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STOCHASTIC PERTURBATION OF NONLINEAR DEGENERATE PARABOLIC PROBLEMS

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Abstract. In this paper, we are interested in nonlinear stochastic partial differential equations. Stochastic perturbations of a class of degenerate parabolic problems with homogeneous Dirichlet boundary conditions are considered. A time-discretization is used to prove the existence of a solution. The pivot-space method leads to the uniqueness. Then, applications to the porous media and the Buckley-Leverett equations are proposed.

1. Introduction

In this paper, we are interested in the following formal stochastic partial differential equation of nonlinear degenerated parabolic type:

$$du - \Delta\phi(u)dt - \operatorname{div}(f(u)\overrightarrow{B})dt + g(.,.,u)dt = h(.,.,u)dw \text{ in } \mathbb{D} \times (0,T) \times \Omega,$$

with homogeneous Dirichlet boundary conditions.

In the sequel, one assumes that $\mathbb{D}$ is a bounded Lipschitz domain of $\mathbb{R}^d$ and T is a positive number and one denotes by $Q = (0,T) \times \mathbb{D}$.

Thereafter, $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the property that $w_0 = 0$. This assumption on W is made for convenience. Our aim is to adapt known methods for nonlinear PDEs to noise perturbed ones. For more general noise, one can consider cylindrical Wiener processes on separable Hilbert spaces (cf. G. Da Prato *et al.* [17]) or space-time noise.

In the sequel, ϕ is assumed to be a non-decreasing Lipschitz-continuous function and the problem degenerates since the set $\Lambda = \{x \in \mathbb{R} : \phi'(x) = 0\}$ is not assumed to be empty (cf. S.N. Antontsev *et al.* [3] for comments). Then, denote by $E = \{r \in R(\phi) : \phi_0^{-1}$ is discontinuous at $r\}$ where ϕ_0^{-1} is the hypo-inverse of ϕ.

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In the deterministic case, if on the one hand ϕ is a strictly increasing function, then E is empty, so the problem is weakly degenerate and a unique variational solution exists in general. See J. Carrillo [13], J. I. Díaz [19], G. Gagneux *et al.* [20], G. Vallet [31] and Ph. Benilan *et al.* [9] for the existence of strong solutions.
If, on the other hand, E is non-empty, the variational solution is not unique in general. Then, one needs Krushkov-entropy inequalities to conclude (see J. Carrillo [13], E. Rouvre *et al.* [30] or G. Vallet [32] for example).

Many papers on the viscous Burgers' type stochastic problem can be found in the literature, where, usually, the stochastic convolution is used. Let us mention, without being exhaustive, G. Da Prato *et al.* [15, 17], W. Grecksch *et al.* [21] or I. Gyongy *et al.* [23].

Fewer papers exist concerning stochastic degenerate parabolic problems. Concerning the stochastic porous media equation, V. Barbu *et al.* in [5] and V.I. Bogachev *et al.* in [11] propose the existence and uniqueness of weak solutions, as well as the existence of invariant probability measures. Then, strong solutions are studied by G. Da Prato *et al.* in [16]. Let us mention too the paper of J. Ren *et al.* [28] based on monotone arguments in a variational approach. Concerning the stochastic Stefan problem, an approximation of the operator is used by V. Barbu *et al.* in [6] where the authors take advantage of the special form of the nonlinear function. Then, in [7], the transition semigroup is studied.

Let us mention too the paper of S. Cerrai *et al.* in [14] where a singular perturbation by a second-order hyperbolic operator is proposed. The compactness argument is based on the Skorokhod theorem. We also mention the paper of G. Díaz *et al.* [18] where the authors are interested in the asymptotic study of the stochastic perturbation of a deterministic problem.

In the main result, a time-discretization scheme in the style of a Rothes' method (cf. I. Gyongy *et al.* [22], W. Grecksch *et al.* [21] or G. Vallet *et al.* [33]) is proposed. The main tool is the pivot-changing method (see J.-L. Lions [25] *cf.* page 190 *sqq.*), for the uniqueness as well as the existence.

The same method does not seem to be possible for the stochastic perturbation of the Buckley-Leverett equation[1] when f is only assumed to be Hölder continuous. Then, following J. U. Kim in [24] for example, one uses compactness arguments for a fixed trajectory, dP-almost surely. The uniqueness of the cluster point would lead to the result.

[1] a generalization of Burgers' equation

After giving the assumptions on the data and the definition of a solution, one finds a section devoted to the implicit time discretization of the equation. Then, *a priori* estimates are presented in a new section, before the one dedicated to the existence and the uniqueness of the solution. Then, some applications and extensions are proposed.

As mentioned by J. U. Kim [24], the equation has to be understood in the following way:

$$\frac{\partial}{\partial t}[u - \int_0^t h(s,x,u)dw(s)] - \Delta\phi(u) - \operatorname{div}(f(u)\overrightarrow{B}) + g(t,x,u) = 0, \quad (1.1)$$

where $\int_0^t h(s,x,u)dw(s)$ denotes the Itô integration of $h(s,x,u)$.

In order to simplify, homogeneous Dirichlet boundary conditions are considered on Γ, the boundary of $\mathbb{D}$, and the initial condition u_0 is given in $L^2(\mathbb{D})$.

2. Assumptions and the results

Let us assume in the sequel that

- ϕ is a non-decreasing Lipschitz-continuous function with $\phi(0) = 0$ and denote by G the associated Kirchoff's function: $G(x) = \int_0^x \sqrt{\phi'(s)}\,ds,\ x \in \mathbb{R}$.
- $\overrightarrow{B} \in (L^\infty(\mathbb{D}))^d$ with $\operatorname{div}\overrightarrow{B} = 0$ almost everywhere in $\mathbb{D}$.
- $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : Q\times\mathbb{R} \to \mathbb{R}$ and $h : Q\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, a positive constant $\mathbb{M}$ exists such that, for any real u and v, almost all t in $(0,T)$, and x in $\mathbb{D}$, one has that

$$|f(u) - f(v)|^2 + |g(t,x,u) - g(t,x,v)|^2 + |h(t,x,u) - h(t,x,v)|^2 \leq \mathbb{M}|\phi(u) - \phi(v)|.|u - v|. \quad (2.1)$$

  Note that inevitably f is a Lipschitz-continuous function, as are g and h with respect to their third variable, uniformly in (t,x).

  In a general way, one proposes to denote by $c(f)$ the Lipschitz-constant of any arbitrary Lipschitz-continuous function f.
- For almost every t in $(0,T)$, $|g(t,.,0)| \leq g_0$ with $g_0 \in L^2(\mathbb{D})$ and $|h(t,.,0)| \leq h_0$ with $h_0 \in L^2(\mathbb{D})$. In particular, the following growth condition holds:

$$|g(.,.,u)| \leq \mathbb{M}\sqrt{u\phi(u)} + g_0 \leq \mathbb{M}\sqrt{c(\phi)}|u| + g_0 \leq c_g\sqrt{g_0^2 + u^2}, \quad (2.2)$$

$$|h(.,.,u)| \leq \mathbb{M}\sqrt{u\phi(u)} + h_0 \leq \mathbb{M}\sqrt{c(\phi)}|u| + h_0 \leq c_h\sqrt{h_0^2 + u^2}. \quad (2.3)$$

These assumptions are enough if one considers only the time-discretization scheme. But, more assumptions are needed concerning g and h in order to pass to the limits with the time-step parameter Δt to 0^+. Thus, one assumes moreover that, for any bounded sequence u_n in $L^\infty(0,T;L^2(\Omega\times\mathbb{D}))$,

$$\int_{\Omega\times Q}[g_n(t,x,u_n(t,x))-g(t,x,u_n(t,x))]^2\,dx\,dt\,dP \underset{n\to\infty}{\to} 0,$$

where

$$g_n(t,x,u)=\int_0^T g(s,x,u)\rho_n(t-s)\,ds$$

and where one denotes by ρ_n the standard real-valued mollifier sequence (respectively with h).

This assumption is obvious if g (respectively h) is time-free. We present in the annexe (section 7) particular frameworks for g and h where such a limit is obtained. For example, in addition to the above hypothesis: if $g(t,x,u)=\alpha(t)\beta(x,u)$; if one controls the growth of $\frac{\partial g}{\partial t}$, or the growth of g with respect to u at infinity.

Denote by $V=H_0^1(\mathbb{D})$, endowed with

$$\|u\|_V=\sqrt{\int_{\mathbb{D}}|\nabla u|^2\,dx}$$

the norm of Poincaré (*cf.* R. Adams [1] Theorem 6.28, page 159), by C_p the Poincaré's constant, *i.e.,* for all $v\in V$, $\|v\|_{L^2(\mathbb{D})}\le C_p\|v\|_V$, and by $\theta=\mathbb{M}\max(||\vec{B}||_\infty^2,C_p^2)$.

Our aim is then to give a result of existence and uniqueness of the variational solution to the above-mentioned problem. Let us fix in what sense such a solution is understood.

Definition 1. *Any function u of $L^2(\Omega\times Q)$ such that $\phi(u)$ belongs to $L^2(\Omega\times(0,T);V)$ and $\frac{\partial}{\partial t}[u-\int_0^t h(s,.,u(s))dw(s)]$, taken in the sense of the vectorial V'-valued distributions, belongs to $L^2(\Omega\times(0,T);V')$ is a solution to our stochastic problem if u is an $L^2(\mathbb{D})$-valued process adapted to the filtration $(\mathcal{F}_t)_t$ and if for t almost everywhere in $(0,T)$ and any test-function v of V, the variational formulation holds*

$$\begin{aligned} 0 &= <\frac{\partial}{\partial t}[u-\int_0^t h(s,.,u(s))dw(s)],v>_{V',V} \\ &\quad+\int_{\mathbb{D}}\{\nabla\phi(u).\nabla v+f(u)\overrightarrow{B}.\nabla v+g(.,.,u)v\}\,dx, \end{aligned} \tag{2.4}$$

with the initial condition $u(0,.) = u_0 \in L^2(\mathbb{D})$.

