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EXACT TRAVELING WAVE SOLUTIONS FOR SOME NONLINEAR EVOLUTION EQUATIONS*

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Abstract. In this paper, we obtain many traveling wave solutions for some nonlinear partial differential equations. The modified tanh-coth method with the symbolic computation is implemented for constructing multiple traveling wave solutions for the two dimensional coupled Burger's, ZK-MEW and one dimensional Ostrovsky equations. The results reveal that the implemented technique is very effective and convenient for solving nonlinear partial differential equations arising in mathematical physics.

Key words. Traveling wave solutions, modified tanh-coth method, nonlinear evolution equations, Riccati equations.

AMS subject classifications. 35Q51, 35C05, 35C07, 35C08.

1. Introduction

Most phenomena in real world problems are described through nonlinear partial differential equations. These equations play an important role in various fields such as fluid mechanics, plasma physics, optical fibers, chemical kinematics, nonlinear optics, and so on. In order to better understand the nonlinear phenomena as well as further practical applications, it is important to seek their more exact traveling wave solutions. In the recent years, many powerful methods have been proposed for obtaining traveling solitary wave solutions to nonlinear evolution equations such as Hirota's bilinear method [18], the sine-cosine method [20, 21], the exp-function method [16], the Jacobi elliptic function method [12], the auxiliary ordinary differential equation method [14], the direct algebraic method [17, 22], and so on.

However, practically there is no unified method that can be used to handle all types of nonlinear partial differential equations. One of the most effective and direct method for constructing soliton solutions for nonlinear equations is tanh-function method [15]. The concept of tanh-function method was first proposed in [15] and it was used to obtain the exact traveling wave solutions to many nonlinear problems. A search in the literature have revealed that the various types of tanh-function methods have been proposed and applied to many nonlinear partial differential equations [8, 13, 19, 10, 11]. The generalized tanh-function method is used for constructing exact traveling wave solutions of general Burgers-Fisher and the Kuramoto-Sivashinsky equations [8]. Later, Fan [13] proposed an extended tanh-function method and obtained new traveling wave solutions of some nonlinear problems. More recently, El-Wakil et al. [11] proposed a modified tanh-coth method for constructing soliton and periodic solutions of nonlinear equations. Wazzan [19] applied the modified tanh-coth method for obtaining new traveling wave solutions of KdV and the KdV-Burgers equations. The main idea of this method is that it uses the Riccati equation and its solutions. The modified tanh-coth method is rather heuristic and posses significant

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features that make it practical for the determination of many traveling solutions for a wide class of nonlinear evolution equations. In this paper, we extend the application of the modified tanh-coth function method to construct exact solutions of the two dimensional coupled Burger's equation, ZK-MEW equation and Ostrovsky equation.

2. Basic idea of the modified tanh-coth function method

In this section, we will present the modified tanh-coth function method in its systematized form. The tanh technique is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function. A partial differential equation $P(u,u_t,u_x,u_{xx},u_{xxx},\cdots)=0$ can be converted to an ordinary differential equation (ODE)

$$Q(u,-k\omega u',k^2u'',k^3u''',\cdots)=0 \tag{2.1}$$

upon using a wave variable $\eta=k(x-\omega t)$, where k and ω represent the wave number and velocity of the traveling wave respectively. The ordinary differential equation (2.1) is then integrated as long as all terms contain derivatives, where the integration constants are considered as zeros. The resulting ODE is then solved by the tanh-coth method, which admits the use of a finite series of functions of the form [11]

$$u(\eta)=a_0+\sum_{n=0}^{M}a_nY^n(\eta)+\sum_{n=1}^{M}b_nY^{-n}(\eta), \tag{2.2}$$

and the Riccati equation

$$Y'=A+BY+CY^2, \tag{2.3}$$

where A, B and C are constants to be prescribed later. Here M is a positive integer, in most cases, that will be determined. The parameter M is usually obtained by balancing the linear terms of highest order in the resulting equation with the highest order nonlinear terms. Substituting (2.2) in the ODE (2.1) and using (2.3) results in an algebraic system of equations in powers of Y that will lead to the determination of the parameters $a_n,b_n(n=0,...,M),k$, and ω. In this paper, we consider the special solutions $Y=\coth\eta\pm\operatorname{csch}\eta$ and $Y=\tanh\eta\mp i\ \operatorname{sech}\eta$ of the Riccati equation (2.3).

