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STOCHASTIC HOMOGENIZATION OF HAMILTON-JACOBI AND "VISCOUS"-HAMILTON-JACOBI EQUATIONS WITH CONVEX NONLINEARITIES—REVISITED

PIERRE-LOUIS LIONS* AND PANAGIOTIS E. SOUGANIDIS†

Dedicated to Andy Majda

Abstract. In this note we revisit the homogenization theory of Hamilton-Jacobi and "viscous"-Hamilton-Jacobi partial differential equations with convex nonlinearities in stationary ergodic environments. We present a new simple proof for the homogenization in probability. The argument uses some a priori bounds (uniform modulus of continuity) on the solution and the convexity and coercivity (growth) of the nonlinearity. It does not rely, however, on the control interpretation formula of the solution as was the case with all previously known proofs. We also introduce a new formula for the effective Hamiltonian for Hamilton-Jacobi and "viscous" Hamilton-Jacobi equations.

Key words. Stochastic homogenization, Hamilton-Jacobi equations, viscosity solutions.

AMS subject classifications. 35D40, 35B27.

1. Introduction

There has been considerable interest and progress in the study of the homogenization of fully nonlinear first- and second-order pde in stationary environments. The results obtained so far concern "non-viscous" and "viscous" Hamilton-Jacobi equations (see [13, 12, 9, 6, 7, 14]) such as

$$-\varepsilon\delta\,\mathrm{tr}\,A\left(x,\frac{x}{\varepsilon},\omega\right)D^2u^\varepsilon+H\left(Du^\varepsilon,u^\varepsilon,x,\frac{x}{\varepsilon},\omega\right)=0 \quad\text{in}\quad U, \tag{1.1}$$

and fully nonlinear elliptic second-order equations (see [1]) such as

$$F\left(D^2u^\varepsilon,Du^\varepsilon,u^\varepsilon,x,\frac{x}{\varepsilon},\omega\right)=0 \quad\text{in}\quad U, \tag{1.2}$$

where the nonnegative symmetric matrix A, the Hamiltonian H, which is convex with respect to the gradient, and the uniformly elliptic nonlinearity F are stationary ergodic — the precise definitions are given later.

Up to now there exist two different, although with many points in common, approaches to study the asymptotics, as $\varepsilon\to 0$, of (1.1). Both make strong use of the control interpretation of the solution (a by-product of the convexity of H and the fact that A is independent of the gradient) and yield the a.s. convergence of the u^ε's. The methodology of [13] and [9] (see also [12]) is based on several a priori bounds, the control formula of u^ε, and the subadditive ergodic theorem. The approach of [6], which was developed for the case $A\equiv \mathrm{Id}$, is based on deriving, using the ergodic theorem, a new formula for the effective nonlinearity that agrees, in view of the min-max theorem, with the formula already found in [13], [9], etc.. The homogenization of (1.2) follows from entirely different methods based on nonlinear pde techniques. To our knowledge it has not been possible to use the methods of [1] to study (1.1).

*College de France, 11 Place Marcelini Berthelot, Paris 75005 and CEREMADE, Université de Paris-Dauphine, Place du Marechal de Lattre de Tassigny, Paris Cedex 16, FRANCE (lions@ceremade.dauphine.fr).

†Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA (souganidis@math.uchicago.edu). Partially supported by the National Science Foundation.

In this note we present a new rather simple argument to prove the homogenization of (1.1) in probability. The convexity and coercivity of H with respect to the gradient are again important. The control formula of u^ε plays, however, absolutely no role in the proof.

In addition we assume a, uniform with respect to ε, uniform modulus of continuity for the u^ε's, which can be obtained under some additional assumptions on A and H. We refer, for example, to [9] for apriori Lipschitz estimates and to [6] for a uniform modulus of continuity under a different set of assumptions when A is independent of $(x,x/\varepsilon)$. In a forthcoming paper [11], we give four general and independent groups of hypotheses giving rise to such moduli.

To explain the role of the convexity, coercivity and uniform modulus of continuity it is convenient to introduce, for each fixed fixed $(p,r,x)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$, the auxiliary problem (its role in the homogenization theory for (1.1) is explained later in this note)

$$\varepsilon v_\varepsilon - \delta \operatorname{tr} A(y,\omega) D_y^2 v_\varepsilon + H(D_y v_\varepsilon + p, y, \omega) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.3}$$

The coercivity and convexity of H provide apriori estimates on Dv_ε in L^α, for some $\alpha > 1$, and hence an L^∞-weak $*$ limit, which, in view of the convexity, can pass inside H. It then follows that the $\varepsilon v^\varepsilon(0,\omega)$'s converge, in probability, to a constant. This relies on showing that the smallest possible limit, i.e., the liminf, and the L^∞-weak $*$ of the $\varepsilon v_\varepsilon$'s agree a.s.. The uniform continuity of the v^ε's together with the stationary ergodic structure are used to show that the $\varepsilon v^\varepsilon$'s actually converge uniformly and always in probability in balls of radius $O(\varepsilon^{-1})$. As is discussed later this is enough to prove the homogenization in probability of the solutions of (1.1). As a by-product of this new proof we are also able to obtain a new formula for the effective nonlinearity which is similar to the one obtained in [6] for $A \equiv \mathrm{Id}$. The same proof would work for (1.2) with convex nonlinearity provided we could obtain an appropriate estimate guaranteeing the (weak) convergence, as $\varepsilon \to 0$, of the Hessians.

