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DECAY TRANSFERENCE AND FREDHOLMNESS OF DIFFERENTIAL OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract. We show that, for some family of weights ω, there are corresponding weighted Sobolev spaces $W_\omega^{m,p}$ on $\mathbb{R}^N$ such that whenever $P(x,\partial)$ is a differential operator with L^∞ coefficients and $P(x,\partial): W^{m,p} \to L^p$ is Fredholm for some $p \in (1,\infty)$, then $P(x,\partial): W_\omega^{m,p} \to L_\omega^p$ $(= W_\omega^{0,p})$ remains Fredholm with the same index. We also show that many spectral properties of $P(x,\partial)$ are closely related, or even the same, in the non-weighted and the weighted settings.

The weights ω arise naturally from a feature of independent interest of the Fredholm differential operators in classical Sobolev spaces ("full" decay transference), proved in the preparatory Section 2. A main virtue of the spaces $W_\omega^{m,p}$ is that they are well suited to handle nonlinearities that may be ill-defined or ill-behaved in non-weighted spaces. Together with the invariance results of this paper, this has proved to be instrumental in resolving various bifurcation issues in nonlinear elliptic PDEs.

1. Introduction

It was shown by the author in [13] that if $P(x,\partial)$ is a linear differential operator on $\mathbb{R}^N$ with measurable bounded coefficients and if $P(x,\partial)$ is Fredholm from $W^{m,p} := W^{m,p}(\mathbb{R}^N)$ to $L^p := L^p(\mathbb{R}^N)$ for some $p \in (1,\infty)$ and $m \in \mathbb{N}$, then at least some of the decay of the right-hand side $f \in L^p$ at infinity (if any) is transferred to every solution $u \in W^{m,p}$ of $P(x,\partial)u = f$, provided that the decay of f is no faster than exponential.

The precise definitions and relevant results will be reviewed in the next section but, in layman's words, the above means that if f decays like $e^{-s\rho(x)}$ for some $s > 0$ as $|x| \to \infty$, where ρ is a suitable positive function accounting for the *type* of decay of f and $s > 0$ measures the *amount* of such decay, then the solutions u decay like $e^{-\min(s,s_0)\rho(x)}$ where $s_0 > 0$ is independent of f

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and u. Thus, if $s \leq s_0$, all the decay of f is transferred to the solutions, while this transfer is only partial if $s > s_0$. For example, $\rho(x) = \ln|x|$ corresponds to power-like decay $|x|^{-s}$ while $\rho(x) = |x|$ accounts for exponential decay $e^{-s|x|}$. The decay transference from the right-hand side to solutions stops at the exponential level, but ρ may exhibit a direction-dependent rate of growth to account for a variety of types of decay.

The theory in [13] is valid for abstract Fredholm operators acting in Banach spaces and the decay transference property outlined above is just one concrete application. In this work, we focus on the case of Fredholm differential operators from $W^{m,p}$ to L^p. As a preamble, we show in Section 2 that, based on the main theorem of [13], a sharper result can often be obtained: If the type ρ of decay is restricted to an appropriate subclass, then its amount is transferred to the solutions u of $P(x,\partial)u = f$ *in its entirety* (Theorem 2.2). In other words, if f decays like $e^{-s\rho(x)}$ with $s > 0$ as $|x| \to \infty$, then the solutions u also decay like $e^{-s\rho(x)}$ as $|x| \to \infty$ and not merely like $e^{-\min(s,s_0)\rho(x)}$ for some $s_0 > 0$ as predicted by the general theory. This is in particular true of power-like decay ($\rho(x) = \ln|x|$ for large $|x|$) but *not* of exponential decay ($\rho(x) = |x|$ for large $|x|$).

In general, the parameter s above is only needed to measure how much decay is transferred and it becomes unnecessary when it is known that full transference occurs. As a result, Theorem 2.2 may be rephrased in terms of weight functions ω independent of s but satisfying specific conditions. These "transference weights," as we shall call them, are conveniently arranged in decreasing subclasses T_m ($m \in \mathbb{N} \cup \{0\}$) depending upon the properties of their partial derivatives of order at most m. Regardless of m, the classes T_m include the functions $(1+|x|^2)^{\frac{s}{2}}$ or $(1+x_i^2)^{\frac{s}{2}}$ for every $s > 0$ but, roughly speaking, no function with faster than polynomial growth. The introduction of corresponding weighted Sobolev spaces $W^{m,p}_\omega$ with $\omega \in T_m$ (Section 3) leads to the announced parameter-free rewording of Theorem 2.2 (Theorem 4.1).

With Theorem 4.1 and the properties of the spaces $W^{m,p}_\omega$ obtained in Section 3, it becomes a fairly simple matter to address the main goal of this paper, which is to give positive answers to the following two questions regarding the operator properties of $P(x,\partial)$. Given $\omega \in T_m$ and $p \in (1,\infty)$,

(a) does the Fredholmness of $P(x,\partial)$ from $W^{m,p}$ to L^p alone imply its Fredholmness, with the same index, from $W^{m,p}_\omega$ to L^p_ω? and (loosely speaking)

(b) are the spectral properties of $P(x, \partial)$ preserved when it is viewed as an operator on L^p with domain $W^{m,p}$ or as an operator on L^p_ω with domain $W^{m,p}_\omega$?

In practice, our answers to (a) and (b) in Theorem 4.2 and Corollary 4.1, respectively, ensure that the Fredholm and several important spectral properties of the operator $P(x, \partial)$ can be used in the weighted setting but need to be verified only in the non-weighted one, where a variety of criteria and methods are already available (see below). The converse of (a) is also true under an extra condition about ω (Theorem 4.3), which also provides an even tighter connection between the spectral properties alluded to in (b) (Corollary 4.1).

Even though the questions (a) and (b) have independent interest, the original motivation for studying the spaces $W^{m,p}_\omega$ was provided by the case when $P(x, \partial)$ arises as the linearization of a nonlinear operator. If so, the nonlinearity may possess fundamental features in the weighted setting that fail to hold in the non-weighted one. For example, some very simple operators incorporating a Nemytskii (also known as substitution) operator are not differentiable, or even defined, from $W^{m,p}$ to L^p for any $p \in (1, \infty)$ but become C^1 or better when viewed from $W^{m,p}_\omega$ to L^p_ω for suitable choices of the weight ω.

This suggests studying the nonlinear problem in these weighted spaces. Since Theorem 4.2 and Corollary 4.1 show that the main properties of the linearization are not affected by this change of setting, existence or bifurcation questions can then be investigated by standard methods (e.g. degree theory or implicit function theorem). In particular, bifurcation issues are discussed in detail in [16] along this line of argument.

Over the years, several papers have addressed the questions (a) and/or (b) above, with various weights and by various methods (and with various answers), sometimes in the more general framework of pseudo-differential operators. See for instance [1], [3], [9], [10], [11], [12], [18], [22], among others. All these works place limitations on the asymptotic behavior of the coefficients of $P(x, \partial)$ and often incorporate further restrictions (e.g. $p = 2$, smoothness of the coefficients, etc.) Incidentally, although defined slightly differently, the transference weights of this paper generalize the weights introduced in [18] when $p = 2$ and the coefficients are C^∞.