3. Two-dimensional Burgers equation

The Burgers equation is a nonlinear partial differential equation of second order and was introduced in [1]. This equation has a large variety of applications in modeling of water in unsaturated soil, dynamics of soil water, statistics of flow problems, mixing, and turbulent diffusion [2]. Due to its important applications in science and engineering, in fact it is necessary to find more exact solutions of the Burger's equation. In this paper, we obtain traveling wave solutions of coupled 2-dimensional Burgers equation of the form [3, 6]

$$u_t-2uu_x-u_{xx}-u_{yy}-2vu_y=0 \tag{3.1}$$

$$v_t-2uv_x-v_{xx}-v_{yy}-2vv_y=0 \tag{3.2}$$

where x, y and t denote the differentiation with respect to the same variables. More recently, Biazar and Ayati [4] applied the exp-function method to construct solitary solutions of the two-dimensional Burgers equations. To look for the traveling wave solutions of equations (3.1) and (3.2), we use the transformation $u(x,y,t)=$

$u(\eta), v(x,y,t) = v(\eta), \eta = kx + ly + \omega t$, where k, l and ω are constants, then equations (3.1) and (3.2) reduce to

$$\omega u' - 2kuu' - (k^2 + l^2)u'' - 2lvu' = 0, \tag{3.3}$$

$$\omega v' - 2kuv' - (k^2 + l^2)v'' - 2lvv' = 0. \tag{3.4}$$

Balancing the linear terms of highest order with the nonlinear terms of equations (3.3) and (3.4) yields the required balancing number $M = 1$, therefore the modified tanh-coth method (2.2) admits the solutions of equations (3.3) and (3.4) in the following form

$$u(\eta) = a_0 + a_1 Y + \frac{b_1}{Y}, \tag{3.5}$$

$$v(\eta) = c_0 + c_1 Y + \frac{d_1}{Y}, \tag{3.6}$$

where $a_0, a_1, b_1, c_0, c_1, d_1$ are constants to be determined and Y satisfies equation (2.3).

Substituting equations (3.5) and (3.6) in equations (3.3) and (3.4), and using equation (2.3) collecting the coefficients of Y, yields a system of algebraic equations for $a_0, a_1, b_1, c_0, c_1, d_1, A, B, C, k, l$ and ω.

Case (I): If we set $A = B = 1$, $C = 0$ in equation (2.3), solving the system of algebraic equations using Maple yields the following sets of solutions:

$$\Big\{a_0 = a_0, a_1 = a_1, b_1 = 0, c_0 = c_0, c_1 = c_1, d_1 = 0, k = -\frac{lc_1}{a_1}, l = l, \\ \omega = -\frac{l(2a_0a_1c_1 - lc_1^2 - la_1^2 - 2a_1^2c_0)}{a_1^2}\Big\},$$

$$\Big\{a_0 = a_0, a_1 = 0, b_1 = b_1, c_0 = c_0, c_1 = 0, d_1 = -\frac{kb_1 - k^2 - l^2}{l}, k = k, l = l, \\ \omega = 2ka_0 + 2lc_0 - k^2 - l^2\Big\},$$

$$\{a_0 = a_0, a_1 = \pm c_1 i, b_1 = b_1, c_0 = c_0, c_1 = c_1, d_1 = \mp b_1 i, k = \pm li, l = l, \omega = \pm 2la_0 i + 2lc_0\}.$$

Substituting the above first set of values and $Y = e^\eta - 1$ in equations (3.5) and (3.6), we obtain the following soliton solution

$$u_1(x,y,t) = a_0 + a_1(e^\eta - 1), \tag{3.7}$$

$$v_1(x,y,t) = c_0 + c_1(e^\eta - 1), \tag{3.8}$$

where $\eta = -\frac{l}{a_1^2}\left\{a_1c_1x - a_1^2y + \left(2a_0a_1c_1 - lc_1^2 - la_1^2 - 2a_1^2c_0\right)t\right\}$.

