ON THE ENERGY CONSERVATION BY WEAK SOLUTIONS OF THE RELATIVISTIC VLASOV-MAXWELL SYSTEM[*]

REINEL SOSPEDRA-ALFONSO[†]

Abstract. We show that weak solutions of the relativistic Vlasov-Maxwell system preserve the total energy provided that the electromagnetic field is locally of bounded variation and, for any $\lambda > 0$, the one-particle distribution function has a square integrable λ-moment in the momentum variable.

Key words. Vlasov-Maxwell, weak solutions, conservation of the total energy.

AMS subject classifications. 35Q61, 35Q83.

1. Introduction

Consider an ensemble of relativistic charged particles that interact through their self-induced electromagnetic field. If collisions among the particles are so improbable that they can be neglected, then the ensemble can be modeled by the so-called relativistic Vlasov-Maxwell (RVM) system. At any given time $t \in \,]0,\infty[$, the RVM system is characterized by the one-particle distribution function $f = f(t,x,p)$ with position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. The self-induced electric and magnetic fields are denoted by $E = E(t,x)$ and $B = B(t,x)$, respectively. Setting all physical constants to one, the model equations for a single particle species read

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_p f = 0 \tag{1.1}$$

$$\frac{\partial E}{\partial t} - \nabla \times B = -4\pi j \tag{1.2}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{1.3}$$

$$\nabla \cdot E = 4\pi \rho, \quad \nabla \cdot B = 0, \tag{1.4}$$

where $v := p \left(1 + |p|^2\right)^{-1/2}$ denotes the relativistic velocity. The coupling of the Vlasov (1.1) and Maxwell equations (1.2)–(1.4) is through the charge and current densities, which we denote by $\rho = \rho(t,x)$ and $j = j(t,x)$ respectively. They are defined by

$$\rho := \int_{\mathbb{R}^3} f \, dp, \quad j := \int_{\mathbb{R}^3} v f \, dp. \tag{1.5}$$

We define the Cauchy problem for the RVM system by (1.1)–(1.5) with initial data

$$f_{|t=0} = f_0, \quad E_{|t=0} = E_0, \quad B_{|t=0} = B_0, \tag{1.6}$$

satisfying (1.4) in the sense of distribution. It is not difficult to check that if (1.4) holds at $t = 0$, then it will do so for all time in which the solution exists. Thus, equations (1.4) can be understood as a mere constraint on the initial data.

[†]Department of Mathematics and Statistics, University of Victoria, PO BOX 3045 STN CSC, Victoria BC V8W 3P4, (sospedra@math.uvic.ca).

Now, define

$$L_{kin}^1(\mathbb{R}^6) := \left\{ g \in L^1(\mathbb{R}^6) : g \geq 0, \int\int \sqrt{1+|p|^2}\, g(x,p)dxdp < \infty \right\}.$$

For $T > 0$, we say that (f, E, B) is a weak solution of the RVM system if

$$f \in L^\infty([0,T[;L_{kin}^1 \cap L^\infty(\mathbb{R}^6)), \quad E, B \in \left[L^\infty([0,T[;L^2(\mathbb{R}^3))\right]^3 \tag{1.7}$$

and equations (1.1)–(1.4) are satisfied in the sense of distributions. In particular, we say that the Vlasov equation (1.1) is satisfied in the sense of distributions if for all $\varphi \in C^\infty([0,T] \times \mathbb{R}^6)$ with compact support in $[0,T[\times\mathbb{R}^6$

$$\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,p)\left[\partial_t\varphi + v \cdot \nabla_x\varphi + K \cdot \nabla_p\varphi\right](t,x,p)dtdxdp$$

$$= -\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x,p)\varphi(0,x,p))dxdp. \tag{1.8}$$

We define analogous relations for the Maxwell equations (1.2)–(1.4) as well. The vector field $K := E + v \times B$ in (1.8) denotes the Lorentz force acting on a reference particle of velocity v. Notice that it satisfies $\nabla_p \cdot K \equiv 0$.

