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INTEGRAL OPERATORS IN SPACES OF BOUNDED, ALMOST PERIODIC, AND ALMOST AUTOMORPHIC FUNCTIONS

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Abstract. In this paper we consider integral operators on the real line and derive certain sufficient conditions under which the operators act as bounded linear operators between the spaces of Stepanov bounded functions. Next, we find conditions that insure the operators act between spaces of Stepanov almost periodic, or between spaces of Stepanov almost automorphic, functions. We apply these results to ordinary differential equations and obtain the existence and uniqueness of bounded, almost periodic, and almost automorphic solutions.

1. Introduction

Almost periodic and almost automorphic functions proved the importance of their role in the theory of differential equations and other areas of mathematics. The literature in the field is enormous; we refer, e.g., to monographs [1, 12, 14, 15, 17] and references therein. In many applications, such as differential equations, spaces of Stepanov almost periodic functions are natural analogs of classical L^p spaces. Surprisingly enough, a similar notion in the case of almost automorphy was introduced very recently, in [16], under the name of Stepanov-like almost automorphic functions. Applications studied in [16], [9] and [7] show that this notion is quite useful and, indeed, is a natural counterpart of Stepanov almost periodicity.

We deal in this paper with integral operators of the form

$$(Au)(x) = \int_{-\infty}^{\infty} K(x,y)u(y)dy$$

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acting in BS^p, S^p, AS^p, respectively the spaces of all Stepanov bounded functions, Stepanov almost periodic functions, and Stepanov-like almost automorphic functions, $1 \leq p \leq \infty$ (the case $p = \infty$ corresponds to bounded, Bohr almost periodic, and uniformly almost automorphic functions, respectively). The kernel $K(x, y)$ is assumed to be a measurable function on R^2.

We are motivated by the work of G. Bruno and A. Pankov [2], devoted to convolution operators acting in the spaces of almost periodic functions. Another motivation comes from the paper by G. Bruno, A. Pankov and Y. Tverdokhleb [3]. Let us also point out that classical almost automorphy, as well as asymptotic and pseudo almost periodicity, of convolution operators is considered in [4, 5].

We introduce, in Section 2, certain spaces of kernel functions and derive sufficient conditions for an integral operator to act continuously between BS^p spaces. Actually, the spaces of Stepanov bounded functions are amalgams in the sense of [10] and, therefore, operators between these spaces can be described by means of infinite operator matrices. The spaces of kernel functions we consider reflect this structure of operators though we do not use it explicitly. Then we investigate in Section 3 some of the local properties of the integral operators A, such as local continuity, local convergence and local compactness. The results obtained are subsequently used in Section 5 when we study almost automorphic operators.

Sections 4 and 5 are, to some extent, parallel. For linear operators acting between BS^p spaces, we introduce the notions of almost periodicity and almost automorphy, respectively. We show that almost periodic (respectively, almost automorphic) operators act between corresponding spaces of Stepanov almost periodic (respectively, almost automorphic) functions. Moreover, the norm of such an operator coincides with the norm of its restriction to corresponding spaces of almost periodic or, respectively, almost automorphic functions. We give also sufficient conditions on the kernel K to insure almost periodicity or almost automorphy of the integral operator A. Namely, the kernel has to be in a certain sense almost periodic (respectively, almost automorphic) along the diagonal. The principal difference between almost periodic and almost automorphic operators is that the underlying topologies are different. Almost periodicity is defined in terms of the norm topology in the space of bounded linear operators, while to define almost automorphy we use the topology of local convergence.

In Section 6, we apply previous results to ordinary differential equations in divergence form. Under natural assumptions, we show that if the right-hand side f is a BS^1 function, then there exists a unique bounded solution. If, in addition, f is an S^1-almost periodic (respectively, S^1-almost automorphic) function, then the bounded solution is actually Bohr almost periodic (respectively, uniformly almost automorphic). The proof is based on investigation of the integral operator whose kernel is the Green's function. In particular, we use exponential decay of the Green's function G. Under certain regularity assumptions on the coefficients exponential decay of G is known even for partial differential operators [18], but we assume here only continuity (L^∞ in the case of bounded solutions) of the coefficients.

Finally, let us mention that our sufficient conditions for an integral operator to be bounded, almost periodic, or almost automorphic, are not necessary (see Section 7).

2. Operators in spaces of Stepanov bounded functions

We consider integral operators of the form

$$(Au)(x) = \int_{-\infty}^{\infty} K(x,y)u(y)\,dy \tag{2.1}$$

in the spaces of bounded and Stepanov bounded functions on $\mathbb{R}$. We always suppose that the kernel function $K(x,y)$ is a measurable function on $\mathbb{R}^2$.

We use standard notation $L^p(X)$ and l^p for the Lebesgue spaces on a measure space X and the space of two-sided infinite p-summable sequences, respectively, with the abbreviation $L^p = L^p(\mathbb{R})$. By L^p_{loc} we denote the space of all measurable functions on $\mathbb{R}$ that belong to $L^p([a,b])$ for every finite interval $[a,b]$, endowed with the topology of L^p-convergence on every finite interval. We always denote by p' the conjugate exponent to p:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For any two Banach spaces E and F we denote by $L(E,F)$ the Banach space of all bounded linear operators acting from E into F, with the norm denoted by $\|\cdot\|_{L(E,F)}$. If $E = F$, we write $L(E)$ and $\|\cdot\|_{L(E)}$.

The space BS^p, $1 \le p < \infty$, of Stepanov bounded functions (see, e.g., [17]), with the exponent p, consists of all functions $u \in L^p_{loc}$ that have finite norm

$$\sup_{s\in\mathbb{R}} \Big(\int_s^{s+1} |u(x)|^p \,dx \Big)^{1/p}. \tag{2.2}$$

This norm is equivalent to the norm

$$\|u\|_{BS^p} = \sup_{n\in\mathbb{Z}} \Big(\int_n^{n+1} |u(x)|^p \,dx \Big)^{1/p}. \tag{2.3}$$

Indeed, we have the following obvious inequalities:

$$\|u\|_{BS^p} \le \sup_{s\in\mathbb{R}} \Big(\int_s^{s+1} |u(x)|^p \, dx \Big)^{1/p} \le 2\|u\|_{BS^p}.$$

Endowed with norm (2.3), BS^p is a Banach space that can be interpreted as the amalgam (L^p, l^∞) in the sense of [10]: each function $u \in BS^p$ can be considered as a sequence of functions $(u_n)_{n\in\mathbb{Z}}$, where $u_n \in L^p([n, n+1])$ is defined by $u_n = u_{|[n,n+1]}$, and this defines an isometric isomorphism between BS^p and its corresponding amalgam. Also we set $BS^\infty = L^\infty$.

To describe classes of integral operators that will be considered, we associate to each kernel function $K(x, y)$ the following quantities:

$$\begin{aligned} \|K\|_{1;r,s} &= \sup_{m\in\mathbb{Z}} \Big\{ \sum_{n=-\infty}^{\infty} \Big[\text{ess sup}_{x\in[m,m+1]} \Big(\int_n^{n+1} |K(x,y)|^r \, dy \Big)^{1/r} \Big]^s \Big\}^{1/s} \\ &= \sup_{m\in\mathbb{Z}} \Big[\sum_{n=-\infty}^{\infty} \Big(\text{ess sup}_{x\in[m,m+1]} \int_n^{n+1} |K(x,y)|^r \, dy \Big)^{s/r} \Big]^{1/s} \qquad (2.4) \end{aligned}$$

and

$$\begin{aligned} \|K\|_{2;\sigma,\theta} &= \sup_{m\in\mathbb{Z}} \Big\{ \sum_{n=-\infty}^{\infty} \Big[\text{ess sup}_{y\in[n,n+1]} \Big(\int_m^{m+1} |K(x,y)|^\sigma \, dx \Big)^{1/\sigma} \Big]^\theta \Big\}^{1/\theta} \\ &= \sup_{m\in\mathbb{Z}} \Big[\sum_{n=-\infty}^{\infty} \Big(\text{ess sup}_{y\in[n,n+1]} \int_m^{m+1} |K(x,y)|^\sigma \, dx \Big)^{\theta/\sigma} \Big]^{1/\theta}, \qquad (2.5) \end{aligned}$$

where $0 < r, s, \sigma, \theta \le \infty$. (In cases when some parameters are infinite the right-hand parts in equations (2.4) and (2.5) have to be changed appropriately.) We denote by $\mathcal{K}_{1;r,s}$ (respectively, $\mathcal{K}_{2;\sigma,\theta}$) the set of all kernel functions K for which $\|K\|_{1;r,s}$ (respectively, $\|K\|_{2;\sigma,\theta}$) is finite.

