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# ASYMPTOTIC STABILITY OF A DECAYING SOLUTION TO THE KELLER-SEGEL SYSTEM OF DEGENERATE TYPE

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(Submitted by: Gustavo Ponce)

Dedicated to Professor Mitsuhiro Nakao on his sixties birthday

**Abstract.** We discuss the global behavior of the weak solution of the Keller-Segel system of degenerate type. Asymptotic stability of the Barenblatt-Pattle solution and its convergence rate for the decaying weak solution in $L^1(\mathbb{R}^n)$ is shown for the degenerated case $1<\alpha<2-\frac{2}{n}$. The method is based on the techniques applied to the Fokker-Plank equation due to Carrillo-Toscani [8] deriving from the explicit time decay of the free energy functional and some new estimates for the nonlinear interaction involving the critical type Sobolev inequality. We give the rigorous justification of those procedures via some approximating procedures.

## 1. Degenerated Keller-Segel system

We are concerned with the large time behavior of the global solution of the degenerated parabolic elliptic system

$$\begin{cases}\partial_t u-\Delta u^\alpha+\nabla(u\nabla\psi)=0, & x\in\mathbb{R}^n,\ t>0,\\ -\Delta\psi+\lambda\psi=u, & x\in\mathbb{R}^n,\ t>0,\\ \quad u(0,x)=u_0(x), & x\in\mathbb{R}^n,\end{cases}\tag{1.1}$$

where $\alpha>1$ and $\lambda>0$. We show that the weak solution of (1.1) for $1<\alpha<2-\frac{2}{n}$ decays at an optimal rate and the asymptotic profile is determined by an explicit method.

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The semilinear type of the equation ($\alpha = 1$) is known as a model of chemotaxis and is largely studied. Jäger-Luckhaus [20] introduced the parabolic-elliptic version of the Keller-Segel system (see, for a detailed summary, Horstmann [18]. Namely,

$$\begin{cases} \partial_t u - \Delta u + \nabla(u \nabla \psi) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ -\Delta \psi + \lambda \psi = u, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

The original model was introduced by Keller-Segel [22], where the system is the coupled nonlinear diffusion equations. It has been studied in detail for the behavior of the system by many authors. This system has a strong connection with the self-interacting particles and dynamics of gaseous stars where a very large physical scale is considered and qualitative behavior of the semilinear case has been studied widely by Biller [1], Biler-Hebisch-Nadzieja [3], Nagai [35], [36], Nagai-Senba-Yoshida [39] and references therein (see also [38]). The above system is also connected to the simplest equation of the semiconductor device simulation if $\lambda = 0$ (cf. Mock [32], Jüngel [21]). The semi-conductor device model has a stabler sign of the nonlinearity that makes the system admit the large data global solutions ([27]-[29]). Indeed it has a connection with fluid mechanics and one may find the derivation from the compressible Navier-Stokes-Poisson system ([25]).

The main problem here is the large-time asymptotic behavior of the solution of the degenerated elliptic parabolic system (1.1) when $\alpha > 1$. For the degenerated case (1.1), it is known that finite propagation of the support of the solution occurs (see Díaz-Nagai-Shimarev [12] and Díaz-Galano-Jüngel [11]). Once there is a point where the solution vanishes, the equation essentially requires the notion of weak solution.

**Definition**. Let $\alpha > 1$. Given $u_0 \in L^1 \cap L^\alpha(\mathbb{R}^n)$ with $u_0(x) \geq 0$ for $x \in \mathbb{R}^n$, we call $(u(t,x), \psi(t,x))$ a weak solution of the system (1.1) if there exists $T > 0$ such that

i) $u(t,x) \geq 0$ for almost all $(t,x) \in [0,T) \times \mathbb{R}^n$,
ii) $u \in L^\infty(0,T; L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ with $\nabla u^\alpha \in L^2((0,T) \times \mathbb{R}^n)$, and
iii) for an arbitrary test function $\phi \in C^1([0,T); C_0^\infty(\mathbb{R}^n))$,

$$\begin{aligned} &\int_{\mathbb{R}^n} u(t)\phi(t)dx - \int_{\mathbb{R}^n} u_0 \phi(0) dx \\ &\quad = \int_{t_0}^{t_1} \int_{\mathbb{R}^N} \left( u(\tau)\partial_t \phi(\tau) - \nabla u^\alpha(\tau) \cdot \nabla \phi(\tau) + u(\tau) \nabla \psi(\tau) \cdot \nabla \phi(\tau) \right) \, dx d\tau \end{aligned}$$

for $0 \le t \le T$, where $\psi = E_n * u$ and $E_n(\cdot)$ is the fundamental solution of $-\Delta + \lambda$ in $\mathbb{R}^n$.

The existence of the weak solution is an application of the standard theory of the degenerated parabolic system (see for example Díaz-Galano-Jüngel [10], Sugiyama-Kunii [50]). Note that the equation does not have the comparison principle of solutions of any type nor the semigroup representation; while it is possible for the semilinear case, the proof of the existence of weak solutions requires some approximation procedure and the parabolic regularity theory. The global existence of the weak solution is classified by a threshold exponent $\alpha = 2 - \frac{2}{n}$. We summarize the known results for the existence and nonexistence for global weak solutions as follows ([49]).

**Proposition 1.1.** ([5], [42], [49], [50]) *Let $n \ge 2$. For $\alpha > 1$ and non-negative $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$, there exists a weak solution $(u, \psi)$ of the degenerated Keller-Segel system* (1.1) *that satisfies the following: For $0 < t < T$,*

$$\|u(t)\|_1 = \|u_0\|_1,$$

$$W(t) + \int_0^t \int_{\mathbb{R}^n} u(t)\Big|\nabla \frac{\alpha}{\alpha-1} u^{\alpha-1} - \nabla\psi\Big|^2 dxdt \le W(0),$$

*where*

$$W(t) \equiv \frac{1}{\alpha-1}\|u(t)\|_\alpha^\alpha - \frac{1}{2}\|\Lambda^{-1}u(t)\|_2^2 \tag{1.3}$$

*with $\Lambda = (-\Delta+1)^{1/2}$. In addition:*

(1) *If $\alpha > 2 - \frac{2}{n}$, then the solution exists globally in time. Moreover, the weak solution satisfies the uniform estimate as follows: For any $T > 0$, we have*

$$\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\alpha, \|(-\Delta)^{-1/2}u_0\|_2, T) \tag{1.4}$$

*for $t \in [0, T]$.*

(2) *For $n \ge 3$ and $1 < \alpha \le 2 - \frac{2}{n}$, there exists a constant $C > 0$ only depending on $n$ and $\alpha$ such that if the initial data satisfies $W(0) > 0$ and*

$$\|u_0\|_1^{1-\gamma} W(0)^{\frac{\gamma-\alpha+1}{\alpha}} < C\|E_n\|^{-1}_{L_w^{\frac{n}{n-2}}}, \tag{1.5}$$

*then the weak solution exists globally in time and satisfies the uniform estimate* (1.4)*, where $E_n$ is the fundamental solution of $\Lambda^2 \equiv -\Delta + 1$ in $\mathbb{R}^n$, $L^q_w$ is the weak Lebesgue space and $\gamma + 1 = \frac{\alpha(n-2)}{(\alpha-2)n}$. In* *particular, if* $\alpha = 2 - \frac{2}{n}$ *the above condition is given by*

$$\|u_0\|_1^{2/n} < \frac{2n}{n-2}\|E_n\|^{-1}_{L_w^{\frac{n}{n-2}}}. \tag{1.6}$$

(3) *If* $\lambda = 1$, $n \geq 3$, $\alpha \leq 2 - \frac{2}{n}$, *and the initial data* $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ *with* $|x|^2 u_0 \in L^1(\mathbb{R}^n)$ *satisfies*

$$W(0) \equiv \frac{1}{\alpha - 1}\|u_0\|_\alpha^\alpha - \frac{1}{2}\|(-\Delta + 1)^{-1/2} u_0\|_2^2 < 0, \tag{1.7}$$

*then the weak solution does not exist globally in time. That is, there exists* $T < \infty$ *such that for some initial data* $u_0$ *the weak solution blows up in a finite time* $T$ *in the following sense:*

$$\limsup_{t \to T} \|u(t)\|_q = \infty \qquad \text{for all } q \in [\alpha, \infty].$$

The formal proof of the nonexistence of the weak solution is obtained by Biler-Nadzieja-Stanczy [5] and rigorous justification is seen in [49]. For the semilinear case, $\alpha = 1$ and $n = 2$ corresponds to the critical case $\alpha = 1 = 2 - \frac{2}{n}$ and the solution may blow up in finite time for large initial data. See, for the semilinear cases, Nagai [36], Senba-Suzuki [47], [48], Suzuki [51]. The threshold value of the degenerated case is given by the best possible constant of the Hardy-Littlewood-Sobolev inequality. See, for further discussion, [50].

We should emphasize here that the exponent $\alpha = 2 - \frac{2}{n}$ appears as the corresponding Fujita exponent for the semilinear heat equation (Fujita [16]), however the actual role of the exponent is different from the power nonlinearity case. For instance, the corresponding quasi-linear parabolic equation has a similar exponent:

$$\begin{cases} \partial_t u - \Delta u^\alpha = u^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where $\alpha > 0$ and $p > 1$. The exponent $p = \alpha + \frac{2}{n} = 1 + \frac{\sigma}{n}$ with $\sigma = \alpha(n-2) + 2$ is considered as the threshold for the global existence and finite time blow up for the small data solutions (Friedman-Kamin [15], Kawanago [23], Mochizuki-Suzuki [33], Vázquez [52]). By the scaling balance, the super critical case $p > \alpha + \frac{2}{n}$ corresponds to $\alpha < 2 - \frac{2}{n}$ in our case. However, since the system (1.1) preserves the mass and entropy functional, the large time behavior turns out in *the opposite situation* to the above case (1.8). In this sense, the situation is quite analogous to the result for the nonlinear

Schrödinger equation:

$$\begin{cases} \partial_t u - i\Delta u = i|u|^{p-1}u, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.9}$$

There is a threshold exponent of the global existence and finite time blow up of solutions, crossing the pseudo-conformal exponent $p = 1+\frac{4}{n}$ (see Cazenave [9], Ogawa-Tsutsumi [44]). Since the weak ($H^1$) solution preserves its charge and energy functional the situation is similar to the case for (1.1).

We now discuss our main aim here on the asymptotic behavior of the global small solution when the degeneracy order is less than the critical case. It has been shown that the weak solution for small initial data shows a decay estimate when $\alpha \le 2 - \frac{2}{n}$.

We define the weighted Lebesgue space as

$$L^p_a(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |\cdot|^a f(\cdot) \in L^p(\mathbb{R}^n)\},$$

for $1 \le p \le \infty$ and $a > 0$.

**Definition**. For $\alpha > 1$ let $\sigma = n(\alpha - 1) + 2$. For $A > 0$ and $M > 0$, we define the Barenblatt-Pattle solution as

$$U(t,x) = (1+\sigma t)^{-n/\sigma}\Big(A - \frac{\alpha-1}{2\alpha}\frac{|x|^2}{(1+\sigma t)^{2/\sigma}}\Big)_+^{\frac{1}{\alpha-1}}, \tag{1.10}$$

where $A$ is chosen as $\|U(t)\|_1 = M$ and $(f)_+$ is the positive part of $f$.

The following result, which is our main concern here, shows that if we restrict the exponent of degeneracy $\alpha < 2-\frac{2}{n}$ and the initial data is in some weighted space, then the solution has the uniform stability estimate for the Barenblatt-Pattle profile.

**Theorem 1.2.** (Asymptotic stability) *Let $1 < \alpha < 2 - \frac{2}{n}$ and assume that $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the condition* (1.5)*. Then the corresponding global weak solution $u(t,x)$ of* (1.1) *satisfies the following asymptotic behavior: We have for $M = \|u_0\|_1$ and for some $0 < \nu < 2$,*

$$\|u(t) - U(t)\|_1 \le C(1+\sigma t)^{-\nu},$$

*where $U(t)$ is the Barenblatt-Pattle solution given by* (1.10) *with $\|U(t)\|_1 = M$. In particular, the decay estimate*

$$\|u(t)\|_q \le C(1+\sigma t)^{-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}, \qquad 1 \le q \le \infty, \tag{1.11}$$

*is optimal.*

**Remark.** It should be noted that the decay estimate for the weak solution is considered in Ogawa [42], Sugiyama [49] and Luckhaus-Sugiyama [30]. Indeed, in [30], the asymptotic stability for the degenerate system (1.1) is considered. Their result covers the stability in $L^p(B_{A(t)})$, where $1 < p < \infty$ and $A(t)$ is some parabolic domain depending on $t$. However, the extremal cases $p = 1$ and $\infty$ are excluded, as

$$t^{\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\|u(t) - U(t)\|_{L^q(B_{A(t)})} = o(1), \qquad 1 < q < \infty,$$

and no explicit decay rate for the error term in $L^1$ is shown. Our result shows that the decay of the error term is improved, because the solution is in the weighted space and, moreover, the regularity of the weak solution is improved in the Hölder space.

The corresponding result for the semi-linear case $\alpha = 1$ is also possible to obtain and the asymptotic profile for this case is the Gaussian $\frac{1}{(2\pi(1+2t))^{n/2}}e^{-\frac{|x|^2}{2(1+2t)}}$. See Biler-Dolbeault [2], Nagai-Syukuin-Umesako [40] for slightly weaker results. For the drift-diffusion case (the system (1.2) with $\lambda = 0$), the complete result is due to Kobayashi-Kawashima [24] and [45] (cf. Ishige-Kawakami [19] for the power nonlinearity case).

The essential difference between the previous result and ours is that the convergence holds for the whole $L^1$ norm, where the total mass is preserved and the stronger convergence rate is shown. Our result is based on the observation of the conservation law equation. That is, the proof of the above asymptotic behavior depends on the analysis of the self-similar rescaled equation and on how the free energy density behaves when $t \to \infty$. This method is developed by Carrillo-Toscani [8] for the Fokker-Plank equation and single porous medium equations. There is a variational formulation of the stationary solution of the porous medium equation (see Otto [46]) and the background of the proof relies on this fact. In our case, the equation has the strong drift term in it and the main issue in showing the above asymptotics is how to handle the nonlinear coupling effect. The crucial point to obtaining the asymptotic stability estimate is the fact that it requires *the stronger regularity* for the weak solution, that is, the Hölder continuity of the weak solution.