The global existence result for weak solutions in both relativistic and non-relativistic settings is due to DiPerna and Lions and can be found in [3]. In [7] this result is revisited. The uniqueness problem, on the other hand, remains unsolved. It is also unknown whether weak solutions preserve the total energy at least almost everywhere in time; see [3, Remark 4, p.740]. It is known, however, that the energy is bounded at almost all t by its value at $t = 0$, namely

$$\mathcal{E}(t) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sqrt{1+|p|^2}\, f(t,x,p)dxdp$$

$$+ \frac{1}{8\pi}\int_{\mathbb{R}^3} |E(t,x)|^2 + |B(t,x)|^2\, dx \leq \mathcal{E}(0). \tag{1.9}$$

In the present note we show that if the electric and magnetic fields E and B are locally of bounded variation and, for any $\lambda > 0$, the function

$$\rho_\lambda(t,x) := \int_{\mathbb{R}^3} |p|^\lambda f(t,x,p)dp \tag{1.10}$$

is square integrable, then the relation (1.9) is in fact an equality for almost all $0 \leq t < T$. Precisely, we prove the following result:

THEOREM 1.1.   Let $\lambda > 0$.   Let $f_0 \in L_{kin}^1 \cap L^\infty(\mathbb{R}^6)$, $E_0, B_0 \in \left[L^2(\mathbb{R}^3)\right]^3$ and denote (f,E,B) to be a weak solution of the RVM system satisfying $(f,E,B)|_{t=0} = (f_0,E_0,B_0)$.   If $E, B \in [L_{loc}^1(]0,T[;BV_{loc}(\mathbb{R}^3))]^3$ and ρ_λ as defined in (1.10) is in $L_{loc}^\infty(]0,T[;L^2(\mathbb{R}^3))$, then the total energy defined by (1.9) satisfies $\mathcal{E}(t) = \mathcal{E}(0)$ for almost all $0 \leq t < T$.

The tools we use are basically those introduced by DiPerna and Lions in [4] to study renormalized solutions of transport equations. We shall also refer to [2], where applications to the Vlasov equation are given. We remark that the same result holds for the electromagnetic field in $[L_{loc}^1(]0,T[;W_{loc}^{1,1}(\mathbb{R}^3))]^3$ since we have the (strict) inclusion $W^{1,1}(\Omega) \subset BV(\Omega)$ for any open set $\Omega \subseteq \mathbb{R}^3$. For a detailed account on functions

of bounded variation; see [1]. We would like to include here the reference [5], where the uniqueness of weak solutions for the Vlasov-Poisson system has been obtained under the sole assumption that the *spatial* density is bounded. Similar results would be desirable for the more demanding Vlasov-Maxwell system.

Formally, the law for the conservation of the total energy is derived as follows. Multiply the Maxwell equations (1.2) and (1.3) by E and B respectively and integrate on $\mathbb{R}^3$ to find that

$$\frac{1}{8\pi}\frac{d}{dt}\int_{\mathbb{R}^3}|E|^2+|B|^2\,dx=-\int_{\mathbb{R}^3}j\cdot E\,dx. \qquad (1.11)$$

Multiply the Vlasov equation by $\sqrt{1+|p|^2}$ and integrate on $\mathbb{R}^3\times\mathbb{R}^3$ to find

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}\sqrt{1+|p|^2}f\,dxdp=\int_{\mathbb{R}^3}j\cdot E\,dx. \qquad (1.12)$$

Then the sum of (1.11) and (1.12) provide the desired result.

As for weak solutions, we shall follow the same scheme. We find relations analogous to (1.11) and (1.12) in sections 2 and 3 respectively. The difficulty is in overcoming the lack of regularity and the need to justify operations that are taken for granted when the solutions are smooth.

2. Energy balance for the Maxwell equation

Here we show that if the current j is square integrable for almost all time, then the weak solution of the RVM system satisfies the energy balance associated to the Maxwell equations, i.e., the relation (1.11). This result is reminiscent of the duality theorem for transport equations given by DiPerna and Lions in [4].

