Differential and Integral Equations Volume 21, Numbers 11-12 (2008), 1083–1112

ENERGY DEPENDENT INVERSE SCATTERING ON THE LINE

Yutaka Kamimura
Department of Ocean Sciences
Tokyo University of Marine Science and Technology
4-5-7 Konan, Minato-ku, Tokyo 108-8477, Japan

(Submitted by: Takashi Suzuki)

Abstract. We give a complete solution to the inverse scattering problem for the energy dependent Schrödinger equation on the full line when there are no bound states. This solution characterizes the image of the energy dependent scattering transform, generalizes the Deift-Trubowitz theorem for the standard Schrödinger equation in the absence of bound states, and indicates clearly the inverse transform by a simple inversion formula.

1. Introduction

Studies of the inverse scattering transforms for energy dependent Schrödinger equations were originally motivated by an application to the inverse scattering theory for the S-waves of the Klein-Gordon equation in relativistic quantum physics (see Weiss and Scharf [19], Cornille [3]). Besides this motivation, they have at least two kinds of important applications: Firstly isospectral flows for energy dependent Schrödinger equations arise in a wide variety of applications (see Sattinger-Szmigielski [16]) and the scattering transforms for the equations play a crucial role in solving the Cauchy problems for nonlinear evolution equations related with the flows. Secondly inverse scattering theory for energy dependent Schrödinger equations are applied effectively to a class of inverse problems (see Jaulent [7], Aktosun-Klaus-van der Mee [1, 2], Kamimura [10]) to determine coefficients in hyperbolic or elliptic partial differential equations in mathematical physics. For any applications it is of great significance to establish a rather wide framework on which the scattering transform behaves well and a simple inversion formula with which the inverse transform becomes valid.

Accepted for publication: June 2008.
AMS Subject Classifications: 81U40, 34A55, 45E10.

This paper studies the inverse scattering problem of the following energy dependent Schrödinger equation

$$f'' + [k^2 - (U(x) + 2k\,Q(x))]f = 0, \quad -\infty < x < \infty, \tag{1.1}$$

where $U(x)$ and $Q(x)$ are real functions defined on $\boldsymbol{R}$. This inverse scattering problem was studied by Jaulent-Jean [9] based upon the Marchenko [14]-Faddeev [5] theory for the standard Schrödinger equation. They recovered a pair $(U(x), Q(x))$ of potentials in the absence of bound states by solving a nonlinear differential equation with the solution of a linear system of Marchenko type integral equations. Their recovery algorithm was simplified by Sattinger-Szmigielski [16], which recovered a pair $(U(x), Q(x))$ of potentials algebraically by solving a linear system of integral equations. The purpose of the present paper is to give a more direct recovery algorithm of a pair $(U(x), Q(x))$ only with the solution of a Marchenko type equation and, by means of it, to establish a necessary and sufficient condition for a prescribed matrix to be the scattering matrix. This characterization is more general than that in Jaulent-Jean [9]; it gives the generalization to the energy dependent Schrödinger equation (1.1) of the result in Deift-Trubowitz [4] for the standard Schrödinger equation in the absence of bound states.

Let $L^1_m(I)$ $(m = 0, 1, 2, \cdots)$ denote the set of measurable functions $f(x)$ such that

$$\int_I (1 + |x|^m)|f(x)|dx < \infty.$$

Throughout the paper we shall work on (1.1) with the following conditions on $U(x)$ and $Q(x)$:

(A1) $U(x) \in L^1_2(\boldsymbol{R})$.
(A2) $Q(x) \in L^1_1(\boldsymbol{R})$, $Q(x)$ is absolutely continuous, and $Q'(x) \in L^1_2(\boldsymbol{R})$.
(A3) There are no bound states.

The last condition (A3) means that $W[f_+(x,k), f_-(x,k)]$ has no zeros in the upper half plane $\boldsymbol{C}_+ := \{k : \operatorname{Im} k > 0\}$, where $f_\pm(x,k)$ are the Jost solutions of (1.1), namely solutions having the asymptotics

$$f_\pm(x,k) \sim e^{\pm ikx}, \quad x \to \pm\infty,$$

and $W[f,g]$ denotes the Wronskian $fg' - f'g$.

Let $\psi_\pm(x,k)$ be the scattering solutions, that is, solutions of (1.1) possessing the asymptotics

$$\psi_+(x,k) \sim \begin{cases} e^{ikx} + s_{12}(k)e^{-ikx}, & x \to -\infty, \\ s_{11}(k)e^{ikx}, & x \to +\infty; \end{cases}$$

$$\psi_-(x,k) \sim \begin{cases} e^{-ikx} + s_{21}(k)e^{ikx}, & x \to +\infty, \\ s_{22}(k)e^{-ikx}, & x \to -\infty. \end{cases}$$

The solution $\psi_+(x,k)$ represents scattering of a wave e^{ikx} falling from $-\infty$ and scattered by the potential $U(x)+2kQ(x)$ with the reflecting wave of the probability $s_{12}(k)$ and the transmitting wave of the probability $s_{11}(k)$. Similarly $\psi_-(x,k)$ represents scattering from $+\infty$; the coefficients $s_{12}(k)$, $s_{21}(k)$ and $s_{11}(k)$, $s_{22}(k)$ are called the reflection and transmission coefficients. The matrix

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix},$$

is called the scattering matrix for $(U(x), Q(x))$. Since, in terms of the Jost solutions $f_\pm(x,k)$, the scattering solutions $\psi_+(x,k)$ are written as

$$\psi_+(x,k) = s_{11}(k)f_+(x,k) = \overline{f_-(x,k)} + s_{12}(k)f_-(x,k),$$

the coefficients $s_{11}(k)$ and $s_{12}(k)$ are determined as

$$s_{11}(k) = -\frac{2ik}{W[f_+(x,k), f_-(x,k)]}, \quad s_{12}(k) = -\frac{W[f_+(x,k), \overline{f_-(x,k)}]}{W[f_+(x,k), f_-(x,k)]}. \tag{1.2}$$

Here we have used the identity $W[f_\pm(x,k), \overline{f_\pm(x,k)}] = \mp 2ik$ for real k. If $f_+(x,k)$ and $f_-(x,k)$ are linearly dependent, say $f_-(x,k) = cf_+(x,k)$, then

$$2ik = W[f_-(x,k), \overline{f_-(x,k)}] = |c|^2 W[f_+(x,k), \overline{f_+(x,k)}] = -2ik|c|^2,$$

which is impossible unless $k = 0$. Therefore, $W[f_+(x,k), f_-(x,k)] \neq 0$ for $k \in \boldsymbol{R}^* := \boldsymbol{R} \setminus \{0\}$, and so, $s_{11}(k), s_{12}(k)$ are continuous functions in $\boldsymbol{R}^*$. Actually, as is seen by a familiar method (see Lemma 2.2), these coefficients are also continuous at $k = 0$ provided that (A1), (A2) hold. Also it turns out that

$$s_{21}(k) = -\frac{W[\overline{f_+(x,k)}, f_-(x,k)]}{W[f_+(x,k), f_-(x,k)]}, \quad s_{22}(k) = -\frac{2ik}{W[f_+(x,k), f_-(x,k)]}, \tag{1.3}$$

which are continuous functions on $\boldsymbol{R}$.

Let Π be the set of all pairs $(U(x), Q(x))$ of real functions satisfying (A1), (A2), and (A3). Then we have the correspondence $\mathcal{S}$ referred to as the scattering transform

$$\mathcal{S} : \Pi \ni (U(x), Q(x)) \mapsto S(k).$$

The inverse scattering problem we are concerned with in this paper consists of three parts:

(1) to characterize the image $\mathcal{S}(\Pi)$ of the scattering transform,
(2) to show that $\mathcal{S} : \Pi \to \mathcal{S}(\Pi)$ is injective, and

(3) to indicate clearly the inverse scattering transform $\mathcal{S}^{-1} : \mathcal{S}(\Pi) \to \Pi$.

The answer to the problem (1) is as follows:

Theorem 1.1. (a) *If a matrix*

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}$$

belongs to $\mathcal{S}(\Pi)$, then $S(k)$ is continuous in $k \in \boldsymbol{R}$ and possesses the following properties:

(I) *Symmetry*: $s_{11}(k) = s_{22}(k)$.

(II) *Unitarity*:

$$|s_{11}(k)|^2 + |s_{12}(k)|^2 = |s_{21}(k)|^2 + |s_{22}(k)|^2 = 1,$$
$$s_{11}(k)\overline{s_{21}(k)} + s_{12}(k)\overline{s_{22}(k)} = 0.$$

(III) *Analyticity*: *$s_{11}(k)$ is continued analytically into the upper half-plane $\overline{\boldsymbol{C}_+}$.*

(IV) *Asymptotics*:

$$s_{11}(k) = C + O\Big(\frac{1}{k}\Big) \text{ as } |k| \to \infty \text{ in } \overline{\boldsymbol{C}_+},$$

where C is a complex number with absolute value $|C| = 1$; and

$$s_{21}(k) = O\Big(\frac{1}{k}\Big),\ s_{12}(k) = O\Big(\frac{1}{k}\Big) \text{ as } |k| \to \infty \text{ in } \boldsymbol{R}.$$

(V) *Rate at $k = 0$*: *$s_{11}(k) \neq 0$ in $\overline{\boldsymbol{C}_+} \setminus \{0\}$,*

$$\exists \lim_{k \to 0,\, k \in \boldsymbol{R}} \frac{1 + s_{21}(k)}{s_{11}(k)} \in \boldsymbol{R}, \quad \exists \lim_{k \to 0,\, k \in \boldsymbol{R}} \frac{1 + s_{12}(k)}{s_{11}(k)} \in \boldsymbol{R};$$

and either

(i) $s_{11}(0) \neq 0$, *or*

(ii) $s_{11}(0) = 0$, $\exists \lim_{k \to 0,\, k \in \overline{\boldsymbol{C}_+}} \dfrac{s_{11}(k)}{k} \in i\boldsymbol{R} \setminus \{0\}$.

(VI) *Consistency*: *the functions*

$$F_+(t) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} s_{21}(k) e^{ikt} dk, \quad F_-(t) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} s_{12}(k) e^{-ikt} dk,$$

which are the negatives of the Fourier transforms of $s_{21}(k)$, $s_{12}(k)$ in $L^2(\boldsymbol{R})$, are absolutely continuous with the derivatives $F_+'(t) \in L_2^1(a, \infty)$, $F_-'(t) \in L_2^1(-\infty, a)$ for each $a \in \boldsymbol{R}$.

(b) *Conversely, if* $S(k)$ *is a continuous matrix on* $\boldsymbol{R}$ *with properties* (I)–(VI)*, then one* (*and only one*) *of the matrices*

$$\begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \quad \begin{pmatrix} -s_{11}(k) & s_{12}(k) \\ s_{21}(k) & -s_{22}(k) \end{pmatrix} \tag{1.4}$$

belongs to $\mathcal{S}(\Pi)$*.*

Theorem 1.1 means that $\mathcal{S}(\Pi)$ is the set of all matrices $S(k)$, continuous in k, satisfying conditions (I)-(VI) under the identification of two matrices in (1.4). In other words, if we let Σ denote the quotient set of the set of all continuous matrices with properties (I)-(VI) divided by the identification as an equivalence relation, then we have $\mathcal{S}(\Pi) = \Sigma$.

