SINGULAR LIMITS FOR 4-DIMENSIONAL SEMILINEAR ELLIPTIC PROBLEMS WITH EXPONENTIAL NONLINEARITY ADDING A SINGULAR SOURCE TERM GIVEN BY DIRAC MASSES

Makkia Dammak and Taieb Ouni
Département de Mathématiques, Faculté des Sciences de Tunis
Campus Universitaire 2092 Tunis, Tunisie

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Abstract. We study the existence of solutions having singular limits for some four-dimensional semilinear elliptic problems with exponential nonlinearity and a singular source term given by Dirac masses with Navier boundary condition. In particular we extend the result of [2].

1. Introduction and statement of the results

In this paper, we will use the method of domain decomposition to study the following problem:

$$\begin{cases} \Delta^2 u &= \rho^4 e^u - 32\pi^2\, \alpha\, \delta_0 \quad \text{in} \quad \Omega \\ u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = B_1 \subset \mathbb{R}^4$ the unit ball centered at the origin, ρ is a parameter that tends to 0 and α is a positive function defined in a neighborhood of 0 in $\mathbb{R}$. We denote by ε the smallest positive number satisfying

$$\rho^4 = \frac{384\, \varepsilon^4}{(1+\varepsilon^2)^4}\ . \tag{1.2}$$

We will suppose in the following that the function α satisfies

$$(A_\alpha)\ \alpha\, \varepsilon^{-1/(\alpha+1)(\alpha+2)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

In particular, if we take $\alpha(\varepsilon) = \mathcal{O}(\varepsilon^{1/2})$, then the condition (A_α) is satisfied.

Let G be the Green's function, solution of the problem

$$\begin{cases} \Delta^2 G &= 64\pi^2 \delta_0 \quad \text{in} \quad \Omega \\ G &= \Delta G = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.3}$$

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and denote by $H(x) = G(x) + 8\log r$ its regular part function. Here, $r = |x|$.

Setting $v = u + \frac{1}{2}\,\alpha\,G$, it is clear that u solves (1.1) if and only if v solves the following problem:

$$\left\{\begin{array}{rcll} \Delta^2 v &=& \rho^4 |x|^{4\alpha} e^{-\frac{1}{2}\alpha\,H} e^v & \text{in} \quad \Omega \\ \Delta v &=& v = 0 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{1.4}$$

In the general setting, we will consider the following problem:

$$\left\{\begin{array}{rcll} \Delta^2 u &=& \rho^4 |x|^{4\alpha} f(|x|) e^u & \text{in} \quad \Omega \\ \Delta u &=& u = 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1.5}$$

where $f : [0, +\infty) \longrightarrow \mathbb{R}$ is a smooth positive function. Our main result reads as follows:

Theorem 1. *Let $\Omega = B_1$ be the unit ball in $\mathbb{R}^4$ and assume that α satisfies (A_α). Then there exists a one-parameter family of radial solutions $(u_{\alpha,\rho})_{0<\rho<\rho_0}$ of (1.5) such that*

$$\lim_{\rho\to 0} u_{\alpha,\rho} = G \quad \text{in } C^\infty_{loc}(B_1\backslash\{0\}).$$

Semilinear equations involving fourth-order elliptic operators and exponential nonlinearity appear naturally in conformal geometry. Recently, there has been considerable interest in equations involving the biharmonic operator Δ^2. A particular feature of the biharmonic operator is that it is conformally invariant. More precisely, on $(\mathbb{M}, g)$ a smooth 4-dimensional Riemannian manifold, there exists a fourth-order conformal covariant operator, independently discovered by Paneitz in 1983, with the leading symbol the Bi-Laplace operator

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}\,S_g\,I - 2\,\mathrm{Ric}_g\right)d, \tag{1.6}$$

where δ denotes the divergence, S_g the scalar curvature of g, Ric_g the Ricci tensor of the metric g and d the deRham differential. The Paneitz operator on the 4-manifold satisfies the conformal covariant property

$$e^{4w}\,P_{e^{2w}g} = P_g. \tag{1.7}$$

Also the Paneitz operator applied to a conformal factor determines a fourth-order curvature invariant which we will call the Q-curvature defined as

$$Q_g = \frac{1}{12}\left(-\Delta_g S_g + S_g^2 - 3\,|\mathrm{Ric}_g|^2\right).$$

Recall that under a conformal change of the metric

$$g_w = e^{2w}\,g,$$

we have

$$P_g\, w + 2\, Q_g = 2\, Q_{g_w}\, e^{4w}, \tag{1.8}$$

where Q_{g_w} is the Q-curvature for g_w.

There are two reasons that make this Q-curvature equation attractive to study. The first consideration comes from the analytic point of view, namely that the generic singularities of the Q-curvature equation are isolated points. The second consideration comes from geometry: the Q-curvature prescribed by the Paneitz operator can be viewed as part of the integrand in the Chern-Gauss-Bonnet formula:

$$8\pi^2 \chi(\mathbb{M}) = \int_{\mathbb{M}} (\frac{1}{4}|W_g|^2 + 2Q_{g_w})\, dv,$$

where $\chi(\mathbb{M})$ is the Euler characteristic of $\mathbb{M}$ and W denotes the Weyl tensor. Note that $|W_g|^2 dv$ is a pointwise conformal invariant, thus the integration of Q_{g_w} is conformally invariant. Since the Q-curvature contains information about the Ricci tensor, it influences the geometry of the underlying manifold directly.

In the special case where $(\mathbb{M}, g) = (\mathbb{R}^4, g_{eucl})$, the Paneitz operator is simply given by

$$P_{g_{eucl}} = \Delta^2,$$

so the equation (1.8) reduces to

$$\Delta^2\, w = Q_{g_w}\, e^{4w},$$

the solutions of which give rise to conformal metrics $g_w = e^{2w}\, g_{eucl}$ whose Q-curvature is given by Q_{g_w}.

In dimension two, the analogue of the Q-curvature is the Gauss curvature and the corresponding problem is

$$\begin{cases} -\Delta u &= \rho^2 e^u - 4\pi \displaystyle\sum_{i=1}^{N} \alpha_i \delta_{p_i} \quad \text{in} \quad \mathcal{D} \\ \quad u &= 0 \qquad\qquad\qquad\qquad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1.9}$$

where $\mathcal{D} \subset \mathbb{R}^2$ is a regular bounded domain, ρ is a parameter tending to 0, $\Lambda := \{p_1, \cdots, p_N\} \subset \mathcal{D}$ is the set of singular sources and where δ_{p_i} denotes the Dirac mass at p_i.