The following proposition is a consequence of the general theory of spaces with mixed norms (see, e.g., [11]) and shows that these are, in general, Fréchet spaces, i.e., complete metric vector spaces, or F-spaces in the terminology of [8].

Proposition 2.1. *(a) If $1 \le r, s, \sigma, \theta \le \infty$, then $\mathcal{K}_{1;r,s}$ and $\mathcal{K}_{2;\sigma,\theta}$ are Banach spaces, with norms $\|K\|_{1;r,s}$ and $\|K\|_{2;\sigma,\theta}$, respectively.*

(b) If $0 < s < 1$, $s < r \le \infty$, $0 < \theta < 1$ and $\theta < \sigma \le \infty$, then $\mathcal{K}_{1;r,s}$ and $\mathcal{K}_{2;\sigma,\theta}$ are F-spaces, with F-norms $\|K\|^s_{1;r,s}$ and $\|K\|^\theta_{2;\sigma,\theta}$, respectively.

(c) If $0 < s < 1$, $0 < r \le s$, $0 < \theta < 1$ and $0 < \sigma \le \theta$, then $\mathcal{K}_{1;r,s}$ and $\mathcal{K}_{2;\sigma,\theta}$ are F-spaces, with F-norms $\|K\|^r_{1;r,s}$ and $\|K\|^\sigma_{2;\sigma,\theta}$, respectively.

(*d*) *There are continuous embeddings* $\mathcal{K}_{1;r_1,s_1} \subset \mathcal{K}_{1;r_2,s_2}$ *and* $\mathcal{K}_{2;\sigma_1,\theta_1} \subset \mathcal{K}_{2;\sigma_2,\theta_2}$*, whenever* $r_1 \geq r_2$*,* $s_1 \leq s_2$*,* $\sigma_1 \geq \sigma_2$ *and* $\theta_1 \leq \theta_2$*.*

Now we are ready to give a sufficient condition for an integral operator of the form (2.1) to act as a bounded operator in the spaces of Stepanov bounded functions.

Theorem 2.1. (*a*) *Let* $p \in [1,\infty]$ *and* $q \in [1,\infty)$*, with* $p \leq q$*. Suppose that* $K \in \mathcal{K}_{1;r,s} \cap \mathcal{K}_{2;\sigma,\theta}$*, where* $\sigma < q$*,* $\theta \geq \sigma/q$*,*

$$(1 - \frac{\sigma}{q})p' \leq r \quad \text{and} \quad s = \frac{\theta(q-\sigma)}{\theta q - \sigma}.$$

Then the integral operator A defined by (2.1) *is a bounded operator from* BS^p *into* BS^q *and*

$$\|A\|_{L(BS^p,BS^q)} \leq \|K\|_{1;r,s}^{1-\sigma/q} \|K\|_{2;\sigma,\theta}^{\sigma/q}. \tag{2.6}$$

(*b*) *Let* $p \in [1,\infty]$ *and* $q \in [1,\infty]$*, with* $p \leq q$*. Suppose that* $K \in \mathcal{K}_{2;q,1}$*. Then the operator A is a bounded operator from* BS^p *into* BS^q *and*

$$\|A\|_{L(BS^p,BS^q)} \leq \|K\|_{2;q,1}. \tag{2.7}$$

(*c*) *Let* $p \in [1,\infty]$ *and* $q \in [1,\infty]$*, with* $p \leq q$*. Suppose that* $K \in \mathcal{K}_{1;r,1}$ *with* $r \geq p'$*. Then the operator A is a bounded operator from* BS^p *into* BS^q *and*

$$\|A\|_{L(BS^p,BS^q)} \leq \|K\|_{1;r,1}. \tag{2.8}$$

Proof. We prove case (*a*), other cases being easier. Let

$$c_{1;m,n} = \text{ess sup}_{x\in[m,m+1]} \Big(\int_n^{n+1} |K(x,y)|^r \, dy \Big)^{1/r}$$

and

$$c_{2;m,n} = \text{ess sup}_{y\in[n,n+1]} \Big(\int_m^{m+1} |K(x,y)|^\sigma \, dx \Big)^{1/\sigma}.$$

Denoting by $A_{m,n}$ the integral operator with kernel function $K|_{[m,m+1]\times[n,n+1]}$, we see, by Theorem 1 of Section 11.3, [11], that $A_{m,n}$ acts as a bounded operator from $L^p(n,n+1)$ into $L^q(m,m+1)$ and its norm does not exceed

$$c_{1;m,n}^{1-\sigma/q} \cdot c_{2;m,n}^{\sigma/q}.$$

Letting $u_n = u|_{[n,n+1]}$, we obtain that, for $x \in [m,m+1]$,

$$(Au)(x) = \sum_{n=-\infty}^{\infty} (A_{m,n}u_n)(x)$$

and, hence,

$$\begin{aligned}\|Au\|_{L^q(m,m+1)} &\le \sum_{n=-\infty}^{\infty} c_{1;m,n}^{1-\sigma/q} \cdot c_{2;m,n}^{\sigma/q} \|u_n\|_{L^p(n,n+1)} \\ &\le \Big[\sum_{n=-\infty}^{\infty} (c_{1;m,n}^{1-\sigma/q} \cdot c_{2;m,n}^{\sigma/q}) \Big] \|u\|_{BS^p}.\end{aligned}$$

Thus,

$$\|A\|_{L(BS^p,BS^q)} \le \sup_{m\in\mathbb{Z}} \sum_{n=-\infty}^{\infty} (c_{1;m,n}^{1-\sigma/q} \cdot c_{2;m,n}^{\sigma/q}).$$

Now we apply the Hölder inequality for infinite sums, with exponents $\alpha = q\theta/(q\theta-\sigma)$ and $\alpha' = q\theta/\sigma$. We obtain

$$\|A\|_{L(BS^p,BS^q)} \le \sup_{m\in\mathbb{Z}} \Big[\Big(\sum_{n=-\infty}^{\infty} c_{1;m,n}^{s} \Big)^{1/s} \Big]^{1-\sigma/q} \cdot \Big[\Big(\sum_{n=-\infty}^{\infty} c_{2;m,n}^{\theta} \Big)^{1/\theta} \Big]^{\sigma/q},$$

which implies the required inequality (2.6). □

Remark 2.1. Note that there is a continuous embedding $BS^{p_1} \subset BS^{p_2}$ whenever $p_1 \ge p_2$. Therefore, if $A \in L(BS^p, BS^q)$, then $A \in L(BS^{p_1}, BS^{q_1})$, provided $p_1 \ge p$ and $q_1 \le q$.

Theorem 2.1 with $p = q$ and $r = s = \sigma = \theta = 1$ implies the following.

Corollary 2.1. *Suppose that $K \in \mathcal{K}_{1;1,1} \cap \mathcal{K}_{2;1,1}$. Then the operator A acts as a bounded operator in all spaces BS^p, $1 \le p \le \infty$, and*

$$\|A\|_{L(BS^p)} \le \|K\|_{1;1,1}^{1-1/p} \|K\|_{2;1,1}^{1/p}. \tag{2.9}$$

Another consequence of Theorem 2.1 and Remark 2.1 is the following.