We first observe the rescaled equation by the self-similar scaling (cf. [17], [2]). By the forward self-similar scaling,

$$\begin{cases} t' = \frac{1}{\sigma}\log(1 + \sigma t), \\ x' = x/(1 + \sigma t)^{1/\sigma}, \end{cases}$$

we introduce new scaled variables $(t', x')$ and scaled functions

$$\begin{cases} v(t',x') \equiv e^{nt'} u\big(\frac{1}{\sigma}(e^{\sigma t'}-1), x'e^{t'}\big), \\ \phi(t',x') \equiv e^{nt'} \psi\big(\frac{1}{\sigma}(e^{\sigma t'}-1), x'e^{t'}\big). \end{cases}$$

Then the scaled system is of the following form:

$$\begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t'>0, x'\in\mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \lambda\phi = v, & t'>0, x'\in\mathbb{R}^n, \\ v(0,x') = u_0(x'), \end{cases} \tag{1.12}$$

with $\kappa = n+2-\sigma = n(2-\alpha)$.

In view of the scaled equation (1.12), the asymptotic profile when $t\to\infty$ naturally coincides with the steady state solution of the non-perturbed degenerated parabolic equation

$$\begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^\alpha + x'v) = 0, & t'>0, x'\in\mathbb{R}^n, \\ v(0,x') = u_0(x'), & x'\in\mathbb{R}^n. \end{cases} \tag{1.13}$$

As is well understood, the limiting solution is the Barenblatt-Pattle solution

$$v(\infty, x') = \Big(A - \frac{\alpha-1}{2\alpha}|x'|^2\Big)_+^{\frac{1}{\alpha-1}},$$

where the constant $A$ is chosen as $\|v(\infty)\|_1 = \|u_0\|_1$. To see this, we observe that the scaled equation (1.12) has the scaled entropy; by setting

$$W_s(v,\phi)(t') \equiv \int_{\mathbb{R}^n} \Big(\frac{1}{\alpha-1} v^\alpha(t') + \frac{1}{2}|x|^2 v(t') - \frac{1}{2} e^{-\kappa t'} v(t')\phi(t')\Big)\, dx',$$

$$K(x', v(t'), \phi(t')) = K(x', v, \phi) \equiv \nabla\Big(\frac{\alpha}{\alpha-1} v^{\alpha-1} + \frac{1}{2}|x'|^2 - e^{-\kappa t'}\phi\Big),$$

the following identities hold:

$$\int_{\mathbb{R}^n} v(t')dx = \int_{\mathbb{R}^n} u_0(x')dx', \tag{1.14}$$

$$W_s(t') + \int_0^{t'} \Big[\int_{\mathbb{R}^n} v(\tau)\left|K(x',v,\phi)(\tau)\right|^2 dx' + e^{-(\kappa+2)\tau}\int_{\mathbb{R}^n} |\nabla\phi(\tau)|^2 dx'\Big] d\tau$$
$$= W_s(0). \tag{1.15}$$

Hence, the decay of the solution follows from the analogous estimate for the global existence and uniform bounds for the weak solution. The convergence to the limiting solution is derived from observing that the functional

$$I(v(t')) \equiv \int_{\mathbb{R}^n} v(t')|K(x', v, \phi)|^2 dx' \tag{1.16}$$

appearing in the scaled entropy inequality (1.15) is integrable in the whole time $t > 0$. However this only implies that the solution converges to the Barenblatt-Pattle solution. To obtain the convergence rate in the mass space $L^1(\mathbb{R}^n)$, we need to consider the decay rate of the above functional (1.16). To this end, we compute the time derivative of (1.16) and show that it is decaying at a certain rate. We have to adjust the estimate for our equation with the non-local interaction term, since our equation is degenerated and rigorous justification is required for the computation of the derivative of the above functional. This will be done by parabolic regularization for the equation and cut off procedure. The detailed proof is given in the Appendix A. To obtain the asymptotics, we also require a slightly better regularity of the weak solution, namely Hölder continuity of the solution. The Hölder estimate for the degenerated equation is due to DiBenedetto-Friedman [13], [14] and Wiegner [53] (see also Caffarelli-Evans [7] and Misawa [31]). We emphasize that the critical type of Sobolev inequality found in [43] (see also [6], [26], and [41]) is used for handling the worst term appearing in the nonlinear estimate.

This paper is organized as follows. In the next section, we derive the above entropy and free energy bound for the scaled equation (1.12). Based on these conserved quantities, we show the time a priori estimate as well as the decay of the solution simultaneously. Section 3 is devoted to the proof of the asymptotic profile of the solution. In appendix A, we give the rigorous treatment for the analysis of the functional (1.16). In appendix B, we show the known regularity result for the degenerated parabolic equation. In what follows, $\|f\|_p$ denotes the $L^p = L^p(\mathbb{R}^n)$ norm and $L^p_w(\mathbb{R}^n)$ denotes the weak $L^p$ space. $C^s(\mathbb{R}^n)$ is the Hölder continuous class of order $s > 0$. $L^p_a(\mathbb{R}^n)$ is the weighted Lebesgue space defined by $\|f\|_{L^p_a} = \|f\|_p + \||x|^a f\|_p < \infty$. We denote $\max(f(x)-g(x), 0)$ by $(f(x)-g(x))_+$. The homogeneous Besov space is denoted by $\dot{B}^s_{p,\sigma}$ (see [43] for the definition). $BMO$ denotes the space of functions with bounded mean oscillation and

$$\|f\|_{BMO} = \sup_{R>0, x\in\mathbb{R}^n} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \bar{f}_{B_R}| dy,$$

where $\bar{f}_{B_R}$ is the average of $f$ over the ball $B_R$. $\operatorname{osc}_D f = \sup_D f - \inf_D f$ denotes the oscillation of a function $f(x)$ in a domain $D$.

## 2. Time Decay of Small Solutions

In this section, we show time decay of the global weak solution of the degenerated Keller-Segel system. This is originally shown in [49]; however, we present the method of rescaling which is shown in [42].

### 2.1. **Rescaled equation.** We introduce the new scaled variables $(t', x')$ as

$$\begin{cases} t' = \frac{1}{\sigma}\log(1+\sigma t), \\ x' = x/(1+\sigma t)^{1/\sigma}, \end{cases} \tag{2.1}$$

where $\sigma = n(\alpha-1)+2$ and introduce the new scaled unknown function $u(t', x')$ by

$$u(t,x) = (1+\sigma t)^{-n/\sigma} v\Big(\frac{1}{\sigma}\log(1+\sigma t), \frac{x}{(1+\sigma t)^{1/\sigma}}\Big),$$
$$\psi(t,x) = (1+\sigma t)^{-n/\sigma} \phi\Big(\frac{1}{\sigma}\log(1+\sigma t), \frac{x}{(1+\sigma t)^{1/\sigma}}\Big).$$

Or this may be written as

$$v\left(t', x'\right) \equiv e^{nt'} u\Big(\frac{1}{\sigma}(e^{\sigma t'}-1), x' e^{t'}\Big),$$
$$\phi\left(t', x'\right) \equiv e^{nt'} \psi\Big(\frac{1}{\sigma}(e^{\sigma t'}-1), x' e^{t'}\Big)$$

and the resulting scaling equation of $(v, \phi)$ follows by setting $\kappa = n+2-\sigma = n(2-\alpha)$,

$$\begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^{\alpha} + x' v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t'>0, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t'>0, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x'), & x' \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

In this case, the vanishing exponent as before can be found to be $\alpha = 2$ by

$$0 = \sigma - n - 2 = n(\alpha - 2)$$

and thus the sub-critical case corresponds to $\alpha < 2$. Hereafter we analyze the above rescaled equation (2.2) to see the asymptotic behavior of the solution. We slightly change the outlook of the solution as follows:

The existence of the weak solution of (2.2) may be proven in a way similar to the original equation. Indeed, the scaling does not change any analytical feature of the original weak solution so that the solution can be obtained from the weak solution of (1.1) except for a weighted restriction such as

$v \in C((0,T); L^\alpha \cap L^1_a(\mathbb{R}^n))$ for $a > 2$. Similar to the original system, we consider the approximated system by the parabolic regularization:

$$\begin{cases} \partial_{t'} v - \text{div}_{\,x'}(\nabla_{x'}(v+\varepsilon)^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t' > 0, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t' > 0, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x'), & x' \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

That is, we again consider the nonnegative weak solution $v(t,x)$ as before. Note that for the construction of the weak solution, we need to use the diagonal argument obtaining the weak solutions $(u,\psi)$ and $(v,\phi)$ simultaneously, since we do not know the uniqueness of the weak solution.

In what follows, we only treat the scaled system (2.2) and we use a simpler notation $t' \to t$ and $x' \to x$ if it does not cause any confusion.

2.2. **Rescaled Conservations of Mass, Entropy and Moment.** We revisit the conservation laws and the entropy functional for the rescaled equation (2.2):

**Proposition 2.1.** *Let $\kappa = n(2-\alpha) > 0$ and assume that the initial data $u_0 \in L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$ with $u_0 \geq 0$. Let $(v,\phi)$ be a weak solution of* (2.2) *and set the functional $W_s(v,\phi)$, $H(v(t))$, and $K_s(v,\phi)$ as follows:*

$$W_s(v,\phi)(t) \equiv \frac{1}{\alpha-1}\int_{\mathbb{R}^n} v^\alpha(t)dx + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v(t)dx - \frac{1}{2}e^{-\kappa t}\int_{\mathbb{R}^n} v(t)\phi(t)\,dx, \tag{2.4}$$

$$K(x,v(t),\phi(t)) \equiv \nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi\Big). \tag{2.5}$$

*Then we have the following identities:*

$$\int_{\mathbb{R}^n} v(t)dx = \int_{\mathbb{R}^n} u_0(x)dx, \tag{2.6}$$

$$W_s(t) + \int_0^t \Big[\int_{\mathbb{R}^n} v\,|K(x,v,\phi)(\tau)|^2\,dxd\tau + e^{-(\kappa+2)\tau}\int_{\mathbb{R}^n}|\nabla\phi(\tau)|^2 dx\Big]d\tau \\ \leq W_s(0). \tag{2.7}$$

**Proof.** The mass conservation law (2.6) follows from the definition of the weak solution. We consider the approximating procedure and consider the approximated system equivalent to (2.3), for $\varepsilon > 0$,

$$\begin{cases} \partial_t v - \operatorname{div}\, \big(\nabla(v+\varepsilon)^\alpha + x(v+\varepsilon) - e^{-\kappa t}(v+\varepsilon)\nabla\phi\big) \\ \qquad\qquad = \varepsilon e^{-(\kappa-2)t}(v-\phi) - \varepsilon n, & t>0, x\in\mathbb{R}^n, \\ -e^{-2t}\Delta\phi + \phi = v, & t>0, x\in\mathbb{R}^n, \\ v(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{2.8}$$

In the localized version of the identities, besides the terms appearing in the formal computation, we have extra terms which are involved with the cut off functions. Let $\zeta$ be a smooth cut off function supported in a ball $B_{2R}$ with radius $R>0$ with $\zeta = 1$ over $B_R$ and $\eta$ be a similar function supported in $B_{R'}$ with $\eta = 1$ over $B_{R'/2}$. For $R' < R$, we set $B_\varepsilon \equiv (\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1} + \frac12|x|^2 - e^{-\kappa t}\phi)$ and $\tilde B_\varepsilon \equiv (\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1}\eta + \frac12|x|^2\eta - e^{-\kappa t}\phi)$. Testing the equation with $\tilde B_\varepsilon\zeta^2$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t v\Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1}\eta + \frac12|x|^2\eta - e^{-\kappa t}\phi\Big)\zeta\ dx \\ &= -\int_{\mathbb{R}^n}(v+\varepsilon)\big|\nabla\tilde B_\varepsilon\big|^2\zeta dx - \int_{\mathbb{R}^n}(v+\varepsilon)\nabla\left(\big(B_\varepsilon + e^{\kappa t}\phi\big)(1-\eta)\right)\cdot\nabla\tilde B_\varepsilon\zeta dx \\ &+ \int_{\mathbb{R}^n}(v+\varepsilon)\tilde B_\varepsilon\nabla B_\varepsilon\cdot\nabla\zeta dx + \varepsilon\int_{\mathbb{R}^n}\big(e^{-(\kappa-2)t}(v-\phi) - n\big)\tilde B_\varepsilon\zeta dx. \end{aligned} \tag{2.9}$$

The second term of the right-hand side of (2.9) is indeed

$$\begin{aligned} &-\int_{\mathbb{R}^n}(v+\varepsilon)\Big|\nabla\big(B_\varepsilon + e^{\kappa t}\phi\big)\Big|^2\eta(1-\eta)\zeta dx \\ &\quad -\frac12\int_{\mathbb{R}^n}(v+\varepsilon)\nabla\Big|\big(B_\varepsilon + e^{\kappa t}\phi\big)\Big|^2\cdot\nabla\eta\ (1-\eta)\zeta dx \\ &\quad\quad + e^{-\kappa t}\int_{\mathbb{R}^n}(v+\varepsilon)\nabla\big(B_\varepsilon + e^{\kappa t}\phi\big)\cdot\nabla\phi\ (1-\eta)\zeta dx \\ &\quad\quad\quad + \int_{\mathbb{R}^n}(v+\varepsilon)\big(B_\varepsilon + e^{\kappa t}\phi\big)\nabla\tilde B_\varepsilon\cdot\big(\nabla\eta\zeta - (1-\eta)\nabla\zeta\big)dx, \end{aligned} \tag{2.10}$$

while the left-hand side can be treated as

$$\begin{aligned} &\int_{\mathbb{R}^n}\partial_t v\cdot\Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1}\eta + \frac12|x|^2\eta - e^{-\kappa t}\phi\Big)\zeta\ dx \\ &= \frac{d}{dt}\Big[\frac{1}{\alpha-1}\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\eta\zeta\ dx + \frac12\int_{\mathbb{R}^n}|x|^2 v\eta\zeta dx - \frac12\int_{\mathbb{R}^n}e^{-\kappa t}v\phi\zeta\ dx \\ &+ \frac14\int_{\mathbb{R}^n}e^{-\kappa t}|\phi|^2\Delta\zeta dx\Big] + e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2\zeta dx \end{aligned} \tag{2.11}$$

$$+ \kappa e^{-\kappa t} \int_{\mathbb{R}^n} \frac{1}{2}|\phi|^2 \Delta\zeta dx + e^{-(\kappa+2)t} \int_{\mathbb{R}^n} \nabla\phi \partial_t \phi \nabla\zeta dx.$$