In contrast, we reiterate that we shall assume nothing more than the measurability and boundedness of the coefficients. Of course, the question (a) introduces the additional assumption that $P(x, \partial)$ is Fredholm in the non-weighted setting, which has an imprecise impact on the coefficients but does not mandate any specific smoothness or asymptotic behavior. In fact, the hypothesis that the coefficients of $P(x, \partial)$ are smooth is too restrictive to be relevant when $P(x, \partial)$ is the linearization of a nonlinear operator, because such a property usually fails at generic points. Also, it is notorious that many nonlinearities can only be handled in Sobolev spaces $W^{m,p}$ with $p > 2$, so that the limitation to an L^2-type theory, weighted or not, is too narrow for a number of nonlinear applications.

Lastly, due to markedly different motivations and perspectives, the connection between the weighted spaces and the decay transference property -as defined in this paper- seems impervious to the approaches by other methods and does not seem to have been used or noticed elsewhere[1]. Yet, this connection is crucial to relate some of the spectral properties of $P(x, \partial)$ in the weighted and non-weighted settings and, more generally, crucial to obtain information about the asymptotic behavior of the solutions from the asymptotic behavior of the right-hand sides.

For completeness, an equivalent version of Theorem 4.2 is given in Theorem 5.2 in "other" weighted spaces. Actually, this family of spaces coincides with the spaces $W^{k,p}_{\omega}$, but this is not obvious in the first place and some additional investigation of the transference weights and associated spaces $W^{m,p}_{\omega}$ is involved.

The practical value of the results of this paper depends largely upon the existence of criteria to decide whether a given differential operator $P(x, \partial)$: $W^{m,p} \to L^p$ (non-weighted setting) is Fredholm. A necessary and sufficient condition is given by the Cordes-Illner theory when the coefficients of $P(x, \partial)$ are continuous and bounded and have vanishing oscillation at infinity ([4], [6]; a brief summary and the main result can be found in [15]). For $p = 2$, an earlier proof was given by Taylor [21]. In special cases, the Cordes-Illner criterion has been rediscovered by pseudo-differential operator arguments. See for instance [18] (when $p = 2$) or [22]. The "vanishing oscillation" assumption is of asymptotic type, but it is still (much) weaker than those used in the literature to establish the Fredholmness of $P(x, \partial)$ in weighted Sobolev spaces. (If $N > 1$, it can be shown that the index is 0 for every $p \in (1, \infty)$, even though we are aware of published proofs only in special cases.)

[1]Results reminiscent of the decay transference property for right-hand sides in the Schwartz space or the Gelfan'd-Shilov spaces have been obtained by pseudo-differential operator methods; see [2] and the references therein, but they do not suffice for our purposes.

The Fredholmness of $P(x, \partial) : W^{m,p} \to L^p$ can also be obtained when $m = 2k$ is even, $p = 2$ and $P(x, \partial)$ is a compact perturbation of an isomorphism $P_0(x, \partial)$; for instance when $P(x, \partial)$ and $P_0(x, \partial)$ have the same leading coefficients and the difference between their non-leading coefficients tends to 0 at infinity. (Of course, $P(x, \partial) : W^{2k,2} \to L^2$ has index 0 in this case.) In turn, sufficient conditions for $P_0(x, \partial) : W^{2k,2} \to L^2$ to be an isomorphism are easily found via the Lax-Milgram theorem and elliptic regularity. This does not require the coefficients to have limits, or even vanishing oscillation, at infinity. A typical example with $k = 1$ is provided by $P(x, \partial) = -\Delta + V(x)$ with $V \in L^\infty$ and $\liminf_{|x|\to\infty} V(x) \geq c > 0$. Such examples and others are not covered by the Cordes-Illner criterion. Yet, there are results ensuring that $P(x, \partial) : W^{2k,p} \to L^p$ has p-independent spectrum and index 0 for every $p \in (1, \infty)$ if this is true when $p = 2$. See [14] when $k = 1$ and the coefficients are real or [17] for the more specific case of Schrödinger operators.

The results of this paper remain valid with no change of the arguments when $P(x, \partial)$ is a system of differential operators on $\mathbb{R}^N$. (The Cordes-Illner criterion can be extended to systems; see Sun [19] and, when $p = 2$, Hörmander [5].) Lastly, $\mathbb{R}^N$ could be replaced by other unbounded domains (exterior domain, cylinder, etc.) if suitable boundary conditions are incorporated, but accurate general results for such cases seem to be lacking, even in the non-weighted setting.

Throughout the paper, $||\cdot||_{m,p}$ denotes the norm of the space $W^{m,p}$ and $\alpha, \beta, \gamma \in (\mathbb{N} \cup \{0\})^N$ are multi-indices. As usual, $|\alpha| := \alpha_1 + \cdots + \alpha_N$ and $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. Occasionally, we shall also use $\partial_i := \frac{\partial}{\partial x_i}, 1 \leq i \leq N$. All the function spaces $W^{m,p}, L^p$, etc., consist of complex-valued functions. In particular, the coefficients of $P(x, \partial)$ may be complex valued and the same thing is true of the right-hand sides f and solutions u of $P(x, \partial)u = f$. Lastly, B_r always refers to the open ball of $\mathbb{R}^N$ with center 0 and radius $r > 0$ and B_r^c denotes its complement.

2. Decay transference

Given an integer $m \in \mathbb{N}$, let $\rho \in W_{loc}^{m,\infty}$ be such that $\rho \geq 0$ and $\partial^\alpha \rho \in L^\infty$ for every multi-index α with $1 \leq |\alpha| \leq m$. The boundedness of the first derivatives implies that $\rho(x) \leq C|x|$ for some constant $C > 0$ and $|x|$ large enough. Conversely, if $\rho(x) = |x|$ for $|x|$ large enough, then ρ satisfies the required condition $\partial^\alpha \rho \in L^\infty$ for $1 \leq |\alpha| \leq m$ regardless of m. Other possible choices of ρ include functions that coincide with $\ln |x|$ for $|x|$ large enough and numerous other examples in which the growth of ρ is not the same in all directions. We first recall the (partial) decay transference property from [13, Theorem 3.1]:

Theorem 2.1. *Let $P(x,\partial) := \Sigma_{|\beta|\leq m} a_\beta(x)\partial^\beta$ be a linear differential operator on $\mathbb{R}^N$ with coefficients $a_\beta \in L^\infty$. If $p \in (1,\infty)$ and $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm (of any index), the following property holds: There is $s_0 > 0$ such that if $f \in L^p$ and $e^{s\rho} f \in L^p$ for some $s > 0$, then every solution $u \in W^{m,p}$ of $P(x,\partial)u = f$ satisfies $e^{\min(s,s_0)\rho}u \in W^{m,p}$.*

In Theorem 2.1, the choice $\rho(x) = |x|$ (for $|x|$ large enough) corresponds to the case when f, and hence also u, has exponential decay, while f and u have power-like decay when $\rho(x) = \ln|x|$ (for $|x|$ large enough). Alternatively, the functions $\rho(x) = (1 + |x|^2)^{\frac{1}{2}}$ and $\rho(x) = \ln(1 + |x|^2)^{\frac{1}{2}}$ can be used instead of $|x|$ and $\ln|x|$, respectively, to assess exponential or power-like decay while avoiding repetition of "for $|x|$ large enough."

The concept of decay above differs from the usual ones, when decay is understood either pointwise or through the behavior of some Sobolev norm in the complement of a ball of increasing radius. In several respects, Theorem 2.1 is sharper: Even though it implies the pointwise decay of u only when $mp > N$, the result that $e^{\min(s,s_0)\rho}u \in W^{m,p}$ incorporates information about the behavior of the derivatives of u when $m \in \mathbb{N}$. It is also readily verified that, when ρ is an increasing function of $|x|$, it implies that the $W^{m,p}$ norm of u in the complement of the ball B_r decreases like $e^{-\min(s,s_0)\rho(r)}$.