Corollary 2.2. *Suppose that $K \in \mathcal{K}_{1;\infty,1} = \mathcal{K}_{2;\infty,1}$. Then the integral operator A acts as a bounded operator from BS^p into BS^q for all $p, q \in [1, \infty]$.*

Proposition 2.2. *Let $\varphi(x) \in L^1$ be a nonnegative function, nonincreasing for $x \ge 0$ and nondecreasing for $x \le 0$. Let $K_j(x,y)$ be a sequence of kernel functions such that $|K_j(x,y)| \le \varphi(x-y)$.*

(a) *If $K_j \to K$ in $L^\infty(\mathbb{R}_x; BS^1(\mathbb{R}_y))$, then $K_j \to K$ in $\mathcal{K}_{1;1,1}$.*
(b) *If $K_j \to K$ in $BS^1(\mathbb{R}_x; L^\infty(\mathbb{R}_y))$, then $K_j \to K$ in $\mathcal{K}_{2;1,1}$.*
(c) *If $K_j \to K$ in $L^\infty(\mathbb{R}^2)$, then $K_j \to K$ in $\mathcal{K}_{1;\infty,1} = \mathcal{K}_{2;\infty,1}$.*

Proof. We prove statement (a), the other cases being similar. Passing to a subsequence, we can assume that $K_j \to K$ almost everywhere on $\mathbb{R}^2$. Hence, $|K(x,y)| \leq \varphi(x-y)$. Therefore, without loss of generality we can assume that $K(x,y)=0$.

Let N be a nonnegative integer. Then

$$\sum_{n=-\infty}^{\infty} \Big(\text{ess sup}_{x\in[m,m+1]} \int_n^{n+1} |K_j(x,y)|\,dy\Big)$$

$$= \Big(\sum_{n\leq m-N-1} + \sum_{n\geq m+N+1} + \sum_{n=m-N}^{m+N}\Big)\Big(\text{ess sup}_{x\in[m,m+1]} \int_n^{n+1} |K_j(x,y)|\,dy\Big)$$

$$= a_{1,m,N} + a_{2,m,N} + a_{3,m,N}. \tag{2.10}$$

Now we estimate each of the quantities $a_{i,m,N}$, $i = 1,2,3$. Making use of the monotonicity property of φ, we obtain that

$$a_{1,m,N} \leq \sum_{n\leq m-N-1} \text{ess sup}_{x\in[m,m+1]} \int_n^{n+1} \varphi(x-y)\,dy$$

$$= \sum_{n\leq m-N-1} \text{ess sup}_{x\in[m,m+1]} \int_{x-(n+1)}^{x-n} \varphi(z)\,dz \leq \sum_{n\leq m-N-1} \int_{m-(n+1)}^{m-n} \varphi(z)\,dz$$

$$= \int_N^{\infty} \varphi(z)\,dz. \tag{2.11}$$

Similarly,

$$a_{2,m,N} \leq \int_{-\infty}^{-N} \varphi(z)\,dz. \tag{2.12}$$

Finally, it is easy to see that

$$a_{3,m,N} \leq \text{ess sup}_{x\in[m,m+1]} \sum_{n=m-N}^{m+N} \int_n^{n+1} |K_j(x,y)|\,dy \tag{2.13}$$

$$\leq (2N+1)\|K_j\|_{L^\infty(\mathbb{R}_x, BS^1(\mathbb{R}_y))}.$$

Combining (2.10)–(2.13), we obtain that

$$\|K_j\|_{1;1,1} \leq \Big(\int_N^{\infty} + \int_{-\infty}^{-N}\Big)\varphi(z)\,dz + (2N+1)\|K_j\|_{L^\infty(\mathbb{R}_x, BS^1(\mathbb{R}_y))}. \tag{2.14}$$

Now, given $\varepsilon > 0$, we can fix N independent of j such that the first term in the right-hand side of (2.14) is less than ε. Next, the second term of that bound tends to 0, and we obtain the required result. $\square$

3. Local properties

Now we show that integral operators in spaces of Stepanov bounded functions are locally continuous. More precisely, we say that an operator $A \in L(BS^p, BS^q)$ is *locally continuous* if for any bounded sequence $u_k \in BS^p$ that converges to u_0 in $L^p_{loc}(\mathbb{R})$ (automatically, $u_0 \in BS^p$) we have that $Au_k \to Au_0$ in $L^q_{loc}(\mathbb{R})$.

Theorem 3.1. *Under the assumptions of Theorem* 2.1, (a), (b), *or* (c), *the integral operator A is a locally continuous operator from BS^p into BS^q.*

Proof. It is enough to show that if u_k is a bounded sequence in BS^p and $u_k \to 0$ in $L^p_{loc}(\mathbb{R})$, then $Au_k \to 0$ in $L^q_{loc}(\mathbb{R})$. We consider case (a) only.

Let N and M be positive integers, u_k^N the restriction of u_k to $[-N, N]$, and v_k^N the restriction of u_k to $S_N = \mathbb{R} \setminus [-N, N]$. Then, for $x \in [-M, M]$,

$$(Au_k)(x) = (A_1 u_k^N)(x) + (A_2 v_k^N)(x), \tag{3.1}$$

where A_1 is the integral operator with the kernel $K|_{[-M,M]\times[-N,N]}$ acting from functions on $[-N, N]$ into functions on $[-M, M]$, and A_2 the integral operator with the kernel $K|_{[-M,M]\times S_N}$ acting from functions on S_N into functions on $[-M, M]$.

Arguing as in the proof of Theorem 2.1, we have that, for any integer $m \in [-M, M-1]$,

$$\|A_2 v_k^N\|_{L^q(m,m+1)} \le L_{1,M,N}^{1-\sigma/q} \cdot L_{2,M,N}^{\sigma/q} \sup_{\substack{n\in\mathbb{Z} \\ n\notin[-N,N-1]}} \|v_k^N\|_{L^p(n,n+1)},$$

where

$$L_{1,M,N} = \sup_{\substack{m\in\mathbb{Z} \\ m\in[-M,M-1]}} \Bigg[\sum_{\substack{n\in\mathbb{Z} \\ n\notin[-N,N-1]}} \left(\text{ess sup}_{x\in[m,m+1]} \int_n^{n+1} |K(x,y)|^r \, dy \right)^{s/r} \Bigg]^{1/s}$$

and

$$L_{2,M,N} = \sup_{\substack{m\in\mathbb{Z} \\ m\in[-M,M-1]}} \Bigg[\sum_{\substack{n\in\mathbb{Z} \\ n\notin[-N,N-1]}} \left(\text{ess sup}_{y\in[n,n+1]} \int_m^{m+1} |K(x,y)|^\sigma \, dx \right)^{\theta/\sigma} \Bigg]^{1/\theta}.$$

If $C = \sup_k \|u_k\|_{BS^p}$, then from previous estimates we obtain

$$\|A_2 v_k^N\|_{L^q(-M,M)} \le 2M \cdot C \cdot L_{1,M,N}^{1-\sigma/q} \cdot L_{2,M,N}^{\sigma/q}.$$

Thus, given M and $\varepsilon > 0$, we can choose N such that

$$\|A_2 v_k^N\|_{L^q(-M,M)} \le \varepsilon$$

independently of k.

By Theorem 1 of Section 11.3, [11], A_1 is a bounded operator from $L^p(-N,N)$ into $L^q(-M,M)$. Therefore, due to equation (3.1), given M we can choose first sufficiently large N and then sufficiently large k_ε such that $\|Au_k\|_{L^q(-M,M)} \neq 2\varepsilon$ provided $k \geq k_\varepsilon$. □

In general, integral operators in the spaces BS^p are not compact. Nevertheless, they can be locally compact. We say that an operator $A \in L(BS^p, BS^q)$ is *locally compact* if the image of any bounded subset of BS^p is a precompact subset in $L^p_{loc}(\mathbb{R})$.