Then we firstly drop the non-positive terms

$$-\varepsilon n \int_{\mathbb{R}^n} \Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1} + \frac{1}{2}|x|^2\Big)\zeta^2 dx$$

appearing in (2.9) and after integrating in $t$ we pass to the limit as $R \to \infty$ so that $\zeta \to 1$, then all the terms involving the derivative of the cut off function $\zeta$ vanish. Note that this process is only possible so long as $\eta$ remains. Now letting $\varepsilon \to 0$ we have the convergence of the solution as

$$\begin{aligned} v_{\varepsilon_n} &\to v && \text{in } C((0,T); L^\alpha \cap L^1_2(\mathbb{R}^n)),\\ \nabla v^\alpha_{\varepsilon_n} &\to \nabla v^\alpha && \text{in } w^* - L^\infty((0,T); L^2(\mathbb{R}^n)),\\ \nabla v^{\alpha-1/2}_{\varepsilon_n} &\to \nabla v^{\alpha-1/2} && \text{in } w - L^2((0,T); L^2(\mathbb{R}^n)),\\ \phi_{\varepsilon_n} &\to \phi && \text{in } C((0,T); W^{2,p}(\mathbb{R}^n)), \end{aligned}$$

(see for details [49, Propposituion 6.1]) and we obtain from (2.9)-(2.11) that

$$\begin{aligned} &\frac{1}{\alpha-1}\int_{\mathbb{R}^n} v^\alpha \eta\, dx + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v\eta dx - \frac{1}{2}\int_{\mathbb{R}^n} e^{-\kappa t} v\phi\, dx\\ &+\int_0^t \Big[\int_{\mathbb{R}^n} v\Big|\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1}\eta + \frac{1}{2}|x|^2\eta - e^{-\kappa t}\phi\Big)\Big|^2 dx + e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2 dx\Big] d\tau\\ &\le \frac{1}{\alpha-1}\int_{\mathbb{R}^n} u_0^\alpha \eta\ dx + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 u_0 \eta dx - \frac{1}{2}\int_{\mathbb{R}^n} u_0\phi(0)\ dx\\ &-\int_0^t \Big[\int_{\mathbb{R}^n} v\Big|\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2\Big)\Big|^2 \eta(1-\eta)dx\\ &+\frac{1}{2}\int_{\mathbb{R}^n} v\nabla\Big|\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2\Big|^2 \cdot \nabla\eta\ (1-\eta)dx\\ &- e^{-\kappa t}\int_{\mathbb{R}^n} v\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2\Big)\cdot\nabla\phi\ (1-\eta)dx\\ &-\int_{\mathbb{R}^n}\Big(\frac{\alpha}{\alpha-1}v^\alpha + \frac{1}{2}|x|^2 v\Big)\nabla\Big(\frac{\alpha}{\alpha-1}v^\alpha\eta + \frac{1}{2}|x|^2\eta - e^{-\kappa t}\phi\Big)\cdot\nabla\eta dx.\Big] d\tau. \end{aligned}$$

Finally, by letting $R' \to \infty$, we see $\eta \to 1$ and the last four terms vanish (noting that $|\nabla\eta| \le C(R')^{-1}$ and we obtain the desired inequality.

2.3. **Rescaled uniform bounds.** The following estimate is a direct consequence of the above a priori bound of the rescaled solution.

**Proposition 2.2.** *Let* $1 < \alpha \le 2 - \frac{2}{n}$ *and* $(v(t), \phi(t))$ *be a weak solution of* (2.2) *for the initial data* $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Assume that*

$$\|u_0\|_1^{1-\gamma} W(0)^{\frac{\gamma-\alpha+1}{\alpha}} < C\|\Gamma_n\|^{-1}_{L_w^{\frac{n}{n-2}}} \tag{2.12}$$

*for* $1 < \alpha \le 2 - \frac{2}{n}$ *and* $\gamma + 1 = \frac{\alpha}{\alpha-1}\frac{n-2}{n}$*, where* $\Gamma_n$ *is the fundamental solution to* $-\Delta$ *in* $\mathbb{R}^n$*. Then*

(1) *we have* $\|v(t)\|_q \le C$ *for all* $1 \le q \le \infty$*; and*
(2) *for all* $\frac{n}{n-1} < r \le \infty$*,* $\|\nabla\phi(t)\|_r \le Ce^{2t}$*.*

**Proof.** Let $E_{n,t}$ be the fundamental solution of $(-e^{-2t}\Delta + 1)$ in $\mathbb{R}^n$. Then, for $v \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$, let $\phi = E_{n,t} * v$ be a solution of the second equation of the system (2.2). Then we have

$$\begin{aligned}\frac{e^{-\kappa t}}{2}\left(e^{-2t}\|\nabla\phi\|_2^2 + \lambda\|\phi\|_2^2\right) &= \int_{\mathbb{R}^n} e^{-\kappa t} v\phi dx \\ &\le e^{-\kappa t}\|E_{n,t}\|_{L_w^{\frac{n}{n-2}}}\|v\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma}\end{aligned} \tag{2.13}$$

for any $\gamma + 1 = \frac{\alpha(n-2)}{n(\alpha-1)}$. Indeed, by the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} v\phi dx &\le \|v\|_r\|\phi\|_{r'} \qquad \text{for } \frac{1}{r} + \frac{1}{r'} = 1 \\ &\le \|v\|_1^{1-\gamma}\|v\|_\alpha^\gamma\|\phi\|_{r'} \qquad \left(\frac{1}{r} = 1 - \gamma + \frac{\gamma}{\alpha}\right) \\ &\le \|E_{n,t}\|_{L_w^{\frac{n}{n-2}}}\|v\|_1^{1-\gamma}\|v\|_\alpha^{1+\gamma} \qquad \left(\frac{1}{r'} = \frac{n-2}{n} + \frac{1}{\alpha} - 1\right).\end{aligned} \tag{2.14}$$

Under the assumption $\alpha \le 2 - \frac{2}{n}$,

$$\alpha\left(1 - \frac{1}{\alpha}\right) \le (\gamma+1)\left(1 - \frac{1}{\alpha}\right) = \frac{n-2}{n}$$

and this gives $\gamma + 1 \ge \alpha$. Hence, by the maximum principle,

$$\|E_{n,t}\|_{L_w^{n/(n-2)}} \le e^{2t}\|\Gamma_n\|_{L_w^{\frac{n}{n-2}}},$$

where $\Gamma_n$ denotes the fundamental solution of $-\Delta$ in $\mathbb{R}^n$. It then follows that

$$e^{-\kappa t}\int_{\mathbb{R}^n} v(t)\phi(t)dx \le e^{-(\kappa-2)t}\|\Gamma_n\|_{L_w^{\frac{n}{n-2}}}\|v(t)\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma} \tag{2.15}$$

and

$$\frac{2}{\alpha-1}\|v(t)\|_\alpha^\alpha \le 2W_s(0) + e^{-(\kappa-2)t}\|\Gamma_n\|_{L_w^{\frac{n}{n-2}}}\|u_0\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma},$$

where $\gamma+1 \geq \alpha$. Under the smallness condition

$$e^{-(\kappa-2)t}\|\Gamma_n\|_{L_w^{\frac{n}{n-2}}}\|u_0\|_1^{1-\gamma}W_s(0)^{\frac{\gamma-\alpha+1}{\alpha}} < 2\Big(\frac{1}{\alpha-1}\Big)^{\frac{\gamma+1}{\alpha}}\Big(\frac{\alpha}{\gamma+1}\Big)\Big(\frac{\alpha+\gamma+1}{\gamma+1}\Big)^{\frac{\gamma-\alpha+1}{\alpha}} \tag{2.16}$$

directly gives the uniform boundedness of $\|v(t)\|_\alpha$. Note that this does not give extra restrictions on the initial data beyond those of (2.12) except that $W_s(0)<\infty$. For $\alpha = 2-\frac{2}{n}$, the condition may be reduced to

$$\|\Gamma_n\|_{L_w^{\frac{n}{n-2}}}\|u_0\|_1^{1-\gamma} < \Big(\frac{2}{\alpha-1}\Big)$$

and again this coincides with the condition of the global existence of the solution (1.6). We now conclude that

$$\|v(t)\|_\alpha^\alpha \leq C(n,\alpha,\|u_0\|_1, W_s(0))$$

uniformly in $t$.

For $1 \leq q \leq 2-\frac{2}{n}$ , the estimate (2.15), the $L^1$ conservation law, and the entropy bound $W_s(t) \leq W_s(0)$ imply

$$\frac{1}{\alpha-1}\|v(t)\|_\alpha^\alpha + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v(t)dx \leq C(W_s(0) + C_n\|u_0\|_1^{\frac{\alpha(1-\gamma))}{\alpha-(\gamma+1)}}) \tag{2.17}$$

for all $t\in[0,\infty)$, where $\gamma<\alpha-1$. Here we note that

$$\frac{\alpha(1-\gamma)}{\alpha-(1+\gamma)} = \frac{n(\alpha-2)+2\alpha}{n\alpha+2}.$$

For the case $q \geq 2-\frac{2}{n}$, if $n=2$, the standard elliptic estimate ensures that the solution $\phi(t)$ of the second equation stays bounded so long as $\|v(t)\|_\alpha$ remains bounded in $t$. This gives the boundedness of $v(t)$. For the higher-dimensional case, we apply the standard parabolic estimate that we see for any $q>\alpha$:

$$\frac{d}{dt}\|v(t)\|_q^q + n(q-1)\|v(t)\|_q^q + \frac{2q}{\alpha+q-1}\|\nabla v^\gamma(t)\|_2^2 = (q-1)e^{-\kappa t}\int_{\mathbb{R}^n}\nabla v^q(t)\cdot\nabla\phi(t)dx \leq C\|v(t)\|_{q+1}^{q+1} \tag{2.18}$$

by the positivity of $(v,\phi)$, where $\gamma = \frac{1}{2}(\alpha+q-1)$. Now we invoke the Gagliardo-Nirenberg interpolation inequality (cf. Nakao [34])

$$\|f\|_{(q+1)/\gamma} \leq C\|f\|_{\alpha/\gamma}^{1-\sigma}\|\nabla f\|_2^\sigma,$$

$$\frac{\gamma}{q+1} = \frac{\gamma(1-\sigma)}{\alpha} + \sigma\Big(\frac{1}{2} - \frac{1}{n}\Big).$$

It then follows, by substituting $f = v^\gamma(t)$, that

$$\|v(t)\|_{q+1}^{q+1} \leq C\|v(t)\|_\alpha^{(q+1)\sigma}\|\nabla v^\gamma(t)\|_2^{\sigma(q+1)/\gamma}. \tag{2.19}$$

Combining (2.19) and (2.18), if we assume that $\sigma(q+1)\gamma < 2$, then we obtain

$$\frac{d}{dt}\|v(t)\|_q^q + n(q-1)\|v(t)\|_q^q \leq \|u(t)\|_\alpha^\delta \leq C(W(0), \|u_0\|_1)$$

by (2.17). This assumption can be verified under the condition $q > 2 - \frac{4}{n+2}$ and this is fulfilled by $q > 2 - \frac{2}{n}$. It follows from the Gronwall inequality that

$$\|v(t)\|_q \leq C(W_s(0), \|u_0\|_1, \|u_0\|_q).$$

By letting $q \to \infty$ in the resulting inequality, we obtain the desired a priori estimate for any $t \in [0, \infty)$. One can eliminate the initial restriction $\|u_0\|_q$ by the parabolic regularity argument (for the detailed estimates, see [42, Theorem 2.4]).

The estimate for the potential term $\phi$ directly follows from the estimate for $v(t)$ and the Hardy-Littlewood inequality: since $\nabla\phi = (-e^{-2t}\Delta + 1)^{-1}\nabla v$,

$$\begin{aligned}\|\nabla\phi\|_q &= \|(-e^{-2t}\Delta + 1)^{-1}\nabla v\|_q \leq Ce^{2t}\|(-e^{-2t}\Delta + 1)^{-1}e^{-2t}\Delta v\|_r \\ &\leq Ce^{2t}\|v\|_r \leq Ce^{2t}\end{aligned}$$

with $q > n/(n-1)$ and $\frac{1}{q} = \frac{1}{r} - \frac{1}{n}$. □

Once we obtain the above uniform bound for the rescaled solution, we can immediately obtain the time decay estimate for the solution of the original equation.

$$\int_{\mathbb{R}^n} v^q(t', x')dx' = \int_{\mathbb{R}^n} e^{n(q-1)t'} u^q(t,x)dx = (1+\sigma t)^{(q-1)n/\sigma}\int_{\mathbb{R}^n} u^q(t,x)dx$$

in the original variables $(t, x)$. Hence we obtain the following decay estimate for the original solution as a corollary of Proposition 2.2.

**Proposition 2.3** ([42], [49])**.** *Let $u_0 \in L_2^1(\mathbb{R}^n) \cap L^\infty$ and $(u(t), \psi(t))$ be a weak solution of* (1.1). *If $1 < \alpha \leq 2 - \frac{2}{n}$ with small initial data* (2.12), *we have*

$$\|u(t)\|_q \leq C(1+\sigma t)^{-\frac{n}{\sigma}(1-\frac{1}{q})}$$

*for all $1 \leq q \leq \infty$.*

2.4. **The moment bounds.** In the last part of this section, we show that the second moment of the weak solution remains bounded for $t \in [0,T]$.

