COMMUN. MATH. SCI.
Vol. 7, No. 4, pp. 973–1008

GAUSSIAN BEAMS SUMMATION FOR THE WAVE EQUATION IN A CONVEX DOMAIN*

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Abstract. We consider the scalar wave equation in a bounded convex domain of $\mathbb{R}^n$. The boundary condition is of Dirichlet or Neumann type and the initial conditions have a compact support in the considered domain. We construct a family of approximate high frequency solutions by a Gaussian beams summation. We give a rigorous justification of the asymptotics in the sense of an energy estimate and show that the error can be reduced to any arbitrary power of ε, which is the high frequency parameter.

Key words. Wave equation, high frequency solutions, Gaussian beams summation, reflection at the boundary.

AMS subject classifications. 35L05, 35L20, 81S30, 41A60.

1. Introduction

In this paper, our aim is to provide asymptotic solutions, in a sense to be made more precise later, to the following initial-boundary value problem (IBVP) for the wave equation

$$\begin{cases} Pu_\varepsilon = \partial_t^2 u_\varepsilon - \partial_x \cdot (c^2(x)\partial_x u_\varepsilon) = 0 \text{ in } [0,T]\times\Omega, \\ u_\varepsilon|_{t=0} = u_\varepsilon^I, \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I \text{ in } \Omega, \\ Bu_\varepsilon = 0 \text{ in } [0,T]\times\partial\Omega, \end{cases} \tag{1.1}$$

where B is a Dirichlet or Neumann type boundary operator.

Above, $T>0$ is fixed, and Ω is a bounded domain of $\mathbb{R}^n$, with $n=2$ or $n=3$ for important applications to acoustics or elastodynamics problems.

We assume the boundary $\partial\Omega$ is $\mathcal{C}^\infty$ and the domain is convex for the bicharacteristic curves of P, see more precisely Assumption B1 below. Furthermore, the coefficient c is assumed to be in $\mathcal{C}^\infty(\bar{\Omega})$, though this assumption may be substantially relaxed.

Our initial data will depend on a small parameter $\varepsilon>0$, playing the role of a small wavelength, and our main objective is to study the high frequency limit, corresponding to $\varepsilon\to 0$, i.e., the construction of high frequency solutions. Moreover, we shall assume that u_ε^I, v_ε^I are

A1. uniformly bounded respectively in $H^1(\Omega)$ and $L^2(\Omega)$,

A2. uniformly supported in a fixed compact set $K\subset\Omega$.

The search for such approximate solutions and related notions of parametrices for the wave equation and similar equations has been an intensive area of research. A

*Received: June 30, 2009; accepted (in revised version): October 17, 2009. Communicated by Olof Runborg.

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widely used technique to produce such high frequency solutions is given by geometric optics, also called the WKB method [32]. This technique is well known in the Physics literature [21]. Then, and in the full space case, approximate solutions are constructed under the form

$$\sum_{j=0}^{N} \varepsilon^j a_j e^{i\psi/\varepsilon}, \tag{1.2}$$

with a real phase function ψ and complex amplitudes functions a_j. The presence of a boundary may lead to further terms with reflected phases and amplitudes.

Typically, initial data should have the same form as in (1.2), but solutions for more general initial conditions can be obtained by summing an infinite number of WKB solutions. Mathematically, this technique relies on the well known theory of Fourier Integral Operators (FIOs), see for instance [15], see also the earlier works of Maslov and Fedoruk [32] and the recent lecture notes by Rauch and Markus [38]. In general, the global construction of a FIO breaks down at some time, due to generic existence of caustics, see [9].

The caustics problem is also linked to the local solvability of the eikonal equation for the phase, which is derived by substituting the WKB ansatz in the partial differential equation. Indeed, the eikonal equation is solved using the method of characteristics and the phase therefore cannot be defined near every point of the domain, at the exception of some very particular cases.

To overcome this difficulty, one either uses a collection of local FIOs or, more generally, constructs a global FIO. This is the way chosen by Chazarain to produce a parametrix for the mixed problem of the wave equation in [6]. Though this method is quite satisfying for the mathematical analysis of propagation of singularities, it does not give approximate solutions directly. A computationally oriented alternative to this elaborate mathematical method is the use of Gaussian beams summation.

Gaussian beams are high frequency asymptotic solutions to linear partial differential equations that are concentrated on a single ray. In the mathematical literature, their first use dates back to the 1960s, see [2]. Since then, they have been useful in a variety of problems in mathematical physics such as modelling seismic [14] or electromagnetic [10] wave fields. They also have been used in pure mathematics, such as propagation of singularities [16, 36] and semiclassical measures [35], see [17] and [12] for other methods concerning these problems.

One advantage of this method over the WKB precedure is that an individual Gaussian beam has no singularities at caustics. Note that Gaussian beams summation is naturally linked to FIOs with complex phases [15] (see [4, 23, 24, 43] for recent contributions).