For the second set of values, we obtain the following soliton solution

$$u_2(x,y,t) = a_0 + b_1(e^\eta - 1)^{-1}, \tag{3.9}$$

$$v_2(x,y,t) = c_0 - \frac{kb_1 - k^2 - l^2}{l}(e^\eta - 1)^{-1}, \tag{3.10}$$

where $\eta = kx + ly + \left(2ka_0 + 2lc_0 - k^2 - l^2\right)t$.

Finally, we obtain the following soliton solution for the third set of values

$$u_3(x,y,t) = a_0 \pm c_1 i\,(e^{\eta}-1) + b_1\,(e^{\eta}-1)^{-1}, \tag{3.11}$$
$$v_3(x,y,t) = c_0 + c_1\,(e^{\eta}-1) \mp b_1 i\,(e^{\eta}-1)^{-1}, \tag{3.12}$$

where $\eta = l\{\pm ix + y \pm 2(a_0 i \pm c_0)t\}$.

Case (II): If we take $A=\frac{1}{2}, B=0, C=\frac{1}{2}$ in equation (2.3), solving the system of algebraic equations using Maple we obtain four sets of nontrivial solutions

$$\left\{a_0 = a_0, a_1 = 0, b_1 = \frac{k^2+l^2-2ld_1}{2k}, c_0 = c_0, c_1 = 0, d_1 = d_1, k = k, l = l, \omega = 2ka_0 + 2lc_0\right\},$$

$$\left\{a_0 = a_0, a_1 = \frac{k^2+l^2-2lc_1}{2k}, b_1 = 0, c_0 = c_0, c_1 = c_1, d_1 = 0, k = k, l = l, \omega = 2ka_0 + 2lc_0\right\},$$

$$\left\{a_0 = a_0, a_1 = \frac{k^2+l^2-2ld_1}{2k}, b_1 = a_1, c_0 = c_0, c_1 = d_1, d_1 = d_1, k = k, l = l, \omega = 2ka_0 + 2lc_0\right\},$$

$$\{a_0 = a_0, a_1 = \pm c_1 i, b_1 = \pm d_1 i, c_0 = c_0, c_1 = c_1, d_1 = d_1, k = \pm li, l = l, \omega = \pm 2la_0 i + 2lc_0\}.$$

Substituting $Y = \coth\xi \pm \mathrm{csch}\xi$ and $Y = \tanh\xi \pm i\mathrm{sech}\xi$ in equations (3.5) and (3.6), the first set gives the following soliton solutions

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2ld_1}{2k}\,(\coth\eta \pm \mathrm{csch}\eta)^{-1}, \quad v(x,y,t) = c_0 + d_1\,(\coth\eta \pm \mathrm{csch}\eta)^{-1}$$

and

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2ld_1}{2k}\,(\tanh\eta \pm i\mathrm{sech}\eta)^{-1}, \quad v(x,y,t) = c_0 + d_1\,(\tanh\eta \pm i\mathrm{sech}\eta)^{-1},$$

where $\eta = kx + ly + 2(ka_0 + lc_0)t$.

Moreover, we obtain the following soliton solutions with the use of second set of values

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2lc_1}{2k}\,(\coth\eta \pm \mathrm{csch}\eta), \quad v(x,y,t) = c_0 + c_1\,(\coth\eta \pm \mathrm{csch}\eta),$$

and

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2lc_1}{2k}\,(\tanh\eta \pm i\mathrm{sech}\eta), \quad v(x,y,t) = c_0 + c_1\,(\tanh\eta \pm i\mathrm{sech}\eta),$$

where $\eta = kx + ly + 2(ka_0 + lc_0)t$.