**Proposition 2.4.** *Let $u_0 \in L^1 \cap L^\alpha$ with $|x|^2 u_0(x) \in L^1(\mathbb{R}^n)$. Then the weak solution $(v,\phi)$ of (2.2) satisfies*

$$\int_{\mathbb{R}^n} |x|^2 v(t)dx \le e^{-nt} \int_{\mathbb{R}^n} |x|^2 u_0 dx + \frac{2(n-2)}{n} W_s(0). \tag{2.20}$$

*That is, $|x|^2 v(t) \in L^1(\mathbb{R}^n)$ for almost all $t$. In addition if we assume that $u_0 \in L^1_{2+a}(\mathbb{R}^2)$ with $0 < a < 1/2$, then we have*

$$\int_{\mathbb{R}^2} |x|^{2+a} v(t)dx \le e^{-(2+a)t} \int_{\mathbb{R}^2} |x|^{2+a} u_0 dx + C. \tag{2.21}$$

**Proof.** We only give the formal part. It can be justified with an appropriate cut off and approximation procedure as we have seen in the proof of Proposition 2.1. To show (2.20) we test $|x|^2 v(t)$ on the equation and we see

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 v(t)dx = 2n\|v(t)\|_\alpha^\alpha - 2\int_{\mathbb{R}^n} |x|^2 v(t)dx + 2e^{-\kappa t}\int_{\mathbb{R}^n} xv(t)\cdot\nabla\phi(t)dx. \tag{2.22}$$

Next we invoke the Pokhozaev identity for the second equation. We multiply the elliptic part of the system by the generator of the dilation $x \cdot \nabla\psi$ and integrate it by parts; it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} x\cdot\nabla\phi(t)v(t)dx &= e^{-2t}\left(1-\frac{n}{2}\right)\int_{\mathbb{R}^n} |\nabla\phi(t)|^2 dx - \frac{n}{2}\int_{\mathbb{R}^n} |\phi(t)|^2 dx \\ &= \left(1-\frac{n}{2}\right)\int_{\mathbb{R}^n} v(t)\phi(t)dx - \|\phi(t)\|_2^2. \end{aligned}\tag{2.23}$$

Combining (2.22) and (2.23), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 v(t)dx + n\int_{\mathbb{R}^n} |x|^2 v(t)dx &= 2n\|v(t)\|_\alpha^\alpha + (n-2)\int_{\mathbb{R}^n} |x|^2 v(t)dx \\ &\quad + (2-n)e^{-\kappa t}\int_{\mathbb{R}^n} u(t)\phi(t)dx - 2e^{-\kappa t}\|\phi(t)\|_2^2 \\ &= 2(n-2)W_s(t) + 2n\Big(\frac{\alpha-2+\frac{2}{n}}{\alpha-1}\Big)\|v(t)\|_\alpha^\alpha - 2e^{-\kappa t}\|\phi\|_2^2. \end{aligned}\tag{2.24}$$

Thus, under the condition $\alpha \le 2 - \frac{2}{n}$, we see that

$$\int_{\mathbb{R}^n} |x|^2 v(t)dx \le e^{-nt}\int_{\mathbb{R}^n} |x|^2 u_0 dx + \frac{2(n-2)}{n} W_s(0)(1-e^{-nt}). \tag{2.25}$$

For a further weighted estimate, we modify (2.22) to have

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^{2+a}v(t)dx+(2+a)\int_{\mathbb{R}^2}|x|^{2+a}v(t)dx$$
$$=(2+a)^2\int_{\mathbb{R}^2}|x|^a v(t)\ dx+(2+a)e^{-\kappa t}\int_{\mathbb{R}^2}r^a x\cdot\nabla\phi v(t)\ dx. \tag{2.26}$$

It follows that

$$\frac{d}{dt}\Big[e^{(2+a)t}\int_{\mathbb{R}^2}|x|^{2+a}v(t)dx\Big]$$
$$=(2+a)^2e^{(2+a)t}\Big(\|v(t)\|_1+\int_{|x|>1}|x|^{2+a}v(t)dx\Big)$$
$$+(2+a)e^{(2+a)t-\kappa t}\Big(\int_{\mathbb{R}^2}|x|^{2+a}vdx\Big)^{2/3}\|\nabla\phi\|_3\|v(t)\|_\infty^{1/2} \tag{2.27}$$

and the uniform boundedness for $\|v\|_1$ and $\|\nabla\phi\|_q$ and the second moment implies that

$$\int_{\mathbb{R}^2}|x|^{2+a}v(t)dx\le e^{-(2+a)t}\int_{\mathbb{R}^2}|x|^{2+a}u_0dx+C+Ce^{-\kappa t}. \tag{2.28}$$

□

This uniform bound for the second moment of $v(t)$ naturally yields the bound for the second moment of $u(t)$ as

$$\int_{\mathbb{R}^n}|x|^2u(t)dx\le\int_{\mathbb{R}^n}|x|^2u_0dx+\frac{2(n-2)}{n}W_s(0)(1+\sigma t)^{2/\sigma}.$$

## 3. The Asymptotic Profile

**Theorem 3.1.** *Let $1<\alpha\le 2-\frac{2}{n}$. Then for any positive initial data $u_0\in L^1_2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, the corresponding weak solution $u(t,x)$ satisfies the following asymptotic behavior: For $M=\|u_0\|_1$, with the condition* (1.5),

$$\|u(t)-U(t)\|_1\le(1+\sigma t)^{-\nu},$$

*where $\sigma=n(\alpha-1)+2$ and $0<\nu<2$ with $\|U(t)\|_1=M$.*

Applying the method of the transport equation or the Fokker-Planck equation due to Carrillo-Toscani [8], we compute the time derivative of the free energy functional: For a weak solution $u$ and $\psi$ of (2.2) , we let

$$H(v(t))\equiv\frac{1}{\alpha-1}\int_{\mathbb{R}^n}v^\alpha(t)dx+\frac{1}{2}\int_{\mathbb{R}^n}|x|^2v(t)dx,$$

$$J(v(t)) \equiv \int_{\mathbb{R}^n} v(t)\Big|\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1}(t)+\frac{|x|^2}{2}\Big)\Big|^2 dx,$$
$$I(v(t)) \equiv \int_{\mathbb{R}^n} v(t)\Big|\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1}(t)+\frac{|x|^2}{2}-e^{-\kappa t}\phi(t)\Big)\Big|^2 dx.$$

We firstly observe that the entropy functional has a certain relation.

**Proposition 3.2.** *For a weak solution $v$ and $\phi$ of* (2.2) *, we have*

$$\begin{aligned} H(v(t)) &+ \frac{1}{2}e^{-\kappa t}(e^{-2t}\|\nabla\phi(t)\|_2^2+\|\phi(t)\|_2^2)+\int_s^t J(v(\tau))d\tau \\ &\le H(v(s))+\frac{1}{2}e^{-\kappa s}(e^{-2s}\|\nabla\phi(s)\|_2^2+\|\phi(s)\|_2^2) \\ &\quad + \int_s^t e^{-\kappa\tau}\Big[\frac{2-\kappa}{2}e^{-2\tau}\|\nabla\phi(\tau)\|_2^2-\frac{\kappa}{2}\|\phi(\tau)\|_2^2\Big]d\tau \\ &\quad + \int_s^t e^{-2\kappa\tau}\int_{\mathbb{R}^n} v(\tau)|\nabla\phi(\tau)|^2 dx d\tau, \end{aligned} \tag{3.1}$$

*where $\kappa = n(2-\alpha)$. In particular, for $1<\alpha\le 2-\frac{2}{n}$, we have that $H(v(t))$ is uniformly bounded in $t$ under the smallness condition* (2.12) *:*

$$H(v(t)) \le H(u_0) - \frac{1}{2}\int_{\mathbb{R}^n}\phi(0)v(0)dx + C\sup_{\tau>0}\Big[\|v(\tau)\|_\infty + e^{-2\tau}\|\nabla\phi(\tau)\|_2^2\Big] \tag{3.2}$$

*for any $t>0$.*

**Remark.** The restriction on the exponent $\alpha \le 2-\frac{2}{n}$ follows from the restriction on $\kappa \ge 2$ in view of the integrability of the third term of the right-hand side of the above inequality.

**Proof of Proposition 3.2.** The result almost follows from Proposition 2.1. Decomposing $W_s(t)$ into $H(t)$ and terms with $\phi$, we see formally that

$$\begin{aligned} &\frac{d}{dt}\Big[H(v(t))+\frac{1}{2}e^{-\kappa t}(\|\phi(t)\|_2^2+e^{-2t}\|\nabla\phi(t)\|_2^2)\Big]+J(v(t)) \\ &=e^{-2\kappa t}\int_{\mathbb{R}^n} v(t)|\nabla\phi(t)|^2dx + e^{-\kappa t}\Big[\frac{2-\kappa}{2}e^{-2t}\|\nabla\phi(t)\|_2^2-\frac{\kappa}{2}\|\phi(t)\|_2^2\Big]. \end{aligned} \tag{3.3}$$

Integrating (3.3) over $[s,t]$, we obtain (3.1). Under the condition $1<\alpha\le 2-\frac{2}{n}$, we have $\kappa\ge 2$ and by Proposition 2.2 $\|v(t)\|_\infty \le C$ and $e^{-2t}\|\nabla\phi(t)\|_2 \le C$. Therefore it follows that

$$H(v(t))+\int_0^t J(v(\tau))d\tau \le H(v(0))+\frac{1}{2}\Big[\|\nabla\phi(0)\|_2^2+\|\phi(0)\|_2^2\Big]$$

$$+\frac{1}{\kappa}\sup_{t>0}\Big((1+\|v(t)\|_\infty)e^{-2t}\|\nabla\phi(t)\|_2^2\Big)$$
$$\le H(v(0))-\frac{1}{2}\int_{\mathbb{R}^n}v(0)\phi(0)dx+C$$

for all $t>0$. □

For a solution $v$ and $\phi$ of (2.2), we let

$$I(v(t))\equiv\int_{\mathbb{R}^n}v(t,x)\,|K(x,v(t),\phi(t))|^2\,dx,$$

where $K(x,v,\phi)=x+\frac{\alpha}{\alpha-1}\nabla v^{\alpha-1}(t,x)-e^{-\kappa t}\nabla\phi(t)$ and $\kappa=n(2-\alpha)$. This is the functional appearing in the entropy bound (2.7). It is not so difficult to see that the asymptotic profile is given by $J(v(t))\to 0$ from the above inequality. However, to obtain the convergence rate for a weak solution in the weighted class $L^1_2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, we deduce that $I(v(t))$ is exponentially decaying. To this end, we observe the time derivative of the functional $I(v(t))$. We assume that $\kappa>0$ and $\alpha<2$.

Following [8], we formally have

$$\begin{aligned}\frac{d}{dt}I(v(t))&=-2\int_{\mathbb{R}^n}v|K(x,v,\phi)|^2dx-2(\alpha-1)\int_{\mathbb{R}^n}v^\alpha\Big|\text{div }K(x,v,\phi)\Big|^2dx\\&-2\int_{\mathbb{R}^n}v^\alpha\,|\nabla K(x,v,\phi)|^2\,dx+2e^{-\kappa t}\int_{\mathbb{R}^n}vK_i(x,v,\phi)K_j(x,v,\phi)\left(D^2_{ij}\phi\right)dx\\&+2e^{-\kappa t}\int_{\mathbb{R}^n}\text{div }(vK(x,v,\phi))\partial_t\phi dx-2\kappa e^{-\kappa t}\int_{\mathbb{R}^n}vK(x,v,\phi)\nabla\phi dx.\end{aligned}\tag{3.4}$$

Since the weak solution does not have enough regularity, the above identity is not necessarily valid and the actual estimate should be obtained in the form of the integral inequality.

For $t>0$, $x\in\mathbb{R}^n$, let $(v,\phi)$ be a solution of the regularized system

$$\begin{cases}\partial_t v-\text{div }\big((v+\varepsilon)K_\varepsilon(x,v,\phi)\big)=-\varepsilon\big(e^{-(\kappa-2)t}(v-\phi)+n\big),\\ \quad -e^{-2t}\Delta\phi+\phi=v,\\ \qquad v(0,x)=u_0(x),\end{cases}\tag{3.5}$$

where

$$K_\varepsilon(x,v,\phi)\equiv\frac{\alpha}{\alpha-1}\nabla(v+\varepsilon)^{\alpha-1}(t,x)+x-e^{-\kappa t}\nabla\phi(t).$$

Note that the above system (3.5) is equivalent to (2.3). The existence of the smooth and sufficiently fast decaying solution at $|x|\to\infty$ of (2.8) is obtained in a similar manner in [49].

**Proposition 3.3.** *Let $\zeta$ be a smooth cut off function such that $\zeta = 1$ in $B_R$ and whose derivatives are supported in $B_{2R} \setminus B_R$. For a solution $v$ and $\phi$ of* (2.8) *belonging to $L^1$, we let*

$$I_\varepsilon(v(t)) \equiv \int_{\mathbb{R}^n} v(t) \left| K_\varepsilon(x, v(t), \phi(t)) \right|^2 \zeta^2 dx,$$

*where $\kappa = n(2-\alpha)$. Then we have*

$$\begin{aligned}
&\frac{d}{dt} I_\varepsilon(v(t)) \leq -2 \int_{\mathbb{R}^n} (v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2 \zeta dx \\
&- 2(\alpha-1)\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \left|\text{div } K_\varepsilon(x,v,\phi)\right|^2 \zeta dx - 2\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha |\nabla K_\varepsilon(x,v,\phi)|^2 \zeta dx \\
&+ 2e^{-\kappa t} \int_{\mathbb{R}^n} (v+\varepsilon) K_{\varepsilon,i}(x,v,\phi) K_{\varepsilon,j}(x,v,\phi) \left(D_{ij}^2 \phi\right) \zeta dx \\
&+ 2e^{-(\kappa-2)t} \int_{\mathbb{R}^n} \left|(v K_\varepsilon(x,v,\phi))\right|^2 \zeta dx - 2\kappa e^{-\kappa t} \int_{\mathbb{R}^n} v K_\varepsilon(x,v,\phi) \cdot \nabla \phi \zeta dx \\
&\qquad + E_I(x, v, \phi, \varepsilon, \nabla \zeta^2),
\end{aligned} \tag{3.6}$$

*where $E_I(x, v, \phi, \varepsilon, \nabla\zeta^2)$ denotes the error term which will be vanishing when we take the limit $R \to \infty$ and $\varepsilon \to 0$.*

The derivation and rigorous treatment of (3.6) is given in Appendix A. We temporary admit the identity (3.6) and proceed to the following.

**Proposition 3.4.** *Let $(v, \phi)$ be a weak solution of* (2.2)*. We set*

$$I(v(t)) \equiv \int_{\mathbb{R}^n} v(t) |K(x,v,\phi)|^2 dx$$

*with $K(x,v,\phi) \equiv \frac{\alpha}{\alpha-1} u^{\alpha-1} + x - e^{-\kappa t} \nabla \phi$. Then if $1 < \alpha < 2 - \frac{2}{n}$ and the solution $v$ has a uniform estimate $\sup_{t>0} \|v(t)\|_\infty \leq C_*$ for some constant, then there exists $T_0 > 0$ such that, for any $T_0 < t$,*

$$I(v(t)) + \int_{T_0}^t I(v(\tau)) d\tau \leq I(u(T_0)) \tag{3.7}$$

*for all $T_0 < t$. In particular, we have $I(v(t)) \leq Ce^{-\nu t}$, where $\nu < \min(2n(2-\alpha) - 4), 1)$ and $C$ depends on the initial data $u_0$.*

**Remark.** It seems that the estimate derived from $\frac{d}{dt} J(v(t))$ seems not to work in our situation since the regularity of the weak solution is not enough to justify the terms appearing in this case.