Remark 2.1. In Theorem 2.1, $e^{t\rho}f \in L^p$ for every $t \in (0,s]$, so that $e^{\min(t,s_0)\rho}u \in W^{m,p}$. Thus, $e^{\tau\rho}u \in W^{m,p}$ for $\tau \in (0, \min(s,s_0)]$. In particular, $e^{\min(s,s'_0)\rho}u \in W^{m,p}$ for every $0 < s'_0 \leq s_0$. Hence, Theorem 2.1 remains true if s_0 is replaced by a smaller positive value.

The goal of this section will be to establish a stronger version of Theorem 2.1 under additional growth limitations. Specifically, instead of just assuming that $\partial^\alpha\rho \in L^\infty$ when $1 \leq |\alpha| \leq m$, we shall henceforth assume that

$$\partial^\alpha(e^{\sigma\rho}) \in L^\infty \text{ for some } \sigma > 0 \text{ and every } 1 \leq |\alpha| \leq m. \tag{2.1}$$

The function $\rho(x) = (1 + |x|^2)^{\frac{1}{2}}$ does not satisfy (2.1) for any $\sigma > 0$, but if $\rho(x) = \ln(1 + |x|^2)^{\frac{1}{2}}$, then (2.1) holds with $\sigma = 1$ and all $m \in \mathbb{N}$. Thus, the condition (2.1) rules out exponential decay but not power-like decay.

Lemma 2.1. *Assume that (2.1) holds. Then,*

(i) $\partial^\alpha\rho \in L^\infty$ when $1 \leq |\alpha| \leq m$; and

(ii) $\partial^\alpha(e^{s\rho}) \in L^\infty$ for every $s \leq \sigma$ and every $1 \leq |\alpha| \leq m$.

Proof. (i) Since $\rho = \frac{1}{\sigma}\ln(e^{\sigma\rho})$ and $\sigma > 0$, this follows from (2.1) and the remark that all the derivatives of the function $\ln\xi$ are bounded in $[1,\infty)$.

(ii) Since $s \le \sigma$ and $\sigma > 0$, the above argument can be repeated with $\ln\xi$ replaced by $\xi^{\frac{s}{\sigma}}$. □

Part (i) of Lemma 2.1 shows that condition (2.1) suffices for the validity of Theorem 2.1 and part (ii) shows that (2.1) is unaffected by shrinking σ.

Theorem 2.2. *Let $P(x,\partial) := \Sigma_{|\beta|\le m} a_\beta(x)\partial^\beta$ be a linear differential operator on $\mathbb{R}^N$ with coefficients $a_\beta \in L^\infty$ and suppose that the function ρ satisfies (2.1). If $p \in (1,\infty)$ and $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm (of any index), the following property holds: If $f \in L^p$ and $e^{s\rho}f \in L^p$ for some $s > 0$, then every solution $u \in W^{m,p}$ of $P(x,\partial)u = f$ satisfies $e^{s\rho}u \in W^{m,p}$.*

Proof. After replacing both s_0 in Theorem 2.1 and σ in (2.1) by $\min(s_0,\sigma)$ (see Remark 2.1 and Lemma 2.1 (ii)), we may and shall assume $s_0 = \sigma$. Therefore, since Lemma 2.1 (i) ensures that Theorem 2.1 is applicable, it follows that $e^{\min(s,\sigma)\rho}u \in W^{m,p}$. If $s \le \sigma$, the proof is complete, so that we shall henceforth assume that $s > \sigma$. In particular, $e^{\sigma\rho}u \in W^{m,p}$. For future use, note that

$$e^{\sigma\rho}u \in W^{m,p} \Rightarrow e^{\sigma\rho}\partial^\alpha u \in L^p \text{ when } |\alpha| \le m \tag{2.2}$$

(use $\partial^\alpha(e^{\sigma\rho}u) \in L^p$, (2.1), Leibnitz' rule and induction).

The calculation of $P(x,\partial)(e^{\sigma\rho}u)$ yields

$$P(x,\partial)(e^{\sigma\rho}u) = e^{\sigma\rho}f + \sum_{|\beta|\le m} a_\beta \sum_{0\le\alpha<\beta} \binom{\beta}{\alpha}\partial^{\beta-\alpha}(e^{\sigma\rho})\partial^\alpha u. \tag{2.3}$$

Since $e^{s\rho}f \in L^p$ and $e^{\sigma\rho}\partial^\alpha u \in L^p$ (as just noted above) and since $\partial^{\beta-\alpha}(e^{\sigma\rho}) \in L^\infty$ by (2.1) if $\alpha < \beta$, the right-hand side of (2.3) is a function $f_1 \in L^p$ such that $e^{\min(s-\sigma,\sigma)\rho}f_1 \in L^p$. Thus, by Theorem 2.1 (with $s_0 = \sigma$), it follows that $e^{\min(\min(s-\sigma,\sigma),\sigma)\rho}e^{\sigma\rho}u \in W^{m,p}$. Now, $\min(\min(s-\sigma,\sigma),\sigma) = s-\sigma$ if $s \le 2\sigma$ and $\min(\min(s-\sigma,\sigma),\sigma) = \sigma$ if $s > 2\sigma$. Thus, $e^{s\rho}u \in W^{m,p}$ if $s \le 2\sigma$ and $e^{2\sigma\rho}u \in W^{m,p}$ if $s > 2\sigma$. This proves the desired result if $s \le 2\sigma$.

If $s > 2\sigma$ (and hence $e^{2\sigma\rho}u \in W^{m,p}$), replace u by $e^{\sigma\rho}u$ and f by $P(x,\partial)(e^{\sigma\rho}u)$ in (2.3) and next use (2.3) as stated to get

$$\begin{aligned} &P(x,\partial)(e^{2\sigma\rho}u) \\ &= e^{2\sigma\rho}f + \sum_{|\beta|\le m} a_\beta \sum_{0\le\alpha<\beta} \binom{\beta}{\alpha}\partial^{\beta-\alpha}(e^{\sigma\rho})\left(e^{\sigma\rho}\partial^\alpha u + \partial^\alpha(e^{\sigma\rho}u)\right). \end{aligned} \tag{2.4}$$

In (2.2), replace first u by $e^{\sigma\rho}u$ and next σ by 2σ to get $e^{\sigma\rho}\partial^\alpha(e^{\sigma\rho}u) \in L^p$ and $e^{2\sigma\rho}\partial^\alpha u \in L^p$ when $|\alpha| \leq m$. Since $e^{s\rho}f \in L^p$ with $s > 2\sigma$ and since $\partial^{\beta-\alpha}(e^{\sigma\rho}) \in L^\infty$ by (2.1) if $\alpha < \beta$, this shows that the right-hand side of (2.4) is a function $f_2 \in L^p$ such that $e^{\min(s-2\sigma,\sigma)\rho}f_2 \in L^p$. Thus, by Theorem 2.1 (with $s_0 = \sigma$), it follows that $e^{\min(\min(s-2\sigma,\sigma),\sigma)\rho}e^{2\sigma\rho}u \in W^{m,p}$. Now, $\min(\min(s-2\sigma,\sigma),\sigma) = s-2\sigma$ if $s \leq 3\sigma$ and $\min(\min(s-2\sigma,\sigma),\sigma) = \sigma$ if $s > 3\sigma$. Thus, $e^{s\rho}u \in W^{m,p}$ if $s \leq 3\sigma$ and $e^{3\sigma\rho}u \in W^{m,p}$ if $s > 3\sigma$. This proves the desired result if $s \leq 3\sigma$. Otherwise, although the inductive formula is somewhat cumbersome to write down, it should now be clear how the above arguments can be repeated to obtain, given any $k \in \mathbb{N}$, that $e^{s\rho}u \in W^{m,p}$ if $s \leq k\sigma$ and $e^{k\sigma\rho}u \in W^{m,p}$ if $s > k\sigma$. Obviously, $s \leq k\sigma$ if k is large enough, so that $e^{s\rho}u \in W^{m,p}$. This completes the proof. □

As noted earlier, condition (2.1) fails when $\rho(x) = (1+|x|^2)^{\frac{1}{2}}$ (exponential decay). The following simple example shows that indeed Theorem 2.2 is false in this case.