The answer to problem (2) will be given in Section 3. We employ the following integral equation of Marchenko type:

$$\overline{\Delta_+(x,t)} + \int_x^\infty \Delta_+(x,r)F_+(r+t)dr + \int_x^\infty F_+(r+t)dr = 0, \quad x \le t, \tag{1.5}$$

where $F_+(t)$ is the function in $L^2(\boldsymbol{R})$ defined from $s_{21}(k)$ in property (VI). With the aid of the unitarity of $S(k)$, one can show that this integral equation admits a unique solution $\Delta_+(x,\cdot)$ in the space $BC[x,\infty)$ of bounded, continuous functions on $[x,\infty)$. The solution $\Delta_+(x,t)$ has the partial derivative $\partial_t\Delta_+(x,t)$ with respect to t. The essential part in the proof of the injectivity of $\mathcal{S} : \Pi \to \mathcal{S}(\Pi)$ consists in finding the identity

$$\frac{2\,\partial_t\Delta_+(x,x)}{1+\Delta_+(x,x)} = \int_x^\infty [U(r) + Q(r)^2]dr - iQ(x), \quad x \in \boldsymbol{R}, \tag{1.6}$$

by taking the imaginary part, so that $Q(x)$ is uniquely recovered, and, in turn, by taking the derivative, so that $U(x)$ is also uniquely recovered. Rewriting this recovery algorithm we give the answer to the problem (3):

Theorem 1.2. *The scattering transform* $\mathcal{S} : \Pi \to \Sigma$ *is bijective and the inverse transform* $\mathcal{S}^{-1} : \Sigma \to \Pi$ *is given by*

$$Q(x) = -\frac{d}{dx}\arg(1+\Delta_+(x,x)), \tag{1.7}$$

$$U(x) = \Big(\frac{d}{dx}(\mathrm{Re}\,\log(1+\Delta_+(x,x)))\Big)^2 - \frac{d^2}{dx^2}(\mathrm{Re}\,\log(1+\Delta_+(x,x))), \tag{1.8}$$

or/and by

$$Q(x) = \frac{d}{dx}\arg(1+\Delta_-(x,x)), \tag{1.9}$$

$$U(x) = \Big(\frac{d}{dx}(\text{Re}\,\log(1+\Delta_-(x,x)))\Big)^2 - \frac{d^2}{dx^2}(\text{Re}\,\log(1+\Delta_-(x,x))), \tag{1.10}$$

where $\Delta_-(x,t)$ is the solution of the integral equation

$$\overline{\Delta_-(x,t)} + \int_{-\infty}^{x} \Delta_-(x,r)F_-(r+t)dr + \int_{-\infty}^{x} F_-(r+t)dr = 0,\ t \le x, \tag{1.11}$$

with the function $F_-(t) \in L^2(\boldsymbol{R})$ defined from $s_{12}(k)$ in the property (VI).

The two pairs $(U(x), Q(x))$ defined by (1.7), (1.8) and by (1.9), (1.10) are equal. This fact, to show which is a crucial part of the proof to assertion (b) in Theorem 1.1, will be proved in Section 4, based on a simple lemma about the Hardy space established in Deift-Trubowitz [4, Lemma 5.1] and a generalization of their theory.

Theorem 1.1 shows that, for prescribed continuous functions $s_{ij}(k)$, $i,j = 1,2$ with properties (I)–(VI), there exists a unique pair $(U(x), Q(x)) \in \Pi$ for which one and only one of the matrices in (1.4) is the scattering matrix. Which is the scattering matrix is determined by the connection

$$C = e^{-i\int_{-\infty}^{\infty} Q(\eta)d\eta} \tag{1.12}$$

between the constant C in property (IV) and $Q(x)$. Theorem 1.2 shows that $S(k)$ in Σ is uniquely determined from $s_{21}(k)$ or $s_{12}(k)$ (except for the sign of $s_{11}(k)$), since $(U(x), Q(x)) = \mathcal{S}^{-1}(S(k))$ is uniquely determined only by $s_{21}(k)$ or $s_{12}(k)$ through (1.7), (1.8) or (1.9), (1.10).

If $Q(x) \equiv 0$, then the scattering matrix $S(k)$ satisfies $S(k) = \overline{S(-k)}$ and vice versa. In fact, in the case where $Q(x) \equiv 0$, the Jost solution satisfies $f_\pm(x,k) = \overline{f_\pm(x,-k)}$ and so, by (1.2) and (1.3), $S(k) = \overline{S(-k)}$; moreover $S(k) = \overline{S(-k)}$ implies that $F_\pm(t)$ are real, and hence $\Delta_\pm(x,t)$ are also so, which combined with (1.6) leads to $Q(x) \equiv 0$. It follows from this observation that Theorem 1.1 contains the following result by Deift-Trubowitz [4] for the standard Schrödinger equation:

$$f'' + [k^2 - U(x)]f = 0, \quad -\infty < x < \infty, \tag{1.13}$$

without bound states.

Corollary 1.3 (Theorem 5.3 in Deift-Trubowitz [4]). *A matrix*

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}$$

is the scattering matrix for $U(x) \in L^1_2(\boldsymbol{R})$ *in* (1.13) *without bound states if and only if* $S(k)$ *satisfies the conditions* (I)–(VI) *with* $C = 1$ *in* (IV) *and in addition* $S(k) = \overline{S(-k)}$.

We mention that, in view of the additional condition $S(k) = \overline{S(-k)}$, the constant C is necessarily real and so $C = \pm 1$; the ambiguity in (1.4) is removed by determining C as $C = 1$ in view of (1.12). Also notice that the limits in (V) are real and the limit in (ii) of (V) is purely imaginary necessarily under the additional condition $S(k) = \overline{S(-k)}$. Let Σ_0 be a subset of Σ defined by $\Sigma_0 = \{S(k) \in \Sigma : S(k) = \overline{S(-k)}\}$. Also let Π_0 be the set of $U(x) \in L^1_2(\boldsymbol{R})$ with which the standard Schrödinger equation (1.13) has no bound states. The set Π_0 can be considered as a subset of Π. Then what we have done is summarized by the following diagram:

$$
\begin{array}{ccc}
\Pi & \xrightarrow{\ \mathcal{S}\ } & \Sigma \\
\cup & & \cup \\
\Pi_0 & \xrightarrow{\ \mathcal{S}\ } & \Sigma_0
\end{array}
$$

It should be noted (see Theorem 6.1) that the set Π is actually the direct product of Π_0 and the set of functions $Q(x)$ satisfying (A2).

We here present an example. Let a be a complex number in the angular domain $-\frac{\pi}{4} < \arg a < \frac{\pi}{4}$ and let $S(k) \in \Sigma$ be the matrix (see (5.1) in Section 5) whose $(2,1)-$component is given by $s_{21}(k) = -\frac{a^2}{k^2+a^2}$. The functions $F_+(t)$ and $F_-(t)$ in the property (VI) are computed by an elementary calculation, and the solutions $\Delta_\pm(x,t)$ are obtained by solving the integral equations (1.5) and (1.11). The inverse scattering transform of the matrix $S(k)$ is given by a pair of rather complicated, but elementary functions $U(x)$, $Q(x)$. It turns out (see Section 5) that $C = 1$ for this $Q(x)$, and that $S(k) \in \Pi_0$ if and only if $a > 0$.

The recovery algorithm of the potential $U(x) + 2kQ(x)$ in this paper is different from that in Jaulent-Jean [9], Sattinger-Szmigielski [16]. We employ only the solutions $\Delta_\pm(x,t)$ of the single integral equations (1.5), (1.11). This enables us to remove the differentiability assumption on $U(x)$ used in the papers mentioned above. Our key function $\Delta_\pm(x,t)$ is related with the so-called transformation kernel $A_\pm(x,k)$ (see (2.5) in Section 2) given by

$$A_\pm(x,t) = \frac{\partial_t \Delta_\pm(x,t)}{|1 + \Delta_\pm(x,x)|}. \tag{1.14}$$

In the case where $Q(x) \equiv 0$ the inversion formula (1.8) is written as $U(x) = -2\frac{d}{dx}A_\pm(x,x)$ with the aid of (1.6), since $1 + \Delta_\pm(x,x)$ is real.

Inverse scattering theory for the energy dependent Schrödinger equation has been developed by many authors. A recovery algorithm for the radial equation (energy dependent Schrödinger equation on the half-line) was established by Jaulent-Jean [8], Jaulent [6]. A complete solution to the inverse scattering problem for the radial equation was given by the author [12]. Related to inverse problems arising in wave propagation in one-dimensional non-conservative media, inverse scattering theory for the energy dependent Schrödinger equation (1.1) with real $U(x)$ and purely imaginary $Q(x)$ was developed by Jaulent [7], Aktosun-Klaus-van der Mee [1, 2]. In [2], using a pair of uncoupled Marchenko integral equations, real $U(x)$ and purely imaginary $Q(x)$ are recovered from an appropriate set of scattering data including information on bound states. The inverse scattering problem for a more general case where k^2 in (1.1) is replaced by $k^2 + m^2$ with a positive mass parameter m was investigated in connection with a nonlinear evolution equation (a long-wave water equation) by Kaup [13], Sattinger-Szmigielski [17], van der Mee-Pivovarchik [18].

2. The Forward Problem

In this section we shall prove the assertion (a) in Theorem 1.1. Although it is essentially due to Jaulent-Jean [9], we include a proof, because some methods for the proof are made simpler than in the paper.

Property (I), i.e., $s_{11}(k) = s_{22}(k)$, is clear from (1.2) and (1.3). By computing the Wronskian of $s_{11}(k)f_+(x,k)$ $(= \overline{f_-(x,k)} + s_{12}(k)f_-(x,k))$ and its complex conjugate, it follows that $|s_{11}(k)|^2 + |s_{12}(k)|^2 = 1$. Similarly we have $|s_{21}(k)|^2 + |s_{22}(k)|^2 = 1$. Moreover, a direct calculation together with (1.2), (1.3) shows that

$$s_{11}(k)\overline{s_{12}(k)} + s_{21}(k)\overline{s_{22}(k)} = 0. \tag{2.1}$$

We thus obtain property (II).

Provided that $U(x) \in L^1_1(\boldsymbol{R})$ and $Q(x) \in L^1(\boldsymbol{R})$, the Jost solutions $f_\pm(x,k)$ are obtained for $k \in \overline{\boldsymbol{C}_+}$ by applying the method of successive approximations to the integral equations

$$f_\pm(x,k) = e^{\pm ikx} - \int_x^{\pm\infty} \frac{\sin k(x-t)}{k}\,[U(t) + 2k\,Q(t)]f_\pm(t,k)dt,$$

and are holomorphic in the upper half-plane $\boldsymbol{C}_+$ and continuous in its closure $\overline{\boldsymbol{C}_+}$. It follows from these integral equations that, for real $k \neq 0$,

$$f_-(x,k) = e^{-ikx} - \frac{e^{-ikx}}{2ik}\int_{-\infty}^{\infty}[U(t)+2k\,Q(t)]f_-(t,k)e^{ikt}dt$$
$$+\frac{e^{ikx}}{2ik}\int_{-\infty}^{\infty}[U(t)+2k\,Q(t)]f_-(t,k)e^{-ikt}dt + o(1), \quad x\to\infty.$$

Comparing this with $f_-(x,k) = s_{11}(k)^{-1}e^{-ikx} + (s_{21}(k)/s_{11}(k))e^{ikx} + o(1)$, $x \to \infty$, we have

$$\frac{1}{s_{11}(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2k\,Q(x)]f_-(x,k)e^{ikx}dx \tag{2.2}$$

$$\frac{s_{21}(k)}{s_{11}(k)} = \frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2k\,Q(x)]f_-(x,k)e^{-ikx}dx, \tag{2.3}$$

for real $k \neq 0$. This leads to

$$\frac{1+s_{21}(k)}{s_{11}(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2k\,Q(x)]f_-(x,k)e^{ikx}dx$$
$$+\frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2k\,Q(x)]f_-(x,k)e^{-ikx}dx$$
$$= 1 - \int_{-\infty}^{\infty}x\,U(x)f_-(x,k)\frac{\sin\,kx}{kx}dx + i\int_{-\infty}^{\infty}(e^{ikx}-e^{-ikx})\,Q(x)f_-(x,k)dx.$$

Hence, the following limit exists:

$$\lim_{k\to 0}\frac{1+s_{21}(k)}{s_{11}(k)} = 1 - \int_{-\infty}^{\infty}x\,U(x)f_-(x,0)dx, \tag{2.4}$$

which is a real number, because $f_-(x,0)$ is real valued.