To obtain the above proposition, we need the following two ingredients. The first one is the Sobolev type inequality in the critical type originally due to Brezis-Gallouet [6]. We give the generalized version obtained in Ogawa-Taniuchi [43]. Let $\dot{B}^s_{r,\sigma}$ be the Besov space defined by the real interpolation between two Sobolev spaces $H^{s_0,r}$ and $H^{s_1,r}$.

**Proposition 3.5** ([6], [26], [43])**.** *There exists a constant $C$ depending only on $n$ such that for $f \in \dot{B}^s_{r,\sigma} \cap \dot{B}^{-s}_{r,\sigma}$ the following inequality holds. For $1 \le \rho < \nu \le \infty$,*

$$\|f\|_{\dot{B}^0_{\infty,\nu}} \le C\|f\|_{\dot{B}^0_{\infty,\rho}}\Big(1+\Big(\log\Big(e+\frac{\|f\|_{\dot{B}^s_{r,\sigma}}+\|f\|_{\dot{B}^{-s}_{r,\sigma}}}{\|f\|_{\dot{B}^0_{\infty,\infty}}}\Big)\Big)^{1/\nu-1/\rho}\Big). \quad (3.8)$$

*In particular since $BMO \subset \dot{B}^0_{\infty,\infty}$ and $C^s \subset \dot{B}^s_{\infty,\infty}$, by setting $\nu = 1$, $\rho = \infty$, and $\sigma = 1$, we have*

$$\|f\|_\infty \le C\Big(1+\|f\|_{BMO}\log\Big(e+\|f\|_2+\|f\|_{C^s}\Big)\Big). \quad (3.9)$$

**Remark.** For the high regularity part, $f \in \dot{B}^s_{r,\sigma}$ and $f \in \dot{B}^{-s}_{r,\sigma}$ can be reduced for the higher frequency part and lower frequency part, respectively. Therefore if we exchange $f$ into $D^2_{ij}\phi$, $\|D^2_{ij}\phi\|_{C^s}$ can be exchanged by $\|\Delta\phi\|_{C^s}$ since the lower frequency part can be neglected. See for the details [43].

The second important part is to show the Hölder continuity of the solution. It is shown that the weak solution of the degenerated parabolic system like the porous medium equation has the Hölder regularity. This is found by DiBeneditto-Friedmann [14] and Wiegner [53] after the analysis of DiBeneditto-Friedman [13] (cf. Cafferelli-Evans [7]).

**Proposition 3.6** (the Hölder continuity of the weak solution [14], [53])**.** *Let $(v(t), \phi(t))$ be a weak solution of* (2.2) *satisfying* (1.4) *with $\|\nabla v^\alpha(t)\|_2 \in L^2(0,\infty)$. Then for any point $(t,x)$ the solution $u$ is Hölder continuous and there exists $s > 0$ such that we have $\|v(t)\|_{C^s} \le Ce^{(n+s)t}$, where $s$ is given by $\alpha$ and $n$.*

For completeness, we show the proof of the above proposition in Appendix B.

**Proof of Proposition 3.4.** To avoid the complexity of the notation and proof, we treat the estimate only for the essential parts in a rather formal way, namely dropping the regularizer $\varepsilon$ and cut off function $\zeta$ without integration in the $t$ variable. The actual estimates are done for the approximated solution. The rigorous procedure requires that all those estimates proceed before passing to the limit $R \to \infty$ and $\varepsilon \to 0$. We show this in a rigorous way in Appendix A Proposition 4.2. Observing the estimate (3.6), we need to estimate the last four terms in the right-hand side. The fourth error term $E_I(x, v, \phi, \varepsilon, \nabla\zeta)$ is handled in Appendix A since it does not have any effect on the estimation of the other terms. Firstly, the sixth term of the right-hand side of (3.6) can be estimated as follows:

$$
\begin{aligned}
-2\kappa e^{-\kappa t} & \int_{\mathbb{R}^n} vK(x,v,\phi)\nabla\phi dx \\
&\le 2\kappa e^{-\kappa t}\|\nabla\phi(t)\|_\infty \Big(\int_{\mathbb{R}^n} vdx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2 dx\Big)^{1/2} \\
&\le 2\kappa\varepsilon^{-1}e^{-2\kappa t}\|v(t)\|_\infty^2 \int_{\mathbb{R}^n} |\nabla\phi(t)|^2 dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2 dx.
\end{aligned}
$$

Hence, from Proposition 3.3, we obtain that

$$
\begin{aligned}
\frac{d}{dt}I(v(t)) \le & -(2-\varepsilon)I(v(t)) - 2(\alpha-1)\int_{\mathbb{R}^n} v^\alpha \Big|\text{div } K(x,v,\phi)\Big|^2 dx \\
& - 2\int_{\mathbb{R}^n} v^\alpha \left|\nabla K(x,v,\phi)\right|^2 dx + 2e^{-(\kappa-2)t}\int_{\mathbb{R}^n} v^2 \left|K(x,v,\phi)\right|^2 dx \\
& + 2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)\left(D_{ij}^2\phi\right) dx \\
& + C\varepsilon^{-1}e^{-2(\kappa-1)t}\|v(t)\|_\infty^2 \sup_t (e^{-2t}\|\nabla\phi(t)\|_2^2), \qquad (3.10)
\end{aligned}
$$

where $K(x,v,\phi) = x + \frac{\alpha}{\alpha-1}\nabla v^{\alpha-1}(t,x) - e^{-\kappa t}\nabla\phi(t,x)$. We now turn to how to treat the following term:

$$
\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x.v,\phi)D_{ij}^2\phi dx.
$$

Applying the logarithmic interpolation inequality of Brezis-Gallouet type (3.8), and noting

$$
\|D_{ij}^2\phi\|_{BMO} \le C\|\Delta\phi\|_{BMO} = Ce^{2t}\|(-\Delta + e^{2t})^{-1}\Delta\phi\|_{BMO},
$$

we see

$$
\begin{aligned}
\|D_{ij}^2\phi(t)\|_\infty &\le (1 + \|D_{ij}^2\phi(t)\|_{BMO}\log(e + \|\phi(t)\|_2 + \|\Delta\phi(t)\|_{C^s}) \\
&\le C\Big(1 + e^{2t}\|(-\Delta + e^{2t})^{-1}\Delta v(t)\|_\infty \\
&\quad \times \log\Big(e + \|v(t)\|_2 + \|\phi(t)\|_2 + e^{2t}(\|v(t)\|_{C^s} + \|\phi(t)\|_{C^s})\Big)\Big)
\end{aligned}
$$

uniformly in $t$. From the uniform Hölder estimate Proposition 3.6 and Proposition 2.2, $\|D^2_{ij}\phi(t)\|_\infty$ is bounded by $(1+t)\log(e+t)e^{2t}$. This allows us to proceed with the estimate as follows:

$$\begin{aligned}&2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x.v,\phi)D^2_{ij}\phi dx \qquad (3.11)\\ &\le 2e^{-\kappa t}\|D^2_{ij}\phi(t)\|_\infty \int_{\mathbb{R}^n} v|K(x,v,\phi)|^2 dx\\ &\le 2C(1+t)e^{-(\kappa-2)t}\|(-\Delta+e^{2t})^{-1}\Delta v(t)\|_\infty \int_{\mathbb{R}^n} v|K(x,v,\phi)|^2 dx.\end{aligned}$$

Combining (3.10) and (3.11), we obtain that, if $\kappa > 2$; i.e., $\alpha < 2 - \frac{2}{n}$, then

$$\begin{aligned}\frac{d}{dt}I(v(t)) \le &-(2-\varepsilon)I(v(t)) + C\sup_t \|v(t)\|_\infty (1+t)e^{-(\kappa-2)t}I(v(t))\\ &+ C\varepsilon^{-1}e^{-2(\kappa-1)t}\|v(t)\|^2_\infty \sup_t(e^{-2t}\|\nabla\phi(t)\|_2^2). \qquad (3.12)\end{aligned}$$

Note that, at this stage, the inequality (3.12) does not include the higher-order terms so that it is possible to justify it for the weak solution. Since $2(\kappa-2) = 2n(2-\alpha)-4 > 0$ under the condition $\alpha < 2-\frac{2}{n}$, we choose $\eta$ such that $\nu \equiv 2-\eta < \min(2(\kappa-2), 2-2\varepsilon)$ and it follows that for some large $T_0 > 0$ which depends on $C_*$, for any $t \ge T_0$,

$$\frac{d}{dt}\left(e^{\nu t}I(v(t))\right) \le Ce^{(\nu-2(\kappa-2))t}. \qquad (3.13)$$

Immediately we obtain that

$$I(v(t)) \le e^{-\nu t}\Big(I(v(T_0)) + C\int_{T_0}^\infty e^{\nu-2(\kappa-2))\tau}d\tau\Big).$$

Since $T_0$ is only depending on $C_*$ we may conclude that $I(v(t) \le C(T_0)$ for $t \in [0, T_0]$ and this concludes the desired estimate. □

The proof of the asymptotic profile in Theorem 1.2 is completed after proving the convergence of the rescaled solution and rescaling.

**Proposition 3.7.** *Let $1 < \alpha < 2 - \frac{2}{n}$ and $(v, \phi)$ be a solution to (2.2). If the initial data satisfies the condition (1.5), then we have for some $\nu < n(2-\alpha)$ that $\|v(t) - V\|_1 \le Ce^{-\nu t}$, where $V(x) = \left[A - \frac{\alpha-1}{2\alpha}|x|^2\right]_+^{1/(\alpha-1)}$ and the constant $A$ is chosen as $\|V\|_1 = \|u_0\|_1$.*

**Proof.** Due to the result from Proposition 3.4, we immediately obtain that

$$\lim_{t\to\infty} I(v(t)) = 0. \qquad (3.14)$$

On the other hand, since by Proposition 2.2,

$$\begin{aligned} J(v(t)) &\le 2I(v(t)) + 2e^{-2\kappa t}\int_{\mathbb{R}^n} v(t)|\nabla\phi(t)|^2 dx \\ &\le 2I(v(t)) + 2e^{-2\kappa t}\|v(t)\|_\infty \|\nabla\phi(t)\|_2^2 \\ &\le 2I(u_0)e^{-\nu t} + 2e^{-(2\kappa-2)t}C(v,\phi), \end{aligned}$$

we conclude from (3.1) in Proposition 3.2 and Proposition 2.2 that for any $s < t$,

$$\begin{aligned} &|H(v(t)) - H(v(s))| \qquad (3.15) \\ &\le \left|e^{-\kappa t}(e^{-2t}\|\nabla\phi(t)\|_2^2 + \|\phi(t)\|_2^2) - e^{-\kappa s}(e^{-2s}\|\nabla\phi(s)\|_2^2 + \|\phi(s)\|_2^2)\right| \\ &+ \int_s^t J(v(\tau))d\tau + \int_s^t e^{-2\kappa\tau}\Big(\int_{\mathbb{R}^n} v(\rho)|\nabla\phi(\tau)|^2 dx\Big)d\tau \\ &\le C(\|\phi\|_{H^1})e^{-\kappa s}(1 - e^{-\kappa t}) + 2\int_s^t I(v(\tau))d\tau \\ &+ C\int_s^t e^{-2(\kappa-1)\tau}d\tau \sup_{t>0}\|v(t)\|_\infty \sup_{t>0} e^{-2t}\|\nabla\phi(t)\|_2^2 \le Ce^{-\nu s} \to 0,\ \ s,t\to\infty \end{aligned}$$

and this shows that $\{H(v(t_n))\}_n$ is a Cauchy sequence as $t_n \to \infty$. Moreover, since $v^\alpha \in L^1$ with $\nabla v^\alpha \in L^1$, we have

$$\int_{\mathbb{R}^n} |\nabla v^\alpha| dx \le \frac{\alpha^2}{(\alpha-1/2)^2}\Big(\int_{\mathbb{R}^n} u_0 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} |\nabla v^{\alpha-\frac{1}{2}}|^2 dx\Big)^{1/2} \le C.$$

Besides the moment bound (2.21) in Proposition 2.4, we also have $|x|^{2+a}v \in L^1$. Therefore, by the compactness of $W^{1,1} \cap L^1_{2+a}(\mathbb{R}^n) \subset L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$, we have a subsequence $v(t_n)$ that converges strongly to $V(x) \in L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$. A similar argument found in [8, Theorem 3.1] works for our case and we see that there exists a limit function $V$ in $L^1_2(\mathbb{R}^n)$ such that $v(t_n) \to V$, as $t_n \to \infty$ in $L^1(\mathbb{R}^n)$. It turns out that the limit function is also non-negative and bounded. By (3.14), the moment bound from Proposition 2.4, and the natural regularity of the weak solution, we see that

$$J(v(t)) \to J(V) = \int_{\mathbb{R}^n} V\Big|\frac{\alpha}{\alpha-1}\nabla V^{\alpha-1} + x\Big|^2 dx = 0$$

and we obtain either $V = 0$ or $\nabla V^{\alpha-1} = -\frac{\alpha-1}{\alpha}x$ almost everywhere. The argument concludes by recalling $M = \|u_0\|_1$ and $V(x) = \big[A - \frac{\alpha-1}{2\alpha}|x|^2\big]_+^{\frac{1}{\alpha-1}}$, where $A$ is chosen such that the $L^1$ norm of $V(x)$ is normalized as 1. Again the estimate (3.1) in Proposition 3.2 and (3.15) gives

$$|H(v(t)) - H(V)| \le Ce^{-\nu t} \tag{3.16}$$

and the desired estimate follows from the argument in [8, Theorem 4.5]. Namely we see firstly that

$$\int_{v<V} |v(t) - V|dx \le \Big(\frac{1}{\alpha}|H(\chi_{B_M} v(t)) - H(V)|\Big)^{\frac{1}{2}} \Big(\int_{B_M} V(x)^{\frac{2}{n}} dx\Big)^{\frac{1}{2}} \tag{3.17}$$

by the special structure of the Barenblatt solution, where $B_M =$ supp $B \equiv \{|x| \le \frac{2\alpha A}{\alpha-1}\}$ and $\chi_{B_M}$ is the characteristic function on $B_M$. By $M = \|V\|_1 = \|v(t)\|_1$ and $V \ge 0$, we see

$$\int_{v\ge V} |v(t) - V|dx = \int_{v<V} (V - v(t))dx = \int_{v<V} |v(t) - V|dx. \tag{3.18}$$

We note that, over $B_M^c$, $V$ is vanishing and by [8, Lemma 4.4]

$$\frac{1}{\alpha-1}\int_{|x|^2>C} v^\alpha(t)dx + \frac{1}{2}\int_{|x|^2>C} (|x|^2 - D)v(t)dx \le |H(v(t)) - H(V)|, \tag{3.19}$$

$$D\int_{|x|^2>C} v(t)dx \le Ce^{-\gamma t}. \tag{3.20}$$

Combining (3.17), (3.18), and (3.19) with (3.16) we conclude that

$$\|v(t) - V\|_1 \le Ce^{-\nu' t},$$

where $\nu' = \min(\frac{2\sigma}{(n+2)\alpha-n}, \nu)$. □

## 4. Appendix A

In this appendix, we derive the estimate for the functional $I(v(t))$ for the weak solution of (2.2). For $t > 0$, $x \in \mathbb{R}^n$, let $(v, \phi)$ be a solution of the regularized system

$$\begin{cases} \partial_t v - \mathrm{div}\,\big((v+\varepsilon)K_\varepsilon(x, v, \phi)\big) = -\varepsilon\big(e^{-(\kappa-2)t}(v - \phi) + n\big), \\ -e^{-2t}\Delta\phi + \phi = v, \\ v(0, x) = u_0(x), \end{cases} \tag{4.1}$$

where

$$K_\varepsilon(x, v, \phi) \equiv \frac{\alpha}{\alpha-1}\nabla(v+\varepsilon)^{\alpha-1}(t, x) + x - e^{-\kappa t}\nabla\phi(t).$$

Note that the above system (4.1) is equivalent to (2.3). The existence of the smooth and sufficiently fast decaying solution at $|x| \to \infty$ of (4.1) is obtained in a similar manner in [49].