Esposito in [5] has proved the existence of solutions to the problem (1.9) having a prescribed singular set S for the limits. To describe his result we need to introduce some notation. Let $\Gamma(x, x')$ be the Green's function defined on $\mathcal{D}\times\mathcal{D}$, the solution of

$$\begin{cases} -\Delta\Gamma(x,x') &= 8\pi\delta_{x=x'} & \text{in} & \mathcal{D} \\ \Gamma(x,x') &= 0 & \text{on} & \partial\mathcal{D} \end{cases}$$

and let

$$h(x,x') = \Gamma(x,x') + 4\log|x-x'|,$$

be the regular part of Γ. Problem (1.9) is equivalent to solving for

$$v = u + \frac{1}{2}\sum_{i=1}^{N}\alpha_i\Gamma(\cdot,p_i)$$

the equation

$$\begin{cases} -\Delta v &= \rho^2\prod_{i=1}^{N}|x-p_i|^{2\alpha_i}\, e^{-\frac{1}{2}\sum_{i=1}^{N}\alpha_i h(x,p_i)}e^{v} & \text{in} & \mathcal{D} \\ v &= 0 & \text{on} & \partial\mathcal{D}. \end{cases} \tag{1.10}$$

For a given smooth function $f:\mathcal{D}\longrightarrow(0,+\infty)$ consider the following "general model" problem

$$\begin{cases} -\Delta v &= \rho^2\prod_{i=1}^{N}|x-p_i|^{2\alpha_i}f(x)e^{v} & \text{in} & \mathcal{D} \\ v &= 0 & \text{on} & \partial\mathcal{D}, \end{cases} \tag{1.11}$$

where $\Lambda=\{p_1,\cdots,p_N\}\subset\mathcal{D}$ and α_i are positive numbers. For $1\le s\le N$ and $m\in\mathbb{N}$, we denote

$$\begin{aligned} \mathcal{F}(x_1,\cdots,x_m) = {} & \sum_{j=1}^{m}h(x_j,x_j) + \sum_{i\neq j}\Gamma(x_i,x_j) \\ & + 4\sum_{i=1}^{s}\sum_{j=1}^{m}\alpha_i\log(|x_j-p_i|) + 2\sum_{j=1}^{m}\log(f(x_j)) \end{aligned} \tag{1.12}$$

which is well defined for $x_i\neq x_j$ when $i\neq j$. Let

$$\mathcal{G}(x_1,\cdots,x_m,w_1,\cdots,w_s) = \sum_{j=1}^{m}\sum_{i=1}^{s}(1+\alpha_i)\Gamma(x_j,w_i). \tag{1.13}$$

$\mathcal{G}$ is well defined for $x_j\neq w_i$ with $x_j\in\mathcal{D}$, $w_i\in\mathcal{D}$. Esposito in [5] has proved the following.

Theorem 2. [5] *Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth open set, f a smooth positive function and $\{\alpha_1, \cdots, \alpha_N\} \subset (0, +\infty)\backslash\mathbb{N}$ be a set of real numbers. We have the following.*

a) *Let $S = \{p_{j_1}, \cdots, p_{j_s}\} \subset \Lambda$, then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem* (1.11) *such that*

$$v_\rho \longrightarrow \sum_{i=1}^{s}(1+\alpha_{j_i})\Gamma(\cdot, p_{j_i}),$$

as $\rho \longrightarrow 0$, in $\mathcal{C}^{2,\beta}_{loc}(\mathcal{D}\backslash S)$ for $\beta \in (0,1)$.

b) *Let $S = \{q_1, \cdots, q_m\} \subset \mathcal{D}\backslash\Lambda$ and $(q_1, \cdots, q_m)$ be a nondegenerate critical point of $\mathcal{F}$ such that $\Delta \log f(q_1) = \cdots = \Delta \log f(q_m) = 0$; then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem* (1.11) *such that*

$$v_\rho \longrightarrow \sum_{i=1}^{m}\Gamma(\cdot, q_i),$$

as $\rho \longrightarrow 0$, in $\mathcal{C}^{2,\beta}_{loc}(\mathcal{D}\backslash S)$ for $\beta \in (0,1)$.

c) *Let S be such that $S \cap \Lambda = \{p_{j_1}, \cdots, p_{j_s}\}, S\backslash\Lambda = \{q_1, \cdots, q_m\}$ and $(q_1, \cdots, q_m)$ a nondegenerate critical point of the function $\mathcal{F}(q_1, \cdots, q_m) + \mathcal{G}(q_1, \cdots, q_m, p_{j_1}, \cdots, p_{j_s})$ such that $\Delta \log f(q_1) = \cdots = \Delta \log f(q_m) = 0$; then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem* (1.11) *such that*

$$v_\rho \longrightarrow \sum_{k=1}^{s}(1+\alpha_{j_k})\Gamma(\cdot, p_{j_k}) + \sum_{i=1}^{m}\Gamma(\cdot, q_i),$$

as $\rho \longrightarrow 0$, in $\mathcal{C}^{2,\beta}_{loc}(\mathcal{D}\backslash S)$ for $\beta \in (0,1)$.

In order to prove our result, approximate solutions are given and some properties of linearized operators are studied. We will use a matching argument inspired from [2], see also [1]. Throughout the paper, the symbol c denotes always a positive constant independent of ε which might change from one line to another.

2. Rotationally symmetric approximate solutions

Letting $\alpha > 0$ be a real number, we first describe the rotationally symmetric approximate solutions of

$$\Delta^2 v - \rho^4 |x|^{4\alpha}\, e^v = 0 \tag{2.1}$$

in $\mathbb{R}^4$. Note that equation (2.1) is invariant under dilation but not under translation.

