mathbb{R}) : -1 \notin \sigma(-\partial_x^2 - 2\phi)\} \text{ if } s > \frac{1}{2},$$

and small initial data in $H^s(\mathbb{R})$ if $0 \leq s \leq \frac{1}{2}$. We also prove global well posedness for small energy data in $H^1(\mathbb{R})$.

1. INTRODUCTION

We consider the following equation derived by Hirota and Satsuma in [8] (see also [14] and [15]) in the context of shallow-water wave equations.

$$u_t - u_{txx} + u_x - 2uu_t + 2u_x \int_x^\infty u_t dx' = 0, \quad x \in \mathbb{R}, t > 0, \quad (1.1)$$

where u is a real-valued function. This equation models the unidirectional propagation of shallow water waves as the well-known Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} + u_x + 4uu_x = 0, \quad (1.2)$$

and Benjamin-Bona-Mahony equation (BBM)

$$u_t - u_{txx} + u_x + 4uu_x = 0. \quad (1.3)$$

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The initial-value problem (IVP) associated to these last two equations has been extensively studied in recent years. It has been shown using the Fourier transform restriction method, introduced by Bourgain [5] in the context of dispersive equations, that the IVP associated to the KdV equation is locally [12] and globally [7] well posed in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$. This result turns out to be sharp [6] in the sense that, for $s < -\frac{3}{4}$, the flow map associated to the KdV equation cannot be even uniformly continuous in $H^s(\mathbb{R})$. It is worth noticing that for $s > -\frac{3}{4}$ the flow map is smooth and even analytic in $H^s(\mathbb{R})$ as a direct consequence of the Picard iteration scheme used to solve the associated integral equation [16].

In the case of the BBM equation, well-posedness results are in some sense “easier” to obtain since the equation can be rewritten in the following form

$$u_t = -(1 - \partial_x^2)^{-1} \partial_x (u + 2u^2),$$

so that one can use the smoothing properties of the operator $(1 - \partial_x^2)^{-1}$. In [4], Bona and Tzvetkov proved global well posedness in $H^s(\mathbb{R})$ for $s \geq 0$, improving former results by Benjamin, Bona and Mahony [2]. More surprising is the fact that well posedness breaks down in $H^s(\mathbb{R})$ for $s < 0$, if one asks the flow map to be C^2 (see [4]). Thus in this sense, solutions of the KdV equation behave better than solutions of the BBM equation at low regularity.

The objective here is to derive a well-posedness theory for the IVP associated to the Hirota-Satsuma equation (HS)

$$\begin{cases} u_t + u_x - u_{txx} - 2uu_t + 2u_x \int_x^\infty u_t dx' = 0, \\ u(0) = u_0, \quad x \in \mathbb{R}, \end{cases} \quad (1.4)$$

since there does not exist, as far as we know, any existence result regarding strong solutions for this equation. The main idea is to rewrite the equation in a more manageable form (see Section 2) as in the case of BBM. However, in the present situation we get a more complicated equation, namely

$$u_t = -\partial_x (1 - \partial_x^2 - 2u)^{-1} u. \quad (1.5)$$

In this case the associated smoothing operator $(1 - \partial_x^2 - 2u)^{-1}$ may not be well defined whenever -1 belongs to the spectrum of $-\partial_x^2 - 2u$. Therefore, we have to make the additional assumption that the initial data u_0 belongs to the set

$$\Omega_s = \{\phi \in H^s(\mathbb{R}) : -1 \notin \sigma(-\partial_x^2 - 2\phi)\}, \quad s \geq 0.$$

It will be proved that Ω_s is an open set in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$. This allows us to derive a local well-posedness theory in small balls of Ω_s whenever $s > \frac{1}{2}$, using a Picard iteration scheme on the integral equation associated to (1.5). In the case $0 \leq s \leq \frac{1}{2}$, we will show the existence of an open ball in $H^s(\mathbb{R})$ centered at the origin and contained in Ω_s . Then we can derive a similar local well-posedness theory for initial data in this ball.

Next we take advantage of the quantity

$$E(u)(t) = \int_{\mathbb{R}} \frac{1}{2}u(x,t)^2 + \frac{1}{2}\partial_x u(x,t)^2 - \frac{1}{3}u(x,t)^3 dx,$$

conserved along the flow of (1.1), to extend the H^1 -solutions globally in time for initial data u_0 satisfying the conditions

$$\|u_0\|_{H^1} < \frac{1}{2}\|\varphi^*\|_{H^1} \quad \text{and} \quad E(u_0) < E\left(\frac{1}{2}\varphi^*\right), \quad (1.6)$$

where φ^* is a critical point of E (see Proposition 1). For $c > 1$, equation (1.1) admits solitary-wave solutions of the form $u_c(x,t) = \phi_\sigma(x-ct)$, where

$$\phi_\sigma(x) = \frac{3}{4}\sigma^2 \operatorname{sech}^2\left(\frac{\sigma}{2}x\right), \quad \text{and} \quad \sigma = \sqrt{1 - \frac{1}{c}} \in (0,1). \quad (1.7)$$

It is worth noticing that all these travelling waves solutions satisfy conditions (1.6) (see Remark 3). Therefore, it seems natural to conjecture that the solitary waves of the HS equation are stable in shape as with the BBM and KdV equations [1].

The paper is organized as follows. In Section 2, we perform the transformation on the equation and deal with the eigenvalue problem. We establish the local well-posedness theory in Ω_s for $s > \frac{1}{2}$ in Section 3, and the local well-posedness theory for small initial data in $H^s(\mathbb{R})$, $0 \leq s \leq \frac{1}{2}$, in Section 4. Finally global solutions for initial data satisfying conditions (1.6) are obtained in Section 5.