Also, the third set gives the following soliton solutions

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2ld_1}{2k}\left\{\coth\eta \pm \mathrm{csch}\eta + (\coth\eta \pm \mathrm{csch}\eta)^{-1}\right\},$$
$$v(x,y,t) = c_0 + d_1\left\{\coth\eta \pm \mathrm{csch}\eta + (\coth\eta \pm \mathrm{csch}\eta)^{-1}\right\},$$

and

$$u(x,y,t) = a_0 + \frac{k^2+l^2-2ld_1}{2k}\left\{\tanh\eta \pm i\mathrm{sech}\eta + (\tanh\eta \pm i\mathrm{sech}\eta)^{-1}\right\},$$
$$v(x,y,t) = c_0 + d_1\left\{\tanh\eta \pm i\mathrm{sech}\eta + (\tanh\eta \pm i\mathrm{sech}\eta)^{-1}\right\},$$

where $\eta = kx + ly + 2(ka_0 + lc_0)t$.

Finally, we obtain the following soliton solutions by using the fourth set of values

$$u(x,y,t) = a_0 \pm c_1 i(\coth\eta \pm \mathrm{csch}\eta) \pm d_1 i(\coth\eta \pm \mathrm{csch}\eta)^{-1},$$
$$v(x,y,t) = c_0 + c_1(\coth\eta \pm \mathrm{csch}\eta) + d_1(\coth\eta \pm \mathrm{csch}\eta)^{-1},$$

and

$$u(x,y,t) = a_0 \pm c_1 i(\tanh\eta \pm i\mathrm{sech}\eta) \pm d_1 i(\tanh\eta \pm i\mathrm{sech}\eta)^{-1},$$
$$v(x,y,t) = c_0 + c_1(\tanh\eta \pm i\mathrm{sech}\eta) + d_1(\tanh\eta \pm i\mathrm{sech}\eta)^{-1},$$

where $\eta = l\{\pm ix + y + 2(c_0 \pm a_0 i)t\}$.

4. ZK-MEW equation

In this section,we consider the two dimensional ZK-MEW equation [5]

$$u_t + \alpha(u^3)_x + (\beta u_{xt} + \gamma u_{yy})_x = 0 \tag{4.1}$$

where α, β and γ are constants. The ZK-MEW equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [5]. Recently, ZK-MEW equation is solved by exp-function method in [5]. In order to obtain the traveling wave solutions, we use the transformation $u(x,y,t) = u(\eta)$, $\eta = kx + ly + \omega t$. The ZK-MEW equation (4.1) can be converted to the ODE

$$wu + \alpha k u^3 + (\beta k^2\omega + \gamma l^2 k)u^{''} = 0, \tag{4.2}$$

upon using $u(x,y,t) = u(\eta)$, $\eta = kx + ly + \omega t$ and integrating the resulting ODE once and neglecting the constant of integration.

To determine the parameter M, we balance the linear terms of highest order in equation (4.2) with the highest order nonlinear terms. This in turn gives $M = 1$. As a result, the modified tanh-coth method (2.2) admits the use of the finite expansion

$$v(\eta) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{4.3}$$

Substituting equation (4.3) in the reduced ODE (4.2) and using equation (2.3) collecting the coefficients of Y, yields a system of algebraic equations for $a_0, a_1, b_1, A, B, C, k, l$ and ω.