Proposition 3.1. *Under the assumptions of Theorem 2.1* (a)*, suppose in addition that* $(1-\frac{\sigma}{q})p' < r$*. Then the integral operator* A *is a locally compact operator from* BS^p *into* BS^q*.*

Proof. This is similar to the proof of Theorem 3.1 (use (3.1)). □

Now we introduce the notion of local convergence for operators in $L(BS^p, BS^q)$. For any compact interval I denote by χ_I the characteristic function of I. For any $A \in L(BS^p, BS^q)$ we denote by $\chi_I A$ the operator defined by $(\chi_I Au)(x) = \chi_I(x)(Au)(x)$. We say that a sequence of operators $A_k \in L(BS^p, BS^q)$ *converges locally* to $A \in L(BS^p, BS^q)$ if the sequence A_k is bounded and for every compact interval I the sequence $\chi_I A_k$ converges to $\chi_I A$ in $L(BS^p, BS^q)$. Note that in this definition we can restrict ourselves to intervals of the form $I = [m, m+1]$, with $m \in \mathbb{Z}$. On every *closed ball* in $L(BS^p, BS^q)$, this convergence is induced by a complete metric generated by the countable family of seminorms $\|\chi_I A\|_{L(BS^p,BS^q)}$, $I = [m, m+1]$, with $m \in \mathbb{Z}$.

Proposition 3.2. *Suppose that* A_k *converges locally in* $L(BS^p, BS^q)$ *to* A*. If the operators* A_k *are locally continuous (respectively, locally compact), then so is* A*.*

Proof. Suppose that the operators A_k are locally continuous and converge locally to A. Let $u_m \in BS^p$, $u_m \to u_0$ in L^p_{loc}, and $I \subset \mathbb{R}$ be a compact interval. Then

$$\begin{aligned}&\|Au_0 - Au_m\|_{L^q(I)} = \|\chi_I Au_0 - \chi_I Au_m\|_{BS^q}\\ &\leq \|\chi_I(A - A_k)u_0\|_{BS^q} + \|\chi_I A_k(u_0 - u_m)\|_{BS^q} + \|\chi_I(A_k - A)u_m\|_{BS^q},\end{aligned}$$

and the result follows immediately. The case of local compactness is similar. □

Also on the level of kernel functions there is a corresponding notion of local convergence. Let $\mathcal{K}$ denote either $\mathcal{K}_{1;r,s}$, or $\mathcal{K}_{2;\sigma,\theta}$. A sequence $K_j(x,y) \in \mathcal{K}$

converges locally in $\mathcal{K}$ to K if the sequence is bounded in $\mathcal{K}$ and $\chi_I(x)K_j(x,y)$ converges to $\chi_I(x)K(x,y)$ in $\mathcal{K}$ for every compact interval I. As in the case of operators, on every *closed ball* in $\mathcal{K}$ the local convergence of kernels is generated by a countable system of seminorms and, hence, is a metric convergence.

Proposition 3.3. *Let $\mathcal{K}$ be the space of kernel functions from Theorem* 2.1, (a), (b), *or* (c), *with p, q, r, s, σ and θ satisfying the assumptions of that theorem. Suppose that K_j converges locally in $\mathcal{K}$ to K. Then for corresponding integral operators we have that A_j converges to A locally in $L(BS^p, BS^q)$.*

Proof. This follows immediately from the definitions and Theorem 2.1. □

Proposition 3.4. *Let $\varphi(x) \in L^1$ be a nonnegative function, nonincreasing for $x \geq 0$ and nondecreasing for $x \leq 0$. Let $K_j(x,y)$ be a sequence of kernel functions such that $|K_j(x,y)| \leq \varphi(x-y)$ that converges to $K(x,y)$ in $L^\infty_{loc}(\mathbb{R}^2)$. Then K_j converges locally to K in $\mathcal{K}_{1;\infty,1} = \mathcal{K}_{2;\infty,1}$.*

Proof. This is similar to Proposition 2.2 and simpler. □

Remark 3.1. More general sufficient conditions of convergence and local convergence can be given, but Propositions 2.2 and 3.4 are enough for the applications we will study in the sequel.

4. Almost periodic operators

In this section we study the action of integral operators in the spaces S^p, $1 \leq p < \infty$, of Stepanov almost periodic (for short, a.p.) functions, as well as in the space AP of Bohr a.p. functions. To unify the notation, we set $S^\infty = AP$.

Definition 4.1. *A function $f \in BS^p$, $1 \leq p \leq \infty$, belongs to the space S^p if the family of shifts $\{f(\cdot - \tau)\}_{\tau \in \mathbb{R}}$ is precompact in BS^p; i.e., every sequence $f(\cdot - \tau_k)$, $\tau_k \in \mathbb{R}$, contains a subsequence that converges in BS^p.*

This is the so-called Bochner definition of almost periodicity. Obviously, it extends to the case of functions with values in a Banach space, see, e.g., [1, 12, 17]. The space S^p, $1 \leq p < \infty$, consists of Stepanov almost periodic functions with the exponent p, while $S^\infty = AP$ is the space of Bohr almost periodic functions.

For $\tau \in \mathbb{R}$, we introduce the operator of right shift

$$(T_\tau f)(x) = f(x - \tau). \tag{4.1}$$

These operators form a one-parametric group of bounded linear operators in all the spaces BS^p, $1 \le p \le \infty$. Note that this group is *not* strongly continuous. But, being restricted to subspaces S^p, $1 \le p \le \infty$, the group $\{T_\tau\}_{\tau\in\mathbb{R}}$ becomes strongly continuous.

Definition 4.2. *Let A be a bounded linear operator acting from BS^p into BS^q, where $1 \le p \le \infty$, $1 \le q \le \infty$. The operator A is said to be* almost periodic *if the operator valued function $T_{-\tau}AT_\tau$ of $\tau \in \mathbb{R}$ with values in $L(BS^p, BS^q)$ is almost periodic.*

The following proposition allows us to introduce the notion of envelope of an almost periodic operator.

Proposition 4.1. *An operator $A \in L(BS^p, BS^q)$ is almost periodic if and only if the set $\{T_{-\tau}AT_\tau\}_{\tau\in\mathbb{R}}$ is precompact in $L(BS^p, BS^q)$.*

Proof. Assume that the set $\{T_{-\tau}AT_\tau\}_{\tau\in\mathbb{R}}$ is precompact in $L(BS^p, BS^q)$; i.e., its closure $\mathcal{E} = \mathcal{E}(A)$ is a compact subset of $L(BS^p, BS^q)$. It is easily seen that the set $\mathcal{E}$ is invariant with respect to the map $\Lambda_\tau : A \mapsto T_{-\tau}AT_\tau$, $\tau \in \mathbb{R}$. Since $\|T_\tau\| \le C$ for all $\tau \in \mathbb{R}$, the family $\{\Lambda_\tau\}_{\tau\in\mathbb{R}}$ is equicontinuous on $\mathcal{E}$ and, hence, precompact in the space $C(\mathcal{E})$ of all continuous self-maps of $\mathcal{E}$. Thus, for each sequence τ_j there exists a subsequence $\tau_{j'}$ such that $\Lambda_{\tau_{j'}} \to \Lambda$ in $C(\mathcal{E})$. Hence,

$$T_{-(\tau+\tau_{j'})}AT_{\tau+\tau_{j'}} = \Lambda_{\tau_{j'}}(T_{-\tau}AT_\tau)$$

converges to $\Lambda(T_{-\tau}AT_\tau)$ uniformly with respect to $\tau \in \mathbb{R}$. If $\tau_j \to \tau_0 \in \mathbb{R}$, this implies the continuity of the operator-valued function $T_{-\tau}AT_\tau$ because for every $u \in BS^p$ the function $T_{-\tau}AT_\tau u \in BS^q$ is continuous with respect to the L^q_{loc}-topology. The case $\tau_j \to \infty$ implies the almost periodicity. The converse statement is trivial. □

Definition 4.3. *The set $\mathcal{E}(A)$ introduced in the proof of Proposition* 4.1 *is called the* envelope *of the almost periodic operator A.*

Proposition 4.2. *Suppose that $A \in L(BS^p, BS^q)$ is almost periodic. Then A maps S^p into S^q.*

Proof. Suppose that $u \in S^p$. To show that $Au \in S^q$ we use the Bochner criterion of Stepanov almost periodicity. Consider the function $(Au)(\cdot+\tau) = (T_{-\tau}A)u$ of the variable $\tau \in \mathbb{R}$ with values in BS^q. Obviously,

$$(T_{-\tau}A)u = (T_{-\tau}AT_\tau)(T_{-\tau}u). \tag{4.2}$$

Since $u \in S^p$, the function $T_{-\tau}u$ is a continuous function of τ with values in BS^p. Moreover, the operator-valued function $T_{-\tau}AT_\tau$ is a continuous function with values in $L(BS^p, BS^q)$. Hence, by (4.2), $(T_{-\tau}A)u$ is a continuous function with values in BS^q.