We now require the transformation representation of the Jost solution $f_\pm(x,k)$:

$$f_\pm(x,k) = e^{\pm i\int_x^{\pm\infty}Q(\eta)d\eta}e^{\pm ikx} + \int_x^{\pm\infty}A_\pm(x,t)e^{\pm ikt}dt, \quad k\in\overline{\boldsymbol{C}_+}, \tag{2.5}$$

where $A_+(x,\cdot) \in L^1(x,\infty)$ and $A_-(x,\cdot) \in L^1(-\infty,x)$ for each $x \in \boldsymbol{R}$. This representation is established provided that $U(x) \in L^1_1(\boldsymbol{R})$ and $Q(x) \in L^1(\boldsymbol{R})$ (see [11, Lemma 2.1]) by applying the method of successive approximations to the integral equation

$$A_\pm(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\pm\infty}U(s)e^{\pm i\int_s^{\pm\infty}Q(\eta)d\eta}ds \mp \frac{i}{2}Q\Big(\frac{x+t}{2}\Big)e^{\pm i\int_{\frac{x+t}{2}}^{\pm\infty}Q(\eta)d\eta} \tag{2.6}$$

$$+\frac{1}{2}\int_{\frac{x+t}{2}}^{\pm\infty} U(s)ds\int_s^{t+s-x} A_\pm(s,u)du+\frac{1}{2}\int_x^{\frac{x+t}{2}} U(s)ds\int_{t+x-s}^{t+s-x} A_\pm(s,u)du$$
$$\pm i\int_x^{\pm\infty} Q(s)A_\pm(s,t+s-x)ds\mp i\int_x^{\frac{x+t}{2}} Q(s)A_\pm(s,t+x-s)ds,$$

$\pm x<\pm t$. Setting $t=x$ in (2.6) we obtain

$$\begin{aligned}&A_+(x,x)-i\int_x^\infty Q(s)A_+(s,s)ds\\&=\frac{1}{2}\int_x^\infty U(s)e^{i\int_s^\infty Q(\eta)d\eta}ds-\frac{i}{2}Q(x)\,e^{i\int_x^\infty Q(\eta)d\eta}.\end{aligned}$$

This gives an integral equation for $A_+(x,x)$, which is solved (see [11, proof of Lemma 3.1]) as

$$A_+(x,x)=\frac{1}{2}e^{i\int_x^\infty Q(\eta)d\eta}\int_x^\infty[U(r)+Q(r)^2]dr-\frac{i}{2}e^{i\int_x^\infty Q(\eta)d\eta}Q(x). \quad (2.7)$$

Under some additional assumptions on $Q(x)$ and $U(x)$, we have more information on $A_\pm(x,t)$. We provide some of them without the proof.

Lemma 2.1. (1) *If $U(x)\in L_2^1(\boldsymbol{R})$ and $Q(x)\in L_1^1(\boldsymbol{R})$, then $A_+(x,\cdot)\in L_1^1(x,\infty)$, $A_-(x,\cdot)\in L^1(-\infty,x)$. The norms of $A_\pm(x,\cdot)$ are bounded uniformly in x; namely, there exists a constant M independent of x such that*

$$\int_x^\infty |(1+|t|)A_+(x,t)|dt\le M,\quad \int_{-\infty}^x |(1+|t|)A_-(x,t)|dt\le M.$$

(2) *If $U(x)\in L_1^1(\boldsymbol{R})$, $Q(x)\in L^1(\boldsymbol{R})$, and $Q'(x)\in L^1(\boldsymbol{R})$, then $A_\pm(x,t)$ are absolutely continuous with respect to t and, for each $x\in\boldsymbol{R}$, the derivatives $\partial_t A_+(x,\cdot)$ and $\partial_t A_-(x,\cdot)$ belong to $L^1(x,\infty)$ and to $L^1(-\infty,x)$, respectively. Their norms are bounded uniformly in x: there exists a constant M independent of x such that*

$$\int_x^\infty |\partial_t A_+(x,t)|dt\le M,\quad \int_{-\infty}^x |\partial_t A_-(x,t)|dt\le M.$$

(3) *If $U(x)\in L_2^1(\boldsymbol{R})$, $Q(x)\in L_1^1(\boldsymbol{R})$, and $Q'(x)\in L_1^1(\boldsymbol{R})$, then $A_\pm(x,t)$ are absolutely continuous with respect to t and, for each $x\in\boldsymbol{R}$, the derivatives $\partial_t A_+(x,\cdot)$ and $\partial_t A_-(x,\cdot)$ belong to $L_1^1(x,\infty)$ and to $L_1^1(-\infty,x)$, respectively. Their norms are bounded uniformly in x: there exists a constant M independent of x such that*

$$\int_x^\infty |(1+|t|)\partial_t A_+(x,t)|dt\le M,\quad \int_{-\infty}^x |(1+|t|)\partial_t A_-(x,t)|dt\le M.$$

(4) *If $U(x) \in L^1_2(\boldsymbol{R})$, $Q(x) \in L^1_1(\boldsymbol{R})$, and $Q'(x) \in L^1_1(\boldsymbol{R})$, then $A_\pm(x,t)$ are absolutely continuous with respect to x and, for each $x \in \boldsymbol{R}$, the derivatives $\partial_x A_+(x,\cdot)$ and $\partial_x A_-(x,\cdot)$ belong to $L^1_1(x,\infty)$ and to $L^1_1(-\infty,x)$, respectively. Their norms go to zero as $x \to \infty$:*

$$\lim_{x\to\infty}\int_x^\infty |(1+|t|)\partial_x A_+(x,t)|dt = 0, \quad \lim_{x\to-\infty}\int_{-\infty}^x |(1+|t|)\partial_x A_+(x,t)|dt = 0.$$

By Lemma 2.1 together with a familiar method (see Marchenko [15, page 303]), we can prove the continuity of the scattering matrix at $k=0$.

Lemma 2.2. *Let $U(x)$ and $Q(x)$ be real functions such that $U(x) \in L^1_2(\boldsymbol{R})$, $Q(x) \in L^1_1(\boldsymbol{R})$, and $Q'(x) \in L^1_1(\boldsymbol{R})$. Then $s_{ij}(k)$, $i,j=1,2$, are continuous at $k=0$. Moreover, $s_{11}(0)=0$ if and only if $W[f_+(x,0), f_-(x,0)] \neq 0$. In this case $s_{11}(k)/k$ tends to a non-zero, purely imaginary number as $k \to 0$ in $\overline{\boldsymbol{C}_+}$.*

Proof. It is clear from (1.2) that if $W[f_+(x,0), f_-(x,0)] \neq 0$, then $s_{11}(k)$ is continuous at $k=0$ with the value $s_{11}(0)=0$. In this case, by (1.2) and the realness of $f_\pm(x,0)$, $s_{11}(k)/k$ tends to a purely imaginary number as $k \to 0$ in $\overline{\boldsymbol{C}_+}$.

We next consider the case where $W[f_+(x,0), f_-(x,0)] = 0$. Taking the derivative of the transformation representation (2.5) we get

$$x{f_+}'(x,0) = -ie^{i\int_x^\infty Q(\eta)d\eta} xQ(x) - xA_+(x,x) + x\int_x^\infty \partial_x A_+(x,t)dt.$$

By virtue of Lemma 2.1.(4), the last term goes to zero as $x \to \infty$, because, for $x > 0$,

$$\left| x\int_x^\infty \partial_x A_+(x,t)dt\right| \le \int_x^\infty |t\,\partial_x A_+(x,t)|dt.$$

Also, since $Q(x)$ and $Q'(x) \in L^1_1(\boldsymbol{R})$, $xQ(x)$ goes to zero as $x \to \infty$, and, in turn, by (2.7), $xA_+(x,x)$ goes to zero then. This shows that

$$ {f_+}'(x,0) = o(x^{-1}), \quad x \to \infty. \tag{2.8}$$

By the transformation representation and Lemma 2.1.(1), the derivative $\dot{f}_+(x,k)$ of $f_+(x,k)$ with respect to k is written as

$$\dot{f}_+(x,k) = ixe^{i\int_x^\infty Q(\eta)d\eta}e^{ikx} + i\int_x^\infty tA_+(x,t)e^{ikt}dt, \quad \operatorname{Im} k \ge 0.$$

This yields

$$\dot{f}_+(x,0) = ix[1+o(1)], \quad x \to \infty. \tag{2.9}$$

Moreover we have

$$\dot{f_+}'(x,0) = ie^{i\int_x^\infty Q(\eta)d\eta} - ixQ(x)e^{i\int_x^\infty Q(\eta)d\eta} - ixA_+(x,x) + i\int_x^\infty tA_+(x,t)dt,$$

which leads to

$$\dot{f_+}'(x,0) = i[1+o(1)], \quad x\to\infty. \tag{2.10}$$

By the assumption $W[f_+(x,0), f_-(x,0)] = 0$, there exists a constant $c \neq 0$ such that $f_+(x,0) = cf_-(x,0)$. Since $f_\pm(x,0)$ are real functions, the constant c is a real number. It follows from (2.8), (2.9), and (2.10) and a similar estimate for $f_-(x,0)$ that $W[\dot{f}_\pm(x,0), f_\pm(x,0)] = -i$. Hence, we compute

$$\frac{\partial}{\partial k}W[f_+(x,k), f_-(x,k)]\Big|_{k=0} = W[\dot{f}_+(x,0), f_-(x,0)] + W[f_+(x,0), \dot{f}_-(x,0)]$$
$$= c^{-1}W[\dot{f}_+(x,0), f_+(x,0)] + cW[f_-(x,0), \dot{f}_-(x,0)] = -i(c+c^{-1}).$$

Hence, by (1.2), we arrive at

$$s_{11}(0) = -\frac{2i}{\frac{\partial}{\partial k}W[f_+(x,k), f_-(x,k)]\big|_{k=0}} = \frac{2c}{1+c^2}.$$

In this way we have proved that $s_{11}(k)$ is still continuous at $k=0$ in the case where $W[f_+(x,0), f_-(x,0)] = 0$ and that $s_{11}(0) = 0$ if and only if $W[f_+(x,0), f_-(x,0)] \neq 0$. The continuity of the function $s_{21}(k)$ follows from (2.4) as

$$\lim_{k\to0}(1+s_{21}(k)) = \lim_{k\to0}\frac{1+s_{21}(k)}{s_{11}(k)}s_{11}(k) = \Big(1 - \int_{-\infty}^\infty xU(x)f_-(x,0)dx\Big)s_{11}(0).$$

□

We next establish the following asymptotic behavior of $s_{ij}(k)$ as $k\to\infty$:

Lemma 2.3. *Let $U(x)$ and $Q(x)$ be real functions such that $U(x) \in L_1^1(\boldsymbol{R})$, $Q(x) \in L^1(\boldsymbol{R})$, and $Q'(x) \in L^1(\boldsymbol{R})$. Then*
(1) $s_{11}(k) = C + O(1/k)$ as $|k|\to\infty$ in $\overline{\boldsymbol{C}_+}$, where $C = e^{-i\int_{-\infty}^\infty Q(\eta)d\eta}$; and
(2) $s_{21}(k) = O(1/k)$, $s_{12}(k) = O(1/k)$ as $|k|\to\infty$ in $\boldsymbol{R}$.