In a bounded domain of general geometry, both of the WKB and the Gaussian beams ansatzs are inadequate to produce asymptotic solutions. Other models are needed to describe the diffraction phenomena or the gliding of rays along the boundary, such as Fourier-Airy Integral Operators [33] or gliding beams [37]. However, in our precise setting of a convex domain with compactly supported initial data, only the reflection effects at the boundary must be considered.

Dirichlet or Neumann boundary conditions can be taken into account by combining a finite sum of successively reflected Gaussian beams [19, 30]. Using an infinite sum of Gaussian beams, one can then match quite general initial conditions. This summation can be achieved in different ways, see [5, 20, 22] and the more recent [14, 18, 26, 28, 29, 34, 44]. In [28] and [44], superpositions of Gaussian beams are used to solve wave equations with initial data of WKB form. In fact, in Theorem 1.1 below, more general initial conditions are allowed through the use of their FBI transforms, which is also naturally linked with the concept of a Gaussian beam.

The FBI or Fourier-Bros-Iagolnitzer transform (see [8, 31, 42]) is, for a given scale ε, the operator $T_\varepsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ defined by

$$T_\varepsilon(a)(y,\eta) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} a(w) e^{i\eta.(y-w)/\varepsilon - (y-w)^2/(2\varepsilon)} dw, c_n = 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}},\ a \in L^2(\mathbb{R}^n). \tag{1.3}$$

Its adjoint is the operator

$$T_\varepsilon^*(f)(x) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} f(y,\eta) e^{i\eta.(x-y)/\varepsilon - (x-y)^2/(2\varepsilon)} dy d\eta, f \in L^2(\mathbb{R}^{2n}). \tag{1.4}$$

Like the Fourier Transform, the FBI transform is an isometry, satisfying $T_\varepsilon^* T_\varepsilon = Id$. Its main property is to decompose an L_x^2 function over the family of functions $(e^{i\eta.(x-y)/\varepsilon - (x-y)^2/(2\varepsilon)})_{(y,\eta) \in \mathbb{R}^{2n}}$. For instance, FBI transformation was the method used in [39] to construct an approximate solution for the Schrödinger equation with WKB initial conditions. The FBI transform is of course again connected with FIOs with complex phases and an interesting result on their global L^2 boundedness was recently proved in [43], regarding the Hermann Kluck propagator.

In this paper, our approach to find asymptotic solutions to the problem (1.1) is to achieve a superposition of incident and reflected Gaussian beams weighted by the FBI transforms of the initial data, satisfying both the condition at the boundary and the initial conditions. Our main result is given by

THEOREM 1.1. *Under Assumptions A1 and A2, suppose the FBI transforms of the initial data are infinitely small on the complement of some ring*

$$R_\eta = \{\eta \in \mathbb{R}^n, r_0 \le |\eta| \le r_\infty\},\ 0 < r_0 < r_\infty,$$

in the sense that

A3. $\|T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^n \times R_\eta^c)} = O(\varepsilon^s)$ *and* $\|T_\varepsilon v_\varepsilon^I\|_{L^2(\mathbb{R}^n \times R_\eta^c)} = O(\varepsilon^s)$, $\forall s \ge 0$.

Then for any integer $R \ge 2$, *there is an asymptotic solution to* (1.1) *of the form*

$$u_\varepsilon^R(t,x) = \sum_k \int_{\mathbb{R}^{2n}} a_\varepsilon^k(t,x,y,\eta,R) e^{i\psi_k(t,x,y,\eta,R)/\varepsilon} dy d\eta,$$

where $a_\varepsilon^k e^{i\psi_k/\varepsilon}$ *are Gaussian beams and the summation over* k *is finite.*
u_ε^R *is asymptotic to the exact solution of the IBVP* (1.1) *in the following sense:*

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon^R(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$and\ \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon^R(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}).$$

Let us note that construction of asymptotic solutions such as a summation of Gaussian beams is certainly not new, but rigorous justification is the main point of our work, together with precise estimates.

This paper is organized as follows. In section 2 we recall the construction of Gaussian beams for a strictly hyperbolic differential operator as achieved in [36]. Then we study the case of the wave equation and construct the incident and reflected beams, and in a final step we construct approximate solutions for (1.1) by a Gaussian beams summation. Justification of the asymptotics is given in section 3. Therein, we introduce approximation operators acting from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ with a complex phase and compute their norms. We apply these operators on FBI transforms of initial data, and estimate the error of the constructed asymptotic solutions near the boundary, thus taking into account the precise boundary condition, and in the interior set. These estimates are combined with the errors in the initial conditions and yield the justification of the asymptotics by means of energy type estimates.