Given $\varepsilon > 0$, we define the function

$$v_\varepsilon(x) := 4\,\log(1+\varepsilon^2) - 4\,\log(\varepsilon^2 + |x|^2)$$

which is a solution of

$$\Delta^2 v - \rho^4 e^v = 0, \tag{2.2}$$

where ρ is given by (1.2). We notice that equation (2.2) is invariant under the following transformation: if v is a solution of (2.2) and if $\tau > 0$, then $v(\tau\,\cdot) + 4\log\tau$ is also a solution of (2.2). Taking into account this observation, we define for $\tau > 0$ and $\alpha > 0$ the function

$$v_{\varepsilon,\tau,\alpha}(x) := \log\,\frac{(1+\varepsilon^2)^4\tau^4(4\alpha^2+8\alpha+6)(\alpha+1)^2}{6\,(\varepsilon^2+\tau^2\,|x|^{2(1+\alpha)})^4}\,. \tag{2.3}$$

Easy computations show that $v_{\varepsilon,\tau,\alpha}$ satisfies the equation

$$\begin{aligned}
&\Delta^2 v_{\varepsilon,\tau,\alpha} - \rho^4\,|x|^{4\alpha}\,e^{v_{\varepsilon,\tau,\alpha}} \\
&= -\frac{64\alpha(\alpha+2)(\alpha+1)^2\,\tau^2\varepsilon^2|x|^{2(\alpha-1)}}{(\varepsilon^2+\tau^2|x|^{2(1+\alpha)})^4}\left(\tau^4\,|x|^{4(\alpha+1)}+\varepsilon^4\right)
\end{aligned} \tag{2.4}$$

in $\mathbb{R}^4$. We will use it as an approximate solution of (2.1). We notice that in dimension two the equation $\Delta\,u + \rho^2|x|^{2\alpha}e^u = 0$ has an explicit solution on $\mathbb{R}^2$, see [5]. Here we do not have an explicit solution of (2.1) but we will construct a solution by perturbing the approximate solution given by (2.3). We define the linear fourth-order elliptic operator

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+|x|^2)^4}$$

which corresponds to the linearization of $\Delta^2 v - 24\,e^v = 0$ about the approximate solution $v_{1,1,0}$ which has been defined above.

3. Construction of solutions without boundary conditions

We define

$$R_{\varepsilon,\alpha} := (\tau/\varepsilon)^{1/(\alpha+1)} r_{\varepsilon,\alpha} \tag{3.1}$$

with

$$r_{\varepsilon,\alpha} := \max(\sqrt{\alpha}, \varepsilon^{1/(\alpha+2)}). \tag{3.2}$$

Definition 1. *Given $k \in \mathbb{N}$, $\beta \in (0,1)$ and $\mu \in \mathbb{R}$, the Hölder weighted spaces $\mathcal{C}^{k,\beta}_\mu(\mathbb{R}^4)$ are defined as the spaces of functions $w \in \mathcal{C}^{k,\beta}_{loc}(\mathbb{R}^4)$ endowed with the following norm:*

$$\|w\|_{\mathcal{C}^{k,\beta}_\mu(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\beta}(\bar{B}_1)} + \sup_{r\geq 1} \left((1+r^2)^{-\mu/2} \, \|w(r\ .)\|_{\mathcal{C}^{k,\beta}(\bar{B}_1 - B_{1/2})} \right).$$

We then define the subspace of radial functions in $\mathcal{C}^{k,\beta}_\mu(\mathbb{R}^4)$ by

$$\mathcal{C}^{k,\beta}_{rad,\mu}(\mathbb{R}^4) = \{ f \in \mathcal{C}^{k,\beta}_\mu(\mathbb{R}^4) : f(x) = f(|x|), \forall\ x \in \mathbb{R}^4 \}.$$

First, we recall the surjectivity result of $\mathbb{L}$ given in [2].

Proposition 1. [2] *Assume that $\mu > 0$ and $\mu \notin \mathbb{Z}$, then $\mathbb{L} : \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4) \longrightarrow \mathcal{C}^{0,\beta}_{rad,\mu-4}(\mathbb{R}^4)$ is surjective.*

In the following, we denote by $\mathcal{G}_\mu$ a right inverse of $\mathbb{L}$. Notice that finding a solution v of (2.1) in $B_{r_{\varepsilon,\alpha}}$ is equivalent to finding a solution w of

$$\Delta^2 w - 24|x|^{4\alpha}\, e^w = 0 \tag{3.3}$$

in $B_{R_{\varepsilon,\alpha}}$ by simply using the following transformation:

$$w(x) = v((\varepsilon/\tau)^{1/(\alpha+1)}x) + 8\log\varepsilon - 4\log((1+\varepsilon^2)\tau/2). \tag{3.4}$$

If we write in $B_{R_{\varepsilon,\alpha}}$

$$w(x) = v_{1,1,\alpha}(x) + h(x),$$

then (3.3) can be written as

$$\begin{aligned} \mathbb{L}\ h = {} & \frac{C_\alpha |x|^{4\alpha}}{(1+|x|^{2(\alpha+1)})^4}(e^h - h - 1) \\ & + \frac{D_\alpha\, |x|^{2(\alpha-1)}}{(1+|x|^{2(\alpha+1)})^4} \left(|x|^{4(\alpha+1)} + 1 \right) - V_\alpha(x)\, h \end{aligned} \tag{3.5}$$

in $B_{R_{\varepsilon,\alpha}}$, where $C_\alpha = 64(4\alpha^2 + 8\alpha + 6)(\alpha+1)^2$, $D_\alpha = 64\alpha(\alpha+2)(\alpha+1)^2$ and

$$V_\alpha(x) = \frac{384}{(1+|x|^2)^4} - \frac{C_\alpha\, |x|^{4\alpha}}{(1+|x|^{2(\alpha+1)})^4}\ . \tag{3.6}$$

Observe that, for $\alpha > 0$ small enough, there exists $c > 0$ such that

$$|V_\alpha(x)| \leq\ c\, \frac{1 + |\log|x||}{(1+|x|^2)^4}\ \alpha\ . \tag{3.7}$$

We will need the following :

Definition 2. *Given* $\bar{r} \geq \frac{1}{2}$, $k \in \mathbb{N}$, $\beta \in (0,1)$ *and* $\mu \in \mathbb{R}$, *the weighted space* $\mathcal{C}^{k,\beta}_{\mu}(B_{\bar{r}})$ *is defined to be the space of functions* $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ *endowed with the norm*

$$\|w\|_{\mathcal{C}^{k,\beta}_{\mu}(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_{1/2})} + \sup_{1/2 \leq r \leq \bar{r}} \Big(r^{-\mu} \, \|w(r\,\cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_1 - B_{1/2})} \Big).$$

For all $\sigma \geq 1$, we denote by

$$\xi_\sigma : \mathcal{C}^{0,\beta}_{\mu}(\bar{B}_\sigma) \longrightarrow \mathcal{C}^{0,\beta}_{\mu}(\mathbb{R}^4)$$

the extension operator defined by $\xi_\sigma = f$ in $\bar{B}_\sigma$ and

$$\xi_\sigma\,(f)(x) = \chi\Big(\frac{|x|}{\sigma}\Big) f\Big(\sigma\,\frac{x}{|x|}\Big) \quad \text{in } \mathbb{R}^4 - \bar{B}_\sigma,$$

where $t \longmapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \leq 1$ and identically equal to 0 for $t \geq 2$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\xi_\sigma(w)\|_{\mathcal{C}^{0,\beta}_{\mu}(\mathbb{R}^4)} \leq \; c\,\|w\|_{\mathcal{C}^{0,\beta}_{\mu}(\bar{B}_\sigma)}. \tag{3.8}$$