Proof. By substituting (2.5) in (2.2) and observing that

$$1 + i\int_{-\infty}^\infty Q(x)e^{i\int_{-\infty}^x Q(\eta)d\eta}dx = e^{i\int_{-\infty}^\infty Q(\eta)d\eta},$$

we have

$$\frac{1}{s_{11}(k)} = e^{i\int_{-\infty}^\infty Q(\eta)d\eta} + \frac{\beta}{2ik} + \frac{1}{2ik}\int_{-\infty}^0 G_U(t)e^{-ikt}dt + \frac{1}{i}\int_{-\infty}^0 G_Q(t)e^{-ikt}dt, \tag{2.11}$$

where

$$\beta := -\int_{-\infty}^{\infty} U(x) e^{i\int_{-\infty}^{x} Q(\eta)d\eta} dx,$$

$$G_V(t) := \int_{-\infty}^{\infty} V(x) A_-(x, t+x) dx, \quad V = U, Q.$$

Since

$$\int_{-\infty}^{0} |G_Q'(t)| dt \le \text{const.} \int_{-\infty}^{x} |\partial_t A_-(x,t)| dt < \infty$$

by Lemma 2.1.(2), it turns out by an integration by parts that

$$\int_{-\infty}^{0} G_Q(t) e^{-ikt} dt = O(1/k)$$

as $|k| \to \infty$ in $\overline{\boldsymbol{C}_+}$. Therefore, by (2.11), $1/s_{11}(k) = C^{-1} + O(1/k)$ as $|k| \to \infty$ in $\overline{\boldsymbol{C}_+}$. This proves assertion (1).

Similarly, by substituting (2.5) in (2.3), we have

$$\frac{s_{21}(k)}{s_{11}(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} H_U(t) e^{-ikt} dt + \frac{1}{i} \int_{-\infty}^{\infty} H_Q(t) e^{-ikt} dt, \tag{2.12}$$

where

$$H_V(t) := \frac{1}{2} V\left(\tfrac{t}{2}\right) e^{i\int_{-\infty}^{\frac{t}{2}} Q(\eta)d\eta} - \int_{\frac{t}{2}}^{\infty} V(x) A_-(x, t-x) dx, \quad V = U, Q.$$

But, by Lemma 2.1.(2), $H_Q'(t) \in L^1(\boldsymbol{R})$ and hence $\frac{s_{21}(k)}{s_{11}(k)} = O(1/k)$ as $|k| \to \infty$ in $\boldsymbol{R}$. This, together with (1), proves that $s_{21}(k) = O(1/k)$ as $|k| \to \infty$ in $\boldsymbol{R}$. The assertion for $s_{12}(k)$ can be proved in a similar manner. □

By Lemma 2.3.(2), $s_{21}(k) \in L^2(\boldsymbol{R})$. Hence, there exists a function $F_+(t) \in L^2(\boldsymbol{R})$ such that

$$s_{21}(k) = -\int_{-\infty}^{\infty} F_+(t) e^{-ikt} dt. \tag{2.13}$$

Lemma 2.4. *Suppose that* $U(x)$ *and* $Q(x)$ *are real functions satisfying conditions* (A1)–(A3) *described at the beginning of the paper. Then* $F_+(t)$ *in* (2.13) *is related with the transformation kernel* $A_+(x,t)$ *via the following equation:*

$$\overline{A_+(x,t)} - \int_x^{\infty} A_+(x,r) F_+(r+t) dr - e^{i\int_x^{\infty} Q(\eta)d\eta} F_+(x+t) = 0, \quad x < t. \tag{2.14}$$

Proof. By (1.2) and (A3) $s_{11}(k)$ is analytic in $\boldsymbol{C}_+$ and, by Lemma 2.2, it is continuous in $\overline{\boldsymbol{C}_+}$. Hence, by Lemma 2.3.(1), the function $s_{11}(k)-C$ belongs to the Hardy space H^{2+}, which is defined to be the set of analytic functions $v(k)$ in $\boldsymbol{C}_+$ with

$$\sup_{\eta>0}\int_{-\infty}^{\infty}|v(\xi+i\eta)|^2 d\xi<\infty.$$

Since, in general, $v\in H^{2+}$ if and only if $v(k)$ is expressed as

$$v(k)=\int_{-\infty}^{0}\hat{v}(t)e^{-ikt}dt$$

in terms of some function $\hat{v}(t)\in L^2(-\infty,0)$ (see, e.g.,[20, Chapter VI]), there exists a function $T_+(t)\in L^2(-\infty,0)$ such that

$$s_{11}(k)=C+\int_{-\infty}^{0}T_+(t)e^{-ikt}dt. \tag{2.15}$$

In terms of the Jost solutions, the scattering solution $\psi_-(x,k)$ is written as

$$\psi_-(x,k)=s_{11}(k)f_-(x,k)=\overline{f_+(x,k)}+s_{21}(k)f_+(x,k).$$

This identity yields

$$\begin{aligned}&s_{11}(k)f_-(x,k)-e^{-i\int_x^\infty Q(\eta)d\eta}e^{-ikx} \\ &=\overline{f_+(x,k)}+s_{21}(k)f_+(x,k)-e^{-i\int_x^\infty Q(\eta)d\eta}e^{-ikx}.\end{aligned} \tag{2.16}$$

It follows from (2.5) and (2.13) that, for each $x\in\boldsymbol{R}$, the right-hand side of (2.16) is the Fourier transform of the function in the left-hand side of (2.14). On the other hand, it follows from (2.5), (2.15), and Lemma 2.3.(1) that, for each $x\in\boldsymbol{R}$, the left-hand side of (2.16) is the Fourier transform of a function in $L^2(-\infty,x)$. Hence the function in the left-hand side of (2.14) must be zero for $x<t$. The proof is complete. □

Equation (2.14) is the Marchenko equation associated with the energy dependent inverse scattering, deduced firstly by Jaulent-Jean [8]. By means of (2.14) we can draw the following conclusion for $F_\pm(t)$ in (VI).

Lemma 2.5. *Under the same assumption as in Lemma* 2.4*,* $F_+'(t)\in L^1_2(a,\infty)$ *and* $F_-'(t)\in L^1_2(-\infty,a)$ *for each* $a\in\boldsymbol{R}$*.*

Proof. We shall prove only the assertions for $F_+(t)$, since the assertions for $F_-(t)$ are proved in a similar way. We fix $a\in\boldsymbol{R}$ and, for brevity, we abbreviate $F_+(t)$ by $F(t)$ and $A_+(x,t)$ by $A(x,t)$. We observe that if $f(t) \in L_1^1(a,\infty)$, then $t\int_t^\infty f(\eta)d\eta \in BC[a,\infty)$ for each $a \in \boldsymbol{R}$. In fact, for $t \geq 0$,

$$|t\int_t^\infty f(\eta)d\eta| \leq \int_t^\infty |\eta f(\eta)|d\eta < \infty.$$

Moreover, notice that the assumption (A2) implies that $(1+|x|)Q(x) \in BC(\boldsymbol{R})$ since $xQ(x) = -\int_x^\infty (tQ(t))'dt$.

We first prove that $F(t) \in L_1^1(a,\infty)$. Since $Q(x)$ is absolutely continuous, it follows from (2.6) that $A(x,t)$ is continuous at $t = x$. Therefore, by (2.14), we have

$$\overline{A(t,t)} - \int_t^\infty A(t,r)F(r+t)dr - e^{i\int_t^\infty Q(\eta)d\eta}F(2t) = 0.$$

By the transformation $r = 2s - t$, this is recast as the Volterra integral equation for $F(2t)$:

$$F(2t) + 2e^{-i\int_t^\infty Q(\eta)d\eta}\int_t^\infty A(t,2s-t)F(2s)ds = e^{-i\int_t^\infty Q(\eta)d\eta}\overline{A(t,t)}. \quad (2.17)$$

This leads to

$$|F(2t)| \leq |A(t,t)| + 2\int_x^\infty |A(t,2s-t)||F(2s)|ds.$$

We now borrow the following estimate for $A(x,t)$:

$$|A(x,t)| \leq \text{const.}\, \sigma\Big(\frac{x+t}{2}\Big), \quad t \geq x \geq a, \quad (2.18)$$

where σ is a function defined by

$$\sigma(t) = \int_t^\infty (|U(\eta)| + |Q'(\eta)|)d\eta. \quad (2.19)$$

This estimate can be found in [9, Equation (I.4.2)], which holds under the assumption $U(x) \in L_1^1(\boldsymbol{R})$, $Q(x) \in L_1^1(\boldsymbol{R})$, and $Q'(x) \in L_1^1(\boldsymbol{R})$. Note that $\sigma(t) \in L^1(a,\infty)$ for each fixed $a \in \boldsymbol{R}$. Moreover, $(1+|t|)\sigma(t) \in BC[a,\infty)$ since, for $t \geq 0$,

$$t\sigma(t) \leq \int_t^\infty \eta(|U(\eta)| + |Q'(\eta)|)d\eta < \infty.$$

From (2.18) we have

$$|F(2t)| \leq |A(t,t)| + M\int_t^\infty \sigma(r)|F(2r)|dr.$$

Applying the Gronwall inequality to this shows that

$$|F(2t)| \le |A(t,t)| + M\int_t^\infty \sigma(r)e^{M\int_t^r \sigma(\eta)d\eta}|A(r,r)|dr,$$

which yields

$$|F(2t)| \le |A(t,t)| + M_1\int_t^\infty \sigma(r)|A(r,r)|dr, \quad t \ge a, \tag{2.20}$$

where we set $M_1 := Me^{M\int_a^\infty \sigma(\eta)d\eta}$. Hence, it follows that

$$\begin{aligned}
&\int_a^\infty (1+|t|)|F(2t)|dt \\
&\le \int_a^\infty (1+|t|)|A(t,t)|dt + M_1\int_a^\infty (1+|t|)dt\int_t^\infty \sigma(r)|A(r,r)|dr \\
&= \int_a^\infty (1+|t|)|A(t,t)|dt + M_1\int_a^\infty \sigma(r)|A(r,r)|dr\int_a^r (1+|t|)dt \\
&\le \exists M_2\int_a^\infty (1+|t|)|A(t,t)|dt.
\end{aligned}$$

But, by (2.7), we obtain

$$\begin{aligned}
&\int_a^\infty |(1+|t|)|A(t,t)|dt \\
&\le \frac{1}{2}\int_a^\infty [|U(r)| + Q(r)^2]dr\int_a^r (1+|t|)dt + \frac{1}{2}\int_a^\infty (1+|t|)|Q(t)|dt
\end{aligned}$$

and hence, by the assumption $U(x) \in L_2^1(\boldsymbol{R})$, $Q(x) \in L_1^1(\boldsymbol{R})$, and $(1+|x|)Q(x) \in BC[a,\infty)$, it follows that $A(t,t) \in L_1^1(a,\infty)$. This shows that $F(t) \in L_1^1(a,\infty)$.