Remark 1. i) Since ϕ is a Lipschitz-continuous function, $\phi(u)$ is also $L^2(\mathbb{D})$-adapted to the filtration $(\mathcal{F}_t)_t$. Thanks to the pivot-space properties $V \hookrightarrow L^2(\mathbb{D}) \equiv [L^2(\mathbb{D})]' \hookrightarrow V'$ (see J.-L. Lions [25] *cf.* page 190 *sqq.*), $\phi(u)$ is V'-adapted to the filtration $(\mathcal{F}_t)_t$. Given that, t almost everywhere, $\phi(u)$ is V-valued and that V is a separable Banach-space, $\phi(u)$ is t almost everywhere V-adapted to the filtration $(\mathcal{F}_t)_t$ thanks to the theorem of Pettis ([34] page 131). In order to obtain the same result for any t, one should prove that $\phi(u)$ belongs to $L^\infty(0,T;V)$. Then, following J.-L. Lions *et al.* [26], one gets that $\phi(u)$ belongs to $C_s(0,T;V)$, the continuous functions from $[0,T]$ into V, endowed with its weak topology.

ii) Since (2.4) is considered for fixed t, almost everywhere in $(0,T)$, the test-function v can be considered as a time-dependent function, as soon as t almost everywhere in $(0,T)$, $v(t) \in V$.

The results we want to prove in the sequel are the following.

With the above hypothesis, a unique solution exists in the sense of Definition 1. Moreover, if for convenience $g(t,x,u) = g(x,u)$ and $h(t,x,u) = h(x,u)$, then $C > 0$ exists such that $||G(u^{\Delta t}) - G(u)||^2_{L^2(\mathbb{D}\times Q)} \leq C\Delta t$, and we have the following.

If $0 \leq u_0 \leq 1$ and if moreover $g = 0$ and $h(.,.,x) = 0$ if $x < 0$ or $x > 1$, then $0 \leq u \leq 1$.

If $\phi = Id$, $g = 0$, h is independent of u and if f is a $\frac{1}{2}$-Hölder-continuous function, a unique solution exists too.

3. Time discretization

Let us consider a positive integer N and denote by $\Delta t = \frac{T}{N}$. Denote by $\mathcal{V}$ (respectively $\mathcal{H}$) the Hilbert-space of V-valued (respectively H-valued) $\mathcal{F}_{(n+1)\Delta t}$ measurable functions in $L^2(\Omega;V)$ (respectively $L^2(\Omega;H)$).

For technical reasons, assume moreover that Δt is small enough to have

$$4(c_g\sqrt{2} + c_h^2 + \theta)\Delta t \leq 1. \tag{3.1}$$

For convenience, consider, for any n in $\{1,..,N\}$, $g_n : \mathbb{D}\times\mathbb{R} \to \mathbb{R}$ and $h_n : \mathbb{D}\times\mathbb{R} \to \mathbb{R}$ measurable functions such that, for any real u and v and almost all x in $\mathbb{D}$, one has that

$$\begin{aligned}&|f(u) - f(v)|^2 + |g_n(x,u) - g_n(x,v)|^2 + |h_n(x,u) - h_n(x,v)|^2\\ &\leq \mathbb{M}|\phi(u) - \phi(v)|.|u - v|.\end{aligned} \tag{3.2}$$

For any n, $|g_n(.,0)| \leq g_0$ and $|h_n(.,0)| \leq h_0$, so that:

$$|g_n(.,u)| \leq \mathbb{M}\sqrt{u\phi(u)} + g_0 \leq \mathbb{M}\sqrt{c(\phi)}|u| + g_0 \leq c_g\sqrt{g_0^2 + u^2}, \tag{3.3}$$

$$|h_n(.,u)| \leq \mathbb{M}\sqrt{u\phi(u)} + h_0 \leq \mathbb{M}\sqrt{c(\phi)}|u| + h_0 \leq c_h\sqrt{h_0^2 + u^2}. \tag{3.4}$$

The aim of this section is to prove the following proposition.

Proposition 1. *Assume Δt small. For a given u_n in $\mathcal{H}$, $\mathcal{F}_{n\Delta t}$-measurable, u_{n+1} exists in $\mathcal{H}$ with $\phi(u_{n+1})$ in $\mathcal{V}$ such that for any v in V*

$$\int_{\mathbb{D}} \Big[(u_{n+1} - u_n)v + \Delta t\{\nabla\phi(u_{n+1}).\nabla v + f(u_{n+1})\overrightarrow{B}.\nabla v + g_n(x,u_{n+1})v\}\Big]\,dx$$
$$= (w_{n+1} - w_n)\int_{\mathbb{D}} h_n(x,u_n)v\,dx. \tag{3.5}$$

Moreover, the following estimate holds:

$$[2 - 4\Delta t c_g\sqrt{2}]||u_{n+1}||_{\mathcal{H}}^2 + ||u_{n+1} - u_n||_{\mathcal{H}}^2 + \frac{4\Delta t}{c(\phi)}||\phi(u_{n+1})||_{\mathcal{V}}^2$$
$$\leq 4\Delta t||h_n(.,u_n)||_{\mathcal{H}}^2 + 2\Delta t c_g\sqrt{2}||g_0||_{L^2(\mathbb{D})}^2 + 2||u_n||_{\mathcal{H}}^2. \tag{3.6}$$

Let us first prove the *a priori* estimate. Formally, it consists in using $v = u_{n+1}$ in (3.5), but since ϕ^{-1} is not a Lipschitz-continuous function, it is not an available test-function. One would get round this by using a regularization of the graph of ϕ^{-1}.

First, let us give this technical lemma:

Lemma 1. *For any Lipschiz-continuous function ψ from $\mathbb{R}$ to $\mathbb{R}$ and all u in $L^2(\mathbb{D})$ such that $\phi(u)$ belongs to V, one has that*

$$\int_{\mathbb{D}} f(u)\overrightarrow{B}.\nabla\psi(\phi(u))\,dx \quad = \quad 0.$$

Proof. Denote, for any positive M, f_M by $\min[M, \max(-M,f)]$ and ξ by $f_M \circ \phi_0^{-1}\psi'$. Then, since E is countable, one has that $\nabla\phi(u) = 0$ almost everywhere in $\{x \in \mathbb{D} : \phi(u(x)) \in E\}$ and

$$\int_{\mathbb{D}} f(u)\overrightarrow{B}.\nabla\psi(\phi(u))\,dx = \lim_{M\to\infty}\int_{\mathbb{D}} f_M(u)\psi'(\phi(u))\overrightarrow{B}.\nabla\phi(u)\,dx$$
$$= \lim_{M\to\infty}\int_{\mathbb{D}} \xi(\phi(u))\overrightarrow{B}.\nabla\phi(u)\,dx.$$

Then, the conclusion is based on chain rules in Sobolev spaces (see M. Marcus *et al.* [27]) and on the Gauss-Green formula. □

Consider in (3.5) the test-function $v = \beta_k(\phi(u_{n+1}))$, where $\beta_k = (\phi + \frac{Id}{k})^{-1}$. Then

$$\int_{\mathbb{D}} (u_{n+1} - u_n)\beta_k \circ \phi(u_{n+1})\, dx + \Delta t \int_{\mathbb{D}} \beta_k'(\phi(u_{n+1}))|\nabla \phi(u_{n+1})|^2\, dx$$
$$= \int_{\mathbb{D}} \{(w_{n+1} - w_n)h_n(x, u_n) - \Delta t g(x, u_{n+1})\}\beta_k \circ \phi(u_{n+1})\, dx.$$

Since $\beta_k'(\phi(u_{n+1})) \geq \frac{1}{c(\phi)+\frac{1}{k}}$ and as β_k converges uniformly to ϕ^{-1} for any bounded interval of $\phi(\mathbb{R})$ with $|\beta_k'(\phi(u_{n+1}))| \leq |u_{n+1}|$, passing to the limits with respect to k leads to

$$\int_{\mathbb{D}} (u_{n+1} - u_n)u_{n+1}\, dx + \frac{\Delta t}{c(\phi)} \int_{\mathbb{D}} |\nabla \phi(u_{n+1})|^2\, dx$$
$$\leq \int_{\mathbb{D}} \{(w_{n+1} - w_n)h_n(x, u_n) - \Delta t g_n(x, u_{n+1})\}u_{n+1}\, dx$$
$$\leq (w_{n+1} - w_n) \int_{\mathbb{D}} h_n(x, u_n)u_n\, dx + \frac{|w_{n+1} - w_n|^2}{2\beta}||h_n(., u_n)||^2_{L^2(\mathbb{D})}$$
$$+ \frac{\Delta t^2}{2\alpha}||g_n(., u_{n+1})||^2_{L^2(\mathbb{D})} + \frac{\alpha}{2}||u_{n+1}||^2_{L^2(\mathbb{D})} + \frac{\beta}{2}||u_{n+1} - u_n||^2_{L^2(\mathbb{D})},$$

where the last part is obtained for any positive α and β.

Thus, by taking the expectation, the inequality becomes

$$\frac{1}{2}[||u_{n+1}||^2_{\mathcal{H}} + ||u_{n+1} - u_n||^2_{\mathcal{H}} - ||u_n||^2_{\mathcal{H}}] + \frac{\Delta t}{c(\phi)}||\phi(u_{n+1})||^2_{\mathcal{V}}$$
$$\leq \frac{\Delta t}{2\beta}||h_n(., u_n)||^2_{\mathcal{H}} + \frac{\Delta t^2}{\alpha}c_g^2[||u_{n+1}||^2_{\mathcal{H}} + ||g_0||^2_{L^2(\mathbb{D})}]$$
$$+ \frac{\alpha}{2}||u_{n+1}||^2_{\mathcal{H}} + \frac{\beta}{2}||u_{n+1} - u_n||^2_{\mathcal{H}},$$

i.e.,

$$[\frac{1}{2} - \frac{\Delta t^2}{\alpha}c_g^2 - \frac{\alpha}{2}]||u_{n+1}||^2_{\mathcal{H}} + [\frac{1}{2} - \frac{\beta}{2}]||u_{n+1} - u_n||^2_{\mathcal{H}} + \frac{\Delta t}{c(\phi)}||\phi(u_{n+1})||^2_{\mathcal{V}}$$
$$\leq \frac{\Delta t}{2\beta}||h_n(., u_n)||^2_{\mathcal{H}} + \frac{\Delta t^2}{\alpha}c_g^2||g_0||^2_{L^2(\mathbb{D})} + \frac{1}{2}||u_n||^2_{\mathcal{H}}.$$

The result holds by considering $\alpha = \Delta t c_g \sqrt{2}$ and $\beta = \frac{1}{2}$.