Notation. The following notation will be used throughout this article. For any positive functions a and b , the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

When $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the set of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{f}(\xi)|^2 \right)^{\frac{1}{2}} < +\infty,$$

where \widehat{f} is the Fourier transform of f . D_x^s will denote the fractional derivative of order s defined by

$$(D_x^s f)^\wedge(\xi) = |\xi|^s \widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall f \in \mathcal{S}'(\mathbb{R}),$$

so that

$$\|f\|_{H^s} \sim \|f\|_{L^2} + \|D_x^s f\|_{L^2}.$$

We also observe that $D^1 = \mathcal{H}\partial_x$, where \mathcal{H} is the Hilbert transform, defined via Fourier transform by

$$(\mathcal{H}f)^\wedge(\xi) = -i\text{sgn}(\xi)\widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}).$$

2. THE EIGENVALUE PROBLEM

First we must rewrite the equation in a more manageable form. To do this, we note that

$$\partial_x \left(u \int_x^{+\infty} u_t dx' \right) = u_x \int_x^{+\infty} u_t dx' - uu_t, \tag{2.1}$$

so that (1.1) becomes

$$(1 - \partial_x^2)u_t + 2\partial_x \left(u \int_x^{+\infty} u_t dx' \right) = -u_x.$$

Now we integrate from x to $+\infty$ to obtain

$$(1 - \partial_x^2) \int_x^{+\infty} u_t dx' - 2u \int_x^{+\infty} u_t dx' = u.$$

It follows that

$$(1 - \partial_x^2 - 2u) \int_x^{+\infty} u_t dx' = u$$

and

$$\int_x^{+\infty} u_t dx' = (1 - \partial_x^2 - 2u)^{-1}u.$$

Differentiating with respect to x we get equation (1.5). The question of when the operator $(1 - \partial_x^2 - 2u)^{-1}$ makes sense will be one of our main concerns in what follows.

Let us denote $H_0 := -\partial_x^2$. From now on we will assume that $\phi \in L^2(\mathbb{R})$, so that the operator

$$M_\phi : \begin{cases} \mathcal{D}(M_\phi) := \{f \in L^2(\mathbb{R}) : \phi f \in L^2(\mathbb{R})\} \rightarrow L^2(\mathbb{R}), \\ f \mapsto \phi f \end{cases}$$

is H_0 -compact. Moreover, we observe that

$$\mathcal{D}(H_0) = H^2(\mathbb{R}) \subseteq L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \subseteq \mathcal{D}(M_\phi).$$

Thus, the Kato-Rellich theorem [9] implies that the formal differential operator $H_\phi = -\partial_x^2 - 2\phi$ defines an unbounded self-adjoint operator

$$H_\phi : \mathcal{D}(H_\phi) = H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad f \mapsto (H_0 - 2\phi)f.$$

Moreover it can be shown that H_ϕ is bounded from below and that the essential spectrum of H_ϕ is

$$\sigma_e(H_\phi) = \sigma_e(H_0) = [0, +\infty).$$

This means that the spectrum of H_ϕ in $(-\infty, 0)$ consists of eigenvalues with finite multiplicity. It follows that

$$R_\phi(-1) := (1 - \partial_x^2 - 2\phi)^{-1} \tag{2.2}$$

exists if and only if -1 is not an eigenvalue of H_ϕ . For $s \geq 0$, we will denote

$$\Omega_s := \{\phi \in H^s(\mathbb{R}) : -1 \notin \sigma(H_\phi)\} \quad \text{and} \quad \mathcal{F}_s = H^s(\mathbb{R}) \setminus \Omega_s. \tag{2.3}$$

It will be proved in the next section that Ω_s is an open subset of $H^s(\mathbb{R})$ whenever $s > \frac{1}{2}$.

Finally we show that \mathcal{F}_s is a nonempty set for all $s \geq 0$.

Lemma 1. *Let $s \geq 0$. The two curves*

$$: p \in [1, +\infty) \mapsto \varphi_{p,1} \quad \text{and} \quad : p \in [1, +\infty) \mapsto \varphi_{p,2}$$

belong to \mathcal{F}_s , where

$$\varphi_{p,1}(x) = \frac{p+2}{4} \operatorname{sech}^2\left(\frac{px}{2}\right) \tag{2.4}$$

and

$$\varphi_{p,2}(x) = \frac{(p+1)(p+2)}{4} \operatorname{sech}^2\left(\frac{px}{2}\right). \tag{2.5}$$

Proof. Consider the differential equations

$$-\phi_1'' + \phi_1 - 2\phi_1^{p+1} = 0, \quad \phi_1(x) \xrightarrow{|x| \rightarrow +\infty} 0, \tag{2.6}$$

and

$$-\phi_2'' + \phi_2 - \frac{2}{p+1}\phi_2^{p+1} = 0, \quad \phi_2(x) \xrightarrow{|x| \rightarrow +\infty} 0, \tag{2.7}$$

for $p \in [1, +\infty)$. These two equations admit solutions $\phi_{p,1}$ and $\phi_{p,2}$, respectively, in $H^\infty(\mathbb{R})$, unique up to translation (see Berestycki and Lions [3]).