LEMMA 2.1. *Let (f,E,B) be a weak solution of the RVM system with initial data (f_0,E_0,B_0). If j as defined in (1.5) is in $[L^\infty(]0,T[;L^2(\mathbb{R}^3))]^3$, then*

$$\frac{1}{8\pi}(\|E(t)\|_{L_x^2}^2+\|B(t)\|_{L_x^2}^2)+\int_0^t\int_{\mathbb{R}^3}j\cdot E\,dsdx=\frac{1}{8\pi}(\|E_0\|_{L_x^2}^2+\|B_0\|_{L_x^2}^2) \qquad (2.1)$$

for almost all $t\in[0,T[$.

Proof. Let $\epsilon>0$ and let $\kappa\in C_0^\infty(\mathbb{R}^3)$, κ even, be a standard mollifier. Define the regularization kernel $\kappa_\epsilon:=\frac{1}{\epsilon^3}\kappa(\frac{x}{\epsilon})$. Since mollification and distributional differentiation commute, i.e., $(\partial_x u)*\kappa_\epsilon=\partial_x(u*\kappa_\epsilon)$, we can convolute (1.2) and (1.3) with κ_ϵ to obtain

$$\frac{\partial E_\epsilon}{\partial t}-\nabla\times B_\epsilon=-4\pi j_\epsilon \qquad (2.2)$$

$$\frac{\partial B_\epsilon}{\partial t}+\nabla\times E_\epsilon=0, \qquad (2.3)$$

where $j_\epsilon:=j*\kappa_\epsilon$ and $(E_\epsilon,B_\epsilon):=(E,B)*\kappa_\epsilon$.

Consider the family of smooth cut-off functions $\phi_R:=\phi(\frac{\cdot}{R})$, $R\geq 1$ where $\phi\in C_0^\infty(\mathbb{R}^3)$, $\phi\geq 0$ and $\phi\equiv 1$ on the ball $B_1\subset\mathrm{supp}\phi\subset B_2$. The smoothness of the fields B_ϵ and E_ϵ with respect to x imply via (2.2) and (2.3) that $\partial_t E_\epsilon,\partial_t B_\epsilon\in L_{loc}^1\left[(]0,T[\times\mathbb{R}^3)\right]^3$. Thus, $E_\epsilon,B_\epsilon\in\left[W_{loc}^{1,1}(]0,T[\times\mathbb{R}^3)\right]^3$ and we can apply the chain rule in Sobolev spaces, i.e., for almost all $t\in]0,T[$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(|E_\epsilon(t)|^2+|B_\epsilon(t)|^2\right)=E_\epsilon\cdot\frac{\partial E_\epsilon}{\partial t}+B_\epsilon\cdot\frac{\partial B_\epsilon}{\partial t}.$$

Therefore, we can multiply (2.2) and (2.3) by $E_\epsilon \phi_R$ and $B_\epsilon \phi_R$ respectively, sum the resultant equations, and integrate by parts to find that

$$\frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|E_\epsilon(t)|^2 + |B_\epsilon(t)|^2 \right) \phi_R - \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|E_\epsilon(0)|^2 + |B_\epsilon(0)|^2 \right) \phi_R$$
$$= \frac{1}{4\pi} \int_0^t \int_{\mathbb{R}^3} (E_\epsilon \times B_\epsilon) \cdot \nabla \phi_R - \int_0^t \int_{\mathbb{R}^3} j_\epsilon \cdot E_\epsilon \phi_R. \tag{2.4}$$

Let $\epsilon \to 0$. The terms on the left side converge as a consequence of the theorem of smooth approximations [6, Theorem 3, p.196]. Also, the same theorem and the assumption made on the current j easily implies that $\int j_\epsilon \cdot E_\epsilon \to \int j \cdot E$ for almost all $s \in [0, T[$. Thus, we may invoke the Lebesgue dominated convergence theorem and the convergence of the second term in the right side follows as well. Clearly, the same reasoning applies to the remaining term. Then, for almost all $t \in [0, T[$

$$\frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|E(t)|^2 + |B(t)|^2 \right) \phi_R - \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|E(0)|^2 + |B(0)|^2 \right) \phi_R$$
$$= \frac{1}{4\pi} \int_0^t \int_{\mathbb{R}^3} (E \times B) \cdot \nabla \phi_R - \int_0^t \int_{\mathbb{R}^3} j \cdot E \phi_R. \tag{2.5}$$