Let us prove now that a unique solution exists. In order to do it, a determinist version of the problem is used.

Lemma 2. *Assume Δt small and consider ϖ given in $L^2(\mathbb{D})$.*

i) A unique u exists in $L^2(\mathbb{D})$ such that $\phi(u)$ belongs to V, and for any v in V

$$\int_{\mathbb{D}} uv\,dx + \Delta t \int_{\mathbb{D}} \{\nabla\phi(u).\nabla v + f(u)\overrightarrow{B}.\nabla v + g_n(x,u)v\}\,dx = \int_{\mathbb{D}} \varpi v\,dx.$$

ii) If for any positive integer k, one denotes ϕ_k by $\phi + \frac{1}{k}Id$, a unique u^k exists in V, such that for any v in V

$$\int_{\mathbb{D}} [u^k v + \Delta t\{\nabla\phi_k(u^k).\nabla v + f(u^k)\overrightarrow{B}.\nabla v + g_n(x,u^k)v\}]\,dx = \int_{\mathbb{D}} \varpi v\,dx.$$

Moreover, the application $\Psi_k : L^2(\mathbb{D}) \to V$, $\varpi \mapsto u^k$ is continuous.

iii) The sequence (u^k) converges weakly in $L^2(\mathbb{D})$ to u and the sequence $(\phi_k(u^k))$ converges weakly in V to $\phi(u)$.

Proof. Let us note that the operator is a Lipschitz perturbation in V' of the maximal monotone operator $-\Delta\phi(u)$ (Resp. ϕ_k). Then, for Δt small enough, the result of is an immediate consequence of nonlinear maximal monotone operators (see Ph. Bénilan [8] or H. Brézis [12] page 19 *sqq.* for example). □

Then, denote by $\varpi = (w_{n+1} - w_n)h_n(x,u_n) + u_n$ and note that, for ω almost surely in Ω, a unique solution $u(\omega)$ exists in the sense of the previous lemma. Moreover, it is the weak limit, in $L^2(\mathbb{D})$, of the sequence $(u^k(\omega))$ given by part ii) of the lemma.

Since the application Ψ_k is continuous from $L^2(\mathbb{D})$ into V, the function $\omega \in \Omega \mapsto u^k(\omega)$ is an $L^2(\mathbb{D})$-valued, $\mathcal{F}_{(n+1)\Delta t}$ measurable function. Therefore, u is a weakly measurable function and consequently a measurable function thanks to the Pettis theorem (K. Yosida [34] page 131), since $L^2(\mathbb{D})$ is a separable set.

The function $\omega \in \Omega \mapsto \phi(u(\omega))$ is a V-valued, $\mathcal{F}_{(n+1)\Delta t}$ measurable function too, by using the weak convergence of $(\phi_k(u^k))$ in V.

Similarly, to the *a priori* estimate, $C = C(\Delta t) \geq 0$ exists such that

$$||u||^2_{L^2(\mathbb{D})} + ||\phi(u)||^2_V \leq C[||g_0||^2_{L^2(\mathbb{D})} + ||\varpi||^2_{L^2(\mathbb{D})}].$$

This implies that u belongs to $\mathcal{H}$ and that $\phi(u)$ belongs to $\mathcal{V}$ and the proposition holds.

4. A priori estimates

The aim of this section is to give some *a priori* estimates in order to pass to the limits with respect to the time-discretization parameter.

In order to do this, let us first introduce some notations needed in the sequel.

Definition 2. *For any sequence $(x_n) \subset X$, where X is any B-space, let us denote by*

$$x^{\Delta t} = \sum_{k=1}^{N} x_k 1_{[(k-1)\Delta t, k\Delta t[},$$

$$\widetilde{x}^{\Delta t} = \sum_{k=1}^{N} \Big[\frac{x_k - x_{k-1}}{\Delta t}[t-(k-1)h] + x_{k-1}\Big] 1_{[(k-1)\Delta t, k\Delta t[}.$$

Thus, $$\frac{\partial \widetilde{x}^{\Delta t}}{\partial t} = \sum_{k=1}^{N} \frac{x_k - x_{k-1}}{\Delta t} 1_{[(k-1)\Delta t, k\Delta t[},\ t \text{ a.e. in}(0,T),$$

and elementary calculus yields

$$||x^{\Delta t}||^2_{L^2(0,T;X)} = \Delta t \sum_{k=1}^{N} ||x_k||^2_X; ||\widetilde{x}^{\Delta t}||^2_{L^2(0,T;X)} \leq \Delta t \sum_{k=0}^{N} ||x_k||^2_X;$$

$$||x^{\Delta t} - \widetilde{x}^{\Delta t}||^2_{L^2(0,T;X)} = \Delta t \sum_{k=0}^{N-1} ||x_{k+1} - x_k||^2_X;$$

$$||\frac{\partial \widetilde{x}^{\Delta t}}{\partial t}||^2_{L^2(0,T;X)} = \frac{1}{\Delta t} \sum_{k=0}^{N-1} ||x_{k+1} - x_k||^2_X;$$

$$||x^{\Delta t}||_{L^\infty(0,T;X)} = \max_{k=1,..,N} ||x_k||_X; ||\widetilde{x}^{\Delta t}||_{L^\infty(0,T;X)} = \max_{k=0,..,N} ||x_k||_X.$$

Therefore, one gets, for any v in V and any t in $(0,T) - \Delta t.\mathbb{N}$, that

$$\int_{\mathbb{D}} \frac{\partial \widetilde{u}^{\Delta t}}{\partial t}(t) v\, dx + \int_{\mathbb{D}} \nabla\phi(u^{\Delta t}(t)).\nabla v\, dx + \int_{\mathbb{D}} f(u^{\Delta t}(t))\overrightarrow{B}.\nabla v\, dx$$
$$+ \int_{\mathbb{D}} g^{\Delta t}(t,x,u^{\Delta t}(t)) v\, dx = \frac{\partial \widetilde{w}^{\Delta t}}{\partial t} \int_{\mathbb{D}} h^{\Delta t}(t,x,u^{\Delta t}(t-\Delta t)) v\, dx \quad a.s.$$

For any positive integer n, denote by

$$B_n := \sum_{k=1}^{k=n}(w_k - w_{k-1})h_k(x, u_{k-1}) = \int_0^{n\Delta t} h^{\Delta t}(s, x, u^{\Delta t}(t-\Delta t))\, dw(s),$$

with the convention that $u^{\Delta t}(t) = u_0$ if $t < 0$. Thus, one gets that

$$\int_{\mathbb{D}} \Big\{ \frac{\partial[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]}{\partial t} v + \nabla\phi(u^{\Delta t}).\nabla v + f(u^{\Delta t})\overrightarrow{B}.\nabla v + g^{\Delta t}(.,.,u^{\Delta t})v \Big\}\, dx = 0. \tag{4.1}$$

By considering (3.6), one gets that

$$\begin{aligned}
&[2 - 4\Delta t c_g\sqrt{2}]||u_n||^2_{\mathcal{H}} + \sum_{k=0}^{n-1}||u_{k+1} - u_k||^2_{\mathcal{H}} + \frac{4\Delta t}{c(\phi)}\sum_{k=0}^{n-1}||\phi(u_{k+1})||^2_{\mathcal{V}} \\
&\leq 4\Delta t \sum_{k=0}^{n-1}||h_k(., u_k)||^2_{\mathcal{H}} + 2\Delta t n c_g\sqrt{2}||g_0||^2_{L^2(\mathbb{D})} + 2||u_0||^2_{\mathcal{H}} \\
&+ 4\Delta t c_g\sqrt{2}\sum_{k=0}^{n-1}||u_{k+1}||^2_{\mathcal{H}} \\
&\leq 4\Delta t c_h^2 n||h_0||^2_{L^2(\mathbb{D})} + 2\Delta t n c_g\sqrt{2}||g_0||^2_{L^2(\mathbb{D})} + 3||u_0||^2_{\mathcal{H}} \\
&+ [4\Delta t c_g\sqrt{2} + 4\Delta t c_h^2]\sum_{k=0}^{n-1}||u_{k+1}||^2_{\mathcal{H}}.
\end{aligned}$$

By using (3.1), the following estimate holds:

$$||u_n||^2_{\mathcal{H}} + \sum_{k=0}^{n-1}||u_{k+1} - u_k||^2_{\mathcal{H}} + \frac{4\Delta t}{c(\phi)}\sum_{k=0}^{n-1}||\phi(u_{k+1})||^2_{\mathcal{V}} \leq c_1 + \Delta t c_2 \sum_{k=0}^{n-1}||u_{k+1}||^2_{\mathcal{H}},$$

where one denotes by $c_1 = 4Tc_h^2||h_0||^2_{L^2(\mathbb{D})} + 2Tc_g\sqrt{2}||g_0||^2_{L^2(\mathbb{D})} + 3||u_0||^2_{\mathcal{H}}$ and $c_2 = 4c_g\sqrt{2} + 4c_h^2$.

Thanks to the discrete lemma of Gronwall (Cf. D. Bainov *et al.* [4] page 165), one has

$$\begin{aligned}
||u_n||^2_{\mathcal{H}} \leq \frac{c_1}{1 - \Delta t c_2} e^{\frac{c_2 \Delta t}{1 - \Delta t c_2}T} \quad &\leq \quad 2c_1 e^T := c_3, \\
\sum_{k=0}^{n-1}||u_{k+1} - u_k||^2_{\mathcal{H}} + \frac{4\Delta t}{c(\phi)}\sum_{k=0}^{n-1}||\phi(u_{k+1})||^2_{\mathcal{V}} \quad &\leq \quad c_1 + Tc_2c_3 := c_4.
\end{aligned}$$

Since ϕ is a Lipschitz-continuous function with $\phi(0) = 0$, one gets the following.