Example 2.1. Let $N = 1$ and $P\left(x, \frac{d}{dx}\right)u := u' + (\tanh x)u$. It is readily checked that $P\left(x, \frac{d}{dx}\right)$ is Fredholm of index 1 from $W^{1,p}$ to L^p for every $p \in (1,\infty)$ and that $u := \frac{1}{\cosh x} \in W^{1,p}$ solves $P\left(x, \frac{d}{dx}\right)u = 0$. Evidently, the right-hand side $f = 0$ satisfies $e^{s(1+|x|^2)^{\frac{1}{2}}}f \in L^p$ for every $s > 0$. However, it is equally obvious that $e^{s(1+|x|^2)^{\frac{1}{2}}}u = \frac{e^{s(1+|x|^2)^{\frac{1}{2}}}}{\cosh x} \notin W^{1,p}$ if $s \geq 1$.

3. Transference weights and related spaces

Definition 3.1. *If $m \in \mathbb{N} \cup \{0\}$, a (decay) transference weight of order m is a function $\omega \in W^{m,\infty}_{loc}$ with values in $[1,\infty)$ such that*

$$\partial^\alpha \omega^t \in L^\infty \text{ for some } t > 0 \text{ and every } 1 \leq |\alpha| \leq m. \tag{3.1}$$

The set of transference weights of order m will be denoted by T_m.

For example, $\omega(x) = (|x|^2+1)^{\frac{s}{2}}$ with $s > 0$ satisfies the required conditions with $t = s^{-1}$ and any $m \in \mathbb{N} \cup \{0\}$, so that $\omega \in \cap_{m\in\mathbb{N}\cup\{0\}}T_m$. In contrast, $\omega(x) = e^{(|x|^2+1)^{\frac{s}{2}}}$ does not satisfy (3.1) for any $s > 0$ and $|\alpha| = 1$ so that $\omega \notin T_1$. (Note that $T_0 = L^\infty_{loc}$ and $T_m \subset T_k$ if $k \leq m$.)

Remark 3.1. If (3.1) holds, it also holds with t replaced by $s \in (0,t]$ (just use Lemma 2.1 (ii) with $\sigma = t$ and $\rho = \ln \omega$).

Remark 3.1 yields a proof that $\omega_1+\omega_2 \in T_m$ if $\omega_1, \omega_2 \in T_m$. This will not be used later, so we only sketch the argument. By Remark 3.1, choose the same t

for ω_1 and ω_2 in (3.1) and write $(\omega_1+\omega_2)^t = (\omega_1^t+\omega_2^t)f(\omega_1^t,\omega_2^t)$, where $f(\xi,\eta)$ is a smooth function of $\xi \geq 1$ and $\eta \geq 1$, which is homogeneous of degree 0. Then, use Leibnitz' formula and the remark that $||D^k f(\xi,\eta)|| = O((\xi+\eta)^{-k})$ by homogeneity. This implies that every convex combination of weights in T_m is in T_m. In contrast, if $m \in \mathbb{N}$, the product $\omega_1\omega_2$ need not be in T_m. For example, if $N = 2$ and ω_1 and ω_2 are the polynomials $1 + x_1^2$ and $1 + x_2^2$, respectively, then $\omega_1, \omega_2 \in T_m$ for every m but $\omega_1\omega_2 \notin T_1$. Nevertheless, $\omega^s \in T_m$ if $\omega \in T_m$ and $s > 0$, which is obvious from Definition 3.1.

The terminology used in Definition 3.1 will be justified by Theorem 4.1 further below. First, we establish a useful property.

Lemma 3.1. *If $m \in \mathbb{N} \cup \{0\}$ and $\omega \in T_m$, then $\omega^{-1} \in W^{m,\infty}$.*

Proof. Since $\omega \geq 1$, this follows from Lemma 2.1 (ii) with $s = -1$ and $\rho = \ln \omega$, so that (2.1) holds with $\sigma = t$ with t from (3.1). □

If $m \in \mathbb{N} \cup \{0\}$ and $\omega \in T_m$ and if $p \in [1,\infty]$, we define the space

$$W_\omega^{m,p} := \{u \in W^{m,p} : \omega u \in W^{m,p}\}, \tag{3.2}$$

equipped with the norm

$$||u||_{m,p,\omega} := ||\omega u||_{m,p}. \tag{3.3}$$

When $m = 0$, we shall use the notation $L_\omega^p := W_\omega^{0,p}$. Since $T_m \subset T_k$ when $k \leq m$, the spaces $W_\omega^{k,p}$ are defined for $0 \leq k \leq m$ and $\omega \in T_m$.

With a difference in notation, similar spaces have been introduced by Schrohe [18] when $p = 2$ and ω is C^∞ and (3.1) is replaced by the stronger requirement[2] $\lim_{|x|\to\infty} \partial^\alpha \omega^t(x) = 0$ for every $|\alpha| \geq 1$. Thus, the "admissible" weights of [18] are T_m-weights for every $m \in \mathbb{N} \cup \{0\}$. The converse is not true: A counter-example with $N \geq 2$ is given by $\omega(x) := (1 + x_1^2)^{\frac{1}{2}}$.

Observe that the condition $u \in W^{m,p}$ is redundant in (3.2), for it is implied by $\omega u \in W^{m,p}$. Indeed, $\omega^{-1} \in W^{m,\infty}$ by Lemma 3.1, so that $u = \omega^{-1}(\omega u) \in W^{m,p}$. This also shows that there is a constant $C = C(m,p,\omega)$ such that

$$||u||_{m,p} \leq C||u||_{m,p,\omega}, \qquad \forall u \in W_\omega^{m,p}. \tag{3.4}$$

Remark 3.2. If $\omega \in T_m$ is bounded, then $W_\omega^{m,p} = W^{m,p}$. Indeed, together with (3.1), the boundedness of ω implies $\omega^t \in W^{m,\infty}$ and then $\omega = (\omega^t)^{1/t} \in W^{m,\infty}$ by (3.1) since ω^t is bounded from above and from below. Thus, $\omega u \in W^{m,p}$ for every $u \in W^{m,p}$; that is, $W^{m,p} \subset W_\omega^{m,p}$.

[2]In [18], ω is ω^t, so some notational adjustments are needed to make a connection.