We close this introduction by a short discussion on the notations. Throughout this paper, we will use standard multiindex notations. The inner product of two vectors $a,b\in\mathbb{R}^d$ will be denoted by $a\cdot b$. The transpose of a matrix A will be noted A^T. If E is a subset of $\mathbb{R}^d$, we denote $\mathbf{1}_E$ its characteristic function. For a smooth function $f\in\mathcal{C}^\infty(\mathbb{R}^d_x,\mathbb{C})$, we will use the notation $\partial_x f$ to denote its gradient vector $(\partial_{x_b} f)_{1\leq b\leq d}$, $\partial_x^2 f$ to denote its Hessian matrix $(\partial_{x_b}\partial_{x_c} f)_{1\leq b,c\leq d}$ and $\partial_x^r f$, $r>2$ to denote the family $(\partial_{x_{b_1}}\dots\partial_{x_{b_r}} f)_{1\leq b_1,\dots,b_r\leq d}$. For a vector function $F\in\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{C}^p)$, we denote its Jacobian matrix by DF with $(DF)_{j,k}=\partial_k F_j$ and its second derivatives by D^2F with $(D^2F)_{j,k,l}=\partial_j\partial_k F_l$. For $y_\varepsilon, z_\varepsilon\in\mathbb{R}_+$, we use the notation $y_\varepsilon\lesssim z_\varepsilon$ if there exists a constant $c>0$ independent of ε such that $y_\varepsilon\leq c z_\varepsilon$. We write $y_\varepsilon\lesssim\varepsilon^\infty$ or $y_\varepsilon=O(\varepsilon^\infty)$ if $\forall s\geq 0$ there exists $c_s>0$ s.t. $y_\varepsilon\leq c_s\varepsilon^s$ for ε small enough . Finally, the word const denotes a positive constant (different each time it appears).

2. Construction of the asymptotic solutions

In this section we first introduce the notion of Gaussian beams for strictly hyperbolic differential operators, following the presentation of [36]. Then the construction of incident and reflected Gaussian beams in the particular case of the wave equation is explained. Finally, the approximate solution for the IBVP (1.1) is given in the last section as an infinite sum of Gaussian beams.

2.1. Gaussian beams for stricly hyperbolic operators.

This section follows basically the presentation of [36].

Let $P(t,x,\partial_t,\partial_x)$ be a strictly hyperbolic differential operator of order m_P and of principal symbol p. That is, we suppose that the roots τ of $p(t,x,\tau,\xi)=0$ are simple and real for all (t,x) and $\xi\neq 0$. The symbol p is assumed to be real. A Gaussian beam for P is a function of the form

$$w_\varepsilon(t,x)=\sum_{j=0}^{N}\varepsilon^j a_j(t,x)e^{i\psi(t,x)/\varepsilon}, N\in\mathbb{N}, \tag{2.1}$$

satisfying

$$\exists m>0 \text{ s.t. } \|Pw_\varepsilon\|_{L^2_{t,x}}=O(\varepsilon^m).$$

Note that the above expansion is similar to the usual WKB expansion, but it is required here that:

(i) the beam w_ε is concentrated on some fixed ray $(t(s),x(s))$ associated to p. Here s is the "time" parameter of this curve.

(ii) the phase ψ is a complex-valued function, but real-valued on the ray $(t(s),x(s))$.

The exact definition of a ray $(t(s),x(s))$ is as follows. First of all, we introduce the so-called null bicharacteristics, which are the curves, solutions of the Hamiltonian equations

$$\begin{aligned}&\dot t(s)=\partial_\tau p(t(s),x(s),\tau(s),\xi(s)), \quad \dot\tau(s)=-\partial_t p(t(s),x(s),\tau(s),\xi(s)),\\&\dot x(s)=\partial_\xi p(t(s),x(s),\tau(s),\xi(s)), \quad \dot\xi(s)=-\partial_x p(t(s),x(s),\tau(s),\xi(s)),\end{aligned} \tag{2.2}$$

with initial conditions satisfying $p(t(0),x(0),\tau(0),\xi(0))=0$. Note that it follows that $p(t(s),x(s),\tau(s),\xi(s))=0$, for all s. Then by definition, the projection on $\mathbb{R}^{n+1}_{t,x}$ of such a curve $(t(s),x(s),\tau(s),\xi(s))$, that is $(t(s),x(s))$, is called a ray. We suppose the conditions for local existence, uniqueness, and smoothness with respect to initial conditions of solutions to the Hamiltonian system (2.2) to be fulfilled; see [13].