Proposition 2. *Assume Δt is small. A constant C_1 depending on T, $\mathbb{M}$, $c(\phi)$, c_h, c_g, h_0, g_0 and $||u_0||_{\mathcal{H}}$, exists such that, for any n,*

$$||u_n||^2_{\mathcal{H}} + ||\phi(u_n)||^2_{\mathcal{H}} + \sum_{k=0}^{N-1} ||u_{k+1} - u_k||^2_{\mathcal{H}} + \sum_{k=1}^{N} \Delta t ||\phi(u_k)||^2_{\mathcal{V}} \leq C_1;$$

i.e., $u^{\Delta t}$, $\tilde{u}^{\Delta t}$ are bounded sequences in $L^\infty(0,T;\mathcal{H})$; $\phi(u^{\Delta t})$ is a bounded sequence in $L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})$; $||u^{\Delta t} - \tilde{u}^{\Delta t}||^2_{L^2(0,T;\mathcal{H})} \leq C_1 \Delta t$.

Since $f(0) \int_{\mathbb{D}} \overrightarrow{B} \nabla v \, dx = 0$, for any v in V, the following inequalities hold:

$$\begin{aligned}
&\int_{\mathbb{D}} [\frac{(u_{n+1} - u_n)}{\Delta t} - \frac{(w_{n+1} - w_n)}{\Delta t} h_n(x, u_n)] v \, dx \\
&\leq \Big[||\phi(u_{n+1})||_V + ||\overrightarrow{B}||_\infty ||f(u_{n+1}) - f(0)||_{L^2(\mathbb{D})} \Big] ||v||_V \\
&+ C_p ||g_n(., u_{n+1})||_{L^2(\mathbb{D})} ||v||_V \\
&\leq \Big[||\phi(u_{n+1})||_V + \sqrt{2\theta} \sqrt{\int_{\mathbb{D}} \phi(u_{n+1}) u_{n+1} \, dx} + C_p ||g_0||_{L^2(\mathbb{D})} \Big] ||v||_V \\
&\leq \Big[||\phi(u_{n+1})||_V + \sqrt{2\theta c(\phi)} ||u_{n+1}||_{L^2(\mathbb{D})} + C_p ||g_0||_{L^2(\mathbb{D})} \Big] ||v||_V .
\end{aligned}$$

Then

$$\begin{aligned}
&\sup_{v \in V \setminus \{0\}} \frac{\int_{\mathbb{D}} [\frac{(u_{n+1} - u_n)}{\Delta t} - \frac{(w_{n+1} - w_n)}{\Delta t} h_n(x, u_n)] v \, dx}{||v||_V} \\
&\leq ||\phi(u_{n+1})||_V + \sqrt{2\theta c(\phi)} ||u_{n+1}||_{L^2(\mathbb{D})} + C_p ||g_0||_{L^2(\mathbb{D})}
\end{aligned}$$

and we have the following proposition.

Proposition 3. *Assume Δt is small. A constant C_2 depending on C_1, $\overrightarrow{B}$ and C_p exist such that, for any n,*

$$\Delta t E \sum_{k=0}^{N-1} || \frac{u_{n+1} - u_n}{\Delta t} - \frac{w_{n+1} - w_n}{\Delta t} h_n(x, u_n) ||^2_{V'} \leq C_2; \tag{4.2}$$

i.e. $\frac{\partial [\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]}{\partial t}$ is a bounded sequence in $L^2((0,T) \times \Omega; V')$.

5. Existence and uniqueness of the solution

The aim of this section is to prove the following.

Theorem 1. *A unique solution exists in the sense of Definition* 1.

Claim 1: Let us first have a look at the result of uniqueness of the solution.

In order to prove it, consider two solutions u_1 and u_2. Then, the subtraction of the corresponding variational formulations leads, for t almost everywhere in $(0,T)$, and any v in V, to

$$\begin{aligned} 0 =&< \frac{\partial}{\partial t}[(u_1-u_2) - \int_0^t [h(s,.,u_1(s)) - h(s,.,u_2(s))]dw(s)], v >_{V',V} \\ &+ \int_{\mathbb{D}} \{\nabla[\phi(u_1)-\phi(u_2)].\nabla v + [f(u_1)-f(u_2)]\overrightarrow{B}.\nabla v\}\,dx \\ &+ \int_{\mathbb{D}} [g(.,.,u_1)-g(.,.,u_2)]v\,dx. \end{aligned}$$

Following the method of pivot space changing (see J.-L. Lions [25] *cf.* page 190 *sqq.*), for almost all t, one chooses the test function $v = -\Delta^{-1}w$ in V where

$$w = (u_1-u_2) - \int_0^t \Big[h(s,.,u_1(s)) - h(s,.,u_2(s))\Big]dw(s).$$

The Green operator provides an available test-function and one gets that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||v||_V^2 + \int_{\mathbb{D}} \{[\phi(u_1)-\phi(u_2)]w + [f(u_1)-f(u_2)]\overrightarrow{B}.\nabla v\}\,dx& \\ + \int_{\mathbb{D}} [g(.,.,u_1)-g(.,.,u_2)]v\,dx = 0.& \end{aligned}$$

Note that

$$\begin{aligned} \int_{\mathbb{D}} [\phi(u_1)-\phi(u_2)]w\,dx =& \int_{\mathbb{D}} [\phi(u_1)-\phi(u_2)](u_1-u_2)\,dx \\ &- \int_{\mathbb{D}} [\phi(u_1)-\phi(u_2)] \int_0^t \Big[h(s,.,u_1(s)) - h(s,.,u_2(s))\Big]dw(s)\,dx \\ \geq \int_{\mathbb{D}} [\phi(u_1)-\phi(u_2)]&(u_1-u_2)\,dx - \frac{1}{2c(\phi)}\int_{\mathbb{D}} [\phi(u_1)-\phi(u_2)]^2\,dx \\ &- \frac{c(\phi)}{2}\int_{\mathbb{D}} \{\int_0^t \Big[h(s,.,u_1(s)) - h(s,.,u_2(s))\Big]dw(s)\}^2\,dx \end{aligned}$$

$$\geq \frac{1}{2}\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)](u_1-u_2)\,dx$$
$$-\frac{c(\phi)}{2}\int_{\mathbb{D}}\{\int_0^t \Big[h(s,.,u_1(s))-h(s,.,u_2(s))\Big]dw(s)\}^2\,dx.$$

One derives from (2.1) that, for any positive ϵ,

$$\begin{aligned}&|\int_{\mathbb{D}}\{[f(u_1)-f(u_2)]\overrightarrow{B}.\nabla v+[g(.,.,u_1)-g(.,.,u_2)]v\}\,dx|\\ \leq\ & \frac{\epsilon\theta}{2}\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)][u_1-u_2]\,dx+\frac{1}{\epsilon}||v||_V^2.\end{aligned}$$

Then, by choosing $\epsilon=[2\theta]^{-1}$, the above equality leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||v||_V^2+\frac{1}{4}\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)][u_1-u_2]\,dx-2\theta||v||_V^2\\ \leq\ & \frac{c(\phi)}{2}\int_{\mathbb{D}}\big[\int_0^t h(s,.,u_1(s))-h(s,.,u_2(s))dw(s)\big]^2\,dx.\end{aligned}$$

Therefore, the integration with respect to time from 0 to t gives

$$\begin{aligned}&\frac{1}{2}||v||_V^2(t)+\frac{1}{4}\int_0^t\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)][u_1-u_2]\,dxds\\ \leq\ & 2\theta\int_0^t ||v||_V^2\,ds+\frac{1}{2}||v||_V^2(0)\\ &+\frac{c(\phi)}{2}\int_0^t\int_{\mathbb{D}}\big[\int_0^s h(\tau,.,u_1(\tau))-h(\tau,.,u_2(\tau))dw(\tau)\big]^2\,dxds.\end{aligned}$$

Now, note that $v(0)=0$ and that, by taking the expectation,

$$\begin{aligned}&\frac{1}{2}||v||_{\mathcal{V}}^2(t)+\frac{1}{4}\int_0^t E\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)][u_1-u_2]\,dxds-2\theta\int_0^t ||v||_{\mathcal{V}}^2\,ds\\ &\leq \frac{c(\phi)}{2}\int_0^t\int_0^s E\int_{\mathbb{D}}[h(\tau,.,u_1(\tau))-h(\tau,.,u_2(\tau))]^2\,dxd\tau ds\\ &\leq \frac{c(\phi)}{2}\mathbb{M}\int_0^t\int_0^s E\int_{\mathbb{D}}[\phi(u_1)-\phi(u_2)][u_1-u_2]\,dxd\tau ds.\end{aligned}$$

Then, Gronwall's lemma leads, on the one hand, to $\phi(u_1)=\phi(u_2)$ and to $h(.,.,u_1)=h(.,.,u_2)$ thanks to (2.1). On the other hand, it leads to $v=0$ and to the uniqueness of the solution.

Claim 2: Let us prove now that the solution exists. Replace first g and h by

$$g_M(t,x,u) = \int_0^T g(s,x,u)\rho_M(t-s)\,ds$$
$$h_M(t,x,u) = \int_0^T h(s,x,u)\rho_M(t-s)\,ds,$$

where one denotes by ρ_M the standard real-valued mollifier sequence. Thus, it can be assumed in this first part that $\frac{\partial g}{\partial t}$ and $\frac{\partial h}{\partial t}$ exist and satisfy the hypothesis (2.1)-(2.3) too.

If one denotes by $g_n(x,u) = g(n\Delta t, x, u)$ and $h_n(x,u) = h(n\Delta t, x, u)$, the time-discretization and the *a priori* estimates of the previous section hold.