By (2.18) and the fact that $(1+|t|)\sigma(t) \in BC[a,\infty)$, $(1+|t|)A(t,t) \in BC[a,\infty)$. This property is passed onto $F(2t)$ through (2.20): we have $(1+|t|)F(t) \in BC[a,\infty)$.

We shall next prove $F'(t) \in L_2^1(a,\infty)$. Differentiating (2.17), we obtain

$$\begin{aligned}
2F'(2t)e^{i\int_t^\infty Q(\eta)d\eta} &= iQ(t)F(2t)e^{i\int_t^\infty Q(\eta)d\eta} + \overline{\frac{d}{dt}A(t,t)} \\
&\quad + 2A(t,t)F(2t) - 2\int_t^\infty \Big(\frac{\partial}{\partial t}A(t,2r-t)\Big)F(2r)dr.
\end{aligned}$$

Since $(1+|t|)Q(t) \in BC(\boldsymbol{R})$ and $F(t) \in L_1^1(a,\infty)$, we find that $Q(t)F(t) \in L_2^1(a,\infty)$. Moreover it is readily verified by (2.7) and the assumptions $U(x) \in L_2^1(\boldsymbol{R})$ and $Q'(x) \in L_2^1(\boldsymbol{R})$ that $\frac{d}{dt}A(t,t) \in L_2^1(a,\infty)$. At this stage we require the assumption $Q'(x) \in L_2^1(\boldsymbol{R})$. By the fact that $\frac{d}{dt}A(t,t) \in L_2^1(a,\infty)$, we get $A(t,t) \in L^1_1(a,\infty)$, which, combined with the fact that $(1+|t|)F(t) \in BC[a,\infty)$, shows that $A(t,t)F(2t) \in L^1_2(a,\infty)$. By letting $x=t$, $t=2r-t$ in (2.6) and differentiating with respect to t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}A(t,2r-t) = &-\int_t^\infty U(s)A(s,s+2(r-t))ds \\ &- 2i\int_t^\infty Q(s)\partial_t A(s,s+2(r-t))ds,\end{aligned}$$

which leads to

$$\begin{aligned}\int_t^\infty\Big(\frac{\partial}{\partial t}A(t,2r-t)\Big)F(2r)dr = &-\int_t^\infty U(s)ds\int_t^\infty A(s,s+2(r-t))F(2r)dr \\ &- 2i\int_t^\infty Q(s)ds\int_t^\infty \partial_t A(s,s+2(r-t))F(2r)dr.\end{aligned}$$

Since $(1+|t|)F(2t) \in BC[a,\infty)$ and $Q(x) \in L^1_1(\boldsymbol{R})$, from Lemma 2.1.(3), we have

$$\begin{aligned}&\int_a^\infty (1+t^2)dt\int_t^\infty |Q(s)|ds\int_t^\infty |\partial_t A(s,s+2(r-t))||F(2r)|dr \\ &\leq \text{const.}\int_a^\infty (1+|t|)dt\int_t^\infty |Q(s)|ds\int_t^\infty |\partial_t A(s,s+2(r-t))|(1+|r|)|F(2r)|dr \\ &\leq \text{const.}\int_a^\infty (1+|t|)dt\int_t^\infty |Q(s)|ds\int_s^\infty |\partial_t A(s,\eta)|d\eta \\ &\leq \text{const.}\int_a^\infty dt\int_t^\infty |Q(s)|ds\int_s^\infty |(1+|\eta|)\partial_t A(s,\eta)|d\eta \\ &\leq \text{const.}\int_a^\infty dt\int_t^\infty |Q(s)|ds < \infty.\end{aligned}$$

This shows that

$$\int_t^\infty Q(s)ds\int_t^\infty \partial_t A(s,s+2(r-t))F(2r)dr \in L^1_2(a,\infty).$$

Similarly, one can show that

$$\int_t^\infty U(s)ds\int_t^\infty A(s,s+2(r-t))F(2r)dr \in L^1_2(a,\infty).$$

Thus, we have proved $F'(t) \in L^1_2(a,\infty)$ for each $a \in \boldsymbol{R}$. □

3. Recovery Algorithm

In this section we establish the following.

Theorem 3.1. *If $S(k)$ is the scattering matrix for $(U(x), Q(x)) \in \Pi$, then*

(1) *$(U(x), Q(x))$ is determined from $s_{21}(k)$ by* (1.6) *where $\Delta_+(x,t)$ is the solution of* (1.5)*; and*

(2) *$(U(x), Q(x))$ is determined from $s_{12}(k)$ by*

$$\frac{2\,\partial_t \Delta_-(x,x)}{1+\Delta_-(x,x)} = -\int_{-\infty}^{x} [U(r) + Q(r)^2]dr + iQ(x),$$

where $\Delta_-(x,t)$ is the solution of (1.11).

Proof. Let $K_+(x,t)$ be a function defined by $K_+(x,t) = -\int_t^\infty A_+(x,s)ds$. Then, performing an integration by parts to (2.5), we have

$$f_+(x,k) = \Big(e^{i\int_x^\infty Q(\eta)d\eta} - K_+(x,x)\Big)e^{ikt} - ik\int_x^\infty K_+(x,t)e^{ikt}dt.$$

This yields

$$f_+(x,0) = e^{i\int_x^\infty Q(\eta)d\eta} - K_+(x,x). \tag{3.1}$$

Therefore, integrating (2.14) and performing an integration by parts, we obtain

$$\overline{K_+(x,t)} + \int_x^\infty K_+(x,r)F_+(r+t)dr + f_+(x,0)\int_x^\infty F_+(r+t)dr = 0, \quad x \le t. \tag{3.2}$$

Since $F_+'(t) \in L_2^1(x,\infty)$ implies $F_+(t) \in L^1(x,\infty)$, the function $\int_x^\infty F_+(r+t)dr$ belongs to $BC[x,\infty)$. Hence (3.2) is viewed as an equation in the space $BC[x,\infty)$. Clearly, if (3.2) has a solution $K_+(x,\cdot) \in L^\infty(x,\infty)$, then it belongs to $BC[x,\infty)$. Hence, by the Riesz-Schauder theory, the solvability of (3.2) follows from the absence of non-trivial solutions in a real linear space $L^1(x,\infty)$ of the correspondence homogeneous equation

$$\overline{\varphi(t)} + \int_x^\infty \varphi(r)F_+(r+t)dr = 0.$$

Noting $F(t) \in BC[x,\infty)$, we easily see that a solution $\varphi(t)$ is in $L^1(x,\infty) \cap L^\infty(x,\infty) \subset L^2(x,\infty)$. Hence, by a lemma in Jaulent-Jean [9, page 124], we conclude that (3.2) has a unique solution.

This observation shows that $f_+(x,0) \ne 0$. Actually, if $f_+(x,0) = 0$ for some $x \in \boldsymbol{R}$, then by the uniqueness of solutions to (3.2), $K_+(x,t) \equiv 0$, $x \le t$, and hence, by (3.1), $e^{i\int_x^\infty Q(\eta)d\eta} = 0$, which is impossible. Since the Jost solution is unique for each $k \in \overline{\boldsymbol{C}_+}$, the function $f_+(x,0)$ is a real function because of the asymptotics $f_+(x,0) \sim 1$, $x \to \infty$. Accordingly $f_+(x,0)$ is a positive function.

We now set $\Delta_+(x,t) = K_+(x,t)/f_+(x,0)$. Then, by (3.2), $\Delta_+(x,t)$ satisfies the equation (1.5). Moreover, by (3.1),

$$f_+(x,0)(1+\Delta_+(x,x)) = e^{i\int_x^\infty Q(\eta)d\eta}. \tag{3.3}$$

Substituting $A_+(x,t) = \partial_t \Delta_+(x,t) f_+(x,0)$ and (3.3) to (2.7), we arrive at (1.6). The proof of (1) is complete. The proof of (2) is similar. □

4. The Inverse Problem

In this section we shall prove that, for prescribed continuous functions $s_{ij}(k)$, $i,j = 1,2$, satisfying the conditions (I)–(VI), there exists a unique pair $(U(x), Q(x))$ of functions with $U(x)$ and $Q'(x) \in L^1_2(\boldsymbol{R})$ for which

$$\begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix} \text{ or } \begin{pmatrix} -s_{11}(k) & s_{12}(k) \\ s_{21}(k) & -s_{22}(k) \end{pmatrix}$$

is the scattering matrix. By condition (VI), the function $s_{21}(k)$ is the Fourier transform of $-F_+(t)$:

$$s_{21}(k) = -\int_{-\infty}^{\infty} F_+(t)e^{-ikt}dt, \tag{4.1}$$

where $F_+(t)$ is a function in $L^2(\boldsymbol{R})$. Let $x \in \boldsymbol{R}$ be fixed and, following the recovery algorithm in Theorem 3.1, employ the integral equation

$$\overline{\Delta_+(x,t)} + \int_x^\infty \Delta_+(x,r)F_+(r+t)dr + \int_x^\infty F_+(r+t)dr = 0, \quad x \le t. \tag{4.2}$$

As with (3.2), it follows that (4.2) has a unique solution $\Delta_+(x,t) \in BC[x,\infty)$.

We pick out some properties (see Kamimura [12, Section 2]) of the solution $\Delta_+(x,t)$.

Lemma 4.1. *Assume $F'_+(t) \in L^1_1(x,\infty) \cap BC[x,\infty)$ for each $x \in \boldsymbol{R}$ and let $\Delta_+(x,t)$ be the solution of* (4.2) *in $BC[x,\infty)$. Then:*

(1) $|\Delta_+(x,t)| \le \text{const.} \int_{x+t}^\infty |F_+(s)|ds$.
(2) *$1+\Delta_+(x,x) \ne 0$ for each $x \in \boldsymbol{R}$. Moreover, $1+\Delta_+(x,x) \to 0$ as $x \to \infty$.*
(3) *For each $x \in \boldsymbol{R}$, $\Delta_+(x,t)$ has the partial derivative $\partial_t\Delta_+(x,t)$ with respect to t, which belongs to $L^1(x,\infty) \cap BC[x,\infty)$.*
(4) *$\Delta_+(x,x)$ is differentiable and the function*

$$p(x) := \frac{\frac{d}{dx}\Delta_+(x,x)}{1+\Delta_+(x,x)}$$

is expressed as

$$\overline{p(x)} = 2w(x)\Big(F_+(2x) + \int_0^\infty H(x,y)F_+(y+2x)dy\Big), \tag{4.3}$$

where

$$w(x) := \frac{1+\Delta_+(x,x)}{|1+\Delta_+(x,x)|} \tag{4.4}$$

and $H(x,y)$ is estimated as

$$|H(x,y)| \le \text{const.} \int_{2x}^\infty |F_+(t)|^2 dt.$$

Moreover, $H(x,y)$ has the derivative $\partial_x H(x,y)$ with

$$|\partial_x H(x,y)| \le \text{const.}\Big(|F_+(2x)| + \int_{2x}^\infty |F_+(t)|^2 dt\Big)^2.$$