Fixing $\mu \in (0,1)$ and making use of Proposition 1, solving equation (3.5) is then equivalent to finding a fixed point h in a small ball of $\mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$. This will be a solution of the following equation:

$$\begin{aligned} h = \mathcal{G}_\mu \circ \xi_{R_{\varepsilon,\alpha}} \Big(& \frac{C_\alpha |x|^{4\alpha}\,\big(e^h - 1 - h\big)}{(1+|x|^{2(\alpha+1)})^4} \\ & + \frac{D_\alpha |x|^{2(\alpha-1)}(|x|^{4(\alpha+1)}+1)}{(1+|x|^{2(\alpha+1)})^4} - V_\alpha(x)\,h \Big) = \mathcal{G}_\mu \circ \xi_{R_{\varepsilon,\alpha}} \circ S(h). \end{aligned} \tag{3.9}$$

We have that

$$|S(0)| \leq c\, D_\alpha \sup_{r \leq R_{\varepsilon,\alpha}} |x|^{2(\alpha-1)}\,(1+|x|^{2(\alpha+1)})^{-4} \,\Big|\,|x|^{4(\alpha+1)} + 1\Big|;$$

this implies $\sup_{r \leq R_{\varepsilon,\alpha}} (1+|x|^2)^{(4-\mu)/2}\,|S(0)| \leq c\,\alpha$.

Now fix $\alpha_0 > 0$; for $\alpha \in (0,\alpha_0)$ and for $h_1,\ h_2 \in B(0, 2c_\kappa\,\alpha) \subset \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$, we have

$$\begin{aligned} & \sup_{r \leq R_{\varepsilon,\alpha}} (1+r^2)^{(4-\mu)/2}\,|S(h_2) - S(h_1)| \\ & \leq c \sup_{r \leq R_{\varepsilon,\alpha}} (1+r^2)^{-2\alpha-2-\mu/2}\,\Big|e^{h_2} - e^{h_1} + h_1 - h_2\Big| \end{aligned}$$

$$+ c\,\alpha \sup_{r\le R_{\varepsilon,\alpha}} (1+r^2)^{-2-\mu/2}\,|h_2-h_1|$$
$$\le c\,\alpha\,\|h_2-h_1\|_{\mathcal{C}^{4,\beta}_{\mu}(\mathbb{R}^4)}.$$

Thus, applying a classical fixed point argument, taking α small enough, we prove the existence and uniqueness of a solution h_α of (3.9). We summarize this in the following.

Proposition 2. *Given $\mu \in (0,1)$ and $\kappa > 0$ there exist $\alpha_0 > 0$ and $c_\kappa > 0$, independent of ε, such that for all $0 < \alpha < \alpha_0$ there exists a solution $h_\alpha \in \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$ of (3.9) satisfying*

$$\|h_\alpha\|_{\mathcal{C}^{4,\beta}_{\mu}(\mathbb{R}^4)} \le 2c_\kappa\,\alpha.$$

Moreover, $u_{1,1,\alpha}(x) = v_{1,1,\alpha}(x) + h_\alpha(x)$ is a solution of (3.3) in $B_{R_{\varepsilon,\alpha}}$.

4. A linearized operator

We define the linear fourth-order elliptic operator $\mathbb{L}_\alpha$ by

$$\mathbb{L}_\alpha := \Delta^2 - \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(\alpha+1)})^4},$$

which corresponds to the linearization of $\Delta^2 u - 24\,|x|^{4\alpha}\,e^u = 0$ about the approximate solution $v_{1,1,\alpha}$ defined above. This operator can be written as

$$\mathbb{L}_\alpha := \mathbb{L} + V_\alpha(x),$$

where $V_\alpha(x)$ is given by (3.6) satisfying the inequality (3.7). Using a perturbation argument one obtains the following:

Proposition 3. *There exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and for all $\mu > 0$, $\mu \notin \mathbb{N}$, $\mathbb{L}_\alpha : \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4) \longrightarrow \mathcal{C}^{0,\beta}_{rad,\mu-4}(\mathbb{R}^4)$ is surjective. Moreover, if we denote by $\mathcal{G}_{\mu,\alpha}$ a right inverse of $\mathbb{L}_\alpha$ we have that*

$$||\mathcal{G}_{\mu,\alpha}\Phi - \mathcal{G}_\mu\Phi||_{\mathcal{C}^{4,\beta}_{\mu}(\mathbb{R}^4)} \le c_\kappa\,\alpha\,||\Phi||_{\mathcal{C}^{0,\beta}_{\mu-4}(\mathbb{R}^4)} \quad \textit{for every} \quad \Phi \in \mathcal{C}^{0,\beta}_{rad,\mu-4}(\mathbb{R}^4).$$

We define

$$\bar{B}_1^* := \bar{B}_1 - \{0\}.$$

With this notation, we have the following:

Definition 3. *Given $k \in \mathbb{R}$, $\beta \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\beta}_\nu(\bar{B}^*_1)$ as the space of functions $w \in \mathcal{C}^{k,\beta}_{loc}(\bar{B}^*_1)$ such that the norm*

$$\|w\|_{\mathcal{C}^{k,\beta}_\nu(\bar{B}^*_1)} := \sup_{r \in (0,1)} \left(r^{-\nu} \|w(r\,\cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_1 - B_{1/2})} \right)$$

is finite.

When $k \geq 2$, we denote by $[\mathcal{C}^{k,\beta}_\nu(\bar{B}^*_1)]_0$ the subspace of functions $w \in \mathcal{C}^{k,\beta}_\nu(\bar{B}^*_1)$ satisfying $w = \Delta w = 0$ on ∂B^*_1. We recall the analysis of the Bi-Laplace operator in weighted spaces performed in [2].