Note that, thanks to (4.1), for any v in V, α in $L^2(0,T)$ and any β in $L^2(\Omega)$, the following variational equation holds:

$$\int_{\Omega\times Q} \frac{\partial[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]}{\partial t} v dx dt dP + \int_{\Omega\times Q} \nabla\phi(u^{\Delta t}).\nabla v + f(u^{\Delta t})\overrightarrow{B}.\nabla v dx dt dP$$
$$+ \int_{\Omega\times Q} g^{\Delta t}(.,.,u^{\Delta t}) v\ \alpha\beta\, dx dt dP = 0. \tag{5.1}$$

Thanks to proposition 2 and to the Lipschitz property of f, g and h, up to a sub-sequence denoted the same way, u, ϕ_u, f_u, g_u and h_u exist respectively in $L^2(\Omega\times Q)$, $L^2(\Omega\times(0,T);V)$ and $L^2(\Omega\times Q)$ for the last three terms, such that

$$u^{\Delta t} \rightharpoonup u;\ \phi(u^{\Delta t}) \rightharpoonup \phi_u;\ f(u^{\Delta t}) \rightharpoonup f_u;$$
$$g^{\Delta t}(.,.,u^{\Delta t}) \rightharpoonup g_u \text{ and } h^{\Delta t}(.,.,u^{\Delta t}) \rightharpoonup h_u$$

respectively in the above mentioned spaces.

Moreover, thanks to the same proposition, one has, in $L^2(\Omega\times Q)$, that

$$u^{\Delta t} - \widetilde{u}^{\Delta t} \to 0,\ u^{\Delta t} - u^{\Delta t}(.-\Delta t) \to 0 \text{ and } h^{\Delta t}(.,.,u^{\Delta t}(.-\Delta t)) \rightharpoonup h_u.$$

Then, one deduces that $\widetilde{u}^{\Delta t} \rightharpoonup u$ weakly in $L^2(\Omega\times Q)$ too and, since by construction $u^{\Delta t}(.-\Delta t)$ is adapted as an $L^2(\mathbb{D})$-valued function, the same property holds for the limit u. Further, $\widetilde{B}^{\Delta t}$ converges weakly in $L^2(\Omega\times Q)$ to $\int_0^{\cdot} h_u(s)\,dw(s)$ (see W. Grecksch *et al.* [21] Proposition 3.2 page 33 for example).

Thanks to proposition 3, and again, up to a sub-sequence denoted by the same way, one gets that $\frac{\partial[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]}{\partial t}$ converges weakly to $\frac{\partial[u - \int_0^{\cdot} h_u(s)\,dw(s)]}{\partial t}$ in $L^2(\Omega\times(0,T),V')$.

Thus, since the embedding of $\{u \in L^2(0,T;L^2(\mathbb{D})) : \frac{\partial u}{\partial t} \in L^2(0,T;V')\}$ in $C([0,T],V')$ is continuous, it follows that $u_0 = \widetilde{u}^{\Delta t}(0) = [\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}](0)$ converges weakly to $u(0)$ in V' and $u(0) = u_0$.

In conclusion, passing to the limits in (5.1) gives us that, for any v in V, α in $L^2(0,T)$ and any β in $L^2(\Omega)$, the following variational equation holds:

$$\begin{aligned} &\int_{\Omega\times(0,T)} < \frac{\partial[u - \int_0^{\cdot} h_u(s)\,dw(s)]}{\partial t}, v >_{V',V} \alpha\beta\,dtdP \\ &+ \int_{\Omega\times(0,T)} \int_{\mathbb{D}} [\nabla\phi_u.\nabla v + f_u \overrightarrow{B}.\nabla v + g_u v]\,dx \Big\}\alpha\beta\,dtdP \quad = \quad 0. \end{aligned}$$

Therefore, for any v in V, t almost everywhere in $(0,T)$ and P-almost surely in Ω,

$$< \frac{\partial[u - \int_0^{\cdot} h_u(s)\,dw(s)]}{\partial t}, v >_{V',V} + \int_{\mathbb{D}} [\nabla\phi_u.\nabla v + f_u \overrightarrow{B}.\nabla v + g_u v]\,dx \quad = \quad 0.$$

Since V is separable, one gets that t almost everywhere in $(0,T)$ and P-almost surely in Ω, for any v in V,

$$< \frac{\partial[u - \int_0^{\cdot} h_u(s)\,dw(s)]}{\partial t}, v >_{V',V} + \int_{\mathbb{D}} [\nabla\phi_u.\nabla v + f_u \overrightarrow{B}.\nabla v + g_u v]\,dx = 0.$$

In order to conclude, one needs to identify the limits ϕ_u, f_u, g_u and h_u. Such a result would be proved if the sequence $(\phi(u^{\Delta t}))$ converges in $L^2(\Omega\times Q)$. Indeed, since ϕ is a non-decreasing function, a classical principle of monotonicity ensures that $\phi_u = \phi(u)$. Then, thanks to (2.1), $f_u = f(u)$, $g_u = g(.,.,u)$ and $h_u = h(.,.,u)$.

Consider two positive integers N and M and denote by $\Delta h = \frac{T}{N}$ and by $\Delta s = \frac{T}{M}$. Then, for any v in V, one gets that

$$\begin{aligned} &\int_{\mathbb{D}} \Big\{ \frac{\partial[(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}) - (\widetilde{u}^{\Delta s} - \widetilde{B}^{\Delta s})]}{\partial t} v + \nabla[\phi(u^{\Delta t}) - \phi(u^{\Delta s})].\nabla v \Big\}\,dx \\ &= -\int_{\mathbb{D}} \Big\{ [f(u^{\Delta t}) - f(u^{\Delta s})]\overrightarrow{B}.\nabla v + [g^{\Delta t}(.,.,u^{\Delta t}) - g^{\Delta s}(.,.,u^{\Delta s})]v \Big\}\,dx. \end{aligned}$$

Following the method presented in the demonstration of the uniqueness property, one chooses the test function $v = -\Delta^{-1}w$ in V where $w = (\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}) - (\widetilde{u}^{\Delta s} - \widetilde{B}^{\Delta s})$. Since $w = (u^{\Delta t} - u^{\Delta s}) - (u^{\Delta t} - \widetilde{u}^{\Delta t}) - (\widetilde{B}^{\Delta t} - \widetilde{B}^{\Delta s}) - (\widetilde{u}^{\Delta s} - u^{\Delta s})$, one has

$$\frac{1}{2}\frac{d}{dt}||v||_V^2 + \frac{1}{4}\int_{\mathbb{D}} [\phi(u^{\Delta t}) - \phi(u^{\Delta s})](u^{\Delta t} - u^{\Delta s})\,dx - (2\theta+1)||v||_V^2$$

$$\leq \quad \frac{c(\phi)}{2}||(u^{\Delta t}-\widetilde{u}^{\Delta t})+(\widetilde{B}^{\Delta t}-\widetilde{B}^{\Delta s})+(\widetilde{u}^{\Delta s}-u^{\Delta s})||^2_{L^2(\mathbb{D})}$$
$$+\frac{C_p^2}{4}||g^{\Delta t}(.,.,u^{\Delta s})-g^{\Delta s}(.,.,u^{\Delta s})||^2_{L^2(\mathbb{D})}$$
$$\leq \quad \frac{3c(\phi)}{2}\left[||u^{\Delta t}-\widetilde{u}^{\Delta t}||^2_{L^2(\mathbb{D})}+||\widetilde{B}^{\Delta t}-\widetilde{B}^{\Delta s}||^2_{L^2(\mathbb{D})}+||\widetilde{u}^{\Delta s}-u^{\Delta s}||^2_{L^2(\mathbb{D})}\right]$$
$$+\frac{C_p^2}{4}||g^{\Delta t}(.,.,u^{\Delta s})-g^{\Delta s}(.,.,u^{\Delta s})||^2_{L^2(\mathbb{D})}.$$

Let us first integrate with respect to the time variable from 0 to t, then take the expectation, to get:

$$\frac{1}{2}E||v(t)||^2_V+\frac{1}{4}E\int_0^t\int_{\mathbb{D}}[\phi(u^{\Delta t})-\phi(u^{\Delta s})](u^{\Delta t}-u^{\Delta s})\,dxds$$
$$\leq \quad \frac{3c(\phi)}{2}\Big[||u^{\Delta t}-\widetilde{u}^{\Delta t}||^2_{L^2(\Omega\times Q)}+||u^{\Delta s}-\widetilde{u}^{\Delta s}||^2_{L^2(\Omega\times Q)}$$
$$+\int_0^t E||\widetilde{B}^{\Delta t}-\widetilde{B}^{\Delta s}||^2_{L^2(\mathbb{D})}\,ds\Big]$$
$$+(2\theta+1)\int_0^t E||v(s)||^2_V\,ds+\frac{C_p^2}{4}E||g^{\Delta t}(.,.,u^{\Delta s})-g^{\Delta s}(.,.,u^{\Delta s})||^2_{L^2(Q)}.$$

Furthermore, note that thanks to proposition 2, a constant C exists (depending mainly on C_1), such that

$$||u^{\Delta t}-\widetilde{u}^{\Delta t}||^2_{L^2(\Omega\times Q)}+||u^{\Delta s}-\widetilde{u}^{\Delta s}||^2_{L^2(\Omega\times Q)}\leq C(\Delta t+\Delta s).$$

Moreover, by using the assumptions on $\frac{\partial g}{\partial t}$ and $\frac{\partial h}{\partial t}$, a positive C exists such that

$$E||g^{\Delta t}(.,.,u^{\Delta s})-g^{\Delta s}(.,.,u^{\Delta s})||^2_{L^2(Q)}$$
$$\leq 2E||g^{\Delta t}(.,.,u^{\Delta s})-g(.,.,u^{\Delta s})||^2_{L^2(Q)}+2E||g(.,.,u^{\Delta s})-g^{\Delta s}(.,.,u^{\Delta s})||^2_{L^2(Q)}$$
$$\leq (\Delta t^2+\Delta s^2)c_g^2(T||g_0||^2_{L^2(\mathbb{D})}+||u^{\Delta s}||^2_{L^2(\mathbb{D}\times Q)})\leq C(\Delta t^2+\Delta s^2).$$

Similarly, one has

$$\int_0^t E||\widetilde{B}^{\Delta t}-\widetilde{B}^{\Delta s}||^2_{L^2(\mathbb{D})}\,ds$$
$$\leq \quad C(\Delta t+\Delta s)+C\int_0^t E\int_0^s||h(\sigma,.,u^{\Delta t})-h(\sigma,.,u^{\Delta s})||^2_{L^2(\mathbb{D})}\,d\sigma ds$$

$$\leq \quad C(\Delta t+\Delta s)+C\int_0^t E\int_0^s\int_{\mathbb{D}}[\phi(u^{\Delta t})-\phi(u^{\Delta s})](u^{\Delta t}-u^{\Delta s})\,dxd\sigma ds.$$