Now let $\tau_k \to \infty$. Since $u \in S^p$, there exists a subsequence $\tau_{k'}$ such that $T_{-\tau_{k'}}u = u(\cdot + \tau_{k'}) \to v$ in BS^p. By Proposition 4.1, there exist a further subsequence $\tau_{k''}$ and an operator $B \in L(BS^p, BS^q)$ such that $T_{-\tau_{k''}}AT_{\tau_{k''}} \to B$ with respect to the operator norm. Equation (4.2) shows now that

$$(Au)(\cdot + \tau_{k''}) = (T_{-\tau_{k''}}A)u \to Bv$$

in BS^q and, hence, $Au \in S^q$. □

Theorem 4.1. *Assume that $A \in L(BS^p, BS^q)$ is a locally continuous almost periodic operator. Then*

$$\|A\|_{L(BS^p,BS^q)} = \|A_{|S^p}\|_{L(S^p,S^q)}.$$

Proof. Since S^p is a closed subspace of BS^p, the inequality

$$\|A_{|S^p}\|_{L(S^p,S^q)} \le \|A\|_{L(BS^p,BS^q)}$$

is trivial.

To prove the opposite inequality, suppose that $u \in BS^p$. Denote by u_k a uniquely determined $2k$-periodic function such that u_k coincides with u on $[-k, k]$. It is easily seen that

$$\|u_k\|_{S^p} \le \|u\|_{BS^p}$$

and the sequence u_k converges to u in L^p_{loc}. By local continuity, $Au_k \to Au$ in BS^q. Moreover, for every $n \in \mathbb{Z}$

$$\begin{aligned}\Big(\int_n^{n+1} |(Au_k)(x)|^q \, dx\Big)^{1/q} &\le \|Au_k\|_{S^q} \\ &\le \|A_{|S^p}\|_{L(S^p,S^q)}\|u_k\|_{S^p} \le \|A_{|S^p}\|_{L(S^p,S^q)}\|u\|_{BS^p}.\end{aligned}$$

Passing to the limit as $k \to \infty$, we obtain that

$$\Big(\int_n^{n+1} |(Au)(x)|^q \, dx\Big)^{1/q} \le \|A_{|S^p}\|_{L(S^p,S^q)}\|u\|_{BS^p}.$$

Hence,

$$\|Au\|_{BS^q} \le \|A_{|S^p}\|_{L(S^p,S^q)}\|u\|_{BS^p},$$

and the required equality follows. □

Theorem 4.2. *Assume one of the following:*

(*a*) *The assumptions of Theorem* 2.1, (*a*), *are satisfied and the function* $K(\cdot+\tau,\cdot+\tau)$ *is an almost periodic function of* $z\in\mathbb{R}$ *with values either in* $\mathcal{K}_{1;r,s}$, *or in* $\mathcal{K}_{2;\sigma,\theta}$.
(*b*) *The assumptions of Theorem* 2.1, (*b*), *are satisfied and the function* $K(\cdot+\tau,\cdot+\tau)$ *is almost periodic in* $z\in\mathbb{R}$ *with values in* $\mathcal{K}_{2;q,1}$.
(*c*) *The assumptions of Theorem* 2.1, (*c*), *are satisfied and the function* $K(\cdot+\tau,\cdot+\tau)$ *is almost periodic in* $z\in\mathbb{R}$ *with values in* $\mathcal{K}_{1;r,1}$.

Then the integral operator A *given by* (2.1) *is an almost periodic operator from* BS^p *into* BS^q.

Proof. A linear change of variables shows that

$$(T_{-\tau}AT_{\tau}u)(x)=\int_{-\infty}^{\infty}K(x+\tau,y+\tau)u(y)\,dy,$$

and the result follows from Theorem 2.1. □

Remark 4.1. Similarly to Definition 4.3, we can introduce the notion of the envelope $\mathcal{E}(K)$ of the kernel function K as the closure of the set $\{K(\cdot+\tau,\cdot+\tau)\}_{\tau\in\mathbb{R}}$ in the corresponding space of kernel functions. As in Proposition 4.1, the kernel function is almost periodic along the diagonal if and only if its envelope is compact. Therefore, in Theorem 4.2 we can replace the assumption of almost periodicity of kernel function by compactness of the envelope $\mathcal{E}(K)$ in the corresponding space of kernels.

5. Almost automorphic operators

We start this section with the following.

Definition 5.1. *A function* $f\in BS^p$, $1\le p\le\infty$, *belongs to the space* AS^p *if for every sequence* $\tau_k\in\mathbb{R}$ *there exists a subsequence* $\tau_{k'}$ *such that* $f(\cdot-\tau_{k'})$ *converges to some function* g *in* L^p_{loc} *(automatically* $g\in BS^p$*), and*

$$\lim_{k'\to\infty}g(\cdot+\tau_{k'})=f(\cdot)$$

in L^p_{loc}.

In the case when $1\le p<\infty$, the space AS^p consists of Stepanov-like almost automorphic functions introduced recently in [16]. In [16] it is also shown that AS^∞ coincides with the space AA_u of uniformly almost automorphic functions, a closed subspace of the space AA of almost automorphic functions.

Recall that a continuous function f is almost automorphic if for every sequence $\tau_k \in \mathbb{R}$ there exists a subsequence $\tau_{k'}$ such that $f(\cdot - \tau_{k'})$ converges to some function g point-wise and

$$\lim_{k'\to\infty} g(\cdot + \tau_{k'}) = f(\cdot)$$

point-wise. The function f is uniformly almost automorphic if these point-wise limits are actually uniform on compact intervals. Equivalently, all the functions g in the definition of almost automorphy are continuous (see, e.g., [14, 15]).

Now let us introduce the notion of an almost automorphic operator.

Definition 5.2. *Let $A \in L(BS^p, BS^q)$ be a locally continuous operator. The operator A is almost automorphic if the operator-valued function $T_{-\tau}AT_\tau$ of τ is a uniformly almost automorphic function with respect to the local convergence in $L(BS^p, BS^q)$; i.e., for every compact interval $I \subset \mathbb{R}$ the function $\chi_I(T_{-\tau}AT_\tau)$ of τ with value in $L(BS^p, BS^q)$ is uniformly almost automorphic.*

As in the case of almost periodic operators, one can introduce the notion of envelope of an almost automorphic operator. However, this notion is not so useful. Instead, we have the following substitute for Proposition 4.1.