Then we observe, using (2.4) and (2.5), that $\varphi_{p,1} = \phi_{p,1}^p$ and $\varphi_{p,2} = \phi_{p,2}^p$. Moreover equations (2.6) and (2.7) are equivalent to

$$H_{\varphi_{p,1}}\phi_{p,1} = -\phi_{p,1} \quad \text{and} \quad H_{\varphi_{p,2}}\phi_{p,2}' = -\phi_{p,2}',$$

which concludes the proof of Lemma 1. □

3. LOCAL WELL-POSEDNESS IN Ω_s WHEN $s > \frac{1}{2}$

Let $s \geq 0$ and $\phi \in \Omega_s$. First observe that

$$D_x^a R_\phi(-1) \in \mathcal{B}(L^2(\mathbb{R})), \quad \forall 0 \leq a \leq 2,$$

so we will denote

$$r_a(\phi) := \|D_x^a R_\phi(-1)\|_{\mathcal{B}(L^2)}, \quad \forall 0 \leq a \leq 2. \tag{3.1}$$

For $s > \frac{1}{2}$, let us also define

$$B_s(\phi) := \left\{ \psi \in H^s(\mathbb{R}) \mid \|\psi - \phi\|_{H^s} < \frac{1}{2C_s r_0(\phi)} \right\}, \tag{3.2}$$

where C_s is the best constant in the Sobolev embedding

$$\|\phi\|_{L^\infty} \leq C_s \|\phi\|_{H^s}, \quad \text{if } s > \frac{1}{2}. \tag{3.3}$$

Note that $C_1 = \frac{1}{\sqrt{2}}$. Then we have the following.

Lemma 2. *Let $s > \frac{1}{2}$ and $\phi \in \Omega_s$. Then $B_s(\phi) \subset \Omega_s$ and we have for all $\psi \in B_s(\phi)$:*

$$r_a(\psi) = \|D_x^a R_\psi(-1)\|_{\mathcal{B}(L^2)} \leq \frac{r_a(\phi)}{1 - 2C_s \|\psi - \phi\|_{H^s}}, \tag{3.4}$$

for all $0 \leq a \leq 2$. In particular Ω_s is an open subset of $H^s(\mathbb{R})$. Moreover if ψ_1 and ψ_2 are in $B_s(\phi)$, we have

$$\begin{aligned} & \|D_x^a (R_{\psi_1}(-1) - R_{\psi_2}(-1))\|_{\mathcal{B}(L^2)} \\ & \leq \frac{2C_s r_a(\phi) r_0(\phi) \|\psi_1 - \psi_2\|_{H^s}}{(1 - 2C_s r_0(\phi) \|\psi_1 - \phi\|_{H^s})(1 - 2C_s r_0(\phi) \|\psi_2 - \phi\|_{H^s})}, \end{aligned} \tag{3.5}$$

for all $0 \leq a \leq 2$.

Proof. Let $\phi \in \Omega_s$ and $\psi \in B_s(\phi)$. Then we have

$$2\psi - 2\phi = (H_\psi + 1) - (H_\phi + 1),$$

so that

$$(H_\psi + 1)R_\phi(-1) = 1 - (2\psi - 2\phi)R_\phi(-1). \tag{3.6}$$

Now we deduce using estimate (3.3) that

$$\|(2\psi - 2\phi)R_\phi(-1)\|_{\mathcal{B}(L^2)} \leq 2C_s r_0(\phi) \|\psi - \phi\|_{H^s} < 1, \tag{3.7}$$

since $\psi \in B_s(\phi)$. Thus, the right-hand side of (3.6) is invertible due to the Neumann series criterion. This implies that $\psi \in \Omega_s$ and that

$$R_\psi(-1) = R_\phi(-1)(1 - (2\psi - 2\phi)R_\phi(-1))^{-1}. \tag{3.8}$$

Finally, we combine (3.7) and (3.8) to deduce (3.4). Now take ψ_1 and ψ_2 in $B_s(\phi)$; we have

$$2\psi_1 - 2\psi_2 = (H_{\psi_2} + 1) - (H_{\psi_1} + 1).$$

Then we obtain that

$$R_{\psi_1}(-1) - R_{\psi_2}(-1) = 2R_{\psi_2}(-1)(\psi_1 - \psi_2)R_{\psi_1}(-1),$$

which implies, recalling (3.8), that

$$D_x^a(R_{\psi_1}(-1) - R_{\psi_2}(-1)) = 2D_x^a R_\phi(-1)(1 - 2(\psi_2 - \phi)R_\phi(-1))^{-1} \circ (\psi_1 - \psi_2)R_\phi(-1)(1 - 2(\psi_1 - \phi)R_\phi(-1))^{-1}. \quad (3.9)$$

Therefore, we deduce estimate (3.5) by applying estimates (3.4) and (3.7) to (3.9). \square

We also use the classical fractional Leibniz rule due to Kenig, Ponce and Vega [11].

Lemma 3. *Let $0 < a < 1$. Then*

$$\|D_x^a(fg) - gD_x^a f - fD_x^a g\|_{L^2} \lesssim \|f\|_{L^\infty} \|D_x^a g\|_{L^2}. \quad (3.10)$$

We consider the following integral equation associated to the Cauchy problem (1.4) and derived through (1.5):

$$u(t) = u_0 - \int_0^t \partial_x R_{u(t')}(-1)u(t')dt', \quad (3.11)$$

where we used the notation (2.2).

Theorem 1. *Let $s > \frac{1}{2}$ and $u_0 \in \Omega_s := \{\phi \in H^s(\mathbb{R}) : -1 \notin \sigma(H_\phi)\}$. Then there exists a positive time $T = T(u_0)$ and a unique solution u of (3.11) in the space $C([0, T]; H^s(\mathbb{R}))$. Moreover, the flow map solution*

$$S_s : \Omega_s \longrightarrow C([0, T]; H^s(\mathbb{R})), \quad u_0 \longmapsto u,$$

is smooth.