The notation needed to state the main results is too cumbersome to be included in the Introduction. Instead we present it, along with the necessary background and the main homogenization result for (1.1), in section 1. The proofs are presented in section 2. Section 3 is devoted to the derivation and proof of the formula for the effective nonlinearity.

We will not list any of the assumptions needed for (1.1) to have "well behaved" viscosity solutions. We refer instead to the "User's Guide" [2] and the references therein. Here we will state only the assumptions that are necessary for the results we prove.

Finally we emphasize that our goal in this paper is to present the key ideas instead of trying to prove the most general result. Hence in several places we do not make the most general assumptions on A and H.

2. Background and main result

Let $(\Omega,\mathcal{F},\mu)$ be a fixed probability space. A random field $\xi:\mathbb{R}^N\times\Omega\to\mathbb{R}$ is called stationary if, for any finitely many $x_1,\ldots,x_k\in\mathbb{R}^N$ and $h\in\mathbb{R}^N$, the distribution of the random vector $(\xi(x_1+h,\cdot),\xi(x_2+h,\cdot),\ldots,\xi(x_k+h,\cdot))$ is independent of h. It turns out that ξ is stationary if

$$\xi(x,\omega) = \tilde{\xi}(\tau_x\omega)$$

for some random variable $\tilde{\xi}:\Omega\to\mathbb{R}$ and a measure preserving transformation $\tau_x:\Omega\to\Omega$ with $x\in\mathbb{R}^N$.

A group $(\tau_x)_{x\in\mathbb{R}^N}$ of measure preserving transformations in Ω is ergodic if all subsets of Ω, which are invariant with respect to $(\tau_x)_{x\in\mathbb{R}^N}$, have probability either zero or one.

Finally here we say that a random field is stationary ergodic if it is stationary and the underlying group of measure preserving transformations is ergodic.

For $(p,r,x)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$ fixed, let $v_\varepsilon(\cdot,\omega)\in BUC(\mathbb{R}^N)$, the space of bounded uniformly continuous functions in $\mathbb{R}^N$, be the solution of the auxiliary problem (1.3). It is well known (see, for example, [13] and [9]), that (1.1) homogenizes in probability if and only if, for all fixed $(p,r,x)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$, the $\varepsilon v_\varepsilon$'s converge uniformly in balls $B_{R/\varepsilon}$ (B_r is the ball of radius r in $\mathbb{R}^N$ centered at the origin) and in probability to a unique constant $-\overline{H}(p,r,x)$, i.e., for all $R>0$,

$$\lim_{\varepsilon\to 0}\max_{B_{R/\varepsilon}}|\varepsilon v_\varepsilon(\cdot,\omega)+\overline{H}(p,r,x)|=0 \quad \text{in probability.} \tag{2.1}$$

In the following, to keep the notation simple we drop the explicit dependence of (1.3) on (r,x) and we consider the approximate problem

$$\varepsilon v_\varepsilon-\delta\,\mathrm{tr}\,A(y,\omega)D_y^2v_\varepsilon+H(D_yv_\varepsilon+p,y,\omega)=0 \ \text{ in } \ \mathbb{R}^N. \tag{2.2}$$

Observe that, if $v^\varepsilon(y,\omega)=\varepsilon v_\varepsilon(\frac{y}{\varepsilon},\omega)$, then

$$v^\varepsilon-\delta\,\mathrm{tr}\,A(\frac{y}{\varepsilon},\omega)D_y^2v^\varepsilon+H(D_yv^\varepsilon+p,\frac{y}{\varepsilon},\omega)=0 \ \text{ in } \ \mathbb{R}^N.$$

If homogenization takes place in probability, we must have that, as $\varepsilon\to 0$, $v^\varepsilon(\cdot,\omega)\to\bar v$ in $C(\mathbb{R}^N)$ and in probability, where $\bar v\in BUC(\mathbb{R}^N)$ solves

$$\bar v+\overline{H}(D_y\bar v+p)=0 \ \text{ in } \ \mathbb{R}^N.$$

The uniqueness of viscosity solutions yields $\bar v=-\overline{H}(p)$, while the local uniform and in probability convergence of the v^ε's to $\bar v$ is equivalent to (2.1).

The main assumptions on $H:\mathbb{R}^N\times\mathbb{R}^N\times\Omega\to\mathbb{R}$ and $A:\mathbb{R}^N\times\Omega\to S^N$, the space of $N\times N$ symmetric matrices, which are assumed to hold a.s. in ω, are:

$$A \text{ and } H \text{ are stationary ergodic processes,} \tag{2.3}$$

$$\begin{cases} A(y,\omega)=\Sigma(y,\omega)\Sigma(y,\omega)^T \quad \text{where } \ \Sigma(\cdot,\omega)\in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^N) \\ \text{is a Lipschitz continuous } N\times M\text{-matrix,} \end{cases} \tag{2.4}$$

and

$$\begin{cases} H(\cdot,\cdot,\omega)\in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^N\times\mathbb{R}^N),\ \xi\mapsto H(\xi,y,\omega) \text{ is convex for all } y\in\mathbb{R}^N, \\ \sup_{y\in\mathbb{R}^N}|H(\xi,y,\omega)|\leqq C_R \text{ for } |\xi|\leqq R, \ \text{ and there exist } \alpha>1 \text{ and } C_1,C_2>0 \\ \text{such that } H(\xi,y,\omega)\geqq C_1|\xi|^\alpha-C_2 \ \text{ for all } \ y\in\mathbb{R}^N. \end{cases} \tag{2.5}$$