Theorem 3.1. *The following properties hold for every* $p \in [1,\infty]$:

(i) If $m \in \mathbb{N} \cup \{0\}$ *and* $\omega \in T_m$, *the space* $W^{m,p}_\omega$ *in (3.2) equipped with the norm (3.3) is a Banach space and* $u \in W^{m,p}_\omega \mapsto \omega u \in W^{m,p}$ *is an isomorphism.*

(ii) If $m \in \mathbb{N}$ *and* $\omega \in T_m$, *the linear operator* $\partial_i : W^{m,p}_\omega \to W^{m-1,p}_\omega$ *is defined and bounded for every* $1 \leq i \leq N$.

Proof. (i) It suffices to prove the isomorphism property. The injectivity and continuity are obvious, the latter by definition of the norm of $W^{m,p}_\omega$. For the surjectivity, let $v \in W^{m,p}$. Then, $\omega^{-1}v \in W^{m,p}_\omega$ (since the assumption $u \in W^{m,p}$ is redundant in (3.2)). Also, $||\omega^{-1}v||_{m,p,\omega} = ||v||_{m,p}$, which shows the continuity of the inverse.

(ii) If $u \in W^{m,p}_\omega$, then $\omega u \in W^{m,p}$, whence $\partial_i(\omega u) = \omega\partial_i u + u\partial_i\omega \in W^{m-1,p}$. Now, $u\partial_i\omega = \omega u(\omega^{-1}\partial_i\omega) = \omega u\partial_i\rho$ with $\omega = e^\rho$. Then (2.1) holds with $\sigma = t$ and t from (3.1), so that, by Lemma 2.1 (i), $\partial_i\rho \in W^{m-1,\infty}$. It follows that $\omega\partial_i u = \partial_i(\omega u) - (\omega u)\partial_i\rho \in W^{m-1,p}$ and that $||\omega\partial_i u||_{m-1,p} \leq C||\omega u||_{m,p}$ where $C > 0$ is a constant independent of u. Equivalently, $||\partial_i u||_{m-1,p,\omega} \leq C||u||_{m,p,\omega}$. □

4. The Fredholmness of differential operators in the spaces $W^{m,p}_\omega$

We begin with a convenient rewording of Theorem 2.2.

Theorem 4.1. *Given* $m \in \mathbb{N}$, *let* $\omega \in T_m$ *and let* $P(x,\partial) := \Sigma_{|\beta|\leq m} a_\beta(x)\partial^\beta$ *be a linear differential operator on* $\mathbb{R}^N$ *with coefficients* $a_\beta \in L^\infty$. *Assume that* $P(x,\partial) : W^{m,p} \to L^p$ *is Fredholm (of any index) for some* $p \in (1,\infty)$. *If* $f \in L^p_\omega$ *and if* $P(x,\partial)u = f$ *for some* $u \in W^{m,p}$, *then* $u \in W^{m,p}_\omega$.

Proof. This follows from the definition of the spaces $W^{k,p}_\omega$ for $k = 0$ and $k = m$ and from Theorem 2.2 with $s = 1$ and with $\rho := \ln\omega$. (From (3.1), ρ satisfies (2.1) with $\sigma = t$, so that Theorem 2.2 is applicable.) □

In the only case of interest when ω is unbounded (Remark 3.2), membership in L^p_ω or $W^{m,p}_\omega$ embodies some L^p or $W^{m,p}$ type of decay at infinity, respectively. By Theorem 4.1, the L^p type of decay of the right-hand sides f accounted for by ω is transferred to the solutions $u \in W^{m,p}$ of $P(x,\partial)u = f$. This is why the weights ω are called "(decay) transference" weights. Theorem 4.2 below shows that these weights also transfer the Fredholm properties of the operator.

Theorem 4.2. *Given* $m \in \mathbb{N}$, *let* $\omega \in T_m$ *and let* $P(x, \partial) := \Sigma_{|\beta| \leq m} a_\beta(x) \partial^\beta$ *be a linear differential operator on* $\mathbb{R}^N$ *with coefficients* $a_\beta \in L^\infty$. *Assume that* $P(x, \partial) : W^{m,p} \to L^p$ *is Fredholm of index* ν *for some* $p \in (1, \infty)$ *and let* $K \subset W^{m,p}$ *denote the null-space of* $P(x, \partial)$. *Then*

(i) $K \subset W^{m,p}_\omega$.

(ii) $P(x, \partial) : W^{m,p}_\omega \to L^p_\omega$ *is Fredholm with index* ν *and null-space* K.

(iii) More generally, if $j \in \mathbb{N}$ *and* K_j *denotes the null-space of* $P(x, \partial)^j$ *when* $P(x, \partial)$ *is viewed as an unbounded operator on* L^p_ω *with domain* $W^{m,p}_\omega$, *then* K_j *is the null-space of* $P(x, \partial)^j$ *when* $P(x, \partial)$ *is viewed as an unbounded operator on* L^p_ω *with domain* $W^{m,p}_\omega$.

Proof. (i) If $u \in K$, then $u \in W^{m,p}$ and $P(x, \partial)u = 0$, so that $u \in W^{m,p}_\omega$ by Theorem 4.1 with $f = 0$.

(ii) First, it must be verified that $P(x, \partial)$ maps $W^{m,p}_\omega$ continuously into L^p_ω. It is readily checked that multiplication by $a_\beta \in L^\infty$ is well defined and bounded on L^p_ω. Also, by Theorem 3.1 (ii), ∂^β is well defined and bounded from $W^{m,p}_\omega$ to $W^{m-|\beta|,p}_\omega$ -and hence to L^p_ω- for every $|\beta| \leq m$. This proves the claim.

We now turn to the "Fredholm" part. Since $W^{m,p}_\omega \subset W^{m,p}$, the null-space of $P(x, \partial) : W^{m,p}_\omega \to L^p_\omega$ is contained in the null-space K of $P(x, \partial) : W^{m,p} \to L^p$. By (i), the converse is true, so that K is also the null-space of $P(x, \partial) : W^{m,p}_\omega \to L^p_\omega$. To complete the proof, it suffices to show that there is a (finite dimensional) subspace Z of L^p_ω $(\subset L^p)$ such that $L^p = P(x, \partial)(W^{m,p}) \oplus Z$ and $L^p_\omega = P(x, \partial)(W^{m,p}_\omega) \oplus Z$.

Let $d \geq 0$ denote the codimension of $P(x, \partial)(W^{m,p})$ in L^p. If $d = 0$, then $P(x, \partial)$ maps $W^{m,p}$ onto L^p. In particular, for every $f \in L^p_\omega \subset L^p$ there is $u \in W^{m,p}$ such that $P(x, \partial)u = f$. By Theorem 4.1, $u \in W^{m,p}_\omega$. This shows that $P(x, \partial)$ maps $W^{m,p}_\omega$ onto L^p_ω (and $Z := \{0\}$).

Suppose now that $d > 0$ and let $f_1, ..., f_d \in L^p$ be such that $L^p = P(x, \partial)(W^{m,p}) \oplus \operatorname{span}\{f_1, ..., f_d\}$. Since $P(x, \partial)(W^{m,p})$ is closed in L^p and $f_1, ..., f_d$ are linearly independent, the direct sum decomposition remains true if $f_1, ..., f_d$ are replaced by sufficiently small perturbations. Therefore, by the denseness of C^∞_0 in L^p, we may assume that $f_1, ..., f_d \in C^\infty_0$. If so, $Z := \operatorname{span}\{f_1, ..., f_d\} \subset L^p_\omega$, so that $P(x, \partial)(W^{m,p}_\omega) \cap Z \subset P(x, \partial)(W^{m,p}) \cap Z = \{0\}$. It only remains to prove that $L^p_\omega = P(x, \partial)(W^{m,p}_\omega) + Z$.