Proof. Let $u_0 \in \Omega_s$; we want to prove that the map

$$F : u \longmapsto u_0 - \int_0^t \partial_x R_{u(t')}(-1)u(t')dt' \quad (3.12)$$

is a contraction in the Banach space

$$X_T^s(u_0) := \{u \in C([0, T]; H^s(\mathbb{R})) : \sup_{t \in [0, T]} \|u(t) - u_0\|_{H^s} \leq \alpha\}, \quad (3.13)$$

where $\alpha = (4C_s r_0(u_0))^{-1}$ and $T > 0$ will be fixed later.

First we will suppose that $\frac{1}{2} < s \leq 1$. We observe from Lemma 2 that, if $u \in X_T^s(u_0)$, then $u(t) \in \Omega_s$ for all $t \in [0, T]$ so that $F(u)$ is well defined on $[0, T]$. Moreover we have using (3.4) that for all $t \in [0, T]$

$$\begin{aligned} \|F(u)(t) - u_0\|_{H^s} &\sim \|F(u)(t) - u_0\|_{L^2} + \|D_x^s(F(u)(t) - u_0)\|_{L^2} \\ &\lesssim \left\| \int_0^t \partial_x R_{u(t')}(-1)u(t')dt' \right\|_{L^2} + \left\| \int_0^t D_x^s \partial_x R_{u(t')}(-1)u(t')dt' \right\|_{L^2} \\ &\lesssim \int_0^t \frac{r_1(u_0) + r_{1+s}(u_0)}{1 - 2C_s r_0(u_0)\|u(t') - u_0\|_{H^s}} \|u(t')\|_{L^2} dt' \\ &\lesssim T \frac{r_1(u_0) + r_{1+s}(u_0)}{1 - 2C_s r_0(u_0)\alpha} (\alpha + \|u_0\|_{L^2}) \\ &\lesssim T(r_1(u_0) + r_{1+s}(u_0))(\alpha + \|u_0\|_{L^2}). \end{aligned} \tag{3.14}$$

Thus, if we choose T such that

$$\begin{aligned} 0 < T &\sim \frac{1}{r_1(u_0) + r_{1+s}(u_0)} \times \frac{\alpha}{\alpha + \|u_0\|_{L^2}} \\ &\sim \frac{1}{(r_1(u_0) + r_{1+s}(u_0))(1 + r_0(u_0)\|u_0\|_{L^2})}, \end{aligned} \tag{3.15}$$

we obtain that

$$F(X_T^s(u_0)) \subset X_T^s(u_0). \tag{3.16}$$

Now let $u, v \in X_T^s(u_0)$ and $t \in [0, T]$, we have that

$$\begin{aligned} F(v)(t) - F(u)(t) &= \int_0^t \partial_x (R_{u(t')}(-1) - R_{v(t')}(-1))u(t')dt' \\ &\quad + \int_0^t \partial_x R_{v(t')}(-1)(u(t') - v(t'))dt'. \end{aligned} \tag{3.17}$$

We estimate the second term of the right-hand side of (3.17) as in (3.14) and obtain that

$$\begin{aligned} \left\| \int_0^t \partial_x R_{v(t')}(-1)(u(t') - v(t'))dt' \right\|_{H^s} \\ \leq 2T(r_1(u_0) + r_{1+s}(u_0)) \sup_{[0, T]} \|u - v\|_{L^2}. \end{aligned} \tag{3.18}$$

Next we use estimate (3.5) to bound the H^s -norm of the first term:

$$\begin{aligned} \sup_{t \in [0, T]} \left\| \int_0^t \partial_x (R_{u(t')}(-1) - R_{v(t')}(-1))u(t')dt' \right\|_{H^s} \\ \lesssim T r_0(u_0)(r_1(u_0) + r_{1+s}(u_0))(\alpha + \|u_0\|_{L^2}) \sup_{[0, T]} \|u - v\|_{H^s}. \end{aligned} \tag{3.19}$$

Thus we conclude, combining (3.17), (3.18) and (3.19), that

$$\begin{aligned} & \sup_{t \in [0, T]} \|F(u)(t) - F(v)(t)\|_{H^s} \\ & \lesssim T(r_1(u_0) + r_{1+s}(u_0))(1 + r_0(u_0)\|u_0\|_{L^2}) \sup_{[0, T]} \|u - v\|_{H^s}. \end{aligned}$$

Then the condition (3.15) is sufficient for F to be a contraction in the space $X_T^s(u_0)$.

In the case $1 < s < 2$, let us denote $a := s - 1$, so that $0 < a < 1$. First we note that

$$\|F(u)(t) - u_0\|_{H^s} \sim \|F(u)(t) - u_0\|_{L^2} + \|D_x^s(F(u)(t) - u_0)\|_{L^2}. \tag{3.20}$$

We bound the first term of the right-hand side of (3.20) as in (3.14):

$$\|F(u)(t) - u_0\|_{L^2} \leq 2Tr_1(u_0)(\alpha + \|u_0\|_{L^2}). \tag{3.21}$$

In order to estimate the second term, we combine the identity

$$\partial_x^2 R_\phi(-1) = -1 - 2\phi R_\phi(-1) + R_\phi(-1), \tag{3.22}$$

Lemmas 2 and 3, and the Sobolev embedding to obtain that for all $t \in [0, T]$

$$\begin{aligned} \|D_x^s(F(u)(t) - u_0)\|_{L^2} &= \left\| \int_0^t D_x^a \partial_x^2 R_{u(t')}(-1)u(t') dt' \right\|_{L^2} \\ &\lesssim \int_0^T (\|D_x^a u(t')\|_{L^2} + 2\|D_x^a u(t')\|_{L^\infty} \|R_{u(t')}(-1)u(t')\|_{L^2}) dt' \\ &\quad + \int_0^T (2\|u(t')\|_{L^\infty} + 1) \|D_x^a R_{u(t')}(-1)u(t')\|_{L^2} dt' \\ &\lesssim T((1 + r_a(u_0))(\alpha + \|u_0\|_{H^s}) + (r_0(u_0) + r_a(u_0))(\alpha + \|u_0\|_{H^s})^2). \end{aligned}$$

Hence if we choose T small enough as a function of $r_0(u_0)$, $r_a(u_0)$, $r_1(u_0)$ and $\|u_0\|_{H^s}$, we deduce that F is an application from $X_T^s(u_0)$ with values on $X_T^s(u_0)$.