Finally, since for some constant C_T that does not depend on R

$$\left| \int_0^t \int_{\mathbb{R}^3} (E \times B) \cdot \nabla \phi_R \right| \leq \frac{C_T}{R} \|E\|_{L_{t,x}^{\infty,2}} \|B\|_{L_{t,x}^{\infty,2}},$$

it is easy to check that (2.1) follows from (2.5) by letting $R \to \infty$. The proof of the lemma is complete. □

3. Energy Balance for the Vlasov equation

In this section we deduce the duality formula [4] resulting from the Vlasov equation (1.1) and (the identity) $K \cdot \nabla_p \sqrt{1 + |p|^2} \equiv v \cdot E$, which gives the energy balance associated to the Vlasov equation. Since we now face a nonlinear term in (1.1), we need to first prove the following lemma, a particular case of Lemma 3.5 in [2].

LEMMA 3.1. *Let κ_{ϵ_1} and κ_{ϵ_2} be two regularization kernels defined on $\mathbb{R}_x^3$ and $\mathbb{R}_p^3$ respectively. Let (f, E, B) be a weak solution of the RVM system. If $E, B \in \left[L^1(]0, T[; BV_{loc}(\mathbb{R}^3)) \right]^3$, then there exist two sequences $\epsilon_1^n > 0$, $\epsilon_2^n > 0$, $\epsilon_1^n \to 0$, $\epsilon_2^n \to 0$ such that*

$$\nabla_x \cdot \left[v(\kappa_{\epsilon_1^n} \kappa_{\epsilon_2^n} * f) \right] + \nabla_p \cdot \left[K(\kappa_{\epsilon_1^n} \kappa_{\epsilon_2^n} * f) \right]$$
$$- (\nabla_x \cdot [vf]) * \kappa_{\epsilon_1^n} \kappa_{\epsilon_2^n} - (\nabla_p \cdot [Kf]) * \kappa_{\epsilon_1^n} \kappa_{\epsilon_2^n} \tag{3.1}$$

converges to 0 in $L^1(]0, T[; L_{loc}^1(\mathbb{R}^3 \times \mathbb{R}^3))$.

Proof. First we omit the dependence in time and show the corresponding convergence on $\mathbb{R}^3 \times \mathbb{R}^3$. Then we study the convergence in time as well.

Indeed, the compact support of the mollifiers and the divergence theorem allow us to rewrite (3.1) as

$$I^v(x, p) + I^K(x, p)$$
$$:= \int \int [f(x, p) - f(x - y, p - q)][v(p) - v(p - q)] \cdot \nabla_y \kappa_{\epsilon_1}(y) \kappa_{\epsilon_2}(q) dy dq$$
$$+ \int \int [f(x, p) - f(x - y, p - q)][K(x, p) - K(x - y, p - q)] \cdot \nabla_q \kappa_{\epsilon_2}(q) \kappa_{\epsilon_1}(y) dy dq.$$

In addition, since we have that

$$K(x,p)-K(x-y,p-q)=E(x)-E(x-y)+v(p)\times[B(x)-B(x-y)]$$
$$+[v(p)-v(p-q)]\times B(x-y), \qquad (3.2)$$

we may decompose the second integral by

$$I^K = I^{K,x}+I^{K,p}$$

where $I^{K,x}$ involves the first two terms in the right side of (3.2) and $I^{K,p}$ involves the third term. Now, let $R>0$ and define the set $B_R \times B_R =: \Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$ such that $\overline{\Omega} +$ supp $\kappa_{\epsilon_1}\kappa_{\epsilon_2} \subset B_{R+1} \times B_{R+1}$. In view of the assumptions of the lemma

$$\|E(x)-E(x-y)\|_{L_x^1(B_R)} \le \|\nabla_x E\|_{\mathcal{M}(B_{R+1})}|y|, \qquad |y| < \epsilon_1,$$