Case (I): If we set $A = B = 1$, $C = 0$ in equation (2.3), solving the system of algebraic equations using Maple yields the following nontrivial solution:

$$\left\{a_0 = \frac{b_1}{2}, a_1 = 0, b_1 = b_1, k = \pm\frac{\sqrt{2}}{b_1}\sqrt{\frac{\alpha b_1^2 + 2\gamma l^2}{\alpha\beta}}, l = l, \omega = \mp\frac{\alpha b_1}{2\sqrt{2}}\sqrt{\frac{\alpha b_1^2 + 2\gamma l^2}{\alpha\beta}}\right\}.$$

Substituting the above set of values and $Y=e^{\eta}-1$ in equation (4.3), we get the soliton solution $u_1(x,y,t)=\frac{b_1}{2}+b_1(e^{\eta}-1)^{-1}=\frac{b_1}{2}\coth\frac{\eta}{2}$, where $\eta=\pm\frac{\sqrt{2}}{b_1}\sqrt{\frac{\alpha b_1^2+2\gamma l^2}{\alpha\beta}}x+ly\mp\frac{\alpha b_1}{2\sqrt{2}}\sqrt{\frac{\alpha b_1^2+2\gamma l^2}{\alpha\beta}}t$.

Case (II): If we set $A=\frac{1}{2},B=0,C=-\frac{1}{2}$ in equation (2.3), solving the system of algebraic equations using Maple we obtain the following four sets of solutions:

$$\left\{a_0=0,a_1=0,b_1=b_1,k=\pm\frac{1}{b_1}\sqrt{\frac{2\alpha b_1^2+\gamma l^2}{\alpha\beta}},l=l,\omega=\mp\alpha b_1\sqrt{\frac{2\alpha b_1^2+\gamma l^2}{\alpha\beta}}\right\},$$

$$\left\{a_0=0,a_1=a_1,b_1=0,k=\pm\frac{1}{a_1}\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}},l=l,\omega=\mp\alpha a_1\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}\right\},$$

$$\left\{a_0=0,a_1=a_1,b_1=b_1,k=\pm\frac{1}{2a_1}\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}},l=l,\omega=\mp2\alpha a_1\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}\right\},$$

$$\left\{a_0=0,a_1=a_1,b_1=-a_1,k=\pm\frac{1}{\sqrt{2}a_1}\sqrt{-\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}},l=l,\right.$$
$$\left.\omega=\pm\sqrt{2}\alpha a_1\sqrt{-\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}\right\}.$$

Substituting $Y=\coth\xi\pm\mathrm{csch}\xi$ and $Y=\tanh\xi\pm i\mathrm{sech}\xi$ in equation (4.3), the first set gives the soliton solutions

$$u_2(x,y,t)=b_1(\coth\eta\pm\mathrm{csch}\eta)^{-1},\quad u_3(x,y,t)=b_1(\tanh\eta\pm i\mathrm{sech}\eta)^{-1},$$

where $\eta=\pm\frac{1}{b_1}\sqrt{\frac{2\alpha b_1^2+\gamma l^2}{\alpha\beta}}x+ly\mp\alpha b_1\sqrt{\frac{2\alpha b_1^2+\gamma l^2}{\alpha\beta}}t$. Also the second set gives the soliton solutions

$$u_4(x,y,t)=a_1(\coth\eta\pm\mathrm{csch}\eta),\quad u_5(x,y,t)=a_1(\tanh\eta\pm i\mathrm{sech}\eta),$$

where $\eta=\pm\frac{1}{a_1}\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}x+ly\mp\alpha a_1\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}t$.

Furthermore, the third set gives the soliton solutions

$$u_6(x,y,t)=a_1(\coth\eta\pm\mathrm{csch}\eta)+b_1(\coth\eta\pm\mathrm{csch}\eta)^{-1},$$
$$u_7(x,y,t)=a_1(\tanh\eta\pm i\mathrm{sech}\eta)+b_1(\tanh\eta\pm i\mathrm{sech}\eta)^{-1},$$

where $\eta=\pm\frac{1}{2a_1}\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}x+ly\mp2\alpha a_1\sqrt{\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}t$.