One can use the same techniques to prove that F is a contraction. The case $s \geq 2$ follows using the same arguments. This finishes the proof of Theorem 1. □

4. LOCAL WELL POSEDNESS FOR SMALL INITIAL DATA IN $H^s(\mathbb{R})$ WHEN $0 \leq s \leq \frac{1}{2}$

In the case $0 \leq s \leq \frac{1}{2}$, we do not know if Ω_s is an open subset of $H^s(\mathbb{R})$. Nevertheless, it will be shown that Ω_s contains an open ball of $H^s(\mathbb{R})$ centered at the origin. We first derive a technical lemma.

Lemma 4. *Let $\phi, \psi \in L^2(\mathbb{R})$ such that $\|\phi\|_{L^2} < \frac{1}{\sqrt{2}}$. Then $\psi R_\phi(-1)$ belongs to $\mathcal{B}(L^2(\mathbb{R}))$ and the following inequality holds:*

$$\|\psi R_\phi(-1)\|_{\mathcal{B}(L^2)} \leq \frac{\|\psi\|_{L^2}}{2(\frac{1}{\sqrt{2}} - \|\phi\|_{L^2})}. \tag{4.1}$$

Proof. Take $f \in L^2(\mathbb{R})$ and define $g = R_\phi(-1)f$; then we have

$$\|\psi R_\phi(-1)f\|_{L^2} \leq \|g\|_{L^\infty} \|\psi\|_{L^2}. \tag{4.2}$$

On the other hand,

$$g = R_\phi(-1)f \iff -g'' + g - 2\phi g = f.$$

Thus, we deduce, multiplying by g , integrating by parts and using the Cauchy-Schwarz inequality that

$$\|g\|_{H^1}^2 = \int_{\mathbb{R}} (f + 2\phi g)g dx \leq (\|f\|_{L^2} + 2\|g\|_{L^\infty} \|\phi\|_{L^2}) \|g\|_{L^2},$$

which leads to

$$\|g\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|f\|_{L^2} + \sqrt{2} \|g\|_{L^\infty} \|\phi\|_{L^2},$$

using the Sobolev embedding (3.3) with $s = 1$. This implies, recalling the hypothesis on ϕ , that

$$\|g\|_{L^\infty} \leq \frac{\|f\|_{L^2}}{2(\frac{1}{\sqrt{2}} - \|\phi\|_{L^2})}. \tag{4.3}$$

Therefore, we deduce, combining (4.2) and (4.3), that

$$\|\psi R_\phi(-1)\|_{\mathcal{B}(L^2)} = \sup_{\|f\|_{L^2} \leq 1} \|\psi R_\phi(-1)f\|_{L^2} \leq \frac{\|\psi\|_{L^2}}{2(\frac{1}{\sqrt{2}} - \|\phi\|_{L^2})}.$$

This finishes the proof of Lemma 4. □

We are now able to state the principal lemma of this section.

Lemma 5. *Let $0 \leq s \leq \frac{1}{2}$. Then*

$$B_s(0, \frac{1}{\sqrt{2}}) := \{\phi \in H^s(\mathbb{R}) : \|\phi\|_{H^s} < \frac{1}{\sqrt{2}}\} \subseteq \Omega_s.$$

Moreover if $\phi \in B_s(0, \frac{1}{\sqrt{2}})$, define $\delta(\phi) := \frac{1}{\sqrt{2}} - \|\phi\|_{H^s}$. Then we have that

$$r_a(\psi) = \|D_x^a R_\psi(-1)\|_{\mathcal{B}(L^2)} \leq \frac{\delta(\phi) r_a(\phi)}{\delta(\phi) - \|\phi - \psi\|_{H^s}}, \tag{4.4}$$

for all $0 \leq a \leq 2$, $\psi \in B_s(\phi, \delta(\phi)) := \{\psi \in H^s(\mathbb{R}) : \|\psi - \phi\|_{H^s} < \delta(\phi)\}$ and

$$\|D_x^a (R_{\psi_1}(-1) - R_{\psi_2}(-1))\|_{\mathcal{B}(L^2)} \tag{4.5}$$

$$\leq \frac{\delta(\phi)r_0(\phi)r_a(\phi)\|\psi_1 - \psi_2\|_{H^s}}{(\delta(\phi) - \|\psi_1 - \phi\|_{H^s})(\delta(\phi) - \|\psi_2 - \phi\|_{H^s})},$$

for all $0 \leq a \leq 2$ and $\psi_1, \psi_2 \in B_s(\phi, \delta(\phi))$.