The construction of a Gaussian beam w_ε is achieved by making Pw_ε vanish to a certain order on a fixed and given ray $(t(s),x(s))$. For this purpose, applying P to the form (2.1) of a Gaussian beam, we obtain a similar form

$$Pw_\varepsilon=\sum_{j=0}^{N+m_P}\varepsilon^{j-m_P}c_j e^{i\psi/\varepsilon}, \tag{2.3}$$

where

$$\begin{aligned}&c_0=p(t,x,\partial_t\psi,\partial_x\psi)a_0,\\&c_j=La_{j-1}+p(t,x,\partial_t\psi,\partial_x\psi)a_j+g_j,\, j\geq 1.\end{aligned} \tag{2.4}$$

Above, $a_j=0$ for $j>N$, $g_1=0$, and g_j is a function of $\psi,a_0,\dots,a_{j-2}$ for $j\geq 2$. Furthermore, L is a linear differential operator with coefficients depending on ψ. Using p', the symbol of the terms of order m_P-1 of P, L can be written in an explicit way as

$$L=\frac{1}{i}\partial_{\tau,\xi}p(t,x,\partial_t\psi,\partial_x\psi)\cdot\partial_{t,x}+\frac{1}{2i}Tr(\partial^2_{\tau,\xi}p(t,x,\partial_t\psi,\partial_x\psi)\partial^2_{t,x}\psi)+p'(t,x,\partial_t\psi,\partial_x\psi). \tag{2.5}$$

For the construction of a Gaussian beam adapted to P, the first step, and by far the most important one, is to build a phase ψ satisfying the eikonal equation

$$p(t,x,\partial_t\psi(t,x),\partial_x\psi(t,x))=0 \text{ on } (t,x)=(t(s),x(s)) \text{ up to order } R \text{ only}, \tag{2.6}$$

with $R\geq 2$, which means

$$\partial^\alpha_{t,x}[p(t,x,\partial_t\psi(t,x),\partial_x\psi(t,x))]|_{(t(s),x(s))}=0 \text{ for } |\alpha|\leq R.$$

Compare this with the usual eikonal equation $p(t,x,\partial_t\psi(t,x),\partial_x\psi(t,x))=0$ required by the WKB method in full space.

Order 0 of the eikonal equation (2.6)

$$p(t(s),x(s),\partial_t\psi(t(s),x(s)),\partial_x\psi(t(s),x(s)))=0,$$

is fulfilled by setting

$$\big(\partial_t\psi,\partial_x\psi\big)|_{(t(s),x(s))}=\big(\tau(s),\xi(s)\big). \tag{P.a}$$

This constraint ensures that $\frac{d}{ds}\psi(t(s),x(s))$ is real, which leads by choosing

$$\psi(t(0),x(0)) \text{ a real quantity},$$

to the required property

$$\psi(t(s),x(s)) \text{ is real.} \tag{P.b}$$

Replacing $\partial_{\tau,\xi}p|_{(t(s),x(s),\tau(s),\xi(s))}$ by $(\dot t(s),\dot x(s))$ yields in the differentiation of (2.6) to the compatibility condition

$$\partial^2_{t,x}\psi|_{(t(s),x(s))}\begin{pmatrix}\dot t(s)\\ \dot x(s)\end{pmatrix} = -\begin{pmatrix}\partial_t p\\ \partial_x p\end{pmatrix}\Bigg|_{(t(s),x(s),\tau(s),\xi(s))} = \begin{pmatrix}\dot\tau(s)\\ \dot\xi(s)\end{pmatrix}. \tag{2.7}$$

It also gives for every function $f\in\mathcal{C}^\infty(\mathbb{R}_t\times\mathbb{R}^n_x,\mathbb{C})$,

$$\partial_{\tau,\xi}p|_{(t(s),x(s),\tau(s),\xi(s))}\cdot\partial_{t,x}f|_{(t(s),x(s))} = \frac{d}{ds}f|_{(t(s),x(s))}. \tag{2.8}$$

Using this relation on $\partial^\alpha_{t,x}\psi$, $|\alpha|=2$, we may write order 2 of the eikonal equation (2.6) as

$$\frac{d}{ds}\partial^2_{t,x}\psi|_{(t(s),x(s))} + H_{12}(s)^T\partial^2_{t,x}\psi|_{(t(s),x(s))} + \partial^2_{t,x}\psi|_{(t(s),x(s))}H_{12}(s) + \partial^2_{t,x}\psi|_{(t(s),x(s))}H_{22}(s)\partial^2_{t,x}\psi|_{(t(s),x(s))} + H_{11}(s) = 0,$$

where $H_{11}(s)=\partial^2_{t,x}p|_{(t(s),x(s),\tau(s),\xi(s))}$, $(H_{12})_{bc}(s)=(\partial_{\tau,\xi})_b(\partial_{t,x})_c p|_{(t(s),x(s),\tau(s),\xi(s))}$, and $H_{22}(s)=\partial^2_{\tau,\xi}p|_{(t(s),x(s),\tau(s),\xi(s))}$. One can substitute for $\partial_t\partial_x\psi|_{(t(s),x(s))}$ and $\partial^2_t\psi|_{(t(s),x(s))}$ from the compatibility condition (2.7), since $\dot t(s)\neq 0$ by the strict hyperbolicity of P. The previous Riccati equation then yields a similar Riccati equation on $\partial^2_x\psi|_{(t(s),x(s))}$. Although non-linear, this equation has a unique global symmetric solution which satisfies the fundamental property

$$\operatorname{Im}\partial^2_x\psi|_{(t(s),x(s))} \text{ is positive definite,} \tag{P.c}$$

given an initial symmetric matrix $\partial^2_x\psi|_{(t(0),x(0))}$ with a positive definite imaginary part (see the proof of Lemma 2.56 p.101 in [19]).