Finally, the fourth set gives the soliton solutions

$$u_8(x,y,t)=a_1\left\{\coth\eta\pm\mathrm{csch}\eta-(\coth\eta\pm\mathrm{csch}\eta)^{-1}\right\},$$
$$u_9(x,y,t)=a_1\left\{\tanh\eta\pm i\mathrm{sech}\eta-(\tanh\eta\pm i\mathrm{sech}\eta)^{-1}\right\},$$

where $\eta=\pm\frac{1}{\sqrt{2}a_1}\sqrt{-\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}x+ly\pm\sqrt{2}\alpha a_1\sqrt{-\frac{2\alpha a_1^2+\gamma l^2}{\alpha\beta}}t$.

4.1. Ostrovsky equation. In this section, we consider the nonlinear evolution equation

$$(u_t + c_0 u_x + p u u_x + q u_{xxx})_x = \nu u. \tag{4.4}$$

The equation (4.4) presented in [7] is used to describe nonlinear surface and internal waves in rotating ocean. Here c_0 is the velocity of dispersionless linear waves, p is the nonlinear coefficient, q is the Boussinesq dispersion, and ν is the Coriolis dispersion coefficients. Later Vakhnenko and Parkes [9] demonstrated that the Ostrovsky equation (4.4) can be transformed to the new integrable equation of the form

$$u u_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{4.5}$$

To look for the traveling wave solutions of equation (4.5), we consider the transformation $u(x,t) = u(\eta)$, $\eta = k(x - \omega t)$, then equation (4.5) becomes

$$k^2 u u^{'''} - k^2 u' u'' + u^2 u' = 0. \tag{4.6}$$

By balancing $uu^{'''}$ and $u^2 u'$ gives the desired balancing number $M = 2$. In this case, the modified tanh-coth method (2.2) admits the use of the finite expansion

$$v(\eta) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{4.7}$$

Substituting equation (4.7) in the reduced ODE (4.6) and using equation (2.3) collecting the coefficients of Y yields a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, A, B, C, k$, and ω.

Case (I): If we set $A = B = 1$, $C = 0$ in equation (2.3) and by the same calculation as in previous sections, the following sets of solutions are obtained:

$$\left\{a_0 = 0, a_1 = 0, a_2 = 0, b_1 = -6k^2, b_2 = -6k^2, k = k, \omega = \omega\right\}.$$

$$\left\{a_0 = -k^2, a_1 = 0, a_2 = 0, b_1 = -6k^2, b_2 = -6k^2, k = k, \omega = \omega\right\}.$$

Substituting the above values and $Y = e^\eta - 1$ in equation (4.7), we obtain the soliton solution

$$u_1(x,t) = -6k^2 \left\{(e^\eta - 1)^{-1} + (e^\eta - 1)^{-2}\right\} = \frac{3k^2}{1 - \cosh\eta}, \quad \text{where } \eta = k(x - \omega t).$$

$$u_2(x,t) = -k^2 \left\{1 + 6(e^\eta - 1)^{-1} + 6(e^\eta - 1)^{-2}\right\} = \frac{k^2(2 + \cosh\eta)}{1 - \cosh\eta}, \quad \eta = k(x - \omega t).$$

Case (II): If we set $A = \frac{1}{2}, B = 0, C = -\frac{1}{2}$ in equation (2.3), solving the system of algebraic equations using Maple we obtain the following six sets of nontrivial solutions:

$$\left\{a_0 = \frac{k^2}{2}, a_1 = 0, a_2 = -\frac{3k^2}{2}, b_1 = 0, b_2 = 0, k = k, \omega = \omega\right\}, \tag{4.8}$$

$$\left\{a_0 = \frac{3k^2}{2}, a_1 = 0, a_2 = -\frac{3k^2}{2}, b_1 = 0, b_2 = 0, k = k, \omega = \omega\right\}, \tag{4.9}$$

$$\left\{a_0 = \frac{k^2}{2}, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -\frac{3k^2}{2}, k = k, \omega = \omega\right\}, \tag{4.10}$$