Proposition 5.1. *Let $A \in L(BS^p, BS^q)$ be a locally continuous operator. The operator A is almost automorphic if and only if the operator-valued function $T_{-\tau}AT_\tau$ of τ is continuous with respect to the local convergence of operators, and for every sequence $\tau_j \to \infty$ there exists a subsequence τ'_j such that $T_{-\tau'_j}AT_{\tau'_j}$ converges locally to an operator $B \in L(BS^p, BS^q)$ and $T_{\tau'_j}BT_{-\tau'_j}$ converges locally to A.*

Proof. Let I be a closed interval and χ_I its characteristic function. A straightforward calculation shows that

$$\chi_I(T_{-\tau-\tau'_j}AT_{\tau+\tau'_j} - T_{-\tau}BT_\tau) = T_{-\tau}[\chi_{I+\tau}(T_{-\tau'_j}AT_{\tau'_j} - B)]T_\tau.$$

Let J be another closed interval. Since $\|T_\tau\| \le C$ for all $\tau \in \mathbb{R}$, we obtain that

$$\begin{aligned}
&\|\chi_I(T_{-\tau-\tau'_j}AT_{\tau+\tau'_j} - T_{-\tau}BT_\tau)\|_{L(BS^p,BS^q)} \\
&= \|T_{-\tau}[\chi_{I+\tau}(T_{-\tau'_j}AT_{\tau'_j} - B)]T_\tau\|_{L(BS^p,BS^q)} \\
&\le C^2\|\chi_{I+\tau}(T_{-\tau'_j}AT_{\tau'_j} - B)\|_{L(BS^p,BS^q)}.
\end{aligned}$$

For $\tau \in J$, we have $I + \tau \subset I + J$. Hence,

$$\|\chi_I(T_{-\tau-\tau'_j} A T_{\tau+\tau'_j} - T_{-\tau} B T_\tau)\|_{L(BS^p,BS^q)}$$
$$\leq C^2 \|\chi_{I+J}(T_{-\tau'_j} A T_{\tau'_j} - B)\|_{L(BS^p,BS^q)}.$$

This implies immediately that $T_{-\tau-\tau'_j} A T_{\tau+\tau'_j}$ converges locally to $T_{-\tau} B T_\tau$ uniformly with respect to $\tau \in J$.

A similar argument shows that $T_{-\tau+\tau'_j} B T_{\tau-\tau'_j}$ converges locally to $T_{-\tau} A T_\tau$ uniformly with respect to τ in every closed interval J. The converse statement is trivial. □

Proposition 5.2. *Suppose that $A \in L(BS^p, BS^q)$ is almost automorphic. Then A maps AS^p into AS^q.*

Proof. Suppose $u \in AS^p$. To show that $Au \in AS^q$, consider the function $(Au)(\cdot + \tau) = (T_{-\tau} A)u$ of the variable τ with values in BS^q. It follows from [16] that $T_{-\tau} u$ is a continuous function of τ with values in BS^p. Equation (4.2) shows now that $(T_\tau A)u$ is a continuous function of τ with respect to the topology of L^q_{loc}.

Now suppose that $\tau_k \to \infty$ is such that $T_{-\tau_k} u = u(\cdot + \tau_k) \to v$ and $T_{\tau_k} v = v(\cdot - \tau_k) \to u$ in L^p_{loc}. By Proposition 5.1, passing to a subsequence we can also assume that there is an operator $B \in L(BS^p, BS^q)$ such that $T_{-\tau_k} A T_{\tau_k} \to B$ and $T_{\tau_k} B T_{-\tau_k} \to A$ with respect to the local convergence of operators. The identity

$$(T_{-\tau_k} A)u - Bv = (T_{-\tau_k} A T_{\tau_k} - B) T_{-\tau_k} u + B(T_{-\tau_k} u - v)$$

shows that $(T_{\tau_k} A)u \to Bv$ in L^q_{loc}. Similarly, $(T_{-\tau_k} B)v \to Au$ in L^q_{loc}. Thus, $v \in AS^q$, and the proof is complete. □

Theorem 5.1. *Assume that $A \in L(BS^p, BS^q)$ is an almost automorphic operator. Then*

$$\|A\|_{L(BS^p,BS^q)} = \|A_{|AS^p}\|_{L(AS^p,AS^q)}.$$

Proof. This is similar to the proof of Theorem 4.1 □

As a direct consequence of Theorem 2.1, we obtain the following.

Theorem 5.2. *Assume one of the following:*

(*a*) *The assumptions of Theorem* 2.1 (*a*) *are satisfied and the function $K(\cdot + \tau, \cdot + \tau)$ is a uniformly almost automorphic function of $z \in \mathbb{R}$ with values either in $\mathcal{K}_{1;r,s}$, or in $\mathcal{K}_{2;\sigma,\theta}$ with respect to the local convergence.*

(b) *The assumptions of Theorem* 2.1 (b) *are satisfied and the function* $K(\cdot+\tau,\cdot+\tau)$ *is uniformly almost automorphic in* $z\in\mathbb{R}$ *with values in* $\mathcal{K}_{2;q,1}$ *with respect to the local convergence.*

(c) *The assumptions of Theorem* 2.1 (c) *are satisfied and the function* $K(\cdot+\tau,\cdot+\tau)$ *is uniformly almost automorphic in* $z\in\mathbb{R}$ *with values in* $\mathcal{K}_{1;r,1}$ *with respect to the local convergence.*

Then the integral operator A *given by* (2.1) *is an almost automorphic operator from* BS^p *into* BS^q.

Remark 5.1. It is easily seen that there is a version of Proposition 5.1 for kernel functions. Hence, in Theorem 5.2 we can replace the almost automorphy of $K(\cdot+\tau,\cdot+\tau)$ in the corresponding space $\mathcal{K}$ of kernel functions by the following condition: $K(\cdot+\tau,\cdot+\tau)$ is a continuous function of τ with values in $\mathcal{K}$, and for every sequence $\tau_j\to\infty$ there exists a subsequence τ_j' such that $K(\cdot+\tau_j',\cdot+\tau_j')$ converges locally in $\mathcal{K}$ to some kernel function $\overline{K}$ and $\overline{K}(\cdot-\tau_j',\cdot-\tau_j')$ converges to $K(\cdot,\cdot)$ locally in $\mathcal{K}$.

6. Application

As an application we consider the following differential equation:

$$-(a(x)u')'+b_1(x)u'+(b_2(x)u)'+c(x)u=f(x),\quad x\in\mathbb{R}. \tag{6.1}$$

Here and later on we always assume that the functions a, b_i $(i=1,2)$ and c belong to L^∞, and

$$a(x)\geq a_0>0, \tag{6.2}$$

for some a_0 independent of x. All derivatives are understood in the weak sense (see, e.g., [13]). Moreover, the left-hand side of (6.1) defines a linear continuous operator $L:H^1_{loc}\to H^{-1}_{loc}$ that also acts from H^1 into H^{-1}. We recall that the norm in H^1 is given by

$$\|u\|_{H^1}=\Big(\int_{-\infty}^{\infty}[u'(x)^2+u(x)^2]\,dx\Big)^{1/2}, \tag{6.3}$$

and H^{-1} is the dual space to H^1.

We associate to the operator L the bilinear form

$$L[u,v]=\int_{-\infty}^{\infty}\Big(a(x)u'(x)v'(x)+b_1(x)u'(x)v(x) \\ -b_2(x)u(x)v'(x)+c(x)u(x)v(x)\Big)\,dx, \tag{6.4}$$

defined on $H^1 \times H^1$. In what follows we suppose that the operator L satisfies

$$L[u,u] \geq \alpha_0 \|u\|_{H^1}^2, \quad u \in H^1, \tag{6.5}$$

for some $\alpha_0 > 0$. This assumption is satisfied if, in addition to (6.2),

$$c(x) \geq c_0 > 0 \tag{6.6}$$

for some sufficiently large c_0. (If $b_1 = b_2 = 0$, then it is enough to assume (6.6) for some $c_0 > 0$.) It is convenient to introduce the set $\mathcal{L} = \mathcal{L}(a_0, m, \alpha_0)$ that consists of operators L satisfying (6.2), (6.5), and

$$a(x) \leq m, \quad |b_i(x)| \leq m \ \ (i = 1, 2), \quad |c(x)| \leq m. \tag{6.7}$$

The Lax-Milgram lemma implies that for every $L \in \mathcal{L}$ there exists an inverse operator $L^{-1} : H^{-1} \to H^1$ and

$$\|L^{-1}\| \leq \alpha_0^{-1}. \tag{6.8}$$

Let $L \in \mathcal{L}$. The Green's function $G(x,y) = G_L(x,y)$ satisfies the equation

$$L_x G(x,y) = \delta(x-y),$$

where L_x is the operator L acting with respect to the variable $x \in \mathbb{R}$, $y \in \mathbb{R}$ is a parameter, and δ is the Dirac δ-function. Note that $\delta \in H^{-1}$, $\delta(\cdot - y)$ is a continuous function of $y \in \mathbb{R}$, and $\|\delta(\cdot - y)\|_{H^{-1}}$ is independent of y. The Green's function $G(x,y)$ is a bounded continuous function of $(x,y) \in \mathbb{R}^2$. Indeed,

$$G(\cdot, y) = L_x^{-1} \delta(\cdot - y)$$

is a bounded continuous function of y with values in H^1. Hence, by the Sobolev embedding theorem, $G(\cdot, y)$ is a bounded continuous function of y with values in the space of bounded continuous functions, which implies the required result.