**Proposition 4.1.** *For a solution $v$ and $\phi$ of* (4.1) *belonging to $L^1$, we let*

$$I_\varepsilon(v(t)) \equiv \int_{\mathbb{R}^n} v(t)\,|K_\varepsilon(x, v(t), \phi(t))|^2\,dx,$$

*where $\kappa = n(2-\alpha)$. Then we have*

$$\begin{aligned}
\frac{d}{dt}I_\varepsilon(v(t)) \le& -2\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta dx \\
&- (2\alpha-1)\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\Big|\text{div } K_\varepsilon(x,v,\phi)\Big|^2\zeta dx \\
&- 2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\nabla K_\varepsilon(x,v,\phi)|^2\zeta dx \\
&+ 2e^{-\kappa t}\int_{\mathbb{R}^n}(v+\varepsilon)K_{\varepsilon,i}(x,v,\phi)K_{\varepsilon,j}(x,v,\phi)\left(D^2_{ij}\phi\right)\zeta dx \\
&+ 2e^{-(\kappa-2)t}\int_{\mathbb{R}^n}|(vK_\varepsilon(x,v,\phi))|^2\,\zeta dx - 2\kappa e^{-\kappa t}\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\cdot\nabla\phi\zeta dx \\
&- 2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\text{div } K_\varepsilon(x,v,\phi)K_\varepsilon(x,v,\phi)\cdot\nabla\zeta dx \\
&- 2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\nabla|K_\varepsilon(x,v,\phi)|^2\cdot\nabla\zeta dx \\
&+ \varepsilon\int_{\mathbb{R}^n}(e^{-(\kappa-2)t}(\phi-v)-n)|K_\varepsilon(x,v,\phi)|^2\zeta dx \\
&- \int_{\mathbb{R}^n}v|K_\varepsilon(x,v,\phi)|^2K_\varepsilon(x,v,\phi)\cdot\nabla\zeta dx \\
&- 2\alpha\int_{\mathbb{R}^n}K_{\varepsilon,i}(x,v,\phi)K_{\varepsilon,j}(x,v,\phi)(v+\varepsilon)^{\alpha-1}\nabla_i v\nabla_j\zeta dx \\
&+ \int_{\mathbb{R}^n}|(vK_\varepsilon(x,v,\phi))|^2\,\nabla\zeta dx + 2e^{-\kappa t}\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\partial_t\phi\cdot\nabla\zeta dx \\
&- 2e^{-\kappa t}\int_{\mathbb{R}^n}(vK(x,v,\phi))(-e^{-2t}\Delta+1)^{-1}(n-e^{-(\kappa-2)t}(v-\phi))\nabla\zeta dx.
\end{aligned} \tag{4.2}$$

**Remark.** We note that the last eight terms in the right-hand side of (3.6) are considered as error terms. We are going to eliminate those terms to justify (3.1).

**Proof of Proposition 4.1.** For simplicity we drop the suffix $\varepsilon$ on $K_\varepsilon(x,v,\phi)$. Let $\zeta$ be a smooth cut off function with $\zeta = 1$ on $B_R$ and $\zeta = 0$ in $B_{2R}^c$. By differentiating,

$$\begin{aligned}
&\frac{d}{dt}I_\varepsilon(v(t)) = \int_{\mathbb{R}^n} \partial_t v|K(x,v,\phi)|^2\zeta dx \\
&- 2\alpha \int_{\mathbb{R}^n} \text{div } (vK(x,v,\phi))(v+\varepsilon)^{\alpha-2}(t)\partial_t v(t)\zeta dx \\
&- 2\alpha \int_{\mathbb{R}^n} vK(x,v,\phi)(v+\varepsilon)^{\alpha-2}(t)\partial_t v(t)\nabla\zeta dx \\
&+ 2e^{-\kappa t}\int_{\mathbb{R}^n} \text{div } (vK(x,v,\phi))\partial_t\phi\zeta dx - 2\kappa e^{-\kappa t}\int_{\mathbb{R}^n} vK(x,v,\phi)\cdot\nabla\phi\zeta dx \\
&+ 2e^{-\kappa t}\int_{\mathbb{R}^n} vK(x,v,\phi)\partial_t\phi\cdot\nabla\zeta dx. \qquad (4.3)
\end{aligned}$$

Then, according to the equation,

$$\partial_t v = \text{div } \big((v+\varepsilon)K_\varepsilon(x,v,\phi)\big) - \varepsilon(n + e^{-(\kappa-2)t}(v-\phi)),$$

the first term of the right-hand side of (4.3) can be expressed as

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_t v|K(x,v,\phi)|^2\zeta dx = -2\int_{\mathbb{R}^n}(v+\varepsilon)K(x,v,\phi)\nabla\big(|K(x,v,\phi)|^2\zeta\big)dx \\
&- \varepsilon\int_{\mathbb{R}^n}(n+e^{-(\kappa-2)t}(v-\phi))|K(x,v,\phi)|^2\zeta dx \\
&= -2\int_{\mathbb{R}^n}(v+\varepsilon)|K(x,v,\phi)|^2\zeta dx - \frac{2\alpha}{\alpha-1}\int_{\mathbb{R}^n}\frac{1}{v+\varepsilon}((v+\varepsilon)K_i(x,v,\phi)) \\
&\qquad \times ((v+\varepsilon)K_j(x,v,\phi))\left(D_{ij}^2(v+\varepsilon)^{\alpha-1}\right)\zeta dx \\
&+ 2e^{-\kappa t}\int_{\mathbb{R}^n}(v+\varepsilon)K_i(x,v,\phi)K_j(x,v,\phi)\left(D_{ij}^2\phi\right)\zeta dx \\
&- \varepsilon\int_{\mathbb{R}^n}(n+e^{-(\kappa-2)t}(v-\phi))|K(x,v,\phi)|^2\zeta dx \\
&- \int_{\mathbb{R}^n}(v+\varepsilon)|K(x,v,\phi)|^2K(x,v,\phi)\cdot\nabla\zeta dx. \qquad (4.4)
\end{aligned}$$

Now the second term of the right-hand side of (4.4) can be written

$$\begin{aligned}
&-\frac{2\alpha}{\alpha-1}\int_{\mathbb{R}^n}\frac{1}{v+\varepsilon}((v+\varepsilon)K_i(x,v,\phi))((v+\varepsilon)K_j(x,v,\phi))\left(D_{ij}^2(v+\varepsilon)^{\alpha-1}\right)\zeta dx \\
&= \frac{2\alpha}{\alpha-1}\int_{\mathbb{R}^n}\Big(-\frac{\nabla v}{(v+\varepsilon)^2}\cdot((v+\varepsilon)K(x,v,\phi))((v+\varepsilon)K_i(x,v))
\end{aligned}$$

$$+\frac{1}{v+\varepsilon}\nabla_j\big((v+\varepsilon)^2K_i(x,v)K_j(x,v)\big)\Big)\nabla_i(v+\varepsilon)^{\alpha-1}\zeta dx$$

$$+\frac{2\alpha}{\alpha-1}\int_{\mathbb{R}^n}\frac{1}{v+\varepsilon}((v+\varepsilon)K_i(x,v,\phi))((v+\varepsilon)K_j(x,v,\phi))\nabla(v+\varepsilon)^{\alpha-1}\cdot\nabla\zeta dx$$

$$=-2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-2}\Big(|K(x,v,\phi)$$

$$\cdot\nabla v|^2-2K(x,v,\phi)\cdot\nabla v\ \mathrm{div}\ \big((v+\varepsilon)K(x,v,\phi)\big)\Big)\zeta dx$$

$$+2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-2}\nabla_i v\Big(K(x,v,\phi)\cdot\nabla\big((v+\varepsilon)K_i(x,v,\phi)\big)$$

$$-K_i(x,v,\phi)\mathrm{div}\ \big((v+\varepsilon)K(x,v,\phi)\big)\Big)\zeta dx$$

$$+2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-1}\nabla v\cdot K(x,v,\phi)K(x,v,\phi)\cdot\nabla\zeta dx. \tag{4.5}$$

Now noting that $K$ is a rotation free field,

$$\nabla_i K_j(x,v,\phi)=\delta_{ij}+\frac{\alpha}{\alpha-1}D_{ij}^2(v+\varepsilon)^{\alpha-1}-e^{-\kappa t}D_{ij}^2\phi=\nabla_j K_i(x,v,\phi),$$

and the second term of the right-hand side of (4.5) is

$$2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-2}\nabla_i v\Big(K(x,v,\phi)\cdot\nabla((v+\varepsilon)K_i(x,v,\phi)) \tag{4.6}$$

$$-K_i(x,v,\phi)\mathrm{div}\ ((v+\varepsilon)K(x,v,\phi))\Big)\zeta dx$$

$$=2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-1}\nabla_i v\Big(K(x,v,\phi)\cdot\nabla_i(K(x,v,\phi))$$

$$-K_i(x,v,\phi)\mathrm{div}\ (K(x,v,\phi))\Big)\zeta dx$$

$$=2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-1}\Big(\nabla v\cdot\frac{1}{2}\nabla|K(x,v,\phi)|^2$$

$$-(K(x,v,\phi)\cdot\nabla v)\mathrm{div}\ (K(x,v,\phi))\Big)\zeta dx,$$

it follows from (4.5) and (4.6) that

$$-\frac{2\alpha}{\alpha-1}\int_{\mathbb{R}^n}\frac{1}{v+\varepsilon}((v+\varepsilon)K_i(x,v,\phi))((v+\varepsilon)K_j(x,v,\phi))\left(D_{ij}^2(v+\varepsilon)^{\alpha-1}\right)\zeta dx$$

$$=-2\alpha\int_{\mathbb{R}^n}(v+\varepsilon)^{\alpha-2}\Big(|K(x,v,\phi)\cdot\nabla v|^2-2K(x,v,\phi)$$

$$\cdot\nabla v\ \mathrm{div}\ \big((v+\varepsilon)K(x,v,\phi)\big)\Big)\zeta dx$$

$$- 2\int_{\mathbb{R}^n} \Big(\frac{1}{2}(v+\varepsilon)^\alpha \Delta|K(x,v,\phi)|^2 + (K(x,v,\phi)\cdot\nabla(v+\varepsilon)^\alpha)\text{div } (K(x,v,\phi))\Big)\zeta dx$$
$$- 2\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \nabla|K(x,v,\phi)|^2\cdot\nabla\zeta dx$$
$$+ 2\alpha\int_{\mathbb{R}^n} (v+\varepsilon)^{\alpha-1}\nabla v\cdot K(x,v,\phi)K(x,v,\phi)\cdot\nabla\zeta dx. \tag{4.7}$$

Combining (4.4) and (4.7), and noting the relation

$$\text{div } (K\text{div } K) = |\text{div } K|^2 + (K\cdot\nabla(\text{div } K)),$$

it follows that

$$\int_{\mathbb{R}^n} \partial_t v|K(x,v,\phi)|^2 dx = -2\int_{\mathbb{R}^n} (v+\varepsilon)|K(x,v,\phi)|^2\zeta dx \tag{4.8}$$
$$- (2\alpha-2)\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \Big|\text{div } K(x,v,\phi)\Big|^2\zeta dx$$
$$+ 2\alpha\int_{\mathbb{R}^n} (v+\varepsilon)^{\alpha-2}|\text{div } (vK(x,v,\phi))|^2\zeta dx$$
$$- 2\int_{\mathbb{R}^n} \Big(\frac{1}{2}(v+\varepsilon)^\alpha\Delta|K(x,v,\phi)|^2$$
$$- (v+\varepsilon)^\alpha(K(x,v,\phi)\nabla\text{div } (K(x,v,\phi)))\Big)\zeta dx$$
$$+ 2\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \text{div } K(x,v,\phi)K(x,v,\phi)\cdot\nabla\zeta dx$$
$$- 2\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha\nabla|K(x,v,\phi)|^2\cdot\nabla\zeta dx$$
$$+ 2\alpha\int_{\mathbb{R}^n} K_i(x,v,\phi)K_j(x,v,\phi)(v+\varepsilon)^{\alpha-1}\nabla v\cdot\nabla\zeta dx$$
$$+ 2e^{-\kappa t}\int_{\mathbb{R}^n} (v+\varepsilon)K_i(x,v,\phi)K_j(x,v,\phi)\left(D^2_{ij}\phi\right)\zeta dx$$
$$- \varepsilon\int_{\mathbb{R}^n} (n + e^{-(\kappa-2)t}(v-\phi))|K(x,v,\phi)|^2\zeta dx$$
$$- \int_{\mathbb{R}^n} v|K(x,v,\phi)|^2 K(x,v,\phi)\cdot\nabla\zeta dx.$$