We also assume that

$$\begin{cases} \text{there exists a modulus } \omega:[0,\infty)\to\mathbb{R} \text{ such that } \lim_{r\to 0}\omega(r)=0 \text{ and,} \\ \text{for all } \varepsilon>0,\ y,\hat y\in\mathbb{R}^N \text{ and a.s. in } \omega, \\ \qquad |v_\varepsilon(y,\omega)-v_\varepsilon(\hat y,\omega)|\leqq\omega(|y-\hat y|). \end{cases} \tag{2.6}$$

The result is:

Theorem A. *Assume* (2.3), (2.4), (2.5), *and* (2.6). *Then, for all* $R>0$, (2.1) *holds.*

The key step of the proof of Theorem A is that, using the assumptions on H and A, it is possible to construct a.s. in ω a strictly sublinear at infinity solution $\underline{v}$ of

$$H(D_y v+p,y,\omega)\leqq\lambda \text{ in } \mathbb{R}^N,$$

where λ is the L^∞-weak $*$ limit of $-\varepsilon v_\varepsilon(0,\omega)$, which, in view of the stationarity, ergodicity and the uniform modulus of continuity of the v_ε's, is constant a.s. in ω.

The existence of this subsolution allows to show that,

$$\varliminf_{\varepsilon\to 0}\varepsilon v_\varepsilon(0,\omega)\geqq -\lambda \text{ a.s. in } \omega.$$

It then follows from a simple real analysis lemma that actually the limit $\lim_{\varepsilon\to 0}\varepsilon v_\varepsilon(0,\omega)$ exists in probability. The uniform convergence on balls of radius $O(\varepsilon^{-1})$ is a consequence of a standard result in ergodic theory and (2.6).

We conclude with some basic facts from the theory of viscosity solutions. First we recall the definition of the relaxed half-limits of a family $(W_\varepsilon)_{\varepsilon>0}$ of bounded, uniformly in ε, functions. We have

$$W^*(x)=\limsup_{\varepsilon\to 0,\, y\to x} W_\varepsilon(y) \quad\text{and}\quad W_*(x)=\liminf_{\varepsilon\to 0,\, y\to x} W_\varepsilon(y).$$

Next fix some $W:\mathbb{R}^N\times\Omega\to\mathbb{R}$ such that $W(\cdot,\omega)\in BUC(\mathbb{R}^N)$ a.s. in ω and, for $\theta>0$, consider the (classical) sup- and inf-convolution regularization W^θ and W_θ of W given by

$$W^\theta(x,\omega)=\sup_{y\in\mathbb{R}^N}\left\{W(y,\omega)-\frac{|x-y|^2}{\theta}\right\} \quad\text{and}\quad W_\theta(x,\omega)=\inf_{y\in\mathbb{R}^N}\left\{W(y,\omega)+\frac{|x-y|^2}{\theta}\right\}.$$

It is well known (see, for example [5] and [2]) that, a.s. in ω, $W^\theta(\cdot,\omega)$ and $W_\theta(\cdot,\omega)$ are Lipschitz continuous with a constant depending on θ, and, as $\theta\to 0$, $W^\theta(\cdot,\omega)\to W(\cdot,\omega)$ and $W_\theta(\cdot,\omega)\to W(\cdot,\omega)$ uniformly on $\mathbb{R}^N$. It is also immediate that, if W is stationary, then so do W^θ and W_θ.

3. The proof of Theorem A

Proof. Since p plays absolutely no role in the proof below, we omit it.

It is immediate from (2.5) that there exists $C_3>0$ such that

$$\sup_{\varepsilon\in(0,1)}(\varepsilon\|v_\varepsilon(\cdot,\omega)\|)\leqq C_3 \text{ a.s. in } \omega, \tag{3.1}$$

where $\|f\|$ denotes the L^∞-norm.

The rest of the argument would be considerably simpler had we assumed that the v_ε's were uniformly Lipschitz continuous. Instead it is necessary to work a bit harder introducing another layer of approximations.

For $\theta>0$, consider next the sup-convolution v_ε^θ of the solution v_ε of (1.3). It follows from (2.4), (2.5), and (2.3) (see [2]) that, for each $R>0$ and a.s. in ω, v_ε^θ is a subsolution of

$$\alpha v_\alpha^\theta-\delta\operatorname{tr}A(y,\omega)D^2v_\alpha^\theta+H(Dv_\alpha^\theta,y,\omega)\leqq o_R(1) \text{ in } B_R,$$

where, as $\theta\to 0$, $o_R(1)\to 0$ a.s. in ω.