Thus, a positive constant C exists such that

$$\begin{aligned}&E||v(t)||_V^2+E\int_0^t\{\int_{\mathbb{D}}[\phi(u^{\Delta t})-\phi(u^{\Delta s})](u^{\Delta t}-u^{\Delta s})\,dx-C||v(s)||_V^2\}\,ds\\ \leq \quad & C(\Delta t+\Delta s)+C\int_0^t E\int_0^s\int_{\mathbb{D}}[\phi(u^{\Delta t})-\phi(u^{\Delta s})](u^{\Delta t}-u^{\Delta s})\,dxd\sigma ds.\end{aligned}$$

In particular, Gronwall's lemma ensures that

$$E||v(t)||_V^2+E\int_0^t\int_{\mathbb{D}}[\phi(u^{\Delta t})-\phi(u^{\Delta s})](u^{\Delta t}-u^{\Delta s})\,dxds\leq C(\Delta t+\Delta s)e^{Ct}. \tag{5.2}$$

It can be deduced that $G(u^{\Delta t})$ is a Cauchy sequence in $L^2(\Omega\times Q)$. Thus, it converges in $L^2(\Omega\times Q)$, as well as $\phi(u^{\Delta t})$, since, for any $x,y\in\mathbb{R}$, $|\phi(x)-\phi(y)|\leq\sqrt{c(\phi)}|G(x)-G(y)|$.

Thus, a unique solution exists. It depends on M; let us denote by u_M this solution.

Let us pass now to the limits with respect to M. Since the *a priori* estimates are kept, the same demonstration holds if one is able to prove that the sequence $(\phi(u_M))$ converges in $L^2(\Omega\times Q)$. In order to do this, consider two solutions u_M and u_N associated to g_M,h_M and g_N,h_N respectively. Therefore, the uniqueness method, with $v=-\Delta^{-1}w$, where

$$w=(u_M-u_N)-\int_0^t[h_M(s,.,u_M(s))-h_N(s,.,u_N(s))]dw(s),$$

leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||v||_V^2+\frac{1}{4}\int_{\mathbb{D}}[\phi(u_M)-\phi(u_N)][u_M-u_N]\,dx\\ \leq\quad&\frac{c(\phi)}{2}\int_{\mathbb{D}}\big[\int_0^t h_M(s,.,u_M(s))-h_N(s,.,u_N(s))dw(s)\big]^2\,dx+2\theta||v||_V^2\\ &+\int_{\mathbb{D}}[(g(.,.,u_M)-g_M(.,.,u_M))-(g(.,.,u_N)-g_N(.,.,u_N))]v\,dx\\ \leq\quad&\frac{3c(\phi)}{2}\int_{\mathbb{D}}\big[\int_0^t h(s,.,u_M(s))-h(s,.,u_N(s))dw(s)\big]^2\,dx\\ &+\frac{3c(\phi)}{2}\int_{\mathbb{D}}\big[\int_0^t h_M(s,.,u_M(s))-h(s,.,u_M(s))dw(s)\big]^2\,dx\end{aligned}$$

$$+\frac{3c(\phi)}{2}\int_{\mathbb{D}}\big[\int_0^t h_(s,.,u_N(s)) - h_N(s,.,u_N(s))dw(s)\big]^2\,dx$$
$$+\frac{C_p^2}{2}\int_{\mathbb{D}}[(g(.,.,u_M)-g_M(.,.,u_M))^2+(g(.,.,u_N)-g_N(.,.,u_N))^2]\,dx$$
$$+(2\theta+1)||v||_V^2.$$

Then, thanks to annexe 7, by integration with respect to time from 0 to t, and by taking the expectation, one gets that for all $\epsilon > 0$, there exists $\bar{N} \in \mathbb{N}$, for all $N, M \geq \bar{N}$, for all t,

$$\begin{aligned} & \frac{1}{2}||v||^2_{\mathcal{V}}(t) + \frac{1}{4}\int_0^t E\int_{\mathbb{D}}[\phi(u_M)-\phi(u_N)][u_M-u_N]\,dxds \\ \leq\; & \frac{3c(\phi)}{2}\mathbb{M}\int_0^t\int_0^s E\int_{\mathbb{D}}[\phi(u_M)-\phi(u_N)][u_M-u_N]\,dxd\tau ds \\ & +(2\theta+1)\int_0^t ||v||_V^2\,ds+\epsilon. \end{aligned}$$

Then, Gronwall's lemma ensures that a positive constant C exists such that

$$E\int_Q[\phi(u_M)-\phi(u_N)][u_M-u_N]\,dxds \leq C\epsilon.$$

Thus, $(G(u_M))$ is a Cauchy sequence and the result holds in the same way.

Let us derive now information about the speed of convergence of the method.

Remark 2. Assume that $g(t,x,u) = g(x,u)$ and $h(t,x,u) = h(x,u)$; then a positive constant C exists such that $||G(u^{\Delta t}) - G(u)||^2_{L^2(\Omega\times Q)} \leq C\Delta t$.

This result holds by passing to the limits in (5.2) with respect to Δs.

6. Applications

6.1. The stochastic porous media equation.

In this section, we consider the stochastic perturbation of the porous media equation

$$du - \Delta\phi(u)dt - \operatorname{div}(f(u)\overrightarrow{B})dt = h(.,.,u)dw \quad \text{ in } \mathbb{D}\times(0,T)\times\Omega,$$

where u denotes a saturation and $\overrightarrow{B}$ is a pressure gradient.

One considers that the initial condition is a saturation; *i.e.*, $0 \leq u_0 \leq 1$ is satisfied. Since u is a saturation too, the applications ϕ, f and h are *a priori* defined for u in $[0,1]$ and (2.1) and (2.3) are assumed in $[0,1]$. Usually,

ϕ is injective and the problem degenerates since $\phi'(0) = \phi'(1) = 0$ (see G. Gagneux *et al.* [20] for example).

Many papers exist in the literature concerning the stochastic porous media equation in a very general setting. See, for example, [5], [11], [16] and [28].

Our aim in this application is less to give a result of well posedness than to propose a qualitative study. Indeed, by taking advantage of a stochastic perturbation h depending on u, a maximum principle is proposed in order to have $0 \leq u \leq 1$. Otherwise, such a result does not seem obvious. Thus, one assumes moreover that $h(.,.,x) = 0$ if $x \leq 0$ or $x \geq 1$.

In the sequel, we propose to extend the nonlinear functions ϕ and f to $\mathbb{R}$. Then, a solution u would exist in the sense of the previous part and one has to prove that $0 \leq u \leq 1$. Therefore, u would be a saturation, independent of the extension.

Proposition 4. *A unique stochastic saturation exists to the above porous media equation with the mentioned hypothesis.*

Proof. Let us denote by $\overline{\phi}(x) = \phi(x)$ if $x \in [0,1]$, $\overline{\phi}(x) = x$ if $x < 0$, $\overline{\phi}(x) = x - 1 + \phi(1)$ else; and $\overline{f}(x) = f(x)$ if $x \in [0,1]$, $\overline{f}(x) = f(0)$ if $x < 0$, $\overline{f}(x) = f(1)$ else.

Since (2.1) and (2.3) still hold for $\bar{\phi}$ and $\bar{f}$, thanks to the previous sections, a unique u exists that is a solution to the problem with $\bar{\phi}$ and $\bar{f}$.

Consider $\varphi : \mathbb{R} \to \mathbb{R}^+$ a regular convex function, with Lipschitz conditions for φ' and φ'', such that, for any x in $[0,1]$, $\varphi(x) = 0$.

By construction of $\bar{\phi}$,

$$\varphi'(u) = \varphi'\Big([\bar{\phi}(u) - \phi(1)]^+ + 1\Big) + \varphi'\Big(-[\bar{\phi}(u)]^-\Big) = \psi(\bar{\phi}(u)),$$

where ψ is a non-decreasing Lipschitz-continuous function. Then, $\varphi'(u)$ belongs to V and Ito's formula leads to

$$\begin{aligned} & E\int_{\mathbb{D}} \varphi(u)\, dx + E\int_{]0,t[\times\mathbb{D}} [\nabla\bar{\phi}(u).\nabla\varphi'(u) + \bar{f}(u)\overrightarrow{B}.\nabla\varphi'(u)]\, dxds \\ = \; & \int_{\mathbb{D}} \varphi(u_0)\, dx + \frac{1}{2}E\int_{]0,t[\times\mathbb{D}} \varphi''(u)h^2(.,.,u)\, dxds. \end{aligned}$$

Then, since $0 \leq u_0 \leq 1$, and as, by hypothesis, $\varphi''(x)h^2(.,.,x) = 0$ for any real x, the right-hand side of the above equation is null.

Thanks to lemma 1 and since

$$E\int_{]0,t[\times\mathbb{D}} \psi'[\bar{\phi}(u)]|\nabla\bar{\phi}(u)|^2\, dxds \geq 0,$$

one gets that

$$E \int_{\mathbb{D}} \varphi(u)\, dx = 0.$$

By using a regularization argument,

$$E \int_{\mathbb{D}} (u^-)^2\, dx + E \int_{\mathbb{D}} [(u-1)^+]^2\, dx = 0$$

can be proved and one gets that $0 \leq u \leq 1$. The function u is then a saturation and the result holds with ϕ and f. □

6.2. **The stochastic viscous Buckley-Leverett equation.**

In this section we consider the formal stochastic viscous Buckley-Leverett equation

$$du - \Delta u\, dt - \operatorname{div}(f(u)\overrightarrow{B})dt = hdw \quad \text{ in } \mathbb{D} \times (0,T) \times \Omega.$$

This model corresponds to a generalization to $d > 1$ of Burgers' equation, *i.e.,* $d = 1$ and $f(x) = x^2$.

One assumes that $\phi = Id$, $g = 0$ and that h is independent of u.

Burgers' equation has been intensively studied in the literature with many extensions. Recall the papers cited in the introduction [15], [17], [21], [23] and M. Röckner *et al.* in [29] for a generalization of the classical case $f(x) = x^2$.