(similarly for B), where $\|\nabla_x E\|_{\mathcal{M}(B_{R+1})} < \infty$ denotes the norm of the measure $\nabla_x E$ (resp. $\nabla_x B$), which coincides with the variation of E (resp. B) on the ball B_{R+1}. Hence, since the relativistic velocity $v \in [C_b^\infty(\mathbb{R}^3)]^3$ satisfies $|v| \le 1$, we find that for some positive constant C_R that depends on R

$$\|I^{K,x}\|_{L_{x,p}^1(\Omega)} \le C_R \frac{\epsilon_1}{\epsilon_2}\left(\|\nabla_x E\|_{\mathcal{M}(B_{R+1})}+\|\nabla_x B\|_{\mathcal{M}(B_{R+1})}\right)$$
$$\times \left(\int |\epsilon_2 \nabla_q \kappa_{\epsilon_2}|\right) \sup_{|y|\le\epsilon_1,|q|\le\epsilon_2} \|f(x,p)-f(x-y,p-q)\|_{L_{x,p}^\infty(\Omega)}. (3.3)$$

Similarly, we find the estimates

$$\|I^{K,p}\|_{L_{x,p}^1(\Omega)} \le \|B\|_{L_x^2}\|\nabla_p v\|_{L_p^2(B_{R+1})}$$
$$\times \left(\int |\epsilon_2 \nabla_q \kappa_{\epsilon_2}|\right) \sup_{|y|\le\epsilon_1,|q|\le\epsilon_2} \|f(x,p)-f(x-y,p-q)\|_{L_{x,p}^2(\Omega)} (3.4)$$

and

$$\|I^v\|_{L_{x,p}^1(\Omega)} \le C_R \frac{\epsilon_2}{\epsilon_1}\left(\int |\epsilon_1 \nabla_y \kappa_{\epsilon_1}|\right)\|\nabla_p v\|_{L_p^2(B_{R+1})}$$
$$\times \sup_{|y|\le\epsilon_1,|q|\le\epsilon_2} \|f(x,p)-f(x-y,p-q)\|_{L_{x,p}^2(\Omega)}. \qquad (3.5)$$

Now, we have $\left(\int |\epsilon \nabla \kappa_\epsilon|\right) \le C$, and we also have that

$$\sup_{|y|\le\epsilon_1,|q|\le\epsilon_2} \|f(x,p)-f(x-y,p-q)\|_{L_{x,p}^2(\Omega)} \to 0, \quad \text{as} \quad \epsilon_1, \epsilon_2 \to 0.$$

Hence, we can choose two sequences $\epsilon_1^n, \epsilon_2^n \to 0$ with $\epsilon_1^n/\epsilon_2^n = 1/n$ such that for some n sufficiently large the right-hand sides of (3.4) and (3.5) are less than $1/n$. Therefore, since we also have $f \in L^\infty(\Omega)$, it follows that (3.3), (3.4), and (3.5) go to zero as $n \to \infty$, and so does (3.1) in $L_{loc}^1(\mathbb{R}^3 \times \mathbb{R}^3)$.

Finally, we consider the dependence in time. The difficulty here seems to arise because the sequences ϵ_1^n and ϵ_2^n may also depend on t. Otherwise we could just invoke the Lebesgue dominated convergence theorem as (3.3) and (3.5) suggest. In particular, we must be careful with the estimate (3.5). Nevertheless, if we keep track of the time dependence along the calculation, we find that

$$\|I^v\|_{L_{t,x,p}^{1,1}((0,T)\times\Omega)} \le C_R \frac{\epsilon_2}{\epsilon_1} \sup_{|y|\le\epsilon_1,|q|\le\epsilon_2} \|f(t,x,p)-f(t,x-y,p-q)\|_{L_{t,x,p}^{1,2}((0,T)\times\Omega)}$$

and we can reason as above. This concludes the proof of the lemma.   □

We now turn to the energy balance (1.12) associated to the Vlasov equation.