Higher order derivatives of the phase on the ray are determined recursively. For $3\le r\le R$, order r of the eikonal equation (2.6) combined with the relation (2.8) leads to linear inhomogeneous ordinary differential equations (ODEs) on $\partial^r_x\psi|_{(t(s),x(s))}$. They have a unique solution for a fixed initial condition $\partial^r_x\psi|_{(t(0),x(0))}$.

The second step of the construction is to make c_j, for $1\le j\le N+1$, vanish on the ray up to the order $R-2j$. The choice of the order $R-2j$ is related to the quadratic imaginary part in the phase and the study of estimates in Sobolev spaces. This will appear clearly in the justification of the approximation in Lemma 2.2. In any case, the equations on the amplitudes $c_j=0$ can be solved on the ray at most up to the order $R-2$, due to the term $\partial^2_{t,x}\psi$ in the operator L (2.5).

Taking into account the eikonal equation (2.6), one gets the following evolution equations on a_j, $0\le j\le N$

$$\frac{1}{i}\partial_{\tau,\xi}p(t,x,\partial_t\psi,\partial_x\psi)\cdot\partial_{t,x}a_j + \left[\frac{1}{2i}Tr(\partial^2_{\tau,\xi}p(t,x,\partial_t\psi,\partial_x\psi)\partial^2_{t,x}\psi) + p'(t,x,\partial_t\psi,\partial_x\psi)\right]a_j + g_{j+1} = 0$$

$$\text{on } (t,x)=(t(s),x(s)) \text{ up to order } R-2j-2. \tag{2.9}$$

This equation *uniquely* determines the Taylor series of a_j on $(t(s),x(s))$ up to the order $R-2j-2$, given the values of their spatial derivatives at $(t(0),x(0))$ up to the same order.

REMARK 2.1. The number N of amplitudes in the ansatz (2.1) and the order R up to which the eikonal equation (2.6) is solved are not independent. Indeed, the computations of the amplitudes' derivatives require

$$R-2N-2\geq 0.$$

Another condition ([36, p.219]) is assumed to ensure that the remainder terms c_j, $N+2\leq j\leq N+m_P$, contribute with the right power of ε (see [45] for an alternative justification)

$$R-2N-3\leq 0. \tag{2.10}$$

An essential point for the use of Gaussian beams is the smoothness of the phase and the amplitudes with respect to (w.r.t.) $(t(0),x(0))$. To this aim, the needed initial values of the derivatives of the phase $\partial_x^r\psi|_{(t(0),x(0))}$, $2\leq r\leq R$, and of the amplitudes $\partial_x^r a_j|_{(t(0),x(0))}$, $0\leq r\leq R-2j-2$, are chosen to be smooth w.r.t. $(t(0),x(0))$. The phase and the amplitudes are then prescribed to be equal to their Taylor developments (truncated up to fixed orders) on the ray.

The final step of the construction is to multiply the amplitudes by a cutoff equal to 1 near the ray.

2.2. Incident and reflected beams for the wave equation. The preceding results will now be applied and detailed for the particular case of the wave equation and the construction of reflected beams. The computations rely on the results of [30] and [36].

We extend c in a smooth way outside $\bar{\Omega}$. Let $p(x,\tau,\xi)=c^2(x)|\xi|^2-\tau^2$ be the principal symbol of the wave operator $P=\partial_t^2-\partial_x\cdot(c^2\partial_x)$. Then $\tau(s)=\tau(0)$ from the Hamiltonian equations (2.2). Writing

$$p=-p_+p_-, \text{ with } p_+(x,\tau,\xi)=c(x)|\xi|+\tau \text{ and } p_-(x,\tau,\xi)=-c(x)|\xi|+\tau,$$

shows that null bicharacteristics $s\mapsto(t(s),x(s),\tau(0),\xi(s))$ for p s.t. $\tau(0)\neq 0$ are either null bicharacteristics for p_+ if $\tau(0)<0$ or for p_- if $\tau(0)>0$, by using the parametrization $s'=-2\tau s$.