Usually, Lipschitz or local-Lipschitz conditions are assumed on the function f. In this application we consider that f is a $\frac{1}{2}$-Hölder-continuous function (with $f(0) = 0$ since $\operatorname{div}\overrightarrow{B} = 0$). The method consists in using a Lipschitz-approximation of f and passing to the limits with respect to this approximation.

Proposition 5. *A unique solution in the sense of Definition* 1 *exists to the above Buckley-Leverett equation with the mentioned hypothesis.*

For any positive M, consider $f_M = (f * \rho_M) \circ T_M$ where ρ_M denotes the usual mollifier sequence of support $\frac{1}{M}$ and $T_M(x) = \max[\min(x, M), -M]$. Then, f_M is a Lipschitz-continuous function and the existence and uniqueness result of the previous section holds. Let us denote by u_M this solution.

Thanks to the stochastic-energy equality, one has that a positive constant C exists such that, for any t,

$$E \int_{\mathbb{D}} u^2(t)\, dx + 2E \int_0^t \int_{\mathbb{D}} |\nabla u|^2\, dxds + 2E \int_0^t \int_{\mathbb{D}} f(u)\overrightarrow{B}.\nabla u\, dxds$$

$$= \int_0^t \int_{\mathbb{D}} h^2 \, dxds.$$

Thus, one deduces that

$$E \int_{\mathbb{D}} u^2(t) \, dx + 2E \int_Q |\nabla u|^2 \, dxds \leq C(h).$$

Moreover, for any v in V,

$$\frac{\Big| < \frac{\partial}{\partial t}[u - \int_0^t h \, dw(s)], v >_{V',V} \Big|}{||v||_V} \leq ||\nabla u||_{L^2(\mathbb{D})} + ||\overrightarrow{B}||_\infty c_P ||f_M(u)||_{L^2(\mathbb{D})}.$$

Since

$$\begin{aligned} |f_M(u)|^2 &= |\int_{\mathbb{R}} f(T_M(u) - y)\rho_M(y) \, dy|^2 \leq \int_{\mathbb{R}} |f(T_M(u) - y)|^2 \rho_M(y) \, dy \\ &\leq c(f) \int_{\mathbb{R}} |T_M(u) - y|\rho_M(y) \, dy \leq c(f)(|T_M(u)| + 1) \leq c_1 u^2 + c_2 \end{aligned}$$

one deduces that

$$\begin{aligned} &\frac{\Big| < \frac{\partial}{\partial t}[u - \int_0^t h \, dw(s)], v >_{V',V} \Big|^2}{||v||_V^2} \\ \leq \; & 2||\nabla u||^2_{L^2(\mathbb{D})} + 2||\overrightarrow{B}||^2_\infty c_P^2 [c_1 ||u||^2_{L^2(\mathbb{D})} + c_2 mes(\mathbb{D})] \end{aligned}$$

and that

$$E \int_0^T ||\frac{\partial}{\partial t}[u - \int_0^t h \, dw(s)]||^2_{V'} \, dt \leq C(h).$$

Following the idea developed in particular by H. W. Alt *et al.* in [2], based on Kolmogorov's theorem of compactness in L^1, for a given ϵ in $(0, T)$, let us integrate the equation from t to $t + \epsilon$ where t belongs to $]0, T - \epsilon[$. Let us choose $v = u_M(t + \epsilon) - u_M(t)$ and integrate again t from 0 to $T - \epsilon$. Then, one gets that

$$\begin{aligned} 0 = & \int_0^{T-\epsilon} \int_{\mathbb{D}} \Big\{ [u_M(t+\epsilon) - u_M(t)] - \int_t^{t+\epsilon} h \, dw \Big\} \Big(u_M(t+\epsilon) - u_M(t) \Big) \, dxdt \\ & + \int_0^{T-\epsilon} \int_{\mathbb{D}} \nabla \Big(\int_t^{t+\epsilon} u_M(s) \, ds \Big) . \nabla \Big(u_M(t+\epsilon) - u_M(t) \Big) \, dxdt \\ & + \int_0^{T-\epsilon} \int_{\mathbb{D}} \Big(\int_t^{t+\epsilon} f_M(u_M(s)) \, ds \Big) \overrightarrow{B} . \nabla \Big(u_M(t+\epsilon) - u_M(t) \Big) \, dxdt \\ = \; & I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon). \end{aligned}$$

i) Note that

$$\begin{aligned} I_2(\epsilon) &= \frac{1}{2}\int_0^{T-\epsilon}\frac{d}{dt}\int_{\mathbb{D}}|\nabla\int_t^{t+\epsilon}u_M(s)\,ds|^2\,dxdt \\ &= \frac{1}{2}\int_{\mathbb{D}}|\nabla\int_{T-\epsilon}^{T}u_M(s)\,ds|^2\,dx-\frac{1}{2}\int_{\mathbb{D}}|\nabla\int_0^{\epsilon}u_M(s)\,ds|^2\,dx \\ &\geq -\frac{\epsilon}{2}\int_0^{\epsilon}\int_{\mathbb{D}}|\nabla u_M|^2(s)\,dxds=-\frac{\epsilon}{2}||u_M||^2_{L^2(0,T;V)}. \end{aligned}$$

ii) Moreover, one gets that

$$I_3(\epsilon)=\int_0^{\epsilon}\int_0^{T-\epsilon}\int_{\mathbb{D}}f_M(u_M(\alpha+t))\overrightarrow{B}.\nabla\Big(u_M(t+\epsilon)-u_M(t)\Big)\,dxdtd\alpha,$$

$$\begin{aligned} |I_3(\epsilon)| &\leq ||\overrightarrow{B}||_\infty\int_0^{\epsilon}\Big[\int_0^{T-\epsilon}||f_M(u_M(\alpha+t))||^2_{L^2(\mathbb{D})}\,dt\Big]^{\frac{1}{2}}\times \\ &\qquad\qquad\times\Big[\int_0^{T-\epsilon}||u_M(t+\epsilon)-u_M(t)||^2_V\,dt\Big]^{\frac{1}{2}}d\alpha \\ &\leq \epsilon||\overrightarrow{B}||_\infty\sqrt{c_1||u_M||^2_{L^2((0,T)\times\mathbb{D})}+c_2}\,||u_M(t+\epsilon)-u_M(t)||_{L^2(0,T-\epsilon;V)} \\ &\leq 2\epsilon||\overrightarrow{B}||_\infty\sqrt{c_1||u_M||^2_{L^2((0,T)\times\mathbb{D})}+c_2}\,||u_M||_{L^2(0,T;V)}. \end{aligned}$$

Thus, $E[\sup_{\epsilon\in(0,T)}\frac{1}{\epsilon}|I_1(\epsilon)|]\leq C||u_M||^2_{L^2(0,T;V)}$.

Following J. U. Kim [24], denote by

$$\Theta(u_M)=||u_M||^2_{L^2(0,T;V)}+||\frac{\partial}{\partial t}[u_M-\int_0^t h\,dw(s)]||^2_{L^2(0,T,V'}+\sup_{\epsilon\in(0,T)}\frac{1}{\epsilon}|I_1(\epsilon)|],$$

and by $\widetilde{\Omega}=\bigcup_{L\geq 2}\bigcap_{m\geq 1}\bigcup_{k\geq m}\{\Theta(u_M)\leq L\}$. Thanks to the above estimations, one deduces that $P(\widetilde{\widetilde{\Omega}})=1$. Then, for P-almost surely ω, a positive constant $L(\omega)$ and a sub-sequence still denoted by u_M exist such that $\Theta(u_M)\leq L(\omega)$ and, moreover, $(t,x)\mapsto\int_0^t h\,dw$ belongs to $L^2(Q)$.

Therefore, $u=u(\omega)$ exists in $L^2(0,T;V)$ with moreover $\frac{\partial}{\partial t}[u-\int_0^t h\,dw(s)]$ in $L^2(0,T,V')$ such that u_M converges weakly to u in $L^2(0,T;V)$ and $\frac{\partial}{\partial t}[u_M-\int_0^t h\,dw(s)]$ converges weakly to $\frac{\partial}{\partial t}[u-\int_0^t h\,dw(s)]$ in $L^2(0,T,V')$.
In particular, $u_0=u_M(0)$ converges to $u(0)$ in V'.

Since $f_M^2(u_M)\leq c_1u_M^2+c_2$, it can be assumed, up to a sub-sequence denoted in the same way, that $f_M(u_M)$ converges weakly to some f_u in

$L^2(Q)$. The problem is then to prove that $f_u = f(u)$. This result would come from the almost everywhere convergence of u_M.

In order to prove this, for any $\epsilon \in (0,T)$, note that

$$\int_0^{T-\epsilon}\int_{\mathbb{D}}[u_M(t+\epsilon)-u_M(t)]^2dxdt \leq \int_0^{T-\epsilon}\int_{\mathbb{D}}[\int_t^{t+\epsilon}hdw]^2dxdt + 2\epsilon L(\epsilon).$$

Thus, uniformly with respect to M, the right-hand side converges to 0 with ϵ. Then, the same result holds for

$$\int_0^{T-\epsilon}\int_{\mathbb{D}}|u_M(t+\epsilon)-u_M(t)|\,dxdt.$$

Then, by previous calculations, the sequence (u_M) is equicontinuous in $L^1(Q)$ and consequently it converges to u in $L^1(Q)$. Then, up to a subsequence still denoted by u_M, it converges almost everywhere in Q and $f_u = f(u)$.