Proposition 4. [2] *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\begin{array}{rccc} \mathcal{L}_\nu & : [\mathcal{C}^{4,\beta}_\nu(\bar{B}^*_1)]_0 & \longrightarrow & \mathcal{C}^{0,\beta}_{\nu-4}(\bar{B}^*_1) \\ & w & \longmapsto & \Delta^2 w \end{array}$$

is surjective

In the following, we denote by $\tilde{\mathcal{G}}_\nu$ a right inverse. Finally, we study the properties of interior and exterior Bi-harmonic extensions. Indeed, for a given real number γ, we define in B_1 the Bi-harmonic function $H^i_\gamma(x) = \gamma\,|x|^2$. This function satisfies $H^i_\gamma = \gamma$ on ∂B_1 and $\Delta H^i_\gamma = 8\gamma$ on ∂B_1. Similarly, for a given real number $\tilde{\gamma}$, we define in $\mathbb{R}^4 - B_1$ the Bi-harmonic function $H^e_{\tilde{\gamma}}(x) = \tilde{\gamma}\,|x|^{-2}$. This function satisfies $H^e_\gamma = \tilde{\gamma}$ on ∂B_1 and $\Delta H^e_{\tilde{\gamma}} = 0$ on ∂B_1.

5. THE NONLINEAR INTERIOR PROBLEM

Given a real number γ, we define

$$\mathbf{v} := u_{1,1,\alpha} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\alpha}). \tag{5.1}$$

We would like to find a solution w of

$$\Delta^2 w - 24|x|^{4\alpha} f((\varepsilon/\tau)^{1/(\alpha+1)}|x|)e^w = 0, \tag{5.2}$$

defined in $B_{R_{\varepsilon,\alpha}}$ as a perturbation of $\mathbf{v}$. Notice that this gives us a solution (of particular form) of the equation

$$\Delta^2 v - \rho^4\,|x|^{4\alpha} f(|x|)\,e^v = 0 \tag{5.3}$$

in $B_{r_{\varepsilon,\alpha}}$ by using the transformation (3.4). Now writing $w = \mathbf{v} + v$ and using the fact that H^i_γ is harmonic, this amounts to solving the equation

$$\begin{aligned}\mathbb{L}_\alpha v &= \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4} e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})+h_\alpha+v}\Big(\frac{f\big((\varepsilon/\tau)^{1/(\alpha+1)}|\cdot|\big)}{f(0)} - 1\Big)\\ &+ \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\, e^{h_\alpha}\,\Big(e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})+v} - v - 1\Big)\\ &+ \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\,\Big(e^{h_\alpha} - 1\Big)\, v,\end{aligned}$$

where $C_\alpha = 64(4\alpha^2 + 8\alpha + 6)(\alpha+1)^2$.

Fix $\mu \in (0,1)$. By Proposition 3, it is sufficient to find a solution $v \in \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$ of

$$\begin{aligned}v = \mathcal{G}_{\mu,\alpha}\circ\mathcal{E}_{R_{\varepsilon,\alpha}}\Big\{&\frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4} e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})+h_\alpha+v}\Big(\frac{f\big((\varepsilon/\tau)^{1/(\alpha+1)}|\cdot|\big)}{f(0)} - 1\Big)\\ &+ \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\, e^{h_\alpha}\,\Big(e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})+v} - v - 1\Big)\\ &+ \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\,\Big(e^{h_\alpha} - 1\Big)\, v\Big\}\end{aligned} \tag{5.4}$$

in $B_{R_{\varepsilon,\alpha}}$. We denote by $\mathcal{N}(=\mathcal{N}_{\varepsilon,\tau,\alpha,\gamma})$ the nonlinear operator appearing on the right-hand side of the above equation.

Given $\kappa > 0$ (whose value will be fixed later) and taking γ so that

$$|\gamma| \le \kappa\, r^2_{\varepsilon,\alpha}, \tag{5.5}$$

we have the following result

Lemma 1. *Under the above assumption, there exists a constant* $c_\kappa > 0$ *such that*

$$\|\mathcal{N}(0)\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} \le c_\kappa\, r^2_{\varepsilon,\alpha}$$

and

$$\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} \le c_\kappa\, r^2_{\varepsilon,\alpha}\, \|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)}$$

provided $v_1, v_2 \in \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$ *satisfy* $\|v_i\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} \le 2\, c_\kappa r^2_{\varepsilon,\alpha}$ *for* $i = 1, 2$.

Proof. The proof of the first estimate follows from the asymptotic behavior of H^i_γ. Indeed, letting c_κ be a constant depending only on κ (provided ε is chosen small enough) it follows from the expression of H^i_γ that

$$\|H^i_\gamma(\,\cdot/R_{\varepsilon,\alpha})\|_{\mathcal{C}^{4,\beta}_2(\bar B_{R_{\varepsilon,\alpha}})} \le c_\kappa\, R^{-2}_{\varepsilon,\alpha}\,|\gamma| \le c_\kappa\, \varepsilon^{2/(\alpha+1)} \le c_\kappa\, r^2_{\varepsilon,\alpha}.$$

Let $\alpha_0 > 0$; for $\alpha \in (0, \alpha_0)$, using the assumption (A_α) and the fact that for $|x| \le R_{\varepsilon,\alpha}/2$, we have $|h_\alpha(x)| \le c_\kappa \alpha\, r^\mu \le c_\kappa \alpha\, \varepsilon^{-\mu/(\alpha+1)(\alpha+2)} \longrightarrow 0$ provided ε is small enough, we then get

$$\Big\|(1+|\cdot|^{2(\alpha+1)})^{-4}\,|\cdot|^{4\alpha}\, e^{h_\alpha}\, \Big(e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})} - 1\Big)\Big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\alpha}})} \le c_\kappa\, r^2_{\varepsilon,\alpha}$$

and

$$\Big\|(1+|\cdot|^{2(\alpha+1)})^{-4}|\cdot|^{4\alpha}\, e^{H^i_\gamma(\cdot/R_{\varepsilon,\alpha})+h_\alpha}\Big(\frac{f((\varepsilon/\tau)^{1/(\alpha+1)}\cdot)}{f(0)} - 1\Big)\Big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\alpha}})}$$
$$\le c_\kappa\, \varepsilon^{1/(\alpha+1)} \le c_\kappa\, r^2_{\varepsilon,\alpha}.$$

Then the first estimate follows taking into account the fact that the extension operator does not modify the estimate.