It is a classical fact in the theory of viscosity solutions (see, for example, [8], [4]) that, for any $\phi\in\mathcal{D}_+(B_{2R})=\{\phi\in C_0^\infty(B_{2R}):\phi\geqq 0\}$ and a.s. in ω,

$$\int \varepsilon v_\varepsilon^\theta \phi\,dy-\delta\int v_\varepsilon^\theta\sum_{i,j=1}^N (A_{ij}\phi)_{y_iy_j}\,dy+\int \phi H(Dv_\varepsilon^\theta,y,\omega)\,dy\leqq o_R(1)\int\phi\,dy,$$

and, since v_α^θ is Lipschitz continuous and H is coercive,

$$\int \varepsilon v_\varepsilon^\theta \phi\,dy+\delta\int \sum_{i,j=1}^N (A_{ij}\phi)_{y_i}(v_\varepsilon^\theta)_{y_j}\,dy+C_1\int |Dv_\varepsilon^\theta|^\alpha\phi\,dy-C_2\int\phi\leqq o_R(1)\int\phi\,dy. \tag{3.2}$$

Moreover, in view of (3.1),

$$\sup_{\varepsilon\in(0,1),\ \theta\in(0,1)} \varepsilon\|v_\varepsilon^\theta(\cdot,\omega)\|\leqq C_3 \quad \text{a.s. in} \quad \omega. \tag{3.3}$$

Next choose ϕ such that $\phi\equiv 1$ on B_R and recall that, a.s. in ω,

$$\int_{B_R}|Dv_\varepsilon^\theta(\cdot,\omega)|\,dy\leqq\left(\int_{B_R}|Dv_\varepsilon^\theta(\cdot,\omega)|^\alpha\,dy\right)^{1/\alpha}|B_R|^{1/\alpha'},$$

where α' is the Hölder dual of α.

It follows from (3.2) and (2.4) that, for some $C_{4,R}>0$,

$$E\int_{B_R}|Dv_\varepsilon^\theta|^\alpha\,dy\leqq C_{4,R}.$$

Since $\alpha>1$, and, as $\theta\to 0$, $v_\varepsilon^\theta\to v_\varepsilon$ locally uniformly and a.s. in ω, we find that, for some other $C_{5,R}>0$,

$$Dv_\varepsilon^\theta\rightharpoonup Dv_\varepsilon \quad\text{in}\quad L^\alpha(B_R\times\Omega) \qquad\text{and}\qquad E\int_{B_R}|Dv_\varepsilon(\cdot,\omega)|^\alpha\leqq C_{5,R}. \tag{3.4}$$

Next we introduce the "normalized" function

$$w_\varepsilon(y,\omega)=v_\varepsilon(y,\omega)-v_\varepsilon(0,\omega),$$

which is a solution of

$$\varepsilon w_\varepsilon-\delta\operatorname{tr}AD^2w_\varepsilon+H(Dw_\varepsilon,y,\omega)=-\varepsilon v_\varepsilon(0,\omega) \quad\text{in}\quad \mathbb{R}^N. \tag{3.5}$$

Given that

$$Dw_\varepsilon=Dv_\varepsilon,$$

it follows from the estimates above and the uniform continuity assumption on the v_ε's that, for all $R>0$,

$$\begin{cases}(w_\varepsilon)_{\varepsilon>0}\text{ bounded in } L^\infty(B_R\times\Omega),\\ (Dw_\varepsilon)_{\varepsilon>0}\text{ bounded in } L^\alpha(B_R\times\Omega),\\ (\varepsilon v_\varepsilon(0,\omega))_{\varepsilon>0}\ \text{ bounded in }\ L^\infty(\Omega),\ \text{ and}\\ |w_\varepsilon(y,\omega)-w_\varepsilon(\hat y,\omega)|\leqq\omega(|y-\hat y|)\ \text{ a.s. in }\ \omega.\end{cases} \tag{3.6}$$

Therefore there exist $w \in L^\infty(B_R \times \Omega)$, for all $R>0$, and $c \in L^\infty(\Omega)$ such that along subsequences, which we still denote by ε, $\varepsilon \to 0$

$$-\varepsilon v_\varepsilon(0,\cdot) \rightharpoonup c \ \text{ in } L^\infty(\Omega)\text{-weak } *,$$

and, for each $R>0$,

$$\begin{cases} w_\varepsilon \rightharpoonup w \ \text{ in } \ L^\infty(B_R \times \Omega)\text{-weak*}, \\ Dw_\varepsilon \rightharpoonup Dw \ \text{ in } \ L^\alpha(B_R \times \Omega), \ \text{ and} \\ |w(y,\omega) - w(\hat{y},\omega)| \leqq \omega(|y-\hat{y}|) \text{ for all } y,\hat{y} \in \mathbb{R}^N \ \text{ and a.s. in } \omega. \end{cases}$$

It also follows from standard arguments from the theory of viscosity solutions that

$$-\delta \operatorname{tr} AD^2 w + H(Dw,y,\omega) \leqq c \ \text{ in } \ \mathbb{R}^N \ \text{ and a.s. in } \omega. \tag{3.7}$$

Finally, since $Dw_\varepsilon = Dv_\varepsilon$ and $EDv_\varepsilon = 0$, we also have

$$EDw = 0.$$

One straightforward consequence of the ergodic theorem is that $w(\cdot,\omega)$ is, a.s. in ω, strictly sublinear at infinity, i.e., it satisfies

$$|y|^{-1} w(y,\omega) \to 0 \ \text{ as } \ |y| \to \infty \ \text{ and a.s. in } \ \omega. \tag{3.8}$$