$$\left\{a_0 = \frac{3k^2}{2}, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -\frac{3k^2}{2}, k = k, \omega = \omega\right\}, \tag{4.11}$$

$$\left\{a_0 = -k^2, a_1 = 0, a_2 = -\frac{3k^2}{2}, b_1 = 0, b_2 = -\frac{3k^2}{2}, k = k, \omega = \omega\right\}, \tag{4.12}$$

$$\left\{a_0 = 3k^2, a_1 = 0, a_2 = -\frac{3k^2}{2}, b_1 = 0, b_2 = -\frac{3k^2}{2}, k = k, \omega = \omega\right\}. \tag{4.13}$$

According to equations (4.8)–(4.10), we obtain the exact traveling wave solution in the following form

$$u_3(x,t) = \frac{k^2}{2}\left\{1 - 3(\coth\eta \pm \operatorname{csch}\eta)^2\right\},\ u_4(x,t) = \frac{k^2}{2}\left\{1 - 3(\tanh\eta \pm i\operatorname{sech}\eta)^2\right\},$$

$$u_5(x,t) = \frac{3k^2}{2}\left\{1 - (\coth\eta \pm \operatorname{csch}\eta)^2\right\}, u_6(x,t) = \frac{3k^2}{2}\left\{1 - (\tanh\eta \pm i\operatorname{sech}\eta)^2\right\},$$

$$u_7(x,t) = \frac{k^2}{2}\left\{1 - 3(\coth\eta \pm \operatorname{csch}\eta)^{-2}\right\},\ u_8(x,t) = \frac{k^2}{2}\left\{1 - 3(\tanh\eta \pm i\operatorname{sech}\eta)^{-2}\right\},$$

where $\eta = k(x - \omega t)$.

Finally equations (4.11)–(4.13) lead the exact traveling wave solutions in the following form

$$u_9(x,t) = \frac{3k^2}{2}\left\{1 - (\coth\eta \pm \operatorname{csch}\eta)^{-2}\right\},\ u_{10}(x,t) = \frac{3k^2}{2}\left\{1 - (\tanh\eta \pm i\operatorname{sech}\eta)^{-2}\right\},$$

where $\eta = k(x - \omega t)$.

$$u_{11}(x,t) = -\frac{k^2}{2}\left\{2 + 3(\coth\eta \pm \operatorname{csch}\eta)^2 + 3(\coth\eta \pm \operatorname{csch}\eta)^{-2}\right\},$$
$$u_{12}(x,t) = -\frac{k^2}{2}\left\{2 + 3(\tanh\eta \pm i\operatorname{sech}\eta)^2 + 3(\tanh\eta \pm i\operatorname{sech}\eta)^{-2}\right\},$$

where $\eta = k(x - \omega t)$.

$$u_{13}(x,t) = \frac{3k^2}{2}\left\{2 - (\coth\eta \pm \operatorname{csch}\eta)^2 - (\coth\eta \pm \operatorname{csch}\eta)^{-2}\right\},$$
$$u_{14}(x,t) = \frac{3k^2}{2}\left\{2 - (\tanh\eta \pm i\operatorname{sech}\eta)^2 - (\tanh\eta \pm i\operatorname{sech}\eta)^{-2}\right\},$$

where $\eta = k(x - \omega t)$. For $k = 1$, the behavior of the obtained traveling wave solutions u_3 and u_9 of Ostrovsky equation are shown graphically, see figures 4.1 and 4.2.

It should be noted that the solution of the ansatz (2.2) goes back to the solutions of standard tanh method once $b_n = 0,\ 1 \le n \le M$. On the other hand in case of $b_n \ne 0$, the corresponding solutions are quite new and cannot be obtained from standard tanh method.