Let then $f \in L^p_\omega \subset L^p$ be given. Since $L^p = P(x, \partial)(W^{m,p}) \oplus Z$, there are $f_Z \in Z \subset L^p_\omega$ and $u \in W^{m,p}$ such that $f = P(x, \partial)u + f_Z$. Thus, $P(x, \partial)u = f - f_Z \in L^p_\omega$, so that $u \in W^{m,p}_\omega$ by Theorem 4.1. This proves that $L^p_\omega = P(x, \partial)(W^{m,p}_\omega) + Z$.

(iii) The case $j = 1$ is part (i). By induction, assume that the result is true for some integer $j \in \mathbb{N}$. By definition, $K_{j+1} := P(x,\partial)^{-1}(K_j) \subset W^{m,p}$. Thus, if $u \in K_{j+1}$, then $u \in W^{m,p}$ and $P(x,\partial)u \in K_j$. By the induction hypothesis, $K_j = \{v \in W_\omega^{m,p} : P(x,\partial)v, ..., P(x,\partial)^{j-1}v \in W_\omega^{m,p}, P(x,\partial)(P(x,\partial)^{j-1}v) = 0\}$, so that $P(x,\partial)u \in W_\omega^{m,p} \subset L_\omega^p$. By Theorem 4.1, this implies $u \in W_\omega^{m,p}$ and so

$$K_{j+1} = \{u \in W_\omega^{m,p} : P(x,\partial)u, ..., P(x,\partial)^j u \in W_\omega^{m,p}, P(x,\partial)(P(x,\partial)^j u) = 0\}.$$

This completes the proof. □

Theorem 4.2 does not ensure that if $P(x,\partial) : W_\omega^{m,p} \to L_\omega^p$ is Fredholm then $P(x,\partial) : W^{m,p} \to L^p$ is also Fredholm. In Theorem 4.3 below, we show that this is true under an extra condition on ω. We shall need the following simple lemma.

Lemma 4.1. *Let $\theta \in L^\infty$ be such that $\lim_{r\to\infty} ||\theta||_{L^\infty(B_r^c)} = 0$. Then, for every weight $\omega \in T_1$ and every $p \in [1,\infty)$, the multiplication operator $u \in W_\omega^{1,p} \mapsto \theta u \in L_\omega^p$ is compact.*

Proof. Let $(u_n) \subset W_\omega^{1,p}$ be a bounded sequence, so that (ωu_n) is bounded in $W^{1,p}$. Since multiplication by θ is compact from $W^{1,p}$ to L^p, there are $v \in L^p$ and a subsequence (ωu_{n_k}) such that $\lim_{k\to\infty} ||\theta\omega u_{n_k} - v||_{0,p} = 0$. By writing $v = \omega(\omega^{-1}v)$, this implies that $\omega^{-1}v \in L_\omega^p$ and $\lim_{k\to\infty} ||\theta u_{n_k} - \omega^{-1}v||_{0,p,\omega} = 0$. □

Theorem 4.3. *Let $m \in \mathbb{N}$ and let $P(x,\partial) := \Sigma_{|\beta|\leq m} a_\beta(x)\partial^\beta$ be a linear differential operator on $\mathbb{R}^N$ with coefficients $a_\beta \in L^\infty$. Assume that $\omega \in T_m$ satisfies the extra condition $\lim_{r\to\infty} ||\omega^{-1}\partial^\alpha\omega||_{L^\infty(B_r^c)} = 0$ for $1 \leq |\alpha| \leq m$ and let $p \in (1,\infty)$. Then, $P(x,\partial) : W_\omega^{m,p} \to L_\omega^p$ is Fredholm of index ν if and only if $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm of index ν .*

Proof. Of course, that $P(x,\partial) : W_\omega^{m,p} \to L_\omega^p$ is Fredholm of index ν if $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm of index ν follows from Theorem 4.2.

Assume now that $P(x,\partial) : W_\omega^{m,p} \to L_\omega^p$ is Fredholm of index ν. For $u \in W^{m,p}$, write

$$\begin{aligned} P(x,\partial)u = P(x,\partial)(\omega(\omega^{-1}u)) &= \sum_{|\beta|\leq m} a_\beta \sum_{\alpha\leq\beta} \binom{\beta}{\alpha} \partial^\alpha\omega\partial^{\beta-\alpha}(\omega^{-1}u) \\ &= \omega P(x,\partial)(\omega^{-1}u) + \sum_{0<|\beta|\leq m} a_\beta \sum_{0<\alpha\leq\beta} \binom{\beta}{\alpha} (\partial^\alpha\omega)\partial^{\beta-\alpha}(\omega^{-1}u). \quad (4.1) \end{aligned}$$

Since multiplication by ω^{-1} is an isomorphism of $W^{m,p}$ to $W_\omega^{m,p}$ and multiplication by ω is an isomorphism of L_ω^p to L^p, and since $P(x,\partial) : W_\omega^{m,p} \to L_\omega^p$ is Fredholm, it follows that the operator

$$u \in W^{m,p} \mapsto \omega P(x,\partial)(\omega^{-1}u) \in L^p$$

is Fredholm. Thus, by (4.1) and since the multiplication by a_β is bounded on L^p, to show that $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm it suffices to prove that $u \in W^{m,p} \mapsto (\partial^\alpha \omega)\partial^{\beta-\alpha}(\omega^{-1}u) \in L^p$ is compact for every $0 < \alpha \leq \beta$.

Rewrite $(\partial^\alpha \omega)\partial^{\beta-\alpha}(\omega^{-1}u)$ as $\omega(\omega^{-1}(\partial^\alpha \omega))\partial^{\beta-\alpha}(\omega^{-1}u)$. Since

$$\partial^{\beta-\alpha} : W_\omega^{m,p} \to W_\omega^{1,p}$$

is continuous by Theorem 3.1 (ii) and since multiplication by $\omega^{-1}(\partial^\alpha \omega)$ is compact from $W_\omega^{1,p}$ to L_ω^p by Lemma 4.1, the claimed compactness follows from the continuity of multiplication by ω^{-1} from $W^{m,p}$ to $W_\omega^{m,p}$ and the continuity of multiplication by ω from L_ω^p to L^p. □

The condition $\lim_{r\to\infty} ||\omega^{-1}\partial^\alpha \omega||_{L^\infty(B_r^c)} = 0$ for every $|\alpha| \geq 1$ holds if $\omega(x) := (1+|x|^2)^{\frac{s}{2}}$ and $s > 0$, but it fails with weights accounting for partial decay (such as $(1+x_1^2)^{\frac{s}{2}}$ when $N \geq 2$). In that regard, see Lemma 5.1 later.

When $p = 2$ and under more restrictive assumptions, but for pseudo-differential operators, Theorem 4.3 was also obtained by Schrohe [18], along with the ω-independence of the Fredholm index (a weaker form of Theorem 4.2). The independence of the (generalized) null-space in Theorem 4.2 depends crucially upon decay transference via Theorem 4.1 and does not seem to have been proved by other arguments, even in special cases.