It follows that, for any v in V,

$$< \frac{\partial}{\partial t}[u - \int_0^t h\,dw], v >_{V',V} + \int_{\mathbb{D}} \nabla u.\nabla v + f(u)\overrightarrow{B}.\nabla v dx = 0.$$

If one denotes by $\hat{u}$ another solution, for any v in V, one gets that

$$< \frac{\partial}{\partial t}[u-\hat{u}], v >_{V',V} + \int_{\mathbb{D}} \nabla[u-\hat{u}].\nabla v + [f(u)-f(\hat{u})]\overrightarrow{B}.\nabla v dx = 0.$$

For a given $\mu > 0$, set $v = p_\mu[u-\hat{u}]$ where $p_\mu(x) = 0$ if $x < \frac{\mu}{e}$, 1 if $x > \mu$ and $\ln(\frac{ex}{\mu})$ else. Note that p_μ is a Lipschitz-continuous function and denote by $P_\mu = \int_0^x p_\mu(s)\,ds$. Then,

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{D}} P_\mu[u-\hat{u}]dx \\ &\quad + \int_{\mathbb{D}} p'_\mu[u-\hat{u}]|\nabla[u-\hat{u}]|^2 + [f(u)-f(\hat{u})]p'_\mu[u-\hat{u}]\overrightarrow{B}.\nabla[u-\hat{u}]dx. \end{aligned}$$

And by construction,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{D}} P_\mu[u-\hat{u}]dx + \frac{1}{2}\int_{\mathbb{D}} p'_\mu[u-\hat{u}]|\nabla[u-\hat{u}]|^2 \\ \leq\ & C\int_{\{\frac{\mu}{e}<u-\hat{u}<\mu\}} |u-\hat{u}|p'_\mu[u-\hat{u}]dx. \end{aligned}$$

Thus,

$$\int_{\mathbb{D}} P_\mu[u-\hat{u}]\,dx \ \leq\ C measure(\{\frac{\mu}{e} < u-\hat{u} < \mu\}) + \int_{\mathbb{D}} P_\mu[0]\,dx.$$

Passing to the limits leads to $u \leq \hat{u}$. Since one is able to prove in the same way that $u \geq \hat{u}$, the solution to the above problem is unique and all the sequence converges, not only a sub-sequence depending on ω.

Then, u_M converges a.s. to u in $L^2(Q)$ and weakly in $L^2(0,T;V)$. One deduces that u is measurable as a $L^2(Q)$-valued function, adapted to the filtration, and weakly measurable as a $L^2(0,T;V)$-valued function. Since this last set is separable, the theorem of Pettis yields to the measurability of u as a $L^2(0,T;V)$-valued function and a solution exists.

For the uniqueness of the solution, one has just to use the same method than the one given above, based on the approximation of the sgn^+ function by p_μ.

7. ANNEXES

Consider a bounded sequence (u_n) in $L^\infty(0,T;L^2(\Omega\times\mathbb{D}))$ and denote by

$$D_n := \int_{\Omega\times Q}[g_n(t,x,u_n(t,x)) - g(t,x,u_n(t,x))]^2\,dxdtdP,$$

where $g_n(t,x,u) = \int_0^T g(s,x,u)\rho_n(t-s)\,ds$ and where one denotes by ρ_n the standard real valued mollifier sequence.

In addition to the hypothesis presented in the introduction, let us consider the following.

Claim 1: Assume that $g(t,x,u) = \alpha(t)\beta(x,u)$. Since α can be assumed nontrivial, $t_0 \in [0,T[$ exists such that $|\beta(x,u)| \leq \frac{c_g\sqrt{g_0^2(x)+u^2}}{|\alpha(t_0)|}$.

$$\begin{aligned} D_n &= \int_{\Omega\times Q}[\int_0^T \alpha(s)\rho_n(t-s)\,ds - \alpha(t)]^2[\beta(x,u_n(t,x))]^2\,dxdtdP \\ &\leq \frac{c_g^2}{|\alpha(t_0)|^2}\int_{\Omega\times Q}[\int_0^T \alpha(s)\rho_n(t-s)\,ds - \alpha(t)]^2[g_0^2+u_n^2]\,dxdtdP \\ &\leq \frac{c_g^2}{|\alpha(t_0)|^2}\int_0^T[\int_0^T \alpha(s)\rho_n(t-s)\,ds - \alpha(t)]^2\,dt\,\times \\ &\qquad\times[||g_0||^2_{L^2(\mathbb{D})} + ||u_n||^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{D}))}]. \end{aligned}$$

Since this last term converges to 0 when n goes to infinity, the result holds.

Claim 2: Assume that g_2 in $L^2(Q)$ is such that

$$|\frac{\partial g}{\partial t}(t,x,u)|^2 \leq C(g_2(t,x)^2 + |u|^2)$$

exists where C is an arbitrary positive constant. Then,

$$\begin{aligned}
D_n &= 2\int_{\Omega\times Q}[\int_0^T \int_s^t \frac{\partial g_n}{\partial t}(\sigma,x,u_n(t,x))\,d\sigma\rho_n(t-s)\,ds]^2\,dxdtdP \\
&\quad +2\int_{\Omega\times Q}[(1-\int_0^T \rho_n(t-s)\,ds)g(t,x,u_n(t,x))]^2\,dxdtdP \\
&\leq 2\int_{\Omega\times Q}\int_0^T [\int_s^t \frac{\partial g_n}{\partial t}(\sigma,x,u_n(t,x))\,d\sigma]^2\rho_n(t-s)\,ds\,dxdtdP \\
&\quad +C\int_0^T (1-1_{(0,T)}*\rho_n)^2\,dt \\
&\leq 2\int_{\Omega\times Q}\int_0^T \int_s^t [\frac{\partial g_n}{\partial t}(\sigma,x,u_n(t,x))]^2\,d\sigma|t-s|\rho_n(t-s)\,ds\,dxdtdP+\epsilon(n) \\
&\leq \frac{C}{n}\int_{\Omega\times Q}\int_0^2\int_s^t [g_2(\sigma,x)^2+|u_n(t,x)|^2]\,d\sigma\rho_n(t-s)\,ds\,dxdtdP+\epsilon(n) \\
&\leq \frac{CT}{n}[||g_2||^2_{L^2(Q)}+||u_n||^2_{L^2(\Omega\times Q)}]+\epsilon(n),
\end{aligned}$$

where $\epsilon(n)$ converges to 0 as n goes to infinity and the result holds.

Claim 3: Assume that the following uniform integrability type assumption holds:

$$\lim_{k\to\infty}\sup_n \int_{|u_n|>k}[g_n(.,.,u_n)-g(.,.,u_n)]^2\,dtdxdP=0.$$

Such an assumption is satisfied if the growth rate of $u\mapsto g(.,.,u)$ at infinity is less than 1; *i.e.*, if γ exists in $(0,2)$ and g_1 in $L^2(\mathbb{D})$ is such that $|g(t,x,u)|^2\leq C(g_1^2(x)+|u|^\gamma)$ where C is an arbitrary positive constant. Indeed, for any positive real M,

$$\begin{aligned}
&\int_{|u_n|>k}[g_n(.,.,u_n)-g(.,.,u_n)]^2\,dtdxdP \\
\leq\;& 2\int_{|u_n|>k}[g_n(.,.,u_n)^2+g(.,.,u_n)^2]\,dtdxdP \\
\leq\;& 4C\int_{|u_n|>k}[g_1^2+u_n^\gamma]\,dtdxdP \\
\leq\;& 4C\int_{|u_n|>k}[g_1^2-\min(M,g_1^2)+\min(M,g_1^2)+u_n^\gamma]\,dtdxdP
\end{aligned}$$

$$\leq \quad 4C\int_{\mathbb{D}}[g_1^2 - \min(M, g_1^2)]\,dx + 4C[\frac{M}{k^2} + \frac{1}{k^{2-\gamma}}]\int_{|u_n|>k} u_n^2\,dtdxdP.$$

Therefore, for any positive M, one gets that

$$\begin{aligned}&\limsup_{k\to\infty}\sup_n \int_{|u_n|>k}[g_n(.,.,u_n) - g(.,.,u_n)]^2\,dtdxdP \\ \leq \quad & 4C\int_{\mathbb{D}}[g_1^2 - \min(M, g_1^2)]\,dx,\end{aligned}$$

and the result holds since the right-hand side of the above inequality converges to 0 as M goes to infinity.

Let us prove now that D_n goes to 0.

Part 1. Consider $a < b$ and assume that the support of $u \mapsto g(t, x, u)$ is a compact set included in $[a, b]$. Such a function satisfies the above assumption and g belongs to $L^2(Q, C_b(\mathbb{R}))$. Indeed, thanks to the growth assumptions on g, one just needs to give information on the measurability of g as a $C_b(\mathbb{R})$-valued function. This last point is detailed in H. Berliocchi *et al.* [10] (Remark 5). Thus, g_n converges to g in $L^2(Q, C_b(\mathbb{R}))$ and one deduces that

$$\begin{aligned}D_n \quad &= \quad \int_{\Omega\times Q}[g_n(t, x, u_n(t, x)) - g(t, x, u_n(t, x))]^2\,dxdtdP \\ &\leq \quad \int_Q ||g_n(t, x, .) - g(t, x, .)||^2_{C_b(\mathbb{R})}\,dxdt.\end{aligned}$$

Then, the result holds since the last term vanishes when n goes to infinity.

Part 2. For any positive number k, denote by φ the even function defined for the positive real values s by $\varphi(s) = \min(1, \max(0, k + 1 - s))$. Then, one has that

$$\begin{aligned}D_n \quad = \quad & \int_{\Omega\times Q}[g_n(t, x, u_n(t, x)) - g(t, x, u_n(t, x))]^2\varphi(u_n)^2\,dxdtdP \\ & + \int_{\Omega\times Q}[g_n(t, x, u_n(t, x)) - g(t, x, u_n(t, x))]^2[1 - \varphi(u_n)]^2\,dxdtdP \\ \leq \quad & \int_{\Omega\times Q}[g_n(t, x, u_n(t, x)) - g(t, x, u_n(t, x))]^2\varphi(u_n)^2\,dxdtdP \\ & + \int_{|u_n|>k}[g_n(t, x, u_n(t, x)) - g(t, x, u_n(t, x))]^2\,dxdtdP.\end{aligned}$$

For any $\epsilon > 0$, consider k such that

$$\int_{|u_n|>k} [g_n(t,x,u_n(t,x)) - g(t,x,u_n(t,x))]^2 \, dxdtdP \leq \epsilon$$

for any n. Since, the support of $u \mapsto g(t,x,u)\varphi(u)$ is a compact set included in $[-k-1, k+1]$, thanks to Part 1, $\limsup_{n\to\infty} D_n \leq \epsilon$ and the result holds.

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