Proof. Let $\phi \in H^s(\mathbb{R})$. Then we have that $2\phi = (H_0 + 1) - (H_\phi + 1)$. Thus, we deduce, since $-1 \notin \sigma(H_0)$, that

$$(H_\phi + 1)R_0(-1) = 1 - 2\phi R_0(-1). \tag{4.6}$$

Moreover, (4.1) implies that

$$\|2\phi R_0(-1)f\|_{\mathcal{B}(L^2)} \leq \sqrt{2}\|\phi\|_{L^2} < 1,$$

whenever $\|\phi\|_{H^s} < \frac{1}{\sqrt{2}}$. Thus the right-hand side of (4.6) is invertible due to the Neumann series criterion and we deduce that $\phi \in \Omega_s$.

Now take $\psi \in B_s(\phi, \delta(\phi))$. We already know that $\psi \in B_s(0, \frac{1}{\sqrt{2}}) \subseteq \Omega_s$ and we obtain arguing as in the proof of Lemma 2 that

$$R_\psi(-1) = R_\phi(-1)(1 - (2\psi - 2\phi)R_\phi(-1))^{-1}.$$

This implies (4.4), recalling (4.1) and the Neumann series criterion. We derive (4.5) the same way. □

We conclude this section with our local well-posedness theorem for small data in $H^s(\mathbb{R})$, whenever $0 \leq s \leq \frac{1}{2}$.

Theorem 2. *Let $0 \leq s \leq \frac{1}{2}$ and*

$$u_0 \in B_s(0, \frac{1}{\sqrt{2}}) = \{\phi \in H^s(\mathbb{R}) : \|\phi\|_{H^s} < \frac{1}{\sqrt{2}}\}.$$

Then there exists a positive time $T = T(u_0)$ and a unique solution u of (3.11) in the space $C([0, T]; H^s(\mathbb{R}))$. Moreover, the flow map solution

$$S_s : \Omega_s \longrightarrow C([0, T]; H^s(\mathbb{R})), \quad u_0 \longmapsto u,$$

is smooth.

Proof. The proof is exactly the same as the proof of Theorem 1 using estimates (4.4) and (4.5) instead of (3.4) and (3.5). □

5. GLOBAL WELL POSEDNESS FOR SMALL INITIAL DATA IN $H^1(\mathbb{R})$

Let u be a smooth solution of (1.4), for instance $u \in C([0, T]; H^{10}(\mathbb{R}))$. Then we multiply (1.4) by u , use (2.1), and integrate to deduce

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 - 2(u \int_x^{+\infty} u_t dx', u_x)_{L^2} + \frac{1}{2} \frac{d}{dt} \|\partial_x u(t)\|_{L^2}^2 = 0. \tag{5.1}$$

Moreover we observe, integrating by parts, that

$$(u \int_x^{+\infty} u_t dx', u_x)_{L^2} = -\frac{1}{2}(u \partial_x \int_x^{+\infty} u_t dx', u)_{L^2} = \frac{1}{6} \frac{d}{dt} \|u(t)\|_{L^3}^3. \tag{5.2}$$

Therefore, we conclude from (5.1) and (5.2) that

$$E(u(t)) = E(u_0), \quad \forall t \in [0, T], \tag{5.3}$$

$$E : H^1(\mathbb{R}) \rightarrow \mathbb{R}, \quad \phi \mapsto \int_{\mathbb{R}} \frac{1}{2} \phi(x)^2 + \frac{1}{2} \phi'(x)^2 - \frac{1}{3} \phi(x)^3 dx \tag{5.4}$$

is well defined using the Gagliardo-Nirenberg inequality.

Next we compute the first and second-order derivatives of E :

$$E'(\phi)\psi = \int_{\mathbb{R}} (-\phi'' + \phi - \phi^2)\psi dx, \tag{5.5}$$

and

$$E''(\phi)(\psi_1, \psi_2) = \int_{\mathbb{R}} \mathcal{L}_\phi \psi_1 \psi_2 dx = (\mathcal{L}_\phi \psi_1, \psi_2)_{L^2}, \tag{5.6}$$

where

$$\mathcal{L}_\phi = -\frac{d^2}{dx^2} + 1 - 2\phi = H_\phi + 1. \tag{5.7}$$

Remark 1. Note that for all $\phi \in H^1(\mathbb{R})$

$$\begin{aligned} \phi \in \Omega_1 &\iff \nexists \psi \neq 0 \text{ such that } \mathcal{L}_\phi \psi = 0 \\ &\iff E''(\phi) \text{ is a nondegenerate bilinear form on } H^1(\mathbb{R}). \end{aligned}$$

Proposition 1. *Let $\varphi^*(x) = \frac{3}{2} \operatorname{sech}^2(\frac{x}{2})$, the critical point of E (which is unique up to translation). Then the best constant C^* in the Sobolev embedding*

$$\|\phi\|_{L^3} \leq C^* \|\phi\|_{H^1}, \quad \forall \phi \in H^1(\mathbb{R}), \tag{5.8}$$

is attained for φ^ .*

Proof. Proposition 1 was proved by Liu in [13]. Moreover, since φ^* is a solution of the equation

$$-\varphi^{*''} + \varphi^* - \varphi^{*2} = 0,$$

we deduce that $\|\varphi^*\|_{H^1}^2 = \|\varphi^*\|_{L^3}^3$, which, combined with $\|\varphi^*\|_{L^3} = C^* \|\varphi^*\|_{H^1}$ implies that

$$E^* := E(\varphi^*) = \frac{1}{6} \|\varphi^*\|_{H^1}^2 \quad \text{and} \quad C^* = \|\varphi^*\|_{H^1}^{-\frac{1}{3}}. \tag{5.9}$$

□

Remark 2. Note that $\varphi^* = \varphi_{1,2}$ and $\frac{1}{2}\varphi^* = \varphi_{1,1}$ belong to \mathcal{F} by Lemma 1.