Denote $h_+(x,\xi)=c(x)|\xi|$ and let $(x_0^t(y,\eta),\xi_0^t(y,\eta))$ (or simply (x_0^t,ξ_0^t)) be the Hamiltonian flow for h_+ starting from the point (y,η), that is

$$\begin{aligned} &\frac{dx_0^t}{dt}=\partial_\xi h_+(x_0^t,\xi_0^t)=c(x_0^t)\frac{\xi_0^t}{|\xi_0^t|}, \quad && \frac{d\xi_0^t}{dt}=-\partial_x h_+(x_0^t,\xi_0^t)=-\partial_x c(x_0^t)|\xi_0^t|,\\ &x_0^t|_{t=0}=y, && \xi_0^t|_{t=0}=\eta,\eta\neq 0. \end{aligned} \tag{2.11}$$

Then the null bicharacteristic curve $(t(s),x(s),\tau(s),\xi(s))$ for p starting at $s=0$ from $(0,y,\mp c(y)|\eta|,\eta)$ is exactly $(t,x_0^{\pm t}(y,\eta),\mp c(y)|\eta|,\xi_0^{\pm t}(y,\eta))$, the null bicharacteristic curve for $p_\pm$.

As in [41], one can prove that the Hamiltonian system (2.11) associated to h_+ has a unique solution global in time (by Cauchy-Lipschitz theorem), which depends smoothly on $(t,y,\eta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$.

The remainder of this section is organized as follows. In section 2.2.1, we explain the construction of incident and reflected beams associated to p_+, then section 2.2.2 is a simple repetition for p_-, and finally in section 2.2.3 we give error estimates for the individual beams gathered in (2.18).

2.2.1. Construction of beams associated to p_+. For the ray $(t,x_0^t(y,\eta))$ associated with p_+, denote $w_\varepsilon^0(t,x,y,\eta)$ to be a Gaussian beam concentrated on that ray, and $\psi_0(t,x,y,\eta)$ and $a_j^0(t,x,y,\eta)$ to be its associated phase and amplitudes. If no confusion is possible, symbols y,η and even t,x,y,η in the notations above will be dropped.

The phase ψ_0 is determined by solving the eikonal equation (2.6) on the ray (t,x_0^t) together with the conditions

$$\partial_t\psi_0(t,x_0^t)=-h_+(x_0^t,\xi_0^t),\, \partial_x\psi_0(t,x_0^t)=\xi_0^t, \qquad (P_0.\text{a})$$

and the choice of

$\psi_0(0,y)$ a real function,

$\partial_x^2\psi_0(0,y)$ a symmetric matrix with a positive definite imaginary part,

$\partial_x^r\psi_0(0,y), 3\le r\le R,$ permutable families.

In particular ψ_0 satisfies the important properties

$$\psi_0(t,x_0^t) \text{ is real}, \qquad (P_0.\text{b})$$

and

$$\operatorname{Im}\partial_x^2\psi_0(t,x_0^t) \text{ is positive definite}. \qquad (P_0.\text{c})$$

The phase ψ_0 is assumed to be equal to its Taylor series up to the order R on $x=x_0^t$

$$\psi_0(t,x)=\sum_{|\alpha|\le R}\frac{1}{\alpha!}(x-x_0^t)^\alpha\partial_x^\alpha\psi_0(t,x_0^t). \qquad (2.12)$$

The amplitudes of $w_\varepsilon^0(t,x)$ are also determined by the requirement that the c_j, $1\le j\le N+1$ in (2.4) are null up to orders $R-2j$ on the ray (t,x_0^t), given their initial spatial derivatives on the ray $\partial_x^r a_j^0(0,y)$, $r=0,\dots,R-2j-2$. We choose them as

$$a_j^0(t,x)=\chi_d(x-x_0^t)\sum_{|\alpha|\le R-2j-2}\frac{1}{\alpha!}(x-x_0^t)^\alpha\partial_x^\alpha a_j^0(t,x_0^t),\, j=0,\dots,N, \qquad (2.13)$$

where $d>0$ and χ_d is a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n,[0,1])$ satisfying

$$\chi_d(x)=1 \text{ if } |x|\le d/2 \text{ and } \chi_d(x)=0 \text{ if } |x|\ge d.$$

Throughout the paper, the parameter d will be adjusted to obtain requested estimates.

This construction leads to a beam $w_\varepsilon^0(t,x,y,\eta)$ called an incident beam for p_+, satisfying

$$\sup_{t\in[0,T]}\|Pw_\varepsilon^0(t,.)\|_{L^2(\Omega)}=O(\varepsilon^m) \text{ for some } m>0.$$

Let $\overset{o}{T^*}\Omega=T^*\Omega\backslash\{\eta=0\}$. To study the reflection on the boundary, we make the following assumptions.

B1. The domain Ω is convex for the bicharacteristic curves of P, that is for every $(y,\eta)\in \overset{o}{T^*}\Omega$, $x_0^t(y,\eta)$ cuts the boundary at only two times of opposite signs and transversally.