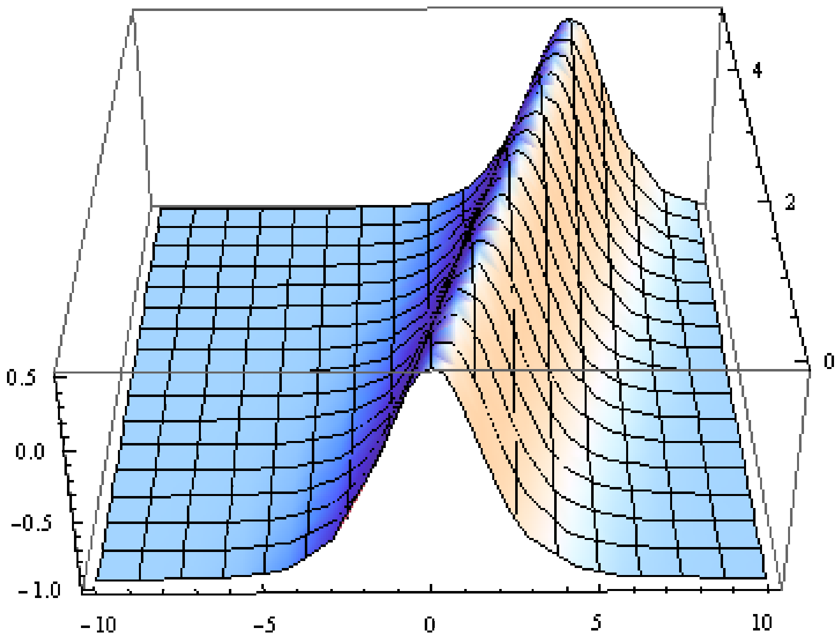

FIG. 4.1. *Solitary wave solution u_3 of Ostrovsky equation (4.4).*

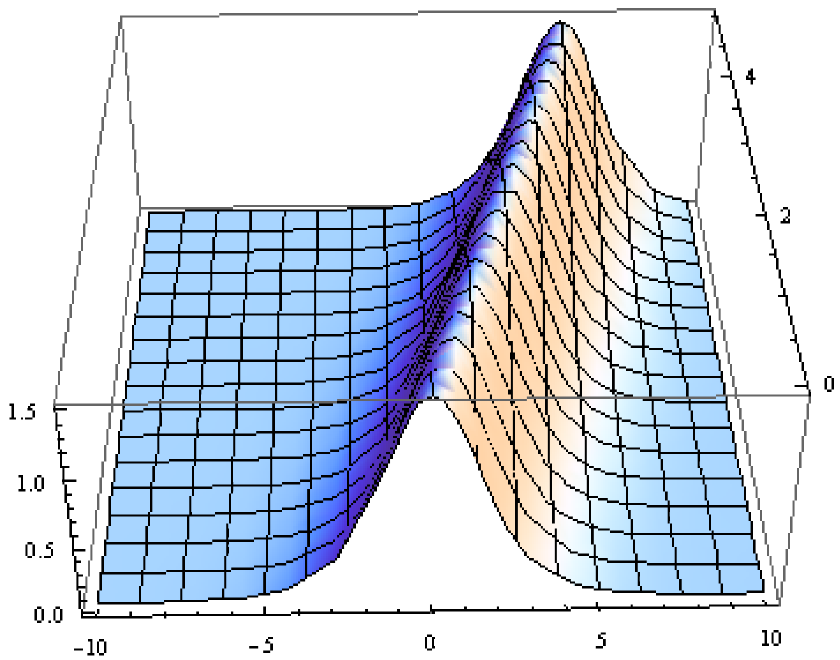

FIG. 4.2. *Solitary wave solution u_9 of Ostrovsky equation (4.4).*

5. Conclusion

In this paper, we implemented a reliable algorithm called modified tanh-coth function method to obtain traveling wave solutions for some nonlinear evolution equations. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations. It is interesting to mention that the sign of the parameters in the Riccati equations can be used to judge the numbers and types of traveling wave solutions. The result reveals that the proposed method is simple and effective, and can be used for many other nonlinear evolutions equations arising in mathematical physics.

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