Since $0 \in \Omega_1$, we know that there exists an open ball of $H^1(\mathbb{R})$ centered at the origin contained in Ω_1 . In the following lemma, we derive the best radius of this ball.

Lemma 6. *Let $B(0, \frac{1}{2}\|\varphi^*\|_{H^1}) := \{\phi \in H^1(\mathbb{R}) : \|\phi\|_{H^1} < \frac{1}{2}\|\varphi^*\|_{H^1}\}$. Then*

$$B(0, \frac{1}{2}\|\varphi^*\|_{H^1}) \subset \Omega_1. \tag{5.10}$$

Proof. Let $\phi \in \mathcal{F}_1$. Then $-1 \in \sigma_d(H_\phi)$. Thus there exists $\psi \in H^2(\mathbb{R}) \setminus \{0\}$ such that

$$-\psi'' + \psi - 2\phi\psi = 0. \tag{5.11}$$

We multiply (5.11) by ψ , integrate by parts and use Hölder's inequality and the Sobolev embedding (5.8) to deduce that

$$\|\psi\|_{H^1}^2 = 2 \int_{\mathbb{R}} \phi\psi^2 dx \leq 2\|\phi\|_{L^3}\|\psi\|_{L^3}^2 \leq 2C^{*3}\|\phi\|_{H^1}\|\psi\|_{H^1}^2. \tag{5.12}$$

This implies that $\|\phi\|_{H^1} \geq \frac{1}{2}\|\varphi^*\|_{H^1}$ by recalling (5.9). Moreover this inequality is sharp since $\frac{1}{2}\varphi^* \in \mathcal{F}_1$ from Remark 2. □

Now we want to control the H^1 -norms of the solutions which have initial data in this ball. For this, we follow the ideas of Kenig and Merle [10] for the H^1 -critical nonlinear Schrödinger equation.

Lemma 7. *Let $0 < \delta_0 < 1$. Assume that $\phi \in B(0, \frac{1}{2}\|\varphi^*\|_{H^1})$ and $E(\phi) \leq (1 - \delta_0)E(\frac{1}{2}\varphi^*)$. Then there exists $0 < \delta = \delta(\delta_0) < \frac{1}{2}$ such that*

$$\|\phi\|_{H^1} \leq (\frac{1}{2} - \delta)\|\varphi^*\|_{H^1} \quad \text{and} \quad E(\phi) \geq 0. \tag{5.13}$$

Proof. Consider the function $f(y) = \frac{1}{2}y - \frac{C^{*3}}{3}y^{\frac{3}{2}}$ and let $\bar{y} = \|\phi\|_{H^1}^2$. Then, we deduce from (5.8) that

$$f(\bar{y}) \leq E(\phi) \leq (1 - \delta_0)E(\frac{1}{2}\varphi^*).$$

Moreover we have $f(0) = 0$, $f(y) \xrightarrow{y \rightarrow +\infty} -\infty$ and a simple calculation gives

$$f'(y) = \frac{1}{2} - \frac{C^{*3}}{2}y^{\frac{1}{2}} = 0 \iff y = y_c := \frac{1}{C^{*6}} = \|\varphi^*\|_{H^1}^2.$$

Therefore f is a continuous function, nonnegative, strictly increasing in $(0, y_c)$ and, since $f(\frac{1}{4}y_c) = \frac{1}{2}E(\varphi^*) = E(\frac{1}{2}\varphi^*)$, we conclude by a continuity argument the existence of $\delta > 0$ such that $\bar{y} \leq (\frac{1}{2} - \delta)^2 y_c$. □

Proposition 2. *Let $0 < \delta_0 < 1$. Assume that $u_0 \in B(0, \frac{1}{2}\|\varphi^*\|_{H^1})$ and $E(u_0) \leq (1 - \delta_0)E(\frac{1}{2}\varphi^*)$. Consider $u \in C([0, T]; H^1(\mathbb{R}))$ the solution of (1.4) satisfying $u(0) = u_0$. Then, there exists $0 < \delta = \delta(\delta_0) < \frac{1}{2}$ such that*

$$\|u(t)\|_{H^1} \leq (\frac{1}{2} - \delta)\|\varphi^*\|_{H^1} \quad \text{and} \quad E(u(t)) \geq 0, \quad \forall t \in [0, T]. \quad (5.14)$$

Proof. Define $I = \{t \in [0, T] : (5.14) \text{ holds}\}$. Then $0 \in I$ and, by continuity, I is closed in $[0, T]$. Now we deduce from (5.3) that

$$0 \leq E(u(t)) = E(u_0) \leq (1 - \delta_0)E(\frac{1}{2}\varphi^*), \quad \forall t \in [0, T],$$

so that Lemma 7 implies that I is open. This proves (5.14) by convexity. \square

We are now able to prove the following global well-posedness theorem

Theorem 3. *Let $u_0 \in B(0, \frac{1}{2}\|\varphi^*\|_{H^1})$ such that $E(u_0) < E(\frac{1}{2}\varphi^*)$. Then the local solution of (1.4) obtained in Theorem 1 extends globally in time.*

Proof. Let $u \in C([0, T_{max}); H^1(\mathbb{R}))$ be the maximally extended solution of (1.4). We know from Proposition 2 and Lemma 6 that there exists $\delta > 0$ such that

$$\|u(t)\|_{H^1} \leq (\frac{1}{2} - \delta)\|\varphi^*\|_{H^1} \quad \text{and} \quad u(t) \in \Omega_1 \quad \forall t \in [0, T_{max}).$$

Since the time existence in the local theory only depends on $\|u_0\|_{L^2}$ and $r_j(u_0) = \|\partial_x^j R_\phi(-1)\|_{\mathcal{B}(L^2)}$, $j = 0, 1, 2$, we only need to derive a priori estimates for these quantities to obtain our global well-posedness result.