We now define functions $Q_+(x)$ and $U_+(x)$ by

$$Q_+(x) = -\frac{d}{dx}(\text{Im}\,\log(1+\Delta_+(x,x))) = -\frac{d}{dx}\arg(1+\Delta_+(x,x)), \tag{4.5}$$

$$U_+(x) = \Big(\frac{d}{dx}(\text{Re}\,\log(1+\Delta_+(x,x)))\Big)^2 - \frac{d^2}{dx^2}(\text{Re}\,\log(1+\Delta_+(x,x))). \tag{4.6}$$

Lemma 4.2. *For each $a \in \boldsymbol{R}$, $Q_+(x) \in L^1_1(a,\infty)$ and $U_+(x) \in L^1_2(a,\infty)$. The function $Q_+(x)$ is absolutely continuous, and the derivative $Q'_+(x)$ belongs to $L^1_2(a,\infty)$.*

Proof. We introduce the function $F^\sharp(x) = \int_x^\infty |F'_+(t)|dt$. This function is a nonnegative, nonincreasing function in $L^1_1(a,\infty)$ because

$$\begin{aligned}\int_a^\infty (1+|x|)F^\sharp(x)dx &= \int_a^\infty |F'_+(t)|dt \int_a^t (1+|x|)dx \\ &\le \text{const.} \int_a^\infty (1+t^2)|F'_+(t)|dt < \infty\end{aligned}$$

by the assumption $F'_+(t) \in L^1_2(a,\infty)$. Since

$$|F_+(2x)| = \left|\int_{2x}^\infty F'_+(t)dt\right| \le F^\sharp(2x),$$

we have, for $x \ge a, y \ge 0$,

$$\begin{aligned}|H(x,y)| &\le \text{const.} \int_{2x}^\infty |F_+(t)|^2 dt \le \text{const.}\, F^\sharp(2x) \int_{2x}^\infty |F_+(t)|dt \\ &\le \text{const.}\, F^\sharp(2x),\end{aligned}$$

$$|\partial_x H(x,y)| \le \text{const.}\left(F^\sharp(2x) + F^\sharp(2x)\int_{2x}^\infty |F_+(t)|dt\right)^2 \le \text{const.}\,(F^\sharp(2x))^2.$$

It follows from (4.3) and $|H(x,y)| \le \text{const.}\,F^\sharp(2x)$ that

$$|p(x)| \le \text{const.}\left(F^\sharp(2x) + F^\sharp(2x)\int_{2x}^\infty |F_+(t)|dt\right) \le \text{const.}\,F^\sharp(2x).$$

Since the definition of $Q_+(x)$ is rewritten as $Q_+(x) = -\mathrm{Im}\,p(x)$, this shows that, for each $a \in \boldsymbol{R}$, $|Q_+(x)| \le \text{const.}\,F^\sharp(2x)$ for $x \ge a$. In particular, $Q_+(x) \in L^1(a,\infty) \cap BC[a,\infty)$. Hence, $w(x)$ defined by (4.4) is written as

$$w(x) = \frac{1 + \Delta_+(x,x)}{|1 + \Delta_+(x,x)|} = e^{i\arg(1+\Delta_+(x,x))} = e^{i\int_x^\infty Q_+(\eta)d\eta}, \tag{4.7}$$

and hence, $w(x)$ is differentiable and $w'(x) = -iw(x)Q_+(x)$. Consequently, by (4.3), $p(x)$ is absolutely continuous and

$$|p'(x)| \le \text{const.}\left\{F'_+(2x) + (F^\sharp(2x))^2\right\}.$$

But, by the fact that

$$|xF^\sharp(x)| \le \int_x^\infty |tF'_+(t)|dt$$

for $x \ge 0$, we find that $xF^\sharp(2x) \in BC[a,\infty)$. Hence, it follows from this estimate that $Q_+(x) = -\mathrm{Im}\,p(x)$ has the derivative $Q'_+(x) \in L^1_2(a,\infty)$, because $F'_+(2x) \in L^1_2(a,\infty)$, $(1+|x|)F^\sharp(2x) \in L^1(a,\infty) \cap BC[a,\infty) \subset L^2(a,\infty)$. Similarly, by $U_+(x) = (\mathrm{Re}\,p(x))^2 - \mathrm{Re}\,p'(x)$, we can prove that $U_+(x) \in L^1_2(a,\infty)$ for each $a \in \boldsymbol{R}$. □

We next define a function $f_+(x,k)$ for $x \in \boldsymbol{R}$, $k \in \overline{\boldsymbol{C}_+}$ by

$$f_+(x,k) = \frac{1}{|1 + \Delta_+(x,x)|}\left(e^{ikx} - ik\int_x^\infty \Delta_+(x,t)e^{ikt}dt\right). \tag{4.8}$$

Then, by the same discussion as that in [12, Lemma 3.3], it follows that $f = f_+(x,k)$ satisfies

$$f'' + [k^2 - (U_+(x) + 2k\,Q_+(x))]f = 0, \quad x \in \boldsymbol{R}. \tag{4.9}$$

Since $\Delta_+(x,x) \to 0$ as $x \to \infty$, $f_+(x,k)$ has the asymptotics $f_+(x,k) \sim e^{ikx}$ as $x \to \infty$. Thus $f_+(x,k)$ is the Jost solution of (4.8).

Performing an integration by parts and taking Lemmas 4.1.(2) and (4.7) into account, we have

$$f_+(x,k) = e^{i\int_x^\infty Q_+(\eta)d\eta}e^{ikx} + \int_x^\infty A_+(x,t)e^{ikt}dt, \quad k \in \overline{\boldsymbol{C}_+}, \tag{4.10}$$

where we set

$$A_+(x,t) := \frac{\partial_t \Delta_+(x,t)}{|1+\Delta_+(x,x)|}.$$

By Lemma 4.1, $A_+(x,\cdot) \in L^1(x,\infty) \cap BC[x,\infty)$ and hence, $A_+(x,\cdot) \in L^2(x,\infty)$ for each $x \in \boldsymbol{R}$. Differentiating (4.2) and using an integration by parts show that $A_+(x,t)$ satisfies the equation (2.14).

In a similar manner, let $\Delta_-(x,t)$ be the solution of the integral equation

$$\overline{\Delta_-(x,t)} + \int_{-\infty}^{x} \Delta_-(x,r)F_-(r+t)dr + \int_{-\infty}^{x} F_-(r+t)dr = 0, \quad t \le x,$$

with the function $F_-(t) \in L^2(\boldsymbol{R})$ determined from $s_{12}(k)$ in condition (VI), and define

$$Q_-(x) = \frac{d}{dx}(\mathrm{Im}\ \log(1+\Delta_-(x,x))) = \frac{d}{dx}\ \arg(1+\Delta_-(x,x)),$$
$$U_-(x) = \Big(\frac{d}{dx}(\mathrm{Re}\ \log(1+\Delta_-(x,x)))\Big)^2 - \frac{d^2}{dx^2}(\mathrm{Re}\ \log(1+\Delta_-(x,x))).$$

Then, by the same method as in the proof of Lemma 4.2, one can show that $U_-(x)$ and $Q_-'(x) \in L_2^1(-\infty,a)$ and $Q_-(x) \in L_1^1(-\infty,a)$ for each $a \in \boldsymbol{R}$. Moreover, the Jost solution $f_-(x,k)$ of the equation

$$f'' + [k^2 - (U_-(x) + 2k\,Q_-(x))]f = 0, \quad x \in \boldsymbol{R} \tag{4.11}$$

with the pair $(U_-(x), Q_-(x))$ is represented as

$$f_-(x,k) = e^{i\int_{-\infty}^{x} Q_-(\eta)d\eta} e^{-ikx} - \int_{-\infty}^{x} A_-(x,t)e^{-ikt}dt, \quad k \in \overline{\boldsymbol{C}_+}, \tag{4.12}$$

in terms of a function $A_-(x,\cdot) \in L^1(-\infty,x) \cap BC(-\infty,x]$. The function $A_-(x,t)$ satisfies the following equation:

$$\overline{A_-(x,t)} - \int_{-\infty}^{x} A_-(x,r)F_-(r+t)dr + e^{i\int_{-\infty}^{x} Q_-(\eta)d\eta} F_-(x+t) = 0, \quad x > t. \tag{4.13}$$

A result similar to Lemma 4.2 is obtained.

Lemma 4.3. *For each $a \in \boldsymbol{R}$, $Q_-(x) \in L_1^1(-\infty,a)$ and $U_-(x) \in L_2^1(-\infty,a)$. The function $Q_-(x)$ is absolutely continuous and the derivative $Q_-'(x)$ belongs to $L_2^1(-\infty,a)$.*

We now set

$$\begin{aligned} &\gamma_+ := e^{i\int_x^{\infty} Q_+(\eta)d\eta}, \quad \gamma_- := e^{i\int_{-\infty}^{x} Q_-(\eta)d\eta}; \\ &u_+(k) := e^{-ikx}f_+(x,k), \quad u_-(k) := e^{ikx}f_-(x,k); \\ &r_+(k) := e^{2ikx}s_{21}(k), \quad r_-(k) := e^{-2ikx}s_{12}(k). \end{aligned} \tag{4.14}$$

Then, by (4.10), we have

$$u_+(k) - \gamma_+ = \int_0^\infty A_+(x, x+t) e^{ikt} dt. \tag{4.15}$$

Because of the fact that $A_+(x, x+\cdot) \in L^2(0,\infty)$ this implies that $u_+(k) - \gamma_+$ belongs to the Hardy space H^{2+}. Moreover, by (4.1), (4.10) and (2.14), we obtain

$$s_{21}(k) f_+(x,k) + \overline{f_+(x,k)} = \overline{\gamma_+} e^{-ikx} - \int_{-\infty}^x \Phi_+(x,t) e^{-ikt} dt,$$

where $\Phi_+(x,t)$ is a function on $t < x$ defined by

$$\Phi_+(x,t) = \gamma_+ F_+(x+t) + \int_x^\infty A_+(x,r) F_+(r+t) dr.$$

This leads to

$$r_+(k) u_+(k) + \overline{u_+(k)} - \overline{\gamma_+} = -\int_0^\infty \Phi(x, x-t) e^{ikt} dt.$$

It follows from the fact that $F_+ \in L^2(\boldsymbol{R})$ and $A_+(x,\cdot) \in L^1(x,\infty)$, and the convolution theorem, that $\Phi(x, x-\cdot) \in L^2(0,\infty)$. Hence, $r_+(k) u_+(k) + \overline{u_+(k)} - \overline{\gamma_+} \in H^{2+}$. Consequently

$$u_+(k) - \gamma_+ \in H^{2+}, \quad r_+(k) u_+(k) + \overline{u_+(k)} - \overline{\gamma_+} \in H^{2+}.$$

Similarly, by (4.12), (4.13), we find that

$$u_-(k) - \gamma_- \in H^{2+}, \quad r_-(k) u_-(k) + \overline{u_-(k)} - \overline{\gamma_-} \in H^{2+}.$$

Observing that, by Lemma 2.1.(1), $\int_t^\infty A_+(x, x+s) ds \in L^1(0,\infty) \cap BC[0,\infty) \subset L^2(0,\infty)$ as a function of t and performing an integration by parts on (4.15) leads to

$$u_+(k) - \gamma_+ = \int_0^\infty A_+(x, x+s) ds + ik \int_0^\infty e^{ikt} dt \int_t^\infty A_+(x, x+s) ds.$$

Since

$$u_+(0) = \gamma_+ + \int_0^\infty A_+(x, x+t) dt = f_+(x, 0) =: L_+$$

is a real number for each fixed $x \in \boldsymbol{R}$,

$$u_+(k) = L_+ + ik \int_0^\infty e^{ikt} dt \int_t^\infty A_+(x, x+s) ds = L_+ + M_+ k + o(1)$$

with a real number L_+ when k tends to zero in $\overline{\boldsymbol{C}_+}$. In a similar way we have $u_-(k) = L_- + M_- k + o(1)$ as $k \to 0$ in $\overline{\boldsymbol{C}_+}$, where $L_- \in \boldsymbol{R}$.