LEMMA 3.2. *Let $\lambda > 0$. In addition to the assumptions of Lemma 3.1, suppose that ρ_λ as defined in (1.10) is in $L^\infty(]0,T[;L^2(\mathbb{R}^3))$. Then*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \sqrt{1+|p|^2} f(t) dx dp = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sqrt{1+|p|^2} f(0) dx dp + \int_0^t \int_{\mathbb{R}^3} E \cdot j \, ds dx \qquad (3.6)$$

for almost all $t \in [0,T[$.

Proof. (f, E, B) is a weak solution of the RVM system. Thus, as a straightforward consequence of Lemma 3.1, there are two sequences $\epsilon_1^n > 0$, $\epsilon_2^n > 0$, $\epsilon_1^n \to 0$, $\epsilon_2^n \to 0$ such that

$$\partial_t f^n + v \cdot \nabla_x f^n + K \cdot \nabla_p f^n = r^n \qquad (3.7)$$

converges to 0 in $L^1(]0,T[;L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3))$ as $n \to \infty$, where $f^n := \kappa_{\epsilon_1^n} \kappa_{\epsilon_2^n} * f$ and r^n is defined by (3.1).

Consider a family of smooth cut-off functions $\phi_R = \phi(\frac{\cdot}{R})$, $R \geq 1$ where $\phi \in C_0^\infty(\mathbb{R}^6)$, $\phi \geq 0$ and $\phi \equiv 1$ on $B_1 \subset \operatorname{supp}\phi \subset B_2$. If we multiply (3.7) by $\sqrt{1+|p|^2}\phi_R$ and integrate by parts, we find that

$$\int_{\mathbb{R}^6} \sqrt{1+|p|^2} f^n(t) \phi_R - \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f^n(0) \phi_R$$
$$= \int_0^t \int_{\mathbb{R}^6} E \cdot v f^n \phi_R + \int_0^t \int_{\mathbb{R}^6} \sqrt{1+|p|^2} \phi_R r^n$$
$$+ \int_0^t \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f^n K \cdot \nabla_p \phi_R + \int_0^t \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f^n v \cdot \nabla_x \phi_R. \qquad (3.8)$$

Here we have used the identity $K \cdot \nabla_p \sqrt{1+|p|^2} \equiv v \cdot E$.

Let $n \to \infty$. In doing so, we notice that the second term in the right-hand side vanishes as a consequence of Lemma 3.1. Also, the convergence of the two terms in the left-hand side and the last term in the right-hand side follow by a straightforward application of the theorem of smooth approximations. Thus, we are led to prove the convergence of the first and third terms in the right-hand side.

Indeed, the reasoning done so far does not preclude us from writing ϕ_R as the product of two suitable functions $\chi_R = \chi(\frac{\cdot}{R})$ and $\zeta_R = \zeta(\frac{\cdot}{R})$ where $\chi \in C_0^\infty(\mathbb{R}_x^3)$ and $\zeta \in C_0^\infty(\mathbb{R}_p^3)$. Hence,

$$\left| \int_{\mathbb{R}^6} E \cdot v(f - f^n) \phi_R \right| \leq \int_{\mathbb{R}^3} |E| \chi_R \int_{\mathbb{R}^3} \zeta_R |f - f^n|$$
$$\leq C_R \|E\|_{L_x^2} \|f - f^n\|_{L_{x,p}^2},$$

which converges to zero as $n \to \infty$. Then, a use of the Lebesgue dominated convergence theorem provides the convergence of the first term in the right-hand side. Since we can do this similarly with the remaining term, we find that as $n \to \infty$, (3.8) converges to

$$\int_{\mathbb{R}^6} \sqrt{1+|p|^2} f(t) \phi_R = \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f(0) \phi_R + \int_0^t \int_{\mathbb{R}^6} E \cdot v f \phi_R$$
$$+ \int_0^t \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f K \cdot \nabla_p \phi_R + \int_0^t \int_{\mathbb{R}^6} \sqrt{1+|p|^2} f v \cdot \nabla_x \phi_R.$$