The third term of the right-hand side is canceled with the second term of the right-hand side of (4.3). Noting $\nabla_i K_j = \nabla_j K_i$ and

$$\frac{1}{2}\Delta|K|^2 - K\nabla\text{div } K = K\cdot\Delta K + |\nabla K|^2 - K\cdot\nabla\text{div } K$$

$$= |\nabla K|^2 + K_j \nabla_i \nabla_i K_j - K_j \nabla_j \nabla_i K_i = |\nabla K|^2,$$

we see that the remaining terms are, from (4.3) and (4.8),

$$\begin{aligned}
\frac{d}{dt} I_\varepsilon(v(t)) &= -2 \int_{\mathbb{R}^n} (v+\varepsilon)|K(x,v,\phi)|^2 \zeta dx \\
&- 2(\alpha-1) \int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \Big| \text{div } K(x,v,\phi) \Big|^2 \zeta dx \\
&- 2 \int_{\mathbb{R}^n} (v+\varepsilon)^\alpha |\nabla K(x,v,\phi)|^2 \zeta dx \\
&+ 2 \int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \text{div } K(x,v,\phi) K(x,v,\phi) \cdot \nabla\zeta dx \\
&- 2 \int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \nabla |K(x,v,\phi)|^2 \cdot \nabla\zeta dx \\
&+ 2\alpha \int_{\mathbb{R}^n} K_i(x,v,\phi) K_j(x,v,\phi) (v+\varepsilon)^{\alpha-1} \nabla v \cdot \nabla\zeta dx \\
&+ 2e^{-\kappa t} \int_{\mathbb{R}^n} (v+\varepsilon) K_i(x,v,\phi) K_j(x,v,\phi) \left(D_{ij}^2 \phi\right) \zeta dx \\
&- \varepsilon \int_{\mathbb{R}^n} (n + e^{-(\kappa-2)t}(v-\phi)) |K(x,v,\phi)|^2 \zeta dx \\
&- \int_{\mathbb{R}^n} v |K(x,v,\phi)|^2 K(x,v,\phi) \cdot \nabla\zeta dx \\
&+ 2e^{-\kappa t} \int_{\mathbb{R}^n} \text{div } (vK(x,v,\phi)) \partial_t \phi \zeta dx - 2\kappa e^{-\kappa t} \int_{\mathbb{R}^n} vK(x,v,\phi) \cdot \nabla\phi \zeta dx \\
&+ 2e^{-\kappa t} \int_{\mathbb{R}^n} vK(x,v,\phi) \partial_t \phi \cdot \nabla\zeta dx.
\end{aligned} \tag{4.9}$$

Since $\phi = (-e^{-2t}\Delta + 1)^{-1} v$, the fifth term of the right-hand side of (4.9) is

$$\begin{aligned}
& 2e^{-\kappa t} \int_{\mathbb{R}^n} \text{div } (vK(x,v,\phi)) \partial_t \phi \zeta dx \\
&= 2e^{-\kappa t} \int_{\mathbb{R}^n} \text{div } \big(vK(x,v,\phi)\big) (-e^{-2t}\Delta + 1)^{-1} \text{div } \big((v+\varepsilon) K(x,v,\phi)\big) \zeta dx \\
&- 2e^{-\kappa t} \int_{\mathbb{R}^n} (vK(x,v,\phi)) (-e^{-2t}\Delta + 1)^{-1} (n - e^{-\kappa-2t}(v-\phi)) \nabla\zeta dx \\
&\le 2e^{-(\kappa-2)t} \Big( \int_{\mathbb{R}^n} |(vK(x,v,\phi))|^2 \zeta dx + \int_{\mathbb{R}^n} |(vK(x,v,\phi))|^2 \nabla\zeta dx \Big)
\end{aligned} \tag{4.10}$$

$$- 2e^{-\kappa t}\int_{\mathbb{R}^n}(vK(x,v,\phi))(-e^{-2t}\Delta+1)^{-1}(n-e^{-(\kappa-2)t}(v-\phi))\nabla\zeta dx.$$

Combining (4.9 ) and (4.10), we obtain (4.2). □

**Proposition 4.2.** *Let* $(v,\phi)$ *be a weak solution of* (2.2)*. We set*

$$I(v(t))\equiv\int_{\mathbb{R}^n}v(t)|K(x,v,\phi)|^2dx$$

*with* $K(x,v,\phi)\equiv\frac{\alpha}{\alpha-1}u^{\alpha-1}+x-e^{-\kappa t}\nabla\phi$*. Then if* $1<\alpha<2-\frac{2}{n}$ *and the solution* $v$ *has uniform estimate* $\sup_{t>0}\|v(t)\|_\infty\le C_*$ *for some constant, it then holds that, for all* $t>T_0$*,*

$$I(v(t))+\int_{T_0}^t I(v(\tau))d\tau\le I(v(T_0)) \tag{4.11}$$

*for almost all* $t>s$ *and* $s\ge 0$*.*

**Proof.** We mainly treat the following two terms appearing in the right-hand side of (4.2):

$$\begin{aligned}&-2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha \text{div } K_\varepsilon(x,v,\phi)K_\varepsilon(x,v,\phi)\cdot\nabla\zeta dx\\&-2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\nabla|K_\varepsilon(x,v,\phi)|^2\cdot\nabla\zeta dx.\end{aligned} \tag{4.12}$$

To handle those terms, we use the Hölder inequality:

$$\begin{aligned}&-2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha \text{div } K_\varepsilon(x,v,\phi)K_\varepsilon(x,v,\phi)\cdot\nabla\zeta dx\\&\le 2\left(\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\text{div } K_\varepsilon(x,v,\phi)|^2\zeta dx\right)^{1/2}\\&\qquad\times\left(\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|K_\varepsilon(x,v,\phi)|^2|\nabla\zeta|^2\zeta^{-1}dx\right)^{1/2}\\&\le(\alpha-1)\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\text{div } K_\varepsilon(x,v,\phi)|^2\zeta dx\\&\qquad+\frac{1}{\alpha-1}\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|K_\varepsilon(x,v,\phi)|^2|\nabla\zeta|^2\zeta^{-1}dx.\end{aligned}$$

The first term is cancelled by the second term in the right-hand side of (4.2). We choose the cut off nicely so that $|\nabla\zeta|\le CR^{-1}\zeta^a$ where $a<1$ and hence we see

$$\begin{aligned}&\frac{1}{\alpha-1}\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|K_\varepsilon(x,v,\phi)|^2|\nabla\zeta|\zeta^{-1}dx \\ &\le\frac{1}{(\alpha-1)R}\int_{R<|x|<2R}(v+\varepsilon)^\alpha|K_\varepsilon(x,v,\phi)|^2\zeta^{2a-1}dx \\ &\le\frac{(\|v(t)\|_\infty+\varepsilon)^{\alpha-1}}{(\alpha-1)R}\int_{R<|x|<2R}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta^{2a-1}dx.\end{aligned}\tag{4.13}$$

Now a similar argument holds for the second term of (4.12) and the remaining term is as in the above:

$$\frac{2}{R}(\|v(t)\|_\infty+\varepsilon)^{\alpha-1}\int_{R<|x|<2R}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta^{2a-1}dx.\tag{4.14}$$

Now we invoke the regularized entropy functional (2.9). Let $\eta$ be a smooth cut off and $\eta=1$ in $|x|<2R$ and vanish over $|x|>3R$ so that it covers the support of $\zeta^2$. Recall

$$\begin{aligned}\frac{d}{dt}W_s(v,\eta)\le&-\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\eta dx \\ &+e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2\eta dx+E_W(x,v,\phi,\varepsilon,\eta),\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}W_s(v,\eta)\equiv&\Big[\frac{1}{\alpha-1}\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\eta\,dx+\frac12\int_{\mathbb{R}^n}|x|^2v\eta dx-\frac12\int_{\mathbb{R}^n}e^{-\kappa t}v\phi\eta\,dx \\ &+\frac14\int_{\mathbb{R}^n}e^{-\kappa t}|\phi|^2\Delta\eta dx\Big]\end{aligned}$$

is the regularized entropy functional and

$$\begin{aligned}&E_W(x,v,\phi,\varepsilon,\eta) \\ &=\frac12\int_{\mathbb{R}^n}\Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1}+\frac12|x|^2-e^{-\kappa t}\phi\Big)^2\nabla((v+\varepsilon)\nabla\zeta^2)dx \\ &+\varepsilon e^{-(\kappa-2)t}\int_{\mathbb{R}^n}(v-\phi)\Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1}+\frac12|x|^2-e^{-\kappa t}\phi\Big)\zeta^2dx \\ &+\varepsilon ne^{-\kappa t}\int_{\mathbb{R}^n}\phi\eta dx+\kappa e^{-\kappa t}\int_{\mathbb{R}^n}\frac12|\phi|^2\Delta\zeta^2dx+e^{-(\kappa+2)t}\int_{\mathbb{R}^n}\nabla\phi\partial_t\phi\nabla\zeta^2dx\end{aligned}$$

is essentially the error term which will be vanishing after letting $\eta\to1$ and $\varepsilon\to0$.

Now, for a small parameter $\delta > 0$, we add the regularized entropy inequality (4.15) multiplied by $\delta$ to (4.2) to have

$$\frac{d}{dt}\big[I_\varepsilon(v(t)) + \delta W_s(v(t))\big] \le -2\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta dx \tag{4.16}$$

$$- (\alpha-1)\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha\big|\text{div }K_\varepsilon(x,v,\phi)\big|^2\zeta dx - \int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\nabla K_\varepsilon(x,v,\phi)|^2\zeta dx$$

$$+ 2e^{-\kappa t}\int_{\mathbb{R}^n}(v+\varepsilon)K_{\varepsilon,i}(x,v,\phi)K_{\varepsilon,j}(x,v,\phi)\left(D^2_{ij}\phi\right)\zeta dx$$

$$+ 2e^{-(\kappa-2)t}\int_{\mathbb{R}^n}|(vK_\varepsilon(x,v,\phi))|^2\,\zeta dx - 2\kappa e^{-\kappa t}\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\cdot\nabla\phi\zeta dx$$

$$- \delta\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\eta dx + \delta e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2\eta dx$$

$$+ \left(\frac{1}{(\alpha-1)R} + \frac{2}{R}\right)\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|K_\varepsilon(x,v,\phi)|^2\zeta^{2a-1}dx$$

$$+ E_I(x,v,\phi,\varepsilon,\zeta) + \delta E_W(x,v,\phi,\varepsilon,\eta),$$

where

$$E_I(x,v,\phi,\varepsilon,\zeta) \equiv \varepsilon e^{-(\kappa-2)t}\int_{\mathbb{R}^n}\phi|K_\varepsilon(x,v,\phi)|^2\zeta dx \tag{4.17}$$

$$- \int_{\mathbb{R}^n}v|K_\varepsilon(x,v,\phi)|^2K_\varepsilon(x,v,\phi)\cdot\nabla\zeta dx$$

$$- 2\alpha\int_{\mathbb{R}^n}K_{\varepsilon,i}(x,v,\phi)K_{\varepsilon,j}(x,v,\phi)(v+\varepsilon)^{\alpha-1}\nabla_i v\nabla_j\zeta dx$$

$$+ \int_{\mathbb{R}^n}|(vK_\varepsilon(x,v,\phi))|^2\,\nabla\zeta dx + 2e^{-\kappa t}\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\partial_t\phi\cdot\nabla\zeta dx$$

$$- 2e^{-\kappa t}\int_{\mathbb{R}^n}(vK(x,v,\phi))(-e^{-2t}\Delta+1)^{-1}(n - e^{-(\kappa-2)t}(v-\phi))\nabla\zeta dx$$

is again the error term from (4.2) and we note that the negative terms involving $\varepsilon$ are dropped off here. Now we have already seen that $\|v(t)\|_\infty$ is uniformly bounded and this is also valid for the approximated solution and by choosing $R > 0$ large enough, the term involving $(\|v(t)\|_\infty + \varepsilon)^{\alpha-1}$ is absorbed by the term from the regularizing entropy

$$-\delta\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\eta dx.$$

Note that $\eta \equiv 1$ over the support of $\zeta$.

Analogously to (3.11) in the formal proof in Proposition 3.4, we have

$$\begin{aligned} 2e^{-\kappa t}\int_{\mathbb{R}^n}&(v+\varepsilon)K_{i,\varepsilon}(x,v,\phi)K_{j,\varepsilon}(x.v,\phi)D^2_{ij}\phi\zeta dx \\ &\le 2e^{-\kappa t}\|D^2_{ij}\phi(t)\|_\infty\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta dx \\ &\le 2C(1+t)e^{-(\kappa-2)t}\|v(t)\|_\infty\int_{\mathbb{R}^n}(v+\varepsilon)|K(x,v,\phi)|^2\zeta dx, \end{aligned} \tag{4.18}$$

$$\begin{aligned} 2e^{-(\kappa-2)t}&\int_{\mathbb{R}^n}\left|(vK_\varepsilon(x,v,\phi))\right|^2\zeta dx \\ &\le 2e^{-(\kappa-2)t}\|v(t)\|_\infty\int_{\mathbb{R}^n}v\left|K_\varepsilon(x,v,\phi)\right|^2\zeta dx \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} -2\kappa e^{-\kappa t}&\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\nabla\phi\zeta dx \\ &\le 2\kappa e^{-\kappa t}\|\nabla\phi(t)\|_2\Big(\int_{\mathbb{R}^n}v\zeta dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}v|K_\varepsilon(x,v,\phi)|^2\zeta dx\Big)^{1/2} \\ &\le 2\kappa\varepsilon^{-1}e^{-2\kappa t}\|v(t)\|_1^2\int_{\mathbb{R}^n}|\nabla\phi(t)|^2dx+\frac{\varepsilon}{2}\int_{\mathbb{R}^n}v|K_\varepsilon(x,v,\phi)|^2\zeta dx. \end{aligned} \tag{4.20}$$