For the next result, recall that the *discrete spectrum* of a closed unbounded linear operator A on a Banach space is the set of isolated eigenvalues λ of A with finite algebraic multiplicity. The latter refers to the dimension of the range of the spectral projection associated with $\{\lambda\}$. It is perhaps less known (except in the finite-dimensional case) that if the multiplicity r of the isolated eigenvalue λ is finite, then $\ker(A-\lambda I)^j$ is independent of j, for j large enough and $r = \dim\ker(A-\lambda I)^j$ (see Taylor [20, Theorems 9.2 and 9.1]). In addition, $A - \lambda I$ is Fredholm ([7, Theorem 5.28]) and hence Fredholm of index 0 by local constancy of the index since λ is isolated in the spectrum of A.

There are varying definitions of the essential spectrum of A in the literature. Here, the *essential spectrum* refers to those $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not Fredholm. The discrete and essential spectra are disjoint, but their union need not equal the whole spectrum of A. For this and related material, see [7].

Corollary 4.1. *Let $P(x,\partial)$ and p be as in Theorem 4.2 and assume that $P(x,\partial)$ is a closed unbounded operator on L^p with domain $W^{m,p}$. Call Σ, Σ_d and Σ_e the spectrum, discrete spectrum and essential spectrum of $P(x,\partial)$, respectively. Then*

(i) If $\Sigma_e \neq \mathbb{C}$ and $\omega \in T_m$, then $P(x,\partial)$ is closed as an unbounded operator on L^p_ω with domain $W^{m,p}_\omega$.

Furthermore, if $\Sigma(\omega), \Sigma_d(\omega)$ and $\Sigma_e(\omega)$ denote the spectrum, discrete spectrum and essential spectrum of $P(x,\partial)$, respectively, as an unbounded operator

(ii) $\Sigma(\omega) \subset \Sigma$ and $\Sigma_e(\omega) \subset \Sigma_e$, whereas

(iii) $\Sigma_d \subset \Sigma_d(\omega)$ and if $\lambda_0 \in \Sigma_d$ has multiplicity r_0, then r_0 is also the multiplicity of $\lambda_0 \in \Sigma_d(\omega)$.

If, in addition, $\lim_{r\to\infty} ||\omega^{-1}\partial^\alpha\omega||_{L^\infty(B_r^c)} = 0$ for $1 \leq |\alpha| \leq m$, then $\Sigma(\omega) = \Sigma, \Sigma_d(\omega) = \Sigma_d$ and $\Sigma_e(\omega) = \Sigma_e$.

Proof. (i) Since $\Sigma_e \neq \mathbb{C}$, there is $\lambda \in \mathbb{C}$ such that $P(x,\partial) - \lambda : W^{2,p} \to L^p$ is Fredholm. By Theorem 4.2, $P(x,\partial) - \lambda : W^{2,p}_\omega \to L^p_\omega$ is Fredholm, which implies that $P(x,\partial) - \lambda$ (and hence also $P(x,\partial)$) is a closed unbounded operator on L^p_ω with domain $W^{2,p}_\omega$.

(ii) If $\lambda \in \Sigma_e(\omega)$, then by Theorem 4.2, $P(x,\partial) - \lambda : W^{2,p} \to L^p$ is not Fredholm; i.e., $\lambda \in \Sigma_e$. Likewise, if $\lambda \in \Sigma(\omega)$, then $P(x,\partial) - \lambda : W^{2,p} \to L^p$ is not invertible: Otherwise, it is Fredholm of index 0 with trivial null-space, so that, by Theorem 4.2, the same thing is true of $P(x,\partial) - \lambda : W^{2,p}_\omega \to L^p_\omega$, contradicting $\lambda \in \Sigma(\omega)$.

(iii) Let $\lambda_0 \in \Sigma_d$ have multiplicity r_0. From the comments preceding the corollary, $P(x,\partial) - \lambda_0 : W^{m,p} \to L^p$ is Fredholm of index 0 and $r_0 = \dim\ker(P(x,\partial) - \lambda_0)^{j_0} > 0$, where j_0 is the smallest integer j such that $\ker(P(x,\partial) - \lambda_0)^{j+1} = \ker(P(x,\partial) - \lambda_0)^j$.

By Theorem 4.2, $P(x,\partial) - \lambda_0 : W^{m,p}_\omega \to L^p_\omega$ is Fredholm of index 0 and $\ker(P(x,\partial) - \lambda_0)^j$ is unchanged for every $j \in \mathbb{N}$. Thus, λ_0 remains an eigenvalue of $P(x,\partial)$ with $\ker(P(x,\partial) - \lambda_0)^{j_0+1} = \ker(P(x,\partial) - \lambda_0)^{j_0}$ in the weighted setting. On the other hand, since $\lambda \notin \Sigma$ for $\lambda \neq \lambda_0$ in the vicinity of λ_0, then $\lambda \notin \Sigma(\omega)$ by (ii). Thus, λ_0 is an isolated point of the spectrum of $P(x,\partial)$ viewed as an unbounded operator on L^p_ω. Since $P(x,\partial) - \lambda_0 : W^{m,p}_\omega \to L^p_\omega$ is Fredholm, it follows from Kato [7, Theorems 5.28 and 5.10] that the range of the spectral projection associated with $\{\lambda_0\}$ is finite dimensional. Thus, $\lambda_0 \in \Sigma_d(\omega)$ and the corresponding (finite) multiplicity of λ_0 is r_0

since, as noted above, $\ker(P(x,\partial)-\lambda_0)^{j_0}$ is the same in the weighted and non-weighted settings.

If it is also assumed that $\lim_{r\to\infty}||\omega^{-1}\partial^\alpha\omega||_{L^\infty(B_r^c)}=0$ for $1\leq|\alpha|\leq m$, then Theorem 4.3 can be used instead of Theorem 4.2 in the above arguments to prove the reverse inclusions in (ii) and (iii). □

5. Fredholmness in other weighted spaces

Even though the spaces $W_\omega^{m,p}$ of Section 3 have direct applications, notably to nonlinear problems (see [16]), it has been more customary to consider Sobolev spaces obtained by changing the Lebesgue measure dx into ωdx where ω is some locally bounded measurable function on $\mathbb{R}^N$. See for instance Kufner [8]. In this section, we give versions of Theorem 4.2 and Corollary 4.1 in such spaces. These versions turn out to be equivalent to the original ones, but perhaps not in a completely obvious way.

We begin with an alternate characterization of the spaces $W_\omega^{m,p}$ (Theorem 5.1) which relies on the following lemma.

Lemma 5.1. *If $m\in\mathbb{N}\cup\{0\}$ and $\omega\in T_m$, then $\omega^{-1}\partial^\alpha\omega\in L^\infty$ for $|\alpha|\leq m$.*

Proof. With no loss of generality, assume $m\in\mathbb{N}$. Start from $\omega=e^\rho$ with $\rho:=\ln\omega$. Then (2.1) holds with $\sigma=t$. By Lemma 2.1 (i), we infer that $\partial^\gamma\rho\in L^\infty$ for $1\leq|\gamma|\leq m$. Next, by induction, $\partial^\alpha e^\psi=e^\psi P_\alpha(D\psi,...,D^{|\alpha|}\psi)$ for $|\alpha|\leq m$, where P_α is a polynomial independent of $\psi\in W_{loc}^{m,\infty}$ (key point: P_α depends only upon the derivatives of ψ of order at least 1; in particular, $P_0=1$). Thus, for $|\alpha|\leq m$,

$$\partial^\alpha\omega=\partial^\alpha\left(e^\rho\right)=e^\rho P_\alpha(D\rho,...,D^{|\alpha|}\rho)=\omega P_\alpha(D\rho,...,D^{|\alpha|}\rho), \tag{5.1}$$

so that $\omega^{-1}\partial^\alpha\omega=P_\alpha(D\rho,...,D^{|\alpha|}\rho)\in L^\infty$. □

Remark 5.1. The argument of the above proof shows at once that, more generally, $\omega^{-s}\partial^\alpha\omega^s\in L^\infty$ for every $|\alpha|\leq m$ and every $s\in\mathbb{R}$.