B2. For every $(y,\eta)\in \overset{o}{T^*}\Omega$, $x_0^t(y,\eta)$ does not remain in a compact of $\mathbb{R}^n$ when t varies in $\mathbb{R}$.

B3. The boundary has no dead-end trajectories, that is infinite number of successive reflections cannot occur in a finite time.

For $(y,\eta)\in \overset{o}{T^*}\Omega$, let $T_1(y,\eta)$ be the instant (that is the exit time) s.t.

$$x_0^{T_1(y,\eta)}(y,\eta)\in\partial\Omega \text{ and } T_1(y,\eta)>0.$$

Note that $\overset{o}{T^*}\Omega$ is an open set, and thanks to B1, the function $(y,\eta)\in \overset{o}{T^*}\Omega\mapsto T_1(y,\eta)$ is well-defined and C^∞, as follows from the implicit function theorem. The reflection involution associated to the considered symbol p is the map

$$\begin{aligned}\mathcal{R}:\overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega} &\to \overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega}\\ (X,\Xi) &\mapsto (X,(Id-2\nu(X)\nu(X)^T)\Xi).\end{aligned}$$

Above ν denotes the exterior normal field to $\partial\Omega$. Let $\varphi_0^t=(x_0^t,\xi_0^t)$ denote the incident Hamiltonian flow solution of (2.11). We define the first reflected flow φ_1^t by the condition

$$\varphi_1^{T_1}=\mathcal{R}o\varphi_0^{T_1},$$

that is the Hamiltonian flow for h_+ having at $t=T_1$, position $x_0^{T_1}$, the direction being given by the reflected vector of $\xi_0^{T_1}$.

Then the broken flow is defined recursively after a finite number of successive reflections as follows (see figure 2.1): for $k>1$, T_k and $\varphi_k^t=(x_k^t,\xi_k^t)$ are determined by:

$$\begin{aligned}&T_k(y,\eta) \text{ is the instant s.t. } x_{k-1}^{T_k(y,\eta)}(y,\eta)\in\partial\Omega \text{ and } T_k(y,\eta)>T_{k-1}(y,\eta),\\ &\varphi_k^{T_k}=\mathcal{R}o\varphi_{k-1}^{T_k}.\end{aligned}$$

The convexity of the boundary B1 implies the non-grazing hypothesis

$$\forall (y,\eta)\in \overset{o}{T^*}\Omega \text{ and } k\geq 1, \dot{x}_{k-1}^{T_k(y,\eta)}(y,\eta)\cdot\nu(x_{k-1}^{T_k(y,\eta)}(y,\eta))>0,$$

where $\dot{x}_{k-1}^t$ denotes $\frac{d}{dt}x_{k-1}^t$. Assumption B3 leads to

$$\forall (y,\eta)\in \overset{o}{T^*}\Omega, T_k(y,\eta) \underset{k\to+\infty}{\to} +\infty. \tag{2.14}$$

It ensures that for a fixed point (y,η) in $\overset{o}{T^*}\Omega$, there is a finite number $q_+(y,\eta)$ of reflections in $[0,T]$.

Following the method of Ralston in [36, p.220], we shall construct reflected beams $w_\varepsilon^1,\dots,w_\varepsilon^{q_+}$ which satisfy the boundary estimate

$$\exists m'>0 \text{ and } s\geq 0 \text{ s.t. } \|B(w_\varepsilon^0+\cdots+w_\varepsilon^{q_+})\|_{H^s([0,T]\times\partial\Omega)}=O(\varepsilon^{m'}),$$

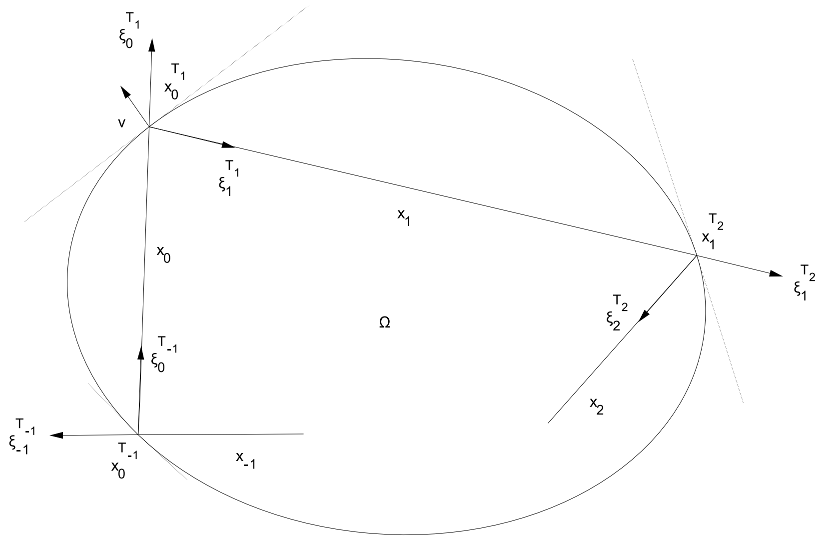

FIG. 2.1. *successive reflections.*

together with the interior estimates

$$\sup_{t\in[0,T]} \|Pw_\varepsilon^k(t,.)\|_{L^2(\Omega)} = O(\varepsilon^m), 1 \le k \le q_+.$$