Here we require the following lemma that is a generalization of Deift-Trubowitz [4, Theorem 5.2]:

Lemma 4.4. *Let the matrix*

$$\begin{pmatrix} t(k) & r_-(k) \\ r_+(k) & t(k) \end{pmatrix}$$

be continuous on $\boldsymbol{R}$ *and have the properties* (I)-(V) *in Theorem* 1.1 *and suppose that, for some complex number* $\gamma_\pm$ *of absolute value* 1, $u_\pm(k) - \gamma_\pm \in H^{2+}$, $r_\pm(k)u_\pm(k) + \overline{u_\pm(k)} - \overline{\gamma_\pm} \in H^{2+}$ *with* $u_\pm(k)$ *continuous and bounded,* $u_\pm(k) = L_\pm + kM_\pm + o(k)$ *as* $k \to 0$ *in* $\overline{\boldsymbol{C}_+}$ *where* $L_\pm \in \boldsymbol{R}$. *Then either*

$$r_+(k)u_+(k) + \overline{u_+(k)} = t(k)u_-(k)$$
$$r_-(k)u_-(k) + \overline{u_-(k)} = t(k)u_+(k);$$

or

$$r_+(k)u_+(k) + \overline{u_+(k)} = -t(k)u_-(k)$$
$$r_-(k)u_-(k) + \overline{u_-(k)} = -t(k)u_+(k).$$

Proof. In case (i), where $t(0) \neq 0$, we define

$$v_-(k) = \frac{1}{t(k)}\Big(r_+(k)u_+(k) + \overline{u_+(k)}\Big).$$

Then

$$v_- - \overline{\gamma_+ C} = \frac{1}{t}\Big(r_+u_+ + \overline{u_+} - \overline{\gamma_+}\Big) + \Big(\frac{1}{t} - \frac{1}{C}\Big)\overline{\gamma_+} \in H^{2+}.$$

Moreover, since

$$r_-v_- + \overline{v_-} = \Big(\frac{r_+r_-}{t} + \frac{1}{\bar{t}}\Big)u_+ = \frac{1-|r_+|^2}{\bar{t}}u_+ = tu_+,$$

we conclude that

$$r_-v_- + \overline{v_-} - \gamma_+C = t(u_+ - \gamma_+) + \gamma_+(t - C) \in H^{2+}.$$

Therefore, it follows from the recast

$$\begin{aligned} &u_-\overline{v_-} - \overline{u_-}v_- - (\gamma_+\gamma_-C - \overline{\gamma_+\gamma_-C}) \\ &= u_-(r_-v_- + \overline{v_-} - \gamma_+C) + \gamma_+C(u_- - \gamma_-) \\ &\quad - (r_-u_- + \overline{u_-} - \overline{\gamma_-})v_- - \overline{\gamma_-}(v_- - \overline{\gamma_+C}) \end{aligned}$$

that the function $u_-\overline{v_-} - \overline{u_-}v_- - (\gamma_+\gamma_-C - \overline{\gamma_+\gamma_-C})$ belongs to H^{2+}. Taking the complex conjugate of this function, we see that it belongs also to H^{2-}, and so, that it must be zero, namely

$$u_-(k)\overline{v_-(k)} - \overline{u_-(k)}v_-(k) = \gamma_+\gamma_-C - \overline{\gamma_+\gamma_-C}.$$

But, by the assumptions $u_+(0) \in \boldsymbol{R}$ and (V),

$$v_-(0) = \frac{1 + r_+(0)}{t(0)} u_+(0)$$

is a real number, and therefore, the purely imaginary number $\gamma_+\gamma_- C - \overline{\gamma_+\gamma_- C}$ must be zero. Thus, we arrive at

$$u_-(k)\overline{v_-(k)} - \overline{u_-(k)}v_-(k) \equiv 0, \quad \gamma_+\gamma_- C = \overline{\gamma_+\gamma_- C}.$$

Since $|\gamma_+\gamma_- C| = 1$, the latter equality means $\gamma_+\gamma_- C = \pm 1$, and so, $\overline{\gamma_+ C} = \pm\gamma_-$. Hence,

$$\pm v_-(k) - \gamma_- \in H^{2+}, \quad \pm r_-(k)v_-(k) \pm \overline{v_-(k)} - \overline{\gamma_-} \in H^{2+}.$$

We now set $h(k) := u_-(k) \mp v_-(k)$. Then

$$h(k) = (u_-(k) - \gamma_-) - (\pm v_-(k) - \gamma_+) \in H^{2+},$$

and

$$\begin{aligned} r_-(k)h(k) + \overline{h(k)} &= (r_-(k)u_-(k) + \overline{u_-(k)} - \overline{\gamma_-}) \\ &\quad - (\pm r_-(k)v_-(k) \pm \overline{v_-(k)} - \overline{\gamma_-}) \in H^{2+}. \end{aligned}$$

Consequently, due to Deift-Trubowitz [4, Lemma 5,1], we have $h \equiv 0$. Hence, by the definition of v_-, we have $r_+(k)u_+(k) + \overline{u_+(k)} = \pm t(k)u_-(k)$. In a similar way we have $r_-(k)u_-(k) + \overline{u_-(k)} = \pm t(k)u_+(k)$.

In case (ii), where $t(0) = 0$, let $w(k) := r_+(k)u_+(k) + \overline{u_+(k)}$. Then, since

$$\frac{\overline{u_+(k)} - u_+(k)}{k} = \overline{M_+} - M_+ + o(1)$$

as $k \to 0$ in $\overline{\boldsymbol{C}_+}$ and (V), the function

$$\frac{w(k)}{k} = \frac{r_+(k) + 1}{k} u_+(k) + \frac{\overline{u_+(k)} - u_+(k)}{k}$$

tends to a purely imaginary number as $k \to 0$ in $\boldsymbol{R}$. By the assumption $w(k) - \overline{\gamma_+} \in H^{2+}$ and the convolution theorem,

$$\frac{w(k + i\varepsilon) - \overline{\gamma_+}}{k + i\varepsilon} = \int_{-\infty}^{0} \Omega_1(t, \varepsilon)e^{-ikt}dt, \quad \varepsilon > 0,$$

where $\Omega_1(\cdot, \varepsilon) \in L^2(-\infty, 0)$, and hence, by setting $\Omega(t, \varepsilon) = -\overline{\gamma_+}ie^{\varepsilon t} + \Omega_1(t, \varepsilon)$, we have

$$\frac{w(k + i\varepsilon)}{k + i\varepsilon} = \int_{-\infty}^{0} \Omega(t, \varepsilon)e^{-ikt}dt \in H^{2+}.$$

Since $w(k)-\overline{\gamma_+}\in H^{2+}$ and $w(k)\in BC(\boldsymbol{R})$, it is readily seen from Poisson's integral representation (see e.g., Yosida [20, Chapter VI]) that the function $w(k)$ is continued analytically into $\overline{\boldsymbol{C}_+}$. Also $\frac{w(k)}{k}$ is continuous on $\boldsymbol{R}$. Therefore, by the Lindelöf theorem, $\frac{w(k)}{k}$ is continuous on $\overline{\boldsymbol{C}_+}$. This shows that $\frac{w(k+i\varepsilon)}{k+i\varepsilon}$ tends to $\frac{w(k)}{k}$ uniformly in k as $\varepsilon\to+0$, and hence $\frac{w(k+i\varepsilon)}{k+i\varepsilon}$ tends to $\frac{w(k)}{k}$ in $L^2(\boldsymbol{R})$. Therefore $\frac{w(k)}{k}$ belongs to H^{2+}, and so

$$v_-(k)-\overline{\gamma_+C}=\frac{w(k)}{k}\Big(\frac{k}{t(k)}-\frac{k}{C}\Big)+\frac{w(k)-\overline{\gamma_+}}{C}$$

belongs to H^{2+}. The rest of the proof is exactly the same as in case (i), since

$$v_-(0)=\lim_{k\to 0}\frac{w(k)}{k}\frac{k}{t(k)}$$

is a real number. □

Using (4.14), it follows from Lemma 4.4 that $f_\pm(x,k)$ satisfy either

$$s_{21}(k)f_+(x,k)+\overline{f_+(x,k)}=s_{22}(k)f_-(x,k),$$
$$s_{12}(k)f_-(x,k)+\overline{f_-(x,k)}=s_{11}(k)f_+(x,k);$$

or

$$s_{21}(k)f_+(x,k)+\overline{f_+(x,k)}=-s_{22}(k)f_-(x,k),$$
$$s_{12}(k)f_-(x,k)+\overline{f_-(x,k)}=-s_{11}(k)f_+(x,k).$$

This implies that $f_\pm(x,k)$ are solutions of both equations (4.9) and (4.11). Therefore, $U_+(x)=U_-(x)=:U(x)$, $Q_+(x)=Q_-(x)=:Q(x)$ for almost every $x\in\boldsymbol{R}$. Hence, by Lemmas 4.2 and 4.3, $Q(x)\in L^1_1(\boldsymbol{R})$, $U(x)\in L^1_2(\boldsymbol{R})$, and $Q'(x)\in L^1_2(\boldsymbol{R})$. Clearly, either

$$\begin{pmatrix} s_{11}(k) & s_{12}(k)\\ s_{21}(k) & s_{22}(k)\end{pmatrix} \text{ or } \begin{pmatrix} -s_{11}(k) & s_{12}(k)\\ s_{21}(k) & -s_{22}(k)\end{pmatrix}$$

is the scattering matrix for $(U(x),Q(x))$.

5. Example

We give an example. Let a be a complex number in the angular domain $-\frac{\pi}{4}<\arg a<\frac{\pi}{4}$ and set

$$S(k)=\begin{pmatrix} \pm\dfrac{k(k+i\sqrt{a^2+\overline{a}^2}\,)}{(k+ia)(k+i\overline{a})} & \dfrac{(k-ia)(k+i\sqrt{a^2+\overline{a}^2})}{(k+ia)(k-i\sqrt{a^2+\overline{a}^2})}\dfrac{\overline{a}^2}{(k+i\overline{a})^2}\\ -\dfrac{a^2}{k^2+a^2} & \pm\dfrac{k(k+i\sqrt{a^2+\overline{a}^2}\,)}{(k+ia)(k+i\overline{a})}\end{pmatrix}. \tag{5.1}$$

Then, as is easily seen, $S(k)$ satisfies conditions (I)–(VI) in Theorem 1.1. The constant C in (IV) is ± 1 according to the signature of $s_{11}(k)$. The function $F_+(t)$ in (VI) is calculated as

$$F_+(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a^2}{k^2+a^2} e^{ikt} dk = \frac{a}{2} e^{-a|t|},$$

and, by an elementary calculation, the solution $\Delta_+(x,t)$ of (1.5) is obtained as follows for $0 \le x \le t$:

$$\Delta_+(x,t) = \frac{1}{2d_+(x)}\Big(\frac{\overline{a}}{2(a+\overline{a})} e^{-2ax} - 1\Big) e^{-\overline{a}(x+t)},$$

where

$$d_+(x) := 1 - \frac{|a|^2}{4(a+\overline{a})^2} e^{-2(a+\overline{a})x}.$$

Hence,

$$1 + \Delta_+(x,x) = \frac{1}{d_+(x)}\Big(1 - \frac{1}{2} e^{-2\overline{a}x} + \frac{\overline{a}^2}{4(a+\overline{a})^2} e^{-2(a+\overline{a})x}\Big).$$

Noting that $d_+(x)$ is a positive function on $x \ge 0$, by (1.7), we have, for $x \ge 0$,

$$Q(x) = -\frac{d}{dx} \arg\Big(1 - \frac{1}{2} e^{-2\overline{a}x} + \frac{\overline{a}^2}{4(a+\overline{a})^2} e^{-2(a+\overline{a})x}\Big). \tag{5.2}$$

Since

$$\operatorname{Re}\Big(1 - \frac{1}{2} e^{-2\overline{a}x} + \frac{\overline{a}^2}{4(a+\overline{a})^2} e^{-2(a+\overline{a})x}\Big) \ge \Big(\frac{1}{2} - \frac{|a|^2}{4(a+\overline{a})^2}\Big) 0, \quad x \ge 0,$$

it follows that

$$\int_0^\infty Q(x)dx = \arg(1 + \Delta_+(0,0)) = \arg\Big(1 + \frac{\overline{a}^2}{2(a+\overline{a})^2}\Big), \tag{5.3}$$

so that $-\frac{\pi}{2} < \int_0^\infty Q(x)dx \le 0$.