The last observation is that c is actually independent of ω. Indeed, in view of the ergodicity assumption, it suffices to show that, for all $y,h \in \mathbb{R}^N$,

$$c(y,\tau_h\omega) = c(y,\omega). \tag{3.9}$$

To this end, recall that the uniqueness of viscosity solutions of (1.3) and (2.3) yield that for each $\varepsilon > 0$ the process v_ε is stationary, and hence, a.s. in ω,

$$v_\varepsilon(0,\tau_h\omega) = v_\varepsilon(h,\omega).$$

The uniform modulus of continuity yields, a.s. in ω,

$$|v_\varepsilon(h,\omega) - v_\varepsilon(0,\omega)| \leqq \omega(|h|),$$

and, hence, (3.9).

We summarize all the above saying that, a.s. in ω, there exists a constant c, the L^∞-weak* limit of the $-\varepsilon v_\varepsilon(0,\omega)$'s, and an a.s. strictly sublinear at infinity uniformly continuous solution w of

$$-\delta \operatorname{tr} A(y,\omega) D^2 w + H(Dw,y,\omega) \leqq c \ \text{ in } \ \mathbb{R}^N \ \text{ and a.s. in } \ \omega.$$

Next we consider the smallest possible local uniform limit $(\varepsilon v_\varepsilon(\cdot,\omega))_*$ of the $\varepsilon v_\varepsilon$'s given by

$$(\varepsilon v_\varepsilon(\cdot,\omega))_*(y) = \liminf_{\varepsilon \to 0,\, z \to y} \varepsilon v_\varepsilon(z,\omega).$$

The uniform modulus of continuity of v_ε yields that actually

$$(\varepsilon v_\varepsilon(\cdot,\omega))_*(y) = \liminf_{\varepsilon \to 0} \varepsilon v_\varepsilon(y,\omega).$$

Moreover, $(\varepsilon v_\varepsilon(\cdot,\omega)_*(0)$ is, a.s. in ω, a constant greater equal than $-c$. Indeed, for $y,h\in\mathbb{R}^N$ and a.s. in ω, we have

$$\varepsilon v_\varepsilon(y,\tau_h\omega)=\varepsilon v_\varepsilon(y+h,\omega) \quad \text{and} \quad |\varepsilon v_\varepsilon(y,\omega)-\varepsilon v_\varepsilon(0,\omega)|\leqq \omega(|y|).$$

Let $\tilde{w}=w-c/\varepsilon$. Then $\tilde{w}$ is an a.s. strictly sublinear at infinity solution of

$$\varepsilon\tilde{w}-\delta\,\mathrm{tr}\,A(y,\omega)D^2\tilde{w}+H(D\tilde{w},y,\omega)\leqq\varepsilon w \text{ in } \mathbb{R}^N.$$

Next we compare $\tilde{w}$ and v_ε. Using the strict sublinearity of $\tilde{w}$ at infinity and the uniform continuity of $\tilde{w}$ and v_ε, we find, employing standard arguments from the theory of viscosity solution (see, for example, [2]), that, for $\lambda,\beta>0$, there exist $o_\lambda(1)\to 0$, as $\lambda\to 0$, depending on ω, and $C_\lambda>0$ such that

$$\begin{aligned}\varepsilon w(0,\omega)-c-\varepsilon v_\varepsilon(0,\omega)&=\varepsilon(\tilde{w}(\cdot,\omega)-\beta(1+|\cdot|^2)^{1/2})(0)-\varepsilon v_\varepsilon(0,\omega)+\varepsilon\beta\\ &\leqq\varepsilon\max_{\mathbb{R}^N}(w(\cdot,\omega)-\beta(1+|\cdot|^2)^{1/2})+o_\lambda(1)+\beta C_\lambda.\end{aligned}$$

Letting first $\varepsilon\to 0$ and then $\beta\to 0$ and, finally, $\lambda\to 0$, we obtain

$$-c\leqq\varliminf_{\varepsilon\to 0}\varepsilon v_\varepsilon(0,\omega).$$

Since $-c$ is the L^∞-weak*-limit of $(\varepsilon v_\varepsilon(0,\omega))_{\varepsilon>0}$, we must also have

$$\varliminf_{\varepsilon\to 0}\varepsilon v_\varepsilon(0,\omega)\leqq -c,$$

and, hence, a.s. in ω,

$$\varliminf_{\varepsilon\to 0}\varepsilon v_\varepsilon(0,\omega)=-c. \tag{3.10}$$

An elementary real analysis lemma (see Lemma 1 below) yields that, as $\varepsilon\to 0$,

$$\varepsilon v_\varepsilon(0,\omega)\to -c \text{ in probability.} \tag{3.11}$$

It now follows once again from the stationarity, the ergodicity and the assumed modulus of continuity (see Lemma 2 below) that (3.11) actually implies that, for each $R>0$, as $\varepsilon\to 0$,

$$\max_{y\in B_{R/\varepsilon}}|\varepsilon v_\varepsilon(y,\cdot)+c|\to 0 \text{ in probability.}$$

□

We continue with the two technical results used in the above proof.