For each $1 \le k \le q_+$, the reflected beam w_ε^k will be written as

$$w_\varepsilon^k = e^{i\psi_k/\varepsilon}(a_0^k + \cdots + \varepsilon^N a_N^k).$$

To ensure the interior estimates, each phase ψ_k and the amplitudes a_j^k $(0 \le j \le N)$ must satisfy equations (2.6) and (2.9) on the reflected ray (t, x_k^t).

As the beams vanish away from their associated rays, the contribution to the boundary norm of $w_\varepsilon^0 + \cdots + w_\varepsilon^{q_+}$ occurs when t is close to some T_k and then from the beams w_ε^{k-1} and w_ε^k. The construction of the reflected beams is completed recursively. Assume that the beam w_ε^{k-1} has been constructed and that its associated phase satisfies

$$\partial_t\psi_{k-1}(t,x_{k-1}^t) = -h_+(x_{k-1}^t,\xi_{k-1}^t), \partial_x\psi_{k-1}(t,x_{k-1}^t) = \xi_{k-1}^t, \quad (P_{k-1}.\text{a})$$

$$\psi_{k-1}(t,x_{k-1}^t) \text{ is real}, \quad (P_{k-1}.\text{b})$$

$$\operatorname{Im}\partial_x^2\psi_{k-1}(t,x_{k-1}^t) \text{ is positive definite}. \quad (P_{k-1}.\text{c})$$

One may write on the boundary $\partial\Omega$

$$B\big(w_\varepsilon^{k-1} + w_\varepsilon^k\big) = \big(\varepsilon^{-m_B} d_{-m_B}^{k-1} + \cdots + \varepsilon^N d_N^{k-1}\big)e^{i\psi_{k-1}/\varepsilon} + \big(\varepsilon^{-m_B} d_{-m_B}^{k} + \cdots + \varepsilon^N d_N^{k}\big)e^{i\psi_{k}/\varepsilon},$$

m_B being the order of B ($m_B = 0$ for Dirichlet and $m_B = 1$ for Neumann).

In order to satisfy the boundary estimate, the first step is to impose on ψ_k to have the same time and tangential derivatives as ψ_{k-1} at $(T_k, x_{k-1}^{T_k})$, up to the order

R. More precisely, let us introduce boundary coordinates near $x_{k-1}^{T_k} = x_k^{T_k}$ as follows. We partition $\partial\Omega$ with a finite number of small open subsets $\mathcal{U}_1,\dots,\mathcal{U}_L$ s.t. there exist $\mathcal{C}^\infty$ parametrizations

$$\sigma_l : \mathcal{N}_l \to \mathbb{R}^n, \, l = 1,\dots,L,$$

where $\mathcal{N}_l$ are open subsets of $\mathbb{R}^{n-1}$, $\sigma_l(\mathcal{N}_l) = \mathcal{U}_l$, and σ_l is a diffeomorphism from $\mathcal{N}_l$ to $\mathcal{U}_l$. Suppose that $x_{k-1}^{T_k}$ belongs to $\mathcal{U}_{l_0}$ and denote $x_{k-1}^{T_k} = \sigma_{l_0}(\hat{z}_k)$. For $x \in \mathbb{R}^n$ close to $x_{k-1}^{T_k}$, we may write

$$x = \sigma_{l_0}(\hat{v}) + v_n \nu(\sigma_{l_0}(\hat{v})),$$

with $\hat{v} \in \mathcal{N}_{l_0}$ and $v_n \in \mathbb{R}$. If we use the notation

$${}^\sigma f(t,\hat{v},v_n) = f(t,x),$$

then we impose

$$\partial_{t,\hat{v}}^\alpha \, {}^\sigma\psi_k(T_k,\hat{z}_k,0) = \partial_{t,\hat{v}}^\alpha \, {}^\sigma\psi_{k-1}(T_k,\hat{z}_k,0), \, |\alpha| \le R. \tag{2.15}$$

Order 0 of (2.15) gives a real value for $\psi_k(T_k,x_{k-1}^{T_k})$. Order 1 of this same constraint and order 0 of the eikonal equation (2.6) on ψ_k are both satisfied by setting

$$\partial_t\psi_k(t,x_k^t) = -h_+(x_k^t,\xi_k^t), \, \partial_x\psi_k(t,x_k^t) = \xi_k^t. \tag{P_k.a}$$

It follows that

$$\psi_k(t,x_k^t) \text{ is real.} \tag{P_k.b}