On the other hand, for $v_1, v_2 \in \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$ satisfying $\|v_i\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} \le 2\, c_\kappa r^2_{\varepsilon,\alpha}$ for $i = 1, 2$ we have that

$$\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}$$
$$\le c_\kappa \varepsilon^{1/(\alpha+1)}\|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} + c_\kappa r^2_{\varepsilon,\alpha}\|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} + c_\kappa \alpha \|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)}$$
$$\le c_\kappa r^2_{\varepsilon,\alpha}\|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)}.$$

This yields the second estimate. □

Observe that these estimates are uniform in τ provided τ remains in a fixed compact subset of $(0, \infty)$. Applying a contraction mapping argument, we obtain the following:

Proposition 5. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ (depending on κ) and $\alpha_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, for all $0 < \alpha < \alpha_0$ and for all τ in some fixed compact subset $[\tau_-, \tau_+] \subset (0, \infty)$, there exists a unique $v_\alpha (= v_\alpha(\varepsilon, \tau, \gamma, \cdot))$ solution of* (5.4) *such that*

$$\|v_\alpha\|_{\mathcal{C}^{4,\beta}_\mu(\mathbb{R}^4)} \le 2\, c_\kappa\, r^2_{\varepsilon,\alpha}.$$

As a conclusion,

$$\mathbf{v} + v_\alpha(\varepsilon, \tau, \gamma, \cdot) = v_{1,1,\alpha} + h_\alpha - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\alpha}) + v_\alpha(\varepsilon, \tau, \gamma, \cdot) \quad (5.6)$$

solves (5.2) in $B_{R_{\varepsilon,\alpha}}$. Since the function v_α is being obtained as a fixed point for a contraction mapping, it depends smoothly on the parameter τ. Moreover, we claim that the mapping $\tau \longrightarrow v_\alpha(\varepsilon, \tau, \gamma, \cdot)|_{B_{R_{\varepsilon,\alpha}}} \in \mathcal{C}^{4,\beta}(B_{R_{\varepsilon,\alpha}})$

is compact. This follows from the fact that the equation we solve is semilinear and in (5.4) the right-hand side belongs to $\mathcal{C}^{8,\beta}(B_{R_{\varepsilon,\alpha}})$.

6. The nonlinear exterior problem

Recall that

$$r_{\varepsilon,\alpha} = \max(\sqrt{\alpha}, \varepsilon^{1/(\alpha+2)}).$$

Let $\lambda \in \mathbb{R}$ and $\tilde{\gamma} \in \mathbb{R}$ be close to 0. We define

$$\tilde{\mathbf{v}}(x) = (1+\alpha+\lambda)\, G(x) + \chi(x)\, H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\alpha})$$

where χ is a cutoff function identically equal to 1 in $B_{1/4}$ and identically equal to 0 outside $B_{1/2}$. We would like to find a solution of the equation

$$\Delta^2\, v - \rho^4 |x|^{4\alpha} f(|x|) e^v = 0 \tag{6.1}$$

in $\bar{B}_1 - B_{r_{\varepsilon,\alpha}}$ which is a perturbation of $\tilde{\mathbf{v}}$. Writing $v = \tilde{\mathbf{v}} + \tilde{v}$, this amounts to solving

$$\Delta^2\, \tilde{v} = \rho^4 |x|^{4\alpha} f(|x|)\, e^{\tilde{\mathbf{v}}}\, e^{\tilde{v}} - \Delta^2\, \tilde{\mathbf{v}}. \tag{6.2}$$

We need to define auxiliary weighted spaces.

Definition 4. *Given $\bar{r} \in (0,1/2)$, $k \in \mathbb{R}$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $C^{k,\beta}_\nu(\bar{B}_1 - B_{\bar{r}})$ as the space of functions $w \in C^{k,\beta}(\bar{B}_1 - B_{\bar{r}})$ endowed with the norm*

$$\|w\|_{C^{k,\beta}_\nu(\bar{B}_1 - B_{\bar{r}})} = \|w\|_{C^{k,\beta}(\bar{B}_1 - B_{1/2})} + \sup_{\bar{r} \le r < 1/2} r^{-\nu} \|w(r.)\|_{C^{k,\beta}(\bar{B}_1 - B_{1/2})}.$$

For $\sigma \in (0,1/2)$, we denote by

$$\tilde{\xi}_\sigma : C^{0,\beta}_\nu(\bar{B}_1 - B_\sigma) \to C^{0,\beta}_\nu(\bar{B}^*_1)$$

the extension operator defined by $\tilde{\xi}_\sigma(f) = f$ in $\bar{B}_1 - B_\sigma$,

$$\tilde{\xi}_\sigma(f)(x) = \tilde{\chi}\Big(\frac{|x|}{\sigma}\Big) f\Big(\sigma \frac{x}{|x|}\Big) \quad \text{in } B_\sigma - B_{\sigma/2}$$

and $\tilde{\xi}_\sigma(f) = 0$ in $B_{\sigma/2}$, where $t \longmapsto \tilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \ge 1$ and identically equal to 0 for $t \le 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on ν such that

$$\|\tilde{\xi}_\sigma(w)\|_{C^{0,\beta}_\nu(\bar{B}^*_1)} \le c\, \|w\|_{C^{0,\beta}_\nu(\bar{B}_1 - B_\sigma)}. \tag{6.3}$$

Fix $\nu \in (-1,0)$; making use of Proposition 4, for solving equation (6.2) it suffices to find a solution $\tilde{v} \in C^{4,\alpha}_\nu(\bar{B}^*_1)$ of the following fixed point problem:

$$\tilde{v} = \tilde{\mathcal{G}}_\nu \circ \tilde{\xi}_{r_{\varepsilon,\alpha}} \left(\rho^4 |x|^{4\alpha} f(|x|)\, e^{\tilde{\mathbf{v}}}\, e^{\tilde{v}} - \Delta^2\, \tilde{\mathbf{v}}\right) = \tilde{\mathcal{G}} \circ \tilde{\xi}_{r_{\varepsilon,\alpha}} \circ \tilde{S}(\tilde{v}). \tag{6.4}$$

We denote by $\tilde{\mathcal{N}}(=\tilde{\mathcal{N}}_{\varepsilon,\alpha,\lambda,\tilde{\gamma}})$ the nonlinear operator appearing on the right-hand side of this equation.

Given $\kappa > 0$ (whose value will be fixed later on), suppose that the parameters λ and $\tilde{\gamma}$ satisfy

$$|\lambda| \leq \kappa\, r_{\varepsilon,\alpha}^2 \tag{6.5}$$

and

$$|\tilde{\gamma}| \leq \kappa\, r_{\varepsilon,\alpha}^2. \tag{6.6}$$

Then the following result holds:

Lemma 2. *Under the above assumptions, there exists a constant* $c_\kappa > 0$ *such that*

$$\|\tilde{\mathcal{N}}(0)\|_{\mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)} \leq\ c_\kappa r_{\varepsilon,\alpha}^3$$

and

$$\|\tilde{\mathcal{N}}(\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1)\|_{\mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)} \leq\ c_\kappa r_{\varepsilon,\alpha}^4\, \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)},$$

provided $\tilde{v}_1, \tilde{v}_2 \in \mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)$ *and satisfy* $\|\tilde{v}_i\|_{\mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)} \leq 2\, c_\kappa r_{\varepsilon,\alpha}^3$ *for* $i = 1, 2$.