Lemma 1. *Let $(X,\mathcal{M},m)$ be an arbitrary measure space with $m(X)<\infty$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable functions such that, for some $C>0$, $|f_n|\leq C$ m-a.e. and $\int_B f_n\,dm\to\int_B \liminf_{n\to\infty} f_n\,dm$ for all $B\in\mathcal{M}$. Then, for all $p\in[1,\infty)$,*

$$f_n\to f=\liminf_{n\to\infty} f_n \text{ in } L^p(X) \text{ and in probability.}$$

Proof. Let $g_n = \inf_{k \geqq n} f_k$. The definition of the liminf yield that, as $n \to \infty$, $g_n \nearrow \liminf_{n\to\infty} f_n$ m-a.e. and in $L^p(X)$ for $p \in [1,\infty)$.

Let $h_n = f_n - g_n$. Then $h_n \geqq 0$ and, as $n \to \infty$, $\int_B f_n \, dm \to 0$ for all $B \in M$ and, in particular, $B = X$.

It follows that, as $n \to \infty$, $f_n - g_n \to 0$ in $L^1(X)$ and in probability. The uniform bound on the $|f_n|$ and the fact that $m(X) < \infty$, then yield as $n \to \infty$, $f_n - g_n \to 0$ in $L^p(X)$ for all $p \in [1,\infty)$. □

Lemma 2. *Let $v_\varepsilon : \mathbb{R}^N \times \Omega \to \mathbb{R}$ be a family of stationary processes which are, uniformly in ε, uniformly continuous in $\mathbb{R}^N$ a.s. in ω. If, for some $C \in \mathbb{R}$, $\varepsilon v_\varepsilon(0,\omega) \to C$ in probability, as $\varepsilon \to 0$, then, for any $r > 0$, as $\varepsilon \to 0$,*

$$\max_{y \in B_{r/\varepsilon}} |\varepsilon v_\varepsilon(y,\cdot) + C| \to 0 \quad \textit{in probability.}$$

Proof. Without any loss of generality we may assume that $C = 0$.

Since $\varepsilon v_\varepsilon(0,\omega) \to 0$ in probability, for each $\delta > 0$ there exists $\varepsilon_\delta > 0$ and $A_\delta \subset \Omega$ such that

$$\operatorname*{ess\,sup}_{\omega \in A_\delta} |\varepsilon v_\varepsilon(0,\omega)| \leq \delta \quad \text{for} \quad \varepsilon \leq \varepsilon_\delta \quad \text{and} \quad \mu(\Omega \setminus A_\delta) \leqq \delta.$$

Applying the ergodic theorem to the characteristic function $\mathbf{1}_{A_\delta}$ of A_δ, we find $\Omega_\delta \subset \Omega$ such that $\mu(\Omega_\delta) = 1$ and, for all $\omega \in \Omega_\delta$,

$$\lim_{R \to 0} |B_R|^{-1} \int_{B_R} \mathbf{1}_{A_\delta}(\tau_x \omega) \, dx = \mu(A_\delta) > 0.$$

If $\Omega_1 = \cap_{\delta \in (0,1)} \Omega_\delta$, then $\mu(\Omega_1) = 1$ and the ergodic theorem holds for $\omega \in \Omega_1$ and all $\delta \in (0,1)$.

Fix $r > 0$. It follows that, given $\theta > 0$, if ε is sufficiently small and $\omega \in \Omega_1$,

$$|\{y : \tau_y \omega \in A_\delta\} \cap B_{r/\varepsilon}| \geqq (1 - 2\theta)|B_{r/\varepsilon}|.$$

The regularity of the Lebesgue measure implies that there exists $\gamma(\theta) > 0$ such that, as $\theta \to 0$, $\gamma(\theta) \to 0$ and, for all $x \in B_{r/\varepsilon}$, there exists $\hat{x} \in \{y : \tau_y \omega \in A_\delta\} \cap B_{r/\varepsilon}$ such that $|x - \hat{x}| \leqq \gamma(\theta)\varepsilon^{-1}$.

Then

$$\begin{aligned}
\operatorname*{ess\,sup}_{\omega \in A_\delta} |\varepsilon v_\varepsilon(x,\omega)| &\leqq \operatorname*{ess\,sup}_{\omega \in A_\delta} |\varepsilon v_\varepsilon(x,\omega) - \varepsilon v_\varepsilon(\hat{x},\omega)| + \operatorname*{ess\,sup}_{\omega \in A_\delta} |\varepsilon v_\varepsilon(\hat{x},\omega)| \\
&\leqq \varepsilon \omega(\gamma(\theta)/\varepsilon) + \operatorname*{ess\,sup}_{\omega \in A_\delta} |\varepsilon v_\varepsilon(0,\tau_{\hat{x}}\omega)| \leqq 2\delta.
\end{aligned}$$