Lemma 5.2. *Let $m\in\mathbb{N}\cup\{0\}$ and $\omega\in T_m$ be given and let $p\in[1,\infty]$. Then $u\in W_\omega^{m,p}$ if and only if $\omega\partial^\alpha u\in L^p$ for every $|\alpha|\leq m$. Furthermore, the norm $||\cdot||_{m,p,\omega}$ is equivalent to the norm*

$$|||u|||_{m,p,\omega}:=\sum_{|\alpha|\leq m}||\omega\partial^\alpha u||_{0,p}. \tag{5.2}$$

Proof. Suppose first that $\omega\partial^\alpha u \in L^p$ for every $|\alpha| \leq m$ and let β be a multi-index with $|\beta| \leq m$. Then,

$$\partial^\beta(\omega u) = \sum_{\alpha\leq\beta} \binom{\beta}{\alpha} \partial^\alpha \omega \partial^{\beta-\alpha} u = \sum_{\alpha\leq\beta} \binom{\beta}{\alpha} (\omega^{-1}\partial^\alpha\omega)(\omega\partial^{\beta-\alpha}u). \tag{5.3}$$

In the last sum, $\omega\partial^{\beta-\alpha}u \in L^p$ by hypothesis and $\omega^{-1}\partial^\alpha\omega \in L^\infty$ by Lemma 5.1, so that $\partial^\beta(\omega u) \in L^p$. This shows that $u \in W^{m,p}_\omega$ and that $||u||_{m,p,\omega}$ is controlled by $|||u|||_{m,p,\omega}$.

Conversely, assume now that $u \in W^{m,p}_\omega$. By Theorem 3.1 (ii), $\partial^\alpha u \in W^{m-|\alpha|,p}_\omega \hookrightarrow L^p_\omega$, so that $\omega\partial^\alpha u \in L^p$ and $|||u|||_{m,p,\omega}$ is controlled by $||u||_{m,p,\omega}$. □

If $m \in \mathbb{N} \cup \{0\}$ and $\omega \in T_m$ and if $p \in [1,\infty)$, we define[3]

$$L^p(\omega dx) := \left\{ u \in L^p : \int |u|^p \omega dx < \infty \right\} \text{ and } ||u||_{L^p(\omega dx)} := \left(\int |u|^p \omega dx \right)^{\frac{1}{p}}$$

and

$$W^{m,p}(\omega dx) := \left\{ u \in L^p(\omega dx) : \partial^\alpha u \in L^p(\omega dx) \text{ for } |\alpha| \leq m \right\}$$

with the norm

$$||u||_{W^{m,p}(\omega dx)} := \sum_{|\alpha|\leq m} ||\partial^\alpha u||_{L^p(\omega dx)}.$$

Theorem 5.1. *Let $m \in \mathbb{N} \cup \{0\}$ and $\omega \in T_m$ be given and let $p \in [1,\infty)$. Then $W^{m,p}_\omega = W^{m,p}(\omega^p dx)$ (algebraically and topologically).*

Proof. Suppose first that $u \in W^{m,p}_\omega$. By Lemma 5.2, $\omega\partial^\alpha u \in L^p$ for $|\alpha| \leq m$ and so $\partial^\alpha u \in L^p(\omega^p dx)$ for $|\alpha| \leq m$, which means that $u \in W^{m,p}(\omega^p dx)$. Conversely, if $u \in W^{m,p}(\omega^p dx)$, then $\omega\partial^\alpha u \in L^p$ for $|\alpha| \leq m$ and so $u \in W^{m,p}_\omega$ by Lemma 5.2. This proves that $W^{m,p}_\omega = W^{m,p}(\omega^p dx)$ as sets. That the topologies are the same follows from the equivalence of the norms $||\cdot||_{m,p,\omega}$ and $|||\cdot|||_{m,p,\omega}$ (Lemma 5.2). □

Remark 5.2. Theorem 5.1 shows that if $\omega_1, \omega_2 \in T_m$ and $\omega_1 \leq \omega_2$, then $W^{m,p}_{\omega_2} \hookrightarrow W^{m,p}_{\omega_1}$ (which is not self-evident if $m > 0$).

By combining Theorem 4.2 and Theorem 5.1, we obtain the following.

[3] The condition $u \in L^p$ is redundant since $\omega \geq 1$ and ω is measurable.

Theorem 5.2. *Let $\omega \in T_m$ with $m \in \mathbb{N}$ and let $P(x,\partial) := \Sigma_{|\beta|\leq m} a_\beta(x)\partial^\beta$ be a linear differential operator on $\mathbb{R}^N$ with coefficients $a_\beta \in L^\infty$. If $p \in (1,\infty)$ and $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm with index ν and null-space $K \subset W^{m,p}$, then*

(i) $K \subset W^{m,p}(\omega dx)$.

(ii) $P(x,\partial) : W^{m,p}(\omega dx) \to L^p(\omega dx)$ is Fredholm with index ν and null-space K.

(iii) More generally, if $j \in \mathbb{N}$ and K_j denotes the null-space of $P(x,\partial)^j$ when $P(x,\partial)$ is viewed as an unbounded operator on L^p with domain $W^{m,p}$, then K_j is the null-space of $P(x,\partial)^j$ when $P(x,\partial)$ is viewed as an unbounded operator on $L^p(\omega dx)$ with domain $W^{m,p}(\omega dx)$.

Proof. Since $\omega \in T_m$, then $\omega^{1/p} \in T_m$ (see Section 3) and $W^{m,p}(\omega dx) = W^{m,p}_{\omega^{1/p}}$, $L^p(\omega dx) = L^p_{\omega^{1/p}}$ by Theorem 5.1, so that the result follows from Theorem 4.2 with ω replaced by $\omega^{1/p}$. □

The analog of Theorem 4.3 in the spaces $W^{m,p}(\omega dx)$ and $L^p(\omega dx)$ is also true. It suffices to combine Theorem 4.3 and Theorem 5.1 (and to use Remark 5.1):

Theorem 5.3. *Let $m \in \mathbb{N}$ and let $P(x,\partial) := \Sigma_{|\beta|\leq m} a_\beta(x)\partial^\beta$ be a linear differential operator on $\mathbb{R}^N$ with coefficients $a_\beta \in L^\infty$. Assume that $\omega \in T_m$ satisfies the extra condition $\lim_{r\to\infty} ||\omega^{-1}\partial^\alpha\omega||_{L^\infty(B_r^c)} = 0$ for $1 \leq |\alpha| \leq m$ and let $p \in (1,\infty)$. Then, $P(x,\partial) : W^{m,p}(\omega dx) \to L^p(\omega dx)$ is Fredholm of index ν if and only if $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm of index ν.*

By the same arguments, Corollary 4.1 remains true when $W^{m,p}_\omega$ and L^p_ω are replaced by $W^{m,p}(\omega dx)$ and $L^p(\omega dx)$, respectively. There should be no need to give an explicit statement.

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