Proof. In $B_{1/2} - B_{r_{\varepsilon,\alpha}}$, we have $\chi = 1$ and $\Delta^2 \tilde{\mathbf{v}} = 0$, thus

$$|\tilde{S}(0)| \leq c_\kappa \varepsilon^4 r^{-4\alpha-8(1+\lambda)}.$$

In $\bar{B}_1 - B_{1/2}$, we have $|H_{\tilde{\gamma}}^e(x/r_{\varepsilon,\alpha})| \leq \kappa r_{\varepsilon,\alpha}^3 r^{-1}$, thus

$$\begin{aligned} |\tilde{S}(0)| &\leq\ c_\kappa\, \varepsilon^4 |x|^{-4\alpha-8(1+\lambda)} + |[\Delta^2, \chi(x)]|\, |H_{\tilde{\gamma}}^{ext}(x/r_{\varepsilon,\alpha})|) \\ &\leq c_\kappa\, (\varepsilon^4 + r^{-1} r_{\varepsilon,\alpha}^3). \end{aligned}$$

Here, we use the notation

$$[\Delta^2, \chi]w = 2\Delta\chi\Delta w + w\Delta^2\chi + 4\nabla\chi \cdot \nabla(\Delta w) + 4\nabla w \cdot \nabla(\Delta\chi) + 4\nabla^2\chi \cdot \nabla^2 w.$$

It follow that

$$||\tilde{S}(0)||_{\mathcal{C}_{\nu-4}^{0,\beta}(\bar{B}_1 - B_{r_{\varepsilon,\alpha}})} \leq c_\kappa r_{\varepsilon,\alpha}^3.$$

Then the proof of the first estimate follows from (6.3).

For the proof of the second estimate, letting $\tilde{v}_1, \tilde{v}_2 \in \mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)$ satisfying $\|\tilde{v}_i\|_{\mathcal{C}_\nu^{4,\beta}(\bar{B}_1^*)} \leq 2\, c_\kappa r_{\varepsilon,\alpha}^3$ for $i = 1, 2$, we have

$$|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)| \quad \leq c_\kappa\, \rho^4 |x|^{4\alpha} |f(|x|)|\, e^{\tilde{\mathbf{v}}}\, (e^{\tilde{v}_2} - e^{\tilde{v}_1})|.$$

This clearly implies

$$|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)| \leq c_\kappa \varepsilon^4 r^{-4\alpha-8(1+\lambda)}\ |\tilde{v}_2 - \tilde{v}_1|.$$

For $\nu \in (-1,0)$ and λ small enough, we get

$$\|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)\|_{\mathcal{C}^{0,\beta}_{\nu-4}(\bar{B}_1 - B_{r_{\varepsilon,\alpha}})} \leq c_\kappa r^4_{\varepsilon,\alpha} \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}^{4,\beta}_{\nu}(\bar{B}^*_1)}.$$

Using also equation (6.3) we obtain the second estimate. Applying a fixed point theorem for contraction mappings we obtain the following result.

Proposition 6. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ and $\alpha_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, for all $\alpha \in (0, \alpha_0)$, for λ satisfying* (6.5) *and a boundary constant $\tilde{\gamma}$ satisfying* (6.6), *there exists a unique solution $\tilde{v}_\alpha (= \tilde{v}_\alpha(\varepsilon, \tau, \tilde{\gamma}, \cdot))$ of* (6.4) *such that*

$$\|\tilde{v}_\alpha\|_{\mathcal{C}^{4,\beta}_{\nu}(\bar{B}^*_1)} \leq 2\, c_\kappa r^3_{\varepsilon,\alpha}.$$

As in the previous section, since the function $\tilde{v}_\alpha$ is being obtained as a fixed point for a contraction mapping, it depends smoothly on the parameter λ. Again this follows from the fact that the equation we solve is semilinear and in (6.4) the right-hand side belongs to $\mathcal{C}^{8,\beta}(\bar{B}^*_1)$.

7. The nonlinear Cauchy-data matching

We gather the results of the previous sections, keeping the notation and applying the result of Section 5 as well as the results of Section 6.

Assume that $\tau \in [\tau_-, \tau_+] \subset (0, \infty)$ is given (the values of τ_- and τ_+ will be fixed later) and consider some set of boundary data γ satisfying (5.5). Given $\kappa > 0$, according to the result of Proposition 5, there exist $\varepsilon_\kappa > 0$ such that, provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find in $B_{r_{\varepsilon,\alpha}}$ a solution of

$$\Delta^2 v - \rho^4 \, |x|^{4\alpha} f(|x|) \, e^v = 0 \tag{7.1}$$

which can be decomposed, by (3.4), as

$$\begin{aligned} v_{int}(x) = v_{\varepsilon,\tau,\alpha}(x) + h_\alpha \left(R_{\varepsilon,\alpha} x / r_{\varepsilon,\alpha}\right) - \log(f(0)) \\ + H^i_\gamma(x/r_{\varepsilon,\alpha}) + v_\alpha(\varepsilon, \tau, \gamma, R_{\varepsilon,\alpha} x / r_{\varepsilon,\alpha}), \end{aligned}$$

where the function $v_\alpha (= v_\alpha(\varepsilon, \tau, \gamma, \cdot)\ ) \in \mathcal{C}^{4,\beta}_{rad,\mu}(\mathbb{R}^4)$ satisfies

$$\|v_\alpha\|_{\mathcal{C}^{4,\beta}_{\mu}(\mathbb{R}^4)} \leq 2\, c_\kappa r^2_{\varepsilon,\alpha}. \tag{7.2}$$

Similarly, given any constant boundary data $\tilde{\gamma}$ satisfying (6.6) and a parameter λ in $\mathbb{R}$ satisfying (6.5), we can use the result of proposition 6 to find a solution v_{ext}, in $\bar{B}_1 - B_{r_{\varepsilon,\alpha}}$ (provided $\varepsilon \in (0, \varepsilon_k)$), of (7.1) which can be decomposed as

$$v_{ext}(x) = (1 + \alpha + \lambda)\, G(x) + \chi(x)\, H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\alpha}) + \tilde{v}_\alpha(\varepsilon, \tau, \tilde{\gamma}, x)$$

where the function $\tilde{v}_\alpha (= \tilde{v}_\alpha(\varepsilon, \tau, \tilde{\gamma}, \cdot)) \in \mathcal{C}^{4,\beta}_\nu(\bar{B}^*_1)$ satisfies

$$\|\tilde{v}_\alpha\|_{\mathcal{C}^{4,\beta}_\nu(\bar{B}^*_1)} \leq 2c_\kappa\, r^3_{\varepsilon,\alpha}\,. \tag{7.3}$$

It remains to choose the parameters $\gamma, \tilde{\gamma}, \lambda$ and τ in such a way that the function which is equal to v_{int} in $B_{r_{\varepsilon,\alpha}}$ and v_{ext} in $\bar{B}_1 - B_{r_{\varepsilon,\alpha}}$ is a smooth function. This amounts to finding these parameters so that

$$v_{int} = v_{ext}, \quad \partial_r v_{int} = \partial_r v_{ext}, \quad \Delta v_{int} = \Delta v_{ext}, \quad \partial_r \Delta v_{int} = \partial_r \Delta v_{ext}, \tag{7.4}$$

near $\partial B_{r_{\varepsilon,\alpha}}$.