This last inequality implies the claim. □

4. New formulae for the effective nonlinearity

Throughout this section we ignore the possible dependence of H on (r,x). It was shown in [9] that the effective Hamiltonian $\overline{H}$ for (1.1) is given, for each $p\in\mathbb{R}^N$, by

$$\overline{H}(p)=\inf_{\Phi\in S}\sup_{y\in\mathbb{R}^N}[-\delta\operatorname{tr}A(y,\omega)D^2\Phi+H(D\Phi+p,y,\omega)], \tag{4.1}$$

where the sup in (4.1) is interpreted in the viscosity sense, and

$$S=\{\Phi:\mathbb{R}^N\times\Omega\to\Omega:\ \Phi(\cdot,\omega)\in C(\mathbb{R}^N),\ |y|^{-1}\Phi(y,\omega)\xrightarrow[|y|\to\infty]{}0\ \text{ and}$$
$$\Phi(y+z,\omega)-\Phi(y,\omega)=\Phi(z,\tau_y\omega)-\Phi(0,\tau_y\omega)\ \text{ for all }\ y,z\in\mathbb{R}^N \text{ and a.s. in } \omega\}.$$

It is worth remarking that, if $\Phi\in S$ is a.e. differentiable with respect to y, then the identity in the definition of S implies that $D_y\Phi(y,\omega)$ is stationary, while the prescribed a.s. behavior at infinity is equivalent to $ED\Phi(y,\cdot)=0$.

Recall that any stationary process $f:\mathbb{R}^N\times\Omega\to\mathbb{R}$ can be written as $f(y,\omega)=\tilde f(\tau_y\omega)$ with $\tilde f(\omega)=f(0,\omega)$ for some $\tilde f:\Omega\to\mathbb{R}$. In what follows given a stationary process f we will denote by $\tilde f$ the random variable it is generated by.

In view of the above it is possible to rewrite (4.1) as

$$\overline{H}(p)=-\inf_{(\tilde X,\tilde q)\in\tilde S}\operatorname*{esssup}_{\Omega}[-\delta\operatorname{tr}(\tilde A(\omega)\tilde X)+\tilde H(\tilde q+p,\omega)], \tag{4.2}$$

where $\tilde S$ consists of random variables $\tilde X$ and $\tilde q$ taking values in S^N and $\mathbb{R}^N$ respectively, such that the pair $(X(\tau_y\omega),q(\tau_y\omega))$ must belong to the superdifferential, (see [2]) in the viscosity sense, of functions $\Phi\in S$ whenever the former is nonempty.

A new formula for $\overline{H}$ was introduced in [6] for (1.1) with $\delta>0$ and $A=\mathrm{Id}$. The equality between the new formula and (4.1) was then used in [6] to prove the homogenization result for (1.1) for $\delta>0$. The fact that A was independent of the space variable as well as uniformly elliptic played a critical role in the analysis and, in particular, the equality between the formulas in [6].

Having proved the homogenization in a different way, either as in [13] and [9], or as in Theorem A, we proceed here to obtain, in a very straightforward way, an extension of the formula of [6] for degenerate elliptic stationary A's and, in particular, for $\delta=0$.

To write the new formula, it is necessary to introduce some additional terminology and notation.

The measure preserving transformation $(\tau_x)_{x\in\mathbb{R}^N}$ gives rise to an isometry on $L^2(\Omega,\mathcal{F},\mu)$ with infinitesimal generators $(\tilde D_i)_{1\leqq i\leqq N}$ in the coordinate directions. For $A\in L^\infty(\Omega;S^N_+)$, where S^N_+ is the set of nonnegative matrices in S^N, and $\tilde b\in L^\infty(\Omega)$, we consider the operator

$$\mathcal{L}_{A,b}=-\delta\operatorname{tr}\tilde A(\omega)\tilde D^2+\tilde b(\omega)\cdot\tilde D.$$

Let $\mathbb{D}$ be the space of probability densities $\tilde\phi:\Omega\to\mathbb{R}$ relative to μ with $\tilde\phi$, $\tilde D\tilde\phi$, $\tilde D^2\tilde\phi\in L^\infty(\Omega)$ and $\inf_\Omega\tilde\phi>0$, and, finally, set

$$\mathcal{E}=\{(\tilde b,\tilde\phi)\in L^\infty(\Omega;\mathbb{R}^N)\times\mathbb{D}:-\delta\operatorname{tr}\tilde D^2(\tilde A(\omega)\tilde\phi)+\widetilde{\operatorname{div}}(\tilde b(\omega)\tilde\phi)=0\},$$

with the equation in the definition of $\mathcal{E}$ satisfied in the weak sense.

Note that $\mathcal{E}$ is always nonempty. Indeed for $\delta=0$, we can always take $\tilde b\equiv 1$ and $\tilde\phi\equiv 1$, while, when $\delta>0$, we choose $\tilde b$ so that $\widetilde{\operatorname{div}}\tilde b=\operatorname{tr}\tilde D^2\tilde A$, in which case $\tilde\phi\equiv 1$ is again admissible.