Combining all those estimates (4.18) - (4.20) and plugging into (4.16), we have

$$\begin{aligned} \frac{d}{dt}\Big[I_\varepsilon(v(t))+\delta W_s(v(t))\Big] &\le -(2-\varepsilon)\int_{\mathbb{R}^n}(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta dx \\ +2C(1+t)e^{-(\kappa-2)t}&\|v(t)\|_\infty\int_{\mathbb{R}^n}(v+\varepsilon)|K(x,v,\phi)|^2\zeta dx \\ &+2\kappa\varepsilon^{-1}e^{-2\kappa t}\|v(t)\|_\infty^2\|\nabla\phi(t)\|_2^2 \\ &+\delta e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2\eta dx+\big(E_I(x,v,\phi,\varepsilon,\zeta)+\delta E_W(x,v,\phi,\varepsilon,\eta)\big), \end{aligned} \tag{4.21}$$

where we drop the non positive terms. For some large time $T_0$, the second term of the right-hand side is absorbed by the first term and then passing to the limit as $R\to\infty$ (so that $\eta\to 1$ and $\zeta\to 1$), we take the limit as $\delta\to 0$ to remove the additional entropy terms. For all error terms remaining here,

$$\begin{aligned} &\lim_{R\to\infty}\big(E_I(x,v,\phi,\varepsilon,\zeta)+\delta E_W(x,v,\phi,\varepsilon,\eta)\big) \\ &\quad\le\varepsilon e^{-(\kappa-2)t}\int_{\mathbb{R}^n}\phi|K_\varepsilon(x,v,\phi)|^2dx \end{aligned}$$

$$+ \varepsilon e^{-(\kappa-2)t} \int_{\mathbb{R}^n} v\Big(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1} + \frac{1}{2}|x|^2\Big)dx$$
$$+ \varepsilon e^{-(2\kappa-2)t} \int_{\mathbb{R}^n} |\phi|^2 dx + \varepsilon n e^{-\kappa t} \int_{\mathbb{R}^n} \phi dx,$$

and they have their meaning after letting $R \to \infty$. By letting $\varepsilon \to 0$, we may choose a subsequence $\{u_{\varepsilon_n}(t,x)\}_{\varepsilon_n}$ such that

$$\begin{aligned} v_{\varepsilon_n} &\to v && \text{in } C((0,T); L^\alpha \cap L^1_2(\mathbb{R}^n)), \\ \nabla v_{\varepsilon_n}^\alpha &\to \nabla v^\alpha && \text{in } w - L^2((0,T); L^2(\mathbb{R}^n)), \\ \nabla v_{\varepsilon_n}^{\alpha-1/2} &\to \nabla v^{\alpha-1/2} && \text{in } w - L^2((0,T); L^2(\mathbb{R}^n)), \\ \phi_{\varepsilon_n} &\to \phi && \text{in } C((0,T); W^{2,p}(\mathbb{R}^n)). \end{aligned}$$

Finally we drop the negative terms involving the higher derivatives in (4.21), namely the second and third terms in the right-hand side. After time integration and letting $\varepsilon \to 0$, we conclude the desired estimate for the weak solution. □

## 5. Appendix B

In this section , we show the uniform a priori estimate Proposition 3.6 for the weak solution of

$$\begin{cases} \partial_t u - \text{div } (\nabla u^\alpha - u\nabla\psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta\psi + \psi = u, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{5.1}$$

in the Hölder space $C^s(\mathbb{R}^n)$.

The original idea is due to the result by DiBeneditto-Friedman [14] and Wiegner [53]. See also for the one-dimensional case Nagai-Mimura [37]. We also notice that, in our case, the equation contains the lower-order terms and a similar but related problem is also treated by Misawa [31] for the $p$-Laplace heat system.

**Theorem 5.1** ([14], [53])**.** *Let $(u,\psi)$ be a weak solution to* (5.1) *and suppose that*

$$\|u(t)\|_\infty + \|u(t)\|_1 + (\|\nabla\psi(t)\|_\infty + \|\nabla\psi(t)\|_2) \le C \tag{5.2}$$

*uniformly in $t \in [0,T]$. Then, for any $t \in (0,T)$, we have $\|u(t)\|_{C^s} \le C$ for some $0 < s < \frac{1}{\alpha-1}$.*

The rescaled version of the above estimate Proposition 3.6 follows by change of variable.

The proof is entirely along the lines of the argument found in [14]. For completeness we give the outlined proof. The crucial step to obtain the proof is to show the local version of the energy inequality and a version of the Harnack principle.

We define notation: For a fixed number $R_0 > 0$ and small $0 < \beta << 2$, we denote a parabolic cylinder around $(t_0, x_0)$ by

$$Q^\beta_{R_0} = \{(t,x) : |x - x_0| < R_0, t_0 - R_0^{2-\beta} < t < t_0\}.$$

Without losing generality, we may assume that $(t_0, x_0)$ can be taken as the origin; if otherwise, we introduce the shifting transformation. Now by the boundedness assumption for the weak solution $u$, we let

$$\mu_- = \inf_{Q^\beta_R} u^\alpha(t,x), \qquad \mu_+ = \sup_{Q^\beta_R} u^\alpha(t,x)$$

and assume, for some $\omega > 0$, that

$$\operatorname*{osc}_{Q^\beta_R} u^\alpha = \mu_+ - \mu_- \le \omega.$$

Then, for $0 < R \le R_0/2$, we assume that

$$\omega^{1/\alpha'} > R^\beta, \qquad \alpha' = \frac{\alpha}{\alpha - 1}. \tag{5.3}$$

We then introduce a smaller cylinder; let

$$Q_R(\omega) = \{(t,x) : |x - x_0| < R, t_0 - \omega^{-1/\alpha'} R^2 < t < t_0\}.$$

In the following, we re-size the above cylinder by changing $\omega$ and $R$. Since $Q_R(\omega) \subset Q^\beta_{R_0}$, we have $\operatorname{osc}_{Q_R(\omega)} u^\alpha \le \omega$. We assume that $\mu_- < \frac{\omega}{4}$. If otherwise we see from (5.3) and the fact that $1/\alpha' < 1$ that $\mu_- \ge \frac{R}{4}$ and the Hölder regularity of the weak solution in $Q_R(\omega)$ follows from the uniform parabolicity of the solution and standard theory of the non-degenerated parabolic equation.

Then, under the assumption $\mu_- < \omega/4$, the following alternative occurs. For some $0 < \rho < 1$, we have either

$$\mu\left(\left\{(t,x) \in Q_R(\omega) : u^\alpha(t,x) < \mu_- + \frac{\omega}{2}\right\}\right) < \rho\mu(Q_R(\omega)),$$

or

$$\mu\left(\left\{(t,x) \in Q_R(\omega) : u^\alpha(t,x) \ge \mu_+ - \frac{\omega}{2_+}\right\}\right) \le (1-\rho)\mu(Q_R(\omega)).$$

**Step 1.** For the first case, we may invoke the following lemma (cf. [14, Lemma 4.1]) which shows the rigidity for the lower level set for the weak solution of (1.12).

**Lemma 5.2** ([13], [14])**.** *If there exists a constant $\rho \in (0,1)$ such that*

$$\mu\left(\left\{(t,x) \in Q_R(\omega) : u^\alpha(t,x) < \mu_- + \frac{\omega}{2}\right\}\right) \le \rho\mu(Q_R(\omega)).$$

*Then we have*

$$\mu\left(\left\{(t,x) \in Q_{R/2}(\omega) : u^\alpha(t,x) < \mu_- + \frac{\omega}{4}\right\}\right) = 0.$$

*That is, we have $v(t,x) \ge \mu_- + \frac{\omega}{4}$ in $Q_{R/2}(\omega)$.*

This is a crucial tool to obtain the regularity of the weak solution and it follows from the localized version of the energy estimate and an iterative argument which is the key part of the proof.

**Proof.** Let $w(t,x)$ denote $u^\alpha(t,x)$ for simplicity. Then from the definition of the weak solution for

$$\begin{cases} \partial_t w^{1/\alpha} - \text{div } (\nabla w - w^{1/\alpha}\nabla\psi) = 0, & t>0, x \in \mathbb{R}^n, \\ -\Delta\psi + \psi = w^{1/\alpha}, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{5.4}$$

we have the following energy estimate: Let $\mu_- < k < \mu_+$, $Q_R(\omega) = (t_0 - \omega^{-1/\alpha'}R^2, t_0) \times B_R$, where $0 < R < R' = (1+a)R$, and we define a cut off function $\zeta(t,x)$ supported in $Q_{R'}$ that is 1 over $Q_R$. Testing the weak form with $\eta^2(w-k)_+$, we have

$$\begin{aligned} &\sup_{t\in I_R} \|(k-w(t))_+\|^2_{L^2(B_R)} + \int_{Q_{R'}} |\nabla(k-w(t))_+|^2\zeta^2 dxdt \\ &\le \int_{Q_{R'}} |w|^{1/\alpha}|\nabla\psi||\nabla(k-w(t))_+|\zeta^2 dxdt \\ &+ \int_{Q_{R'}} |w|^{1/\alpha}|\nabla\psi(t)||(k-w(t))_+||\nabla\zeta^2|dxdt \\ &+ \int_{Q_{R'}} |(k-w(t))_+|^2|\Delta\zeta^2|dxdt + \int_{Q_{R'}} |(k-w(t))_+||w(t)^{1/\alpha}||\partial_t\zeta^2|dxdt \\ &\le \sup_t \|\nabla\phi(t)\|_\infty \|v(t)\|_\infty\ \mu\Big(\{(t,x)\in Q_R; w(t,x) < k_m\}\Big) \\ &+ \frac{1}{2}\int_{Q_{R'}} |\nabla(k-w(t))_+|^2\zeta^2 dxdt + \int_{Q_{R'}} |(k-w(t))_+|^2(|\nabla\zeta^2|^2 + |\Delta\zeta^2|)dxdt \end{aligned}$$

$$+\int_{Q_{R'}} |(k-w(t))_+||w(t)^{1/\alpha}||\partial_t\zeta^2|dxdt.$$

Hence we see from the fact that $(k-w)_+ \le \omega$, $|w| \le \omega$, $|\nabla\zeta^2|^2 + |\Delta\zeta^2| \le C(aR)^{-2}$, and $|\partial_t\zeta^2| \le C\omega^{1/\alpha'}(aR)^{-2}$ and the assumption (5.2) that

$$\sup_{t\in I_R} \|(k-w(t))_+\|^2_{L^2(B_R)} + \int_{Q_{R'}} |\nabla(k-w(t))_+|^2\zeta^2 dxdt \le C\frac{\omega^2}{a^2R^2}\mu\Big(\{(t,x)\in Q_{R'}; w(t,x)<k_m\}\Big),$$

where $k_- = \mu_- + \frac{\omega}{2^s}$. Here we use the uniform boundedness assumption (5.2). Now let $R_m = \frac{R}{2}(1+\frac{1}{2^{m-1}})$ and $k_m = \mu_- + \frac{\omega}{2}\left(\frac{1}{2}-\frac{1}{2^m}\right)$ for $m = 1, 2, \cdots$. We exchange $Q_R(\omega)$ with $Q_{R_{m+1}}(\omega)$ and for simplicity we let $Q_m = Q_{R_m}(\omega)$. By the Ladyzenskaya inequality, we have

$$\begin{aligned}\|(k_{m+1}-w(t))_+\|^2_{L^2(Q_{m+1})} &\le \|(k_{m+1}-w(t))_+\|^2_{L^{\frac{2(n+2)}{n+2}}(Q_m)} \\ &\le C\frac{\omega^2}{a^2R^2}\mu\Big(\{(t,x)\in Q_{R_m}; w(t,x)<k_m\}\Big) \\ &\le \frac{CA^m}{(k_m-k_{m+1})^{\frac{2(n+4)}{n+2}}}\|(k_m-w(t))_+\|^{2+\frac{4}{n+2}}_{L^2(Q_m)}.\end{aligned}$$

Then by setting

$$J_m = \int_{Q_{R_m}} |(k_m-w)_+|^2 dxdt,$$

it follows that

$$J_{m+1} \le CA^m\rho^{-\frac{2(n+4)}{n+2}} J_m^{1+\frac{2}{m+2}}$$

and by the assumption, we conclude that $J_m \to 0$. The conclusion follows from this. □

**Step 2.** For the second case, arguments analogous to the above yield the upper bounds for the solution.

**Lemma 5.3** ([13], [14]). *If there exists a constant $\rho \in (0,1)$ such that*

$$\mu\Big(\Big\{(t,x)\in Q_R(\omega) : u^\alpha(t,x) \ge \mu_+ - \frac{\omega}{\sqrt{2}}\Big\}\Big) \le (1-\rho)\mu(Q_R(\omega)),$$

*then we have $u^\alpha(t,x) \le \mu_+ - \frac{\omega}{4}$ in $Q_{R/2}(\omega)$.*

**Step 3.** Now we take any parabolic cylinder $Q_{R_0}$ and set $\omega_0 = \operatorname{osc}_{Q_{R_0}} w$ and we suppose that

$$\omega^{1/\beta\alpha'} > R_0 \tag{5.5}$$

holds for $\alpha' = \alpha/(\alpha-1)$ initially.

Combining the conclusions from Lemma 5.2 and 5.3, we may conclude that, in either case, we have from the fact that $\omega = \mu_+ - \mu_-$,

$$\operatorname*{osc}_{Q_{R/2}} w \le \begin{cases} \mu_+ - \mu_- - \frac{\omega}{4} \\ \mu_+ - \frac{\omega}{4} - \mu_- \end{cases} \le \big(1 - \tfrac{1}{4}\big)\omega \equiv \eta\omega.$$

As in the proof in [14], we exchange the parameters $\{\omega_m\}_m$ and $\{R_m\}_m$ and the integral regions $Q_m = \{(t,x) : |x - x_0| < R(1 - \frac{1}{2^m}), t_0 - \eta\rho R^2 < t < t_0\}$ one by one as

$$\omega_{m+1} = \eta\omega_m, \qquad R_{m+1} = \Big(\tfrac{1}{2}\sqrt{\eta}^{1/\alpha'}\sqrt{\rho}\Big)R_m,$$

with $\alpha' = \alpha/(\alpha-1)$. For $R_m = \eta^m R_0$, we conclude that

$$\operatorname*{osc}_{Q_{R_m}} w \le \eta^m\omega_0 = \Big(\frac{2}{\sqrt{\rho}}\Big)^{2n\alpha'}\Big(\frac{R_m}{R_0}\Big)^{2\alpha'}\omega_0,$$

which provides the Hölder continuity of order $\sigma$ for the rescaled weak solution $w(t,x) = u^\alpha(t,x)$

$$|u(t,x) - u(t',x')| \le C(|t-t'|^2 + |x-x'|)^{s/2}.$$

for $s = 2/(\alpha-1)$.

If the assumption (5.5) does not hold then we reduce the sides of the parabolic region $R_0$ into $R_0/2$. Again, if (5.5) does not hold, then we reduce the size $R_0$ into $R_0/4$. Repeating this procedure $k$ times we see that

$$\operatorname*{osc}_{Q_{R_0/2^k}} w = \omega < \Big(\frac{R_0}{2^k}\Big)^{\beta\alpha'}.$$

This again yields the Hölder estimate for the weak solution $v(t,x)$ with $s = \beta/(\alpha-1)$.

Since the constant in the right-hand side is independent of the location of $(t_0, x_0)$, we have

$$\sup_{t>0} \|u(t)\|_{C^s} \le C,$$

so the solution satisfies the uniformly bounded conditions (5.2).

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