Let $\phi \in H^1(\mathbb{R})$, satisfying

$$\|\phi\|_{H^1} \leq (\frac{1}{2} - \delta)\|\varphi^*\|_{H^1}, \quad (5.15)$$

and $\lambda > 0$ such that $-\lambda$ is an eigenvalue of H_ϕ ; *i.e.*,

$$-\psi'' + \lambda\psi - 2\phi\psi = 0, \quad \text{for some } \psi \in H^2(\mathbb{R}) \setminus \{0\}.$$

We deduce, multiplying by ψ , integrating by parts, and using the Sobolev embedding (5.8) that

$$\|\psi'\|_{L^2}^2 + \lambda\|\psi\|_{L^2}^2 \leq 2\|\varphi^*\|_{H^1}^{-1}\|\phi\|_{H^1}\|\psi\|_{H^1}^2 \leq (1 - 2\delta)\|\psi\|_{H^1}^2,$$

so that $0 < \lambda \leq 1 - 2\delta$. Thus we deduce that $d(\sigma(H_\phi), -1) \geq 2\delta$. Moreover, it is well known from the spectral theorem and the fact that $R_\phi(-1)$ is a normal operator (see [9] for example) that

$$\|R_\phi(-1)\|_{\mathcal{B}(L^2)} = \text{spr}(R_\phi(-1)) = d(\sigma(H_\phi), -1)^{-1} \leq (2\delta)^{-1}, \quad (5.16)$$

where $\text{spr}(R_\phi(-1))$ is the spectral radius of $R_\phi(-1)$.

Moreover, we deduce from (3.22), (5.15) and (5.16) that

$$\|\partial_x^2 R_\phi(-1)\|_{\mathcal{B}(L^2)} \leq 1 + (2\|\phi\|_{L^\infty} + 1)\|R_\phi(-1)\|_{\mathcal{B}(L^2)} \leq 1 + \frac{\|\varphi^*\|_{H^1} + 1}{2\delta}. \tag{5.17}$$

Finally, let $f \in L^2(\mathbb{R})$. Integrating by parts and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\partial_x R_\phi(-1)f\|_{L^2}^2 &= (\partial_x R_\phi(-1)f, \partial_x R_\phi(-1)f)_{L^2} = -(R_\phi(-1)f, \partial_x^2 R_\phi(-1)f)_{L^2} \\ &\leq \|R_\phi(-1)\|_{\mathcal{B}(L^2)} \|\partial_x^2 R_\phi(-1)\|_{\mathcal{B}(L^2)} \|f\|_{L^2}^2. \end{aligned}$$

Therefore, we deduce that

$$\|\partial_x R_\phi(-1)\|_{\mathcal{B}(L^2)} \leq (\|R_\phi(-1)\|_{\mathcal{B}(L^2)} \|\partial_x^2 R_\phi(-1)\|_{\mathcal{B}(L^2)})^{\frac{1}{2}},$$

which combined with (5.16) and (5.17) concludes the proof of Theorem 3. \square

Remark 3. Note that the solitary-wave solutions of the Hirota-Satsuma equation defined in (1.7) satisfy

$$\|\phi_\sigma\|_{H^1} < \frac{1}{2}\|\varphi^*\|_{H^1} \quad \text{and} \quad E(\phi_\sigma) < \frac{1}{2}E(\varphi^*) = E(\frac{1}{2}\varphi^*), \quad \forall c > 1. \tag{5.18}$$

Proof. Note that

$$\|\phi_\sigma\|_{H^1}^2 = \|\phi_\sigma\|_{L^2}^2 + \|\phi'_\sigma\|_{L^2}^2 = \sigma^3 \|\frac{1}{2}\varphi^*\|_{L^2}^2 + \sigma^5 \|\frac{1}{2}\varphi^{*\prime}\|_{L^2}^2 < \frac{1}{4}\|\varphi^*\|_{H^1}^2,$$

since $0 < \sigma < 1$, whenever $c > 1$. Next, observe that ϕ_σ is a solution of the equation

$$-\phi'' + \sigma^2\phi - 2\phi^2 = 0.$$

Then, we deduce, multiplying by ϕ_σ and integrating by parts that

$$\int_{\mathbb{R}} \phi'_\sigma(x)^2 dx + \sigma^2 \int_{\mathbb{R}} \phi_\sigma(x)^2 dx = 2 \int_{\mathbb{R}} \phi_\sigma(x)^3 dx,$$

so that

$$\begin{aligned} E(\phi_\sigma) &= \frac{1}{2}\|\phi_\sigma\|_{H^1}^2 - \frac{1}{3}\|\phi_\sigma\|_{L^3}^3 = \frac{1}{3}\|\phi'_\sigma\|_{L^2}^2 + \left(\frac{1}{2} - \frac{\sigma^2}{6}\right)\|\phi_\sigma\|_{L^2}^2 \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} - \frac{\sigma^2}{6}\right)\sigma^3\|\varphi^*\|_{L^2}^2 + \frac{1}{6}\sigma^5\|\varphi^{*\prime}\|_{L^2}^2\right). \end{aligned} \tag{5.19}$$

Now observe that

$$\frac{1}{2}\left(\frac{1}{2} - \frac{\sigma^2}{6}\right)\sigma^3 < \frac{1}{6} \iff \sigma^5 - 3\sigma^3 + 2 > 0,$$

which is always true whenever $0 < \sigma < 1$, so that (5.18) follows from (5.9) and (5.19). \square

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