We need the following more precise information about Green's functions.

Lemma 6.1. *Let $L \in \mathcal{L}(a_0, m, \alpha_0)$. Then there exist $C > 0$ and $\varepsilon_0 > 0$ depending only on m and α_0 such that*

$$|G_L(x,y)| \leq C \exp(-\varepsilon_0 |x-y|).$$

Proof. Fix any positive even infinitely differentiable function φ such that $\varphi(0) = 1$ and $\varphi(x) = \exp(-|x|)$ for $|x| \geq 1$. We look for $G = G_L$ in the form

$$G(x,y) = \varphi(\varepsilon(x-y)) G_\varepsilon(x,y),$$

where $\varepsilon > 0$.

Let

$$\varphi_1(x) = \frac{\varphi'(x)}{\varphi(x)}, \quad \varphi_2(x) = \frac{\varphi''(x)}{\varphi(x)}.$$

These functions are infinitely differentiable, bounded, and $\varphi_1(x) = \varphi_2(x) = 1$ if $|x| \geq 1$. Since $\varphi(0) = 1$,

$$\frac{1}{\varphi(\varepsilon(x-y))}\delta(x-y) = \delta(x-y)$$

and we obtain the following equation for G_ε :

$$L_\varepsilon G_\varepsilon(x,y) := LG_\varepsilon(x,y) + \varepsilon L_{1,\varepsilon} G_\varepsilon(x,y) + \varepsilon^2 L_{2,\varepsilon} G_\varepsilon(x,y) = \delta(x-y), \quad (6.9)$$

where

$$L_{1,\varepsilon} u = \varphi_1(\varepsilon(x-y))[(a(x)u)' + a(x)u' + b_1(x)u + b_2 u]$$

and

$$L_{2,\varepsilon}\varphi_2(\varepsilon(x-y))a(x)u.$$

All the operators on the right-hand side of (6.9) act with respect to the variable x. It is easy to verify that the operators $L_{1,\varepsilon}$ and $L_{2,\varepsilon}$ act from H^1 into H^{-1} and are bounded uniformly with respect to ε. Since the operator L is invertible, then so is L_ε with $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of $L \in \mathcal{L}(a_0, m, \alpha_0)$. Moreover, the norm of the inverse operator L_ε^{-1} is bounded by a constant that depends on a_0, m, and α_0 only. Hence, there exists a unique solution $G_\varepsilon(x,y)$ which is a bounded continuous function of y with values in H^1. Furthermore, $\|G_\varepsilon(\cdot,y)\|_{H^1}$ is bounded independently of $L \in \mathcal{L}(a_0, m, \alpha_0)$. As a consequence, $G(x,y)$ is a bounded continuous function on $\mathbb{R}^2$ and its sup-norm is bounded by a constant that depends on a_0, m, and α_0 only. This implies the required result. □

Remark 6.1. Actually, the proof of Lemma 6.1 gives the following. If $\exp(-\varepsilon|x|)f \in H^{-1}$ and $\varepsilon \in (0, \varepsilon_0]$, then equation (6.1) has a unique solution such that $\exp(-\varepsilon|x|)u \in H^1$.

Denote by $\mathcal{G}(a_0, m, \alpha_0)$ the set of Green's functions G_L of all operators $L \in \mathcal{L}(a_0, m, \alpha_0)$.

Lemma 6.2. *The set $\mathcal{G} = \mathcal{G}(a_0, m, \alpha_0)$ is precompact in the space $C(\mathbb{R}^2)$ with respect to the topology of uniform convergence on compact sets.*

Proof. If $G \in \mathcal{G}$, then

$$\begin{aligned}\|G(\cdot,y_1) - G(\cdot,y_2)\|_{H^1} &\leq \|L^{-1}\|\|\delta(\cdot - y_1) - \delta(\cdot - y_2)\|_{H^{-1}} \\ &\leq \alpha_0^{-1}\|\delta(\cdot - y_1) - \delta(\cdot - y_2)\|_{H^{-1}}.\end{aligned}$$

Here $L \in \mathcal{L}(a_0, m, \alpha_0)$ is the operator whose Green's function is G. Since $\delta(\cdot - y)$ is a uniformly continuous function of y with values in H^{-1}, the functions from the set $\mathcal{G}$ are equicontinuous functions of $y \in \mathbb{R}$ with values in H^1. Since for every compact interval I the embedding $H^1(I) \subset C(I)$ is compact, the result follows. □

Theorem 6.1. *Suppose that $L \in \mathcal{L}(a_0, m, \alpha_0)$.*

(*a*) *If $f \in BS^1$, then equation* (6.1) *has a unique bounded solution.*

(*b*) *If the coefficients a, b_1, b_2 and c are almost periodic, and $f \in S^1$, then equation* (6.1) *has a unique almost periodic solution.*

(*c*) *If the coefficients a, b_1, b_2 and c are almost automorphic, and $f \in AS^1$, then equation* (6.1) *has a unique uniformly almost automorphic solution.*

Proof. (*a*) It follows from Lemma 6.1 that $G \in \mathcal{K}_{1;\infty,1} = \mathcal{K}_{2;\infty,1}$. By Corollary 2.2, the function

$$u(x) = \int_{-\infty}^{\infty} G(x, y) f(y)\, dy \tag{6.10}$$

is a bounded solution.

Let us prove the uniqueness. Fix any $\varepsilon > 0$ and an infinitely differentiable function ψ such that $\psi(x) = 1$ on $[m, m+1]$, $\psi(x) = 0$ for $x \notin [m-1, m+2]$, $0 \le \psi(x) \le 1$ and $|\psi'(x)| \le 2$ (such a function obviously exists). A straightforward calculation shows that

$$L(\psi u) = \psi f + g,$$

where

$$g = b_1\psi' u + \psi(b_2 u)' - \psi' a u' - (a\psi' u)'.$$

It is easy to verify that $\|\psi f\|_{H^{-1}} \le C$ and $\|g\|_{H^{-1}} \le C$, where $C > 0$ is independent of m. Hence, $\|\psi u\|_{H^1} \le C_1$, with $C_1 > 0$ independent of m. This implies that $u' \in BS^2$. Therefore, $\exp(-\varepsilon|x|)u'(x) \in L^2$. The boundedness of u implies that $\exp(-\varepsilon|x|)u(x) \in L^2$. Thus, $\exp(-\varepsilon|x|)u(x) \in H^1$. Since $f \in BS^1$, it is easy to see that $\exp(-\varepsilon|x|)f(x) \in L^1 \subset H^{-1}$ for any $\varepsilon > 0$. Now, by Remark 6.1, the uniqueness follows.