Finally, we let $R \to \infty$ and show that the above equality converges to (3.6). The convergences of the term on the left and the first term in the right-hand side are straightforward, since for $t = 0$ and for almost all $t > 0$, $f(t) \in L^1_{kin}(\mathbb{R}^6)$. Also, since

$$\left| \int_0^t \int_{\mathbb{R}^6} \sqrt{1 + |p|^2} f v \cdot \nabla_x \phi_R \right| \leq \frac{C}{R} \int_0^t \int_{\mathbb{R}^6} \sqrt{1 + |p|^2} f \leq \frac{C_T}{R},$$

the last term converges to zero as $R \to \infty$. In order to obtain the convergence of the second term on the right, we first notice that for any $\lambda > 0$,

$$\rho(t, x) = \int_{|p| \leq 1} f(t, x, p) dp + \int_{|p| > 1} f(t, x, p) dp$$
$$\leq \sqrt{4\pi/3} \|f(t, x)\|_{L^2_p} + \rho_\lambda(t, x).$$

Then, the hypothesis made in the lemma implies that

$$\|\rho(t)\|_{L^2_x} \leq \sqrt{4\pi/3} \|f(t)\|_{L^2_{x,p}} + \|\rho_\lambda(t)\|_{L^2_x} < \infty. \tag{3.9}$$

As a result, and since $|v| \leq 1$, we can easily verify that $E \cdot v f \in L^1(]0, T[\times \mathbb{R}^6)$, so the Lebesgue theorem provides the expected convergence. Hence, we are only left to show that the third term in the right-hand side converges to zero. To this end, we first produce

$$\left| \int_0^t \int_{\mathbb{R}^6} \sqrt{1 + |p|^2} f K \cdot \nabla_p \phi_R \right| \leq \frac{C}{R} \int_0^t \int_{\mathbb{R}^3} (|E| + |B|) \int_{R \leq |p| \leq 2R} |p| f. \tag{3.10}$$

To estimate the above inequality we observe that

$$\frac{1}{R} \int_{R \leq |p| \leq 2R} |p| f \leq \begin{cases} 2^{1-\lambda} \rho_\lambda / R^\lambda, & 0 < \lambda < 1 \\ \rho_\lambda / R^\lambda, & 1 \leq \lambda \end{cases}.$$

Thus, for any $\lambda > 0$, there exists a constant $C > 0$ independent of R such that the right-hand side of (3.10) is less or equal than

$$\frac{C}{R^\lambda} \int_0^t \left(\|E(s)\|_{L^2_x} + \|B(s)\|_{L^2_x} \right) \|\rho_\lambda(s)\|_{L^2_x} ds \leq \frac{C_T}{R^\lambda}.$$

Therefore, (3.10) converges to zero as $R \to \infty$ and the proof of the lemma is complete. $\square$

4. Proof of Theorem 1.1

Proof.   Since $|v| \leq 1$, we have $|j| \leq \rho$. Then, in view of (3.9), we can combine Lemmas 2.1 and 3.2 to produce the equality for almost all $0 \leq t < T$ claimed for (1.9). $\square$

REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity*, Oxford University Press, Inc. NY, 2000.

[2] F. Bouchut, *Renormalized solutions to the Vlasov equation with coefficients of bounded variation*, Arch. Rational Mech. Anal., 157, 75–90, 2001.

[3] R.J. DiPerna and P.L. Lions, *Global weak solutions of the Vlasov-Maxwell systems*, Commun. Pure Appl. Math., 42(6), 729–757, 1989.

[4] R.J. DiPerna and P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98, 511–547, 1989.

[5] G. Loeper, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl., 86, 68–79, 2006.

[6] R. McOwen, *Partial Differential Equations. Methods and Applications*, Prentice Hall Inc., Simon-Schuster/A Viacom Company, New Jersey-U.S.A., 1996.

[7] G. Rein, *Global weak solutions to the relativistic Vlasov-Maxwell system revisited*, Commun. Math. Sci., 2(2), 145–158, 2004.