Let $\underline{H}:\mathbb{R}^N\to\mathbb{R}$ be defined by

$$\underline{H}(p)=\sup_{(\tilde b,\tilde\phi)\in\mathcal{E}}[(p,\mathbb{E}(\tilde b\tilde\phi))-\mathbb{E}\tilde H(\tilde b,\cdot)\tilde\phi]. \tag{4.3}$$

Theorem B. *Assume the hypotheses of Theorem A. Then, for all* $p\in\mathbb{R}^N$*,* (4.1) *and* (4.3) *are equal.*

Proof. We begin with the inequality

$$\underline{H}\leqq\overline{H}. \tag{4.4}$$

To this end, let $(\tilde b,\tilde\phi)\in\mathcal{E}$ and recall the equation

$$\varepsilon v_\varepsilon-\delta\,\mathrm{tr}\,A(y,\omega)D^2v_\varepsilon+H(Dv_\varepsilon+p,y,\omega)=0\ \text{ in }\ \mathbb{R}^N.$$

Then, a.s. in ω, v_ε is a viscosity subsolution of

$$\varepsilon v_\varepsilon-\delta\,\mathrm{tr}\,A(y,\omega)Dv_\varepsilon+b(y,\omega)\cdot(Dv_\varepsilon+p)-H^*(b(y,\omega),y,\omega)\leqq 0,$$

and, hence, a subsolution in the sense of distributions.

It follows that $\tilde v_\varepsilon(\omega)=v_\varepsilon(0,\omega)$ is a weak subsolution in $H^1(\Omega)$ of

$$\varepsilon\tilde v_\varepsilon-\delta\,\mathrm{tr}(\tilde A\tilde D^2\tilde v_\varepsilon)+\tilde b(\omega)\cdot(\tilde D\tilde v_\varepsilon+p)-\tilde H^*(\tilde b(\omega),\omega)\leqq 0.$$

Multiplying this last inequality by $\tilde\phi$, integrating with respect to the probability measure μ, and using that $(\tilde b,\tilde\phi)\in\mathcal{E}$, we find

$$\varepsilon\mathbb{E}(\tilde v_\varepsilon\tilde\phi)+\big(p,\mathbb{E}(\tilde b\tilde\phi)\big)-\mathbb{E}(\tilde H(\tilde b,\cdot)\tilde\phi)\leqq 0.$$

Recall that, as $\varepsilon\to 0$,

$$\varepsilon\tilde v_\varepsilon(\cdot)\to-\overline{H}(p)\ \text{ in }\ L^p(\Omega)\ \text{ for all }\ p\in[1,\infty)\ \text{ and in probability,}$$

and, hence, as $\varepsilon\to 0$,

$$\varepsilon\mathbb{E}\tilde v_\varepsilon\phi\to-\overline{H}(p).$$

Since $(\tilde b,\tilde\phi)\in\mathcal{E}$ is a general element of $\mathcal{E}$ we conclude that (4.4) holds.

The inequality $\underline{H}(p)\geqq\overline{H}(p)$ follows from the min-max theorem, the assumed superlinear growth of H and the fact that $A(\cdot,\omega)\in C^{0,1}(\mathbb{R}^N)$ a.s. in ω. The proof of [6], for $A=\mathrm{Id}$, extends easily here. $\square$

We remark that Theorem A played an important role in the above proof to pass in the limit, as $\varepsilon\to 0$, in the term $\varepsilon\mathbb{E}\tilde v_\varepsilon\phi$. When the conclusion of Theorem A is not known a priori, it is necessary to use the ergodic theorem as it is done in [6]. The difficulty, however, encountered, when $\tilde A$ is degenerate, is that the invariant measures $\tilde\phi$ may not be unique.

REFERENCES

[1] L.A. Caffarelli, P.E. Souganidis and L. Wang, *Stochastic homogenization for fully nonlinear, second-order partial differential equations*, Comm. Pure Appl. Math., 30, 319–361, 2005.

[2] M. G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., 27, 1–67, 1992.

[3] M.G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277, 1–42, 1983.

[4] H. Ishii, *On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions*, Funkc. Ekvac., 38, 101–120, 1995.

[5] R. Jensen, P.-L. Lions and P.E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. Amer. Math. Soc., 102, 975–978, 1998.

[6] E. Kosygina, F. Rezakhanlou and S.R.S. Varadhan, *Stochastic homogenization for Hamilton-Jacobi-Bellman equations*, Comm. Pure Appl. Math., 59, 1489–1521, 2006.

[7] E. Kosygina and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium*, preprint.

[8] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part 2: Viscosity solutions and uniqueness*, Comm. Par. Diff. Equ., 8, 1229–1276, 1993.

[9] P.-L. Lions and P.E. Souganidis, *Homogenization for "viscous" Hamilton-Jacobi equations in stationary, ergodic media*, Comm. Par. Diff. Equ., 30, 335–376, 2005.

[10] P.-L. Lions and P.E. Souganidis, *Correctors for the homogenization of Hamilton-Jacobi equations in a stationary ergodic setting*, Comm. Pure Appl. Math., 56, 1–23, 2003.

[11] P.-L. Lions and P.E. Souganidis, in preparation.

[12] F. Rezankhanlou and J. Tarver, *Homogenization for stochastic Hamilton-Jacobi equations*, Arch. Rat. Mech. Anal., 151, 277–309, 2000.

[13] P.E. Souganidis, *Stochastic homogenization of Hamilton-Jacobi equations and some applications*, Asympt. Anal., 20, 1–11, 1999.

[14] R. Schwab, *Homogenization of Hamilton-Jacobi equations in space-time stationary ergodic medium*, Indiana Univ. Math. J., 58, 537–582, 2009.