(*b*) By Theorem 4.2, Proposition 2.2, and Lemma 6.1, it is sufficient to show that $G(\cdot + \tau, \cdot + \tau)$ is an almost periodic function of τ with values in $L^\infty(\mathbb{R}^2)$. Let us denote by $L(\tau) \in \mathcal{L}(a_0, m, \alpha_0)$ the differential operator whose coefficients are $a(\cdot + \tau)$, $b_1(\cdot + \tau)$, $b_2(\cdot + \tau)$ and $c(\cdot + \tau)$. It is easily seen that $L(\tau)$ is an almost periodic function of τ with values on $L(H^1, H^{-1})$. Also,

$$L(\tau) G(\cdot + \tau, y + \tau) = \delta(\cdot - y) \tag{6.11}$$

(here y and τ are parameters). The inverse operator $L^{-1}(\tau)$ is an almost periodic function of τ with values in $L(H^{-1}, H^1)$. Therefore, $G(\cdot+\tau,\cdot+\tau)$ is an almost periodic function of τ with values in $L^\infty(\mathbb{R}_y, H^1(\mathbb{R}_x)$, hence, in $L^\infty(\mathbb{R}^2)$.

(c) Let $\tau_j \in \mathbb{R}$ be a sequence such that $a(\cdot+\tau_j) \to \overline{a}(\cdot)$, $b_1(\cdot+\tau_j) \to \overline{b_1}(\cdot)$, $b_2(\cdot+\tau_j) \to \overline{b_2}(\cdot)$, $c(\cdot+\tau_j) \to \overline{c}(\cdot)$ uniformly on compact intervals, and $\overline{a}(\cdot-\tau_j) \to a(\cdot)$, $\overline{b_1}(\cdot-\tau_j) \to b_1(\cdot)$, $\overline{b_2}(\cdot-\tau_j) \to b_2(\cdot)$, $\overline{c}(\cdot-\tau_j) \to c(\cdot)$ uniformly on compact intervals. By Lemma 6.2, the sequence $G(\cdot+\tau_j,\cdot+\tau_j)$ is precompact with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^2$. Passing to a subsequence, we can assume that $G(\cdot+\tau_j,\cdot+\tau_j) \to g(\cdot,\cdot)$ in that topology. Moreover, this sequence is bounded and uniformly continuous with respect to the first variable with values in in H^1. Since H^1 is a Hilbert space, its balls are weakly compact. Thus, we can also assume that $G(\cdot+\tau_j, y+\tau_j) \to g(y,\cdot)$ weakly in H^1 and uniformly with respect to y in every compact interval, and, hence, $g(y,\cdot)$ is a weakly continuous function with values in H^1.

Now let v be an arbitrary smooth finitely supported function. Multiplying equation (6.11), with $\tau = \tau_j$, by v and integrating, we obtain

$$L(\tau_j)[G(\cdot+\tau_j, y+\tau_j), v] = v(y), \tag{6.12}$$

where $L(\tau)[u,v]$ is the bilinear form associated to the operator $L(\tau)$. Due to the convergence properties of $G(\cdot+\tau_j,\cdot+\tau_j)$ and of the coefficients of L, we can pass to the limit in (6.12) and obtain that

$$\overline{L}[g(\cdot,y), v] = v(y),$$

which means that $g = G_{\overline{L}}$, the Green's function of the operator $\overline{L}$ that corresponds to the coefficients $\overline{a}$, $\overline{b_1}$, $\overline{b_2}$ and $\overline{c}$. The same argument shows that, probably after an additional passage to a subsequence, $G_{\overline{L}}(\cdot-\tau_j,\cdot-\tau_j) \to G(\cdot,\cdot)$ in $L^\infty_{loc}(\mathbb{R}^2)$.

Thus, by Remark 5.1, $G(\cdot+\tau,\cdot+\tau)$ is an almost automorphic function of τ with values in $L^\infty_{loc}(\mathbb{R}^2)$. This fact, Lemma 6.1, and Proposition 3.4 show that $G(\cdot+\tau,\cdot+\tau)$ is an almost automorphic function of τ with respect to the local convergence in $\mathcal{K}_{1;\infty,1} = \mathcal{K}_{2;\infty,1}$. By Theorem 5.2, the solution given by (6.10) belongs to AS^∞. The proof is complete. □

Concluding remarks. The assumptions of Theorem 2.1 are not optimal. Indeed, let

$$K(x,y) = g(x)\varphi(x-y)h(y), \tag{6.13}$$

where $g, h \in L^\infty$ and $\varphi \in L^1$. The corresponding operator A is the composition $A = M_g \circ C_\varphi \circ M_h$, where M_g and M_h are the multiplication operators by g and h, respectively, and $C_\varphi = \varphi\star$ is the convolution operator. The operators M_g and M_h are obviously bounded in BS^p for all $p \in [1, \infty]$. The same is true for the convolution operator (see, e.g., [2]). Hence, $A \in L(BS^p)$. But, in general, the kernel K does not satisfy the assumptions of Theorem 2.1 even in the case when $g = h \equiv 1$. If both functions f and g are almost periodic (respectively, uniformly almost automorphic), then the operator A is almost periodic (respectively, almost automorphic) in $L(BS^p)$. In the case when $h \equiv 1$ these operators are the so-called product-convolution operators considered in [6].

Let us also mention that $K \in L^\infty(\mathbb{R}_x; L^1(\mathbb{R}_y))$ implies $A \in L(L^\infty)$, and this is also not covered by Theorem 2.1.

REFERENCES

[1] L. Amerio and G. Prouse, "Almost Periodic Functions and Functional Equations," Van Nostrand, 1971.

[2] G. Bruno and A. Pankov, *On convolution operators in the spaces of almost periodic functions and L^p spaces,* Z. Anal. Anwend., 19 (2000), 359-367.

[3] G. Bruno, A. Pankov, and Y. Tverdokhleb, *On almost-periodic operators in the spaces of sequences,* Acta Appl. Math., 65 (2001), 153–167.

[4] D. Bugajewski and T. Diagana, *Almost automorphy of the convolution operator and applications to differential and functional-differential equations,* Nonlinear Stud., 13 (2006), 129–140.

[5] D. Bugajewski, T. Diagana, and C.M. Mahop, *Asymptotic and pseudo almost periodicity of the convolution operator and applications to differential and integral equations,* Z. Anal. Anwend., 25 (2006), 327–340.

[6] R.C. Busby and H.A. Smith, *Product-convolution operators and mixed-norm spaces,* Trans. Amer. Math. Soc., 263 (1981), 309–341.

[7] T. Diagana and G. N'Guérékata, *Stepanov-like almost automorphic functions and applications to some semilinear equations,* Applicable Analysis, 86 (2007), 723–733.

[8] N.Dunford and J.T. Schwartz, "Linear operators. General Theory," Interscience Publ., New York-London, 1958.

[9] S. Fatajou, N. Van Minh, G. N'Guérékata, and A. Pankov, *Stepanov-like almost automorphic solutions for nonautonomous evolution equations,* Electr. J. Differ. Equat., 2007 (2007), 1–11.

[10] J.J.F. Fournier and J. Stewart, *Amalgams of L^p and l^q,* Bull. Amer. Math. Soc., 13 (1985), 1–21.

[11] L. I. Kantorovich and G. P. Akilov, "Functional Analysis," Pergamon, Oxford, 1982.

[12] B.M. Levitan and V.V. Zhikov, "Almost Periodic Functions and Differential Equations," L. W. Longdon, Cambridge-New York, 1982.

[13] J.L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems," Vol. 1, Springer, Berlin, 1972.

[14] G.M. N'Guérékata, "Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces," Kluwer Academic/Plenum Publishers, New York-London-Moscow, 2001.

[15] G.M. N'Guérékata, "Topics in Almost Automorphy," Springer, New York, Boston, Dordrecht, Lodon, Moscow 2005.

[16] G.M. N'Guérékata and A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations,* Nonlinear Analysis, 68 (2008), 2658–2667.

[17] A. Pankov, "Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations," Kluwer, Dordrecht, 1990.

[18] M.A. Shubin, *Almost periodic functions and partial d ifferential operators,* Russian Math. Surveys, 33 (1978), 1–52.