To compute $Q(x)$ for $x \le 0$ it is more convenient to work with (1.9). The function $F_-(t)$ in (VI) for $t \le 0$ is calculated as

$$\begin{aligned} F_-(t) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(k-ia)(k+i\sqrt{a^2+\overline{a}^2})}{(k+ia)(k-i\sqrt{a^2+\overline{a}^2})} \frac{\overline{a}^2}{(k+i\overline{a})^2} e^{-ikt} dk \\ &= -\frac{2\sqrt{a^2+\overline{a}^2}\ \overline{a}^4}{|\sqrt{a^2+\overline{a}^2}+a|^4} e^{\sqrt{a^2+\overline{a}^2}t} = \overline{A} e^{bt}, \end{aligned}$$

where we set

$$A = -\frac{2\sqrt{a^2+\overline{a}^2}\, a^4}{|\sqrt{a^2+\overline{a}^2}+a|^4}, \quad b = \sqrt{a^2+\overline{a}^2}.$$

Then the equation (1.11) is solved as, for $t \le x \le 0$,

$$\Delta_-(x,t) = \frac{A}{b\,d_-(x)}\Big(\frac{\overline{A}}{2b}\,e^{2bx} - 1\Big)e^{b(x+t)},$$

where $b = \sqrt{a^2 + \overline{a}^2}$. Note that

$$d_-(x) = 1 - \frac{|A|^2}{4b^2}e^{4bx} \ge 1 - \frac{|A|^2}{4b^2} > 0$$

for $x \le 0$ because of the fact that $|b+a| \ge |a|$. This, together with (1.9), shows that

$$Q(x) := \frac{d}{dx}\arg\Big(1 - \frac{A}{b}e^{2bx} + \frac{|A|^2}{4b^2}e^{4bx}\Big) \tag{5.4}$$

for $x \le 0$. In a similar way, for $\int_0^\infty Q(x)dx$, we have

$$\int_{-\infty}^{0} Q(x)dx = \arg\Big(1 + \frac{2a^4}{|\sqrt{a^2+\overline{a}^2}+a|^4} + \frac{|a|^8}{|\sqrt{a^2+\overline{a}^2}+a|^8}\Big), \tag{5.5}$$

so that

$$0 \le \int_{-\infty}^{0} Q(x)dx < \frac{\pi}{2}.$$

Thus, we arrive at

$$-\frac{\pi}{2} < \int_{-\infty}^{\infty} Q(x)dx < \frac{\pi}{2}.$$

In view of (1.12) this implies that $C = 1$, so that $\int_{-\infty}^{\infty} Q(x)dx = 0$. (This is, of course, verified by a direct calculation from (5.3) and (5.5).) Comparing the asymptotics of $s_{11}(k)$ in (5.1) as $|k| \to \infty$, we conclude that, in our example, the matrix $S(k)$ with plus as the signature is adapted to be the scattering matrix for a pair $(U(x), Q(x))$ in Π. Since $S(k)$ satisfies the realness condition $S(k) = \overline{S(-k)}$ if and only if $a > 0$, the matrix $S(k)$ with $a > 0$ is the scattering matrix for the standard Schrödinger equation (1.13).

6. Bound States

We conclude this paper with a remark on the condition (A3). The following theorem shows that whether (A3) is fulfilled or not is determined only by $U(x)$, independent of $Q(x)$.

Theorem 6.1. *For* (U,Q) *satisfying* (A1) *and* (A2), *let* $n(U,Q)$ *be the number of zeros, counted with multiplicities, of* $W[f_+(x,k), f_-(x,k)]$ *in the upper half-plane* $\boldsymbol{C}_+$. *Then* $n(U,Q_1) = n(U,Q_2)$ *for any* Q_1, Q_2 *satisfying* (A2). *In particular,* $n(U,Q) = n(U,0)$ *for any* Q *satisfying* (A2).

Proof. Let $f_\pm(x,k;\tau)$ be the Jost solutions of

$$f'' + [k^2 - (U(x) + 2k\tau\, Q(x))]f = 0, \quad -\infty < x < \infty, \tag{6.1}$$

for $0 \le \tau \le 1$, and let $s_{11}(k;\tau)$ be the transmission coefficient for (6.1). Note that, by (1.2), $s_{11}(k;\tau)$ is a continuous, non-zero function on $\boldsymbol{R}$ provided that $s_{11}(0,\tau) \neq 0$. Since $s_{11}(0,\tau) = s_{11}(0,0)$, if $s_{11}(0,0) \neq 0$ (this is the same as saying $W[f_+(x,0;0), f_-(x,0,0)] = 0$), then $s_{11}(0,\tau) \neq 0$ for any $\tau \in [0,1]$, and hence we can define the index

$$\operatorname{ind} s_{11}(k;\tau) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d[\arg s_{11}(k;\tau)].$$

By the property (IV), $s_{11}(k;\tau)$ tends to a point $C = C(\tau)$ on the unit circle as $k \to \pm\infty$. Hence the curve $\gamma(\tau)$ defined by $z = s_{11}(k;\tau)$, $-\infty \le k \le \infty$, is a continuous, oriented, closed curve in the complex plane from the point $C = C(\tau)$ to itself, not passing through the origin $z = 0$. The integer $\operatorname{ind} s_{11}(k;\tau)$ indicates how many times the curve $\gamma(\tau)$ winds around the origin in the counterclockwise direction. By the argument principle, this index coincides with minus the number of poles, counted with orders. Thus, $\operatorname{ind} s_{11}(k;\tau) = -n(U,\tau Q)$.

Since, as is easily shown, $s_{11}(k;\tau)$ is continuous in τ, the winding numbers $\operatorname{ind} s_{11}(k;\tau)$ are invariant under small perturbations of τ. Moreover $z = s_{11}(k;\tau)$ cannot cross the origin. It follows from these observations that $\operatorname{ind} s_{11}(k;\tau) = \operatorname{ind} s_{11}(k;0)$ for any $\tau \in [0,1]$. In particular, we have $\operatorname{ind} s_{11}(k;1) = \operatorname{ind} s_{11}(k;0)$. Consequently, if (1.13) with $U(x) \in L^1_2(\boldsymbol{R})$ satisfying (A2) has no bound states, then neither does (1.1) with the same $U(x)$ and any $Q(x)$ satisfying (A2).

If $s_{11}(0;0) = 0$, namely, if $W[f_+(x,0;0), f_-(x,0,0)] \neq 0$, then we employ the function

$$s_{11}(k;\tau)\,\frac{2ik-1}{2ik} = \frac{1-2ik}{W[f_+(x,k;\tau), f_-(x,k,\tau)]}$$

in place of $s_{11}(k;\tau)$ and proceed in the same way as above. Noting that the index of this function equals $-n(U,\tau Q)$ we complete the proof. □

Let Π_0 be the set of $U(x) \in L^1_2(\boldsymbol{R})$ for which the standard Schrödinger equation (1.13) has no bound states. Then Theorem 6.1 implies that the set Π is the direct product of Π_0 and the set of functions $Q(x)$ satisfying (A2).

References

[1] T. Aktosun, M. Klaus, and C. van der Mee, *Wave scattering in one dimension with absorption*, J. Math. Phys., 39 (1998), 1957–1992.

[2] T. Aktosun, M. Klaus, and C. van der Mee, *Inverse scattering in one-dimensional nonconservative media*, Integral Equations and Operator Theory, 30 (1998), 279–316.
[3] H. Cornille, *Existence and uniqueness of crossing symmetric $N/D-$type equations corresponding to the Klein-Gordon equation*, J. Math. Phys., 11 (1970), 79–98.
[4] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure. Appl. Math., 32 (1979), 121–251.
[5] L.D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Trudy Mat. Inst. Steklov, 73 (1964), 314–336; translated in Amer. Math. Soc. Transl. (2), 65 (1967), 139–166.
[6] M. Jaulent, *On an inverse scattering problem with an energy-dependent potential*, Ann. Inst. Henri Poincaré, Sect A, 17 (1972), 363–378.
[7] M. Jaulent, *Inverse scattering problems in absorbing media*, J. Math. Phys., 17 (1976), 1351–1360.
[8] M. Jaulent and C. Jean, *The inverse s-wave scattering problem for a class of potentials depending on energy*, Comm. Math. Phys., 28 (1972), 177–220.
[9] M. Jaulent and C. Jean, *The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential, I,II*, Ann. Inst. Henri Poincaré, Sect. A, 25 (1976), 105–118; 119–137.
[10] Y. Kamimura, *An inverse problem in advection-diffusion*, J. Phys: Conf. Ser., 73 (2007), doi 012012.
[11] Y. Kamimura, *An inversion formula in energy dependent scattering*, J. Int. Eq. Appl., 19 (2007), 473–512.
[12] Y. Kamimura, *Energy dependent inverse scattering*, Funkcialaj Ekvacioj, 51 (2008), 189–219.
[13] D.J. Kaup, *A higher-order water-wave equation and the method for solving it*, Prog. Theor. Phys., 54 (1975), 396–408.
[14] V.A. Marchenko, *Construction of the potential energy from phases of the scattered waves*, Dokl. Akad. Nauk. SSSR, 104 (1955), 695–698, in Russian.
[15] V.A. Marchenko, "Sturm-Liouville operators and applications, Operator Theory," Advances and Applications 22, Birkhäuser-Verlag, Basel, 1986.
[16] D.H. Sattinger and J. Szmigielski, *Energy dependent scattering theory*, Diff. Int. Eq., 8 (1995), 945–959.
[17] D.H. Sattinger and J. Szmigielski, *A Riemann-Hilbert problem for an energy dependent Schrödinger operator*, Inverse Problems, 12 (1996), 1003–1025.
[18] C. van der Mee and V. Pivovarchik, *Inverse scattering for a Schrödinger equation with energy dependent potential*, J. Math. Phys., 42 (2001), 158–181.
[19] R. Weiss and G. Scharf, *The inverse problem of potential scattering according to the Klein-Gordon equation*, Helv. Phys. Acta, 44 (1971), 910–929.
[20] K. Yosida, "Functional analysis," 3rd. ed., Springer-Verlag, Berlin, 1971.