Assuming we have already done so, this provides for each ε and α small enough a function $v_{\varepsilon,\alpha} \in \mathcal{C}^{4,\alpha}(\bar{B}_1)$ (which is obtained by patching together the functions v_{int} and v_{ext}) which is a solution of our equation; elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result since, as ε tends to 0, the sequence of solutions we have obtained satisfies the required properties, namely, away from the 0 the sequence $v_{\varepsilon,\alpha}$ converges to G.

Before we proceed, the following remarks are due. First, it will be convenient to notice that the function $v_{\varepsilon,\tau,\alpha}$ can be expanded as

$$v_{\varepsilon,\tau,\alpha}(x) \;\;=\;\; -4\log\tau - 8(1+\alpha)\log|x| + \mathcal{O}\Big(\frac{\varepsilon^2\tau^{-2}}{|x|^{2(\alpha+1)}}\Big) \tag{7.5}$$

near $\partial B_{r_{\varepsilon,\alpha}}$. Similarly, we can write the function $(1+\alpha+\lambda)\,G(x)$ (which appear in the expression of v_{ext}) as

$$\begin{aligned}(1+\alpha+\lambda)\,G(x) &= -8(1+\alpha+\lambda)\,\log|x| + (1+\alpha+\lambda)H(x)\\ &= -8(1+\alpha+\lambda)\log|x| + H(0) + \mathcal{O}(r^2_{\varepsilon,\alpha})\end{aligned} \tag{7.6}$$

near $\partial B_{r_{\varepsilon,\alpha}}$. Then, one gets

$$\begin{aligned}(v_{int} - v_{ext})(x) = &-4\log\tau + 8\lambda\,\log|x| + H^i_\gamma(x/r_{\varepsilon,\alpha})\\ &- H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\alpha}) - H(0) - \log(f(0)) + \mathcal{O}(r^2_{\varepsilon,\alpha}).\end{aligned}$$

It will be convenient to solve instead of (7.4) the following set of equations:

$$\begin{aligned}(v_{int} - v_{ext})(r_{\varepsilon,\alpha}\,\cdot) = 0, &\qquad \partial_r\left((v_{int} - v_{ext})(r_{\varepsilon,\alpha}\,\cdot)\right) = 0,\\ \Delta((v_{int} - v_{ext})(r_{\varepsilon,\alpha}\,\cdot)) = 0, &\qquad \partial_r\,\Delta((v_{int} - v_{ext})(r_{\varepsilon,\alpha}\,\cdot)) = 0,\end{aligned} \tag{7.7}$$

on S^3.

Here we assume that our functions are defined on S^3 using simply the change of variables $x = r_{\varepsilon,\alpha}\, y$ to parameterize $\partial B_{r_{\varepsilon,\alpha}}$. Then, the set of equations (7.7) yields the system

$$\left\{\begin{array}{r} -4\log\tau - H(0) - \log(f(0)) + \gamma - \tilde\gamma + 8\lambda\,\log r_{\varepsilon,\alpha} + \mathcal{O}(r_{\varepsilon,\alpha}^2) = 0 \\ 8\lambda + 2\,\gamma + 2\,\tilde\gamma + \mathcal{O}(r_{\varepsilon,\alpha}^2) = 0 \\ 16\lambda + 8\,\gamma + \mathcal{O}(r_{\varepsilon,\alpha}^2) = 0 \\ -32\lambda + \mathcal{O}(r_{\varepsilon,\alpha}^2) = 0. \end{array}\right. \tag{7.8}$$

Here and below the terms $\mathcal{O}(r_{\varepsilon,\alpha}^2)$ depend nonlinearly on α, λ, γ and $\tilde\gamma$ but are bounded (in the appropriate norm) by a constant (independent of ε and α) time $r_{\varepsilon,\alpha}^2$. Let us comment briefly on how these equations are obtained. These equations simply come from (7.7) when expansions (7.5) and (7.6) are used, together with the expression of H^i_γ and $H^e_{\tilde\gamma}$ and also the estimates (7.2) and (7.3). This system can be readily simplified into

$$\frac{1}{\log r_{\varepsilon,\alpha}}\Big[4\log\tau + H(0) + \log(f(0))\Big] = \mathcal{O}(r_{\varepsilon,\alpha}^2), \quad \lambda = \mathcal{O}(r_{\varepsilon,\alpha}^2),$$

$$\gamma = \mathcal{O}(r_{\varepsilon,\alpha}^2), \quad \tilde\gamma = \mathcal{O}(r_{\varepsilon,\alpha}^2).$$

We are now in a position to define τ_- and τ_+ since, according to the above, as ε tends to 0 we expect that τ will converge to τ^* satisfying

$$-4\log\tau^* = H(0) + \log(f(0))$$

and hence it is enough to choose τ_- and τ_+ so that

$$4\log\tau_- < -\Big[H(0) + \log(f(0))\Big] < 4\log\tau_+.$$

If we define

$$t = \frac{1}{\log r_{\varepsilon,\alpha}}\Big[4\log\tau + H(0) + \log(f(0))\Big],$$

then our system (7.8) reads

$$(t, \alpha, \lambda, \gamma, \tilde\gamma) = \mathcal{O}(r_{\varepsilon,\alpha}^2). \tag{7.9}$$

The nonlinear term which appears on the right-hand side of (7.9) is continuous and compact. In addition, this nonlinear term sends the ball of radius $\kappa\, r_{\varepsilon,\alpha}^2$ into itself, provided κ is large enough. Applying Schauder's fixed point theorem in the ball of radius $\kappa\, r_{\varepsilon,\alpha}^2$ in the product space, (7.9) can then be solved and the proof of Theorem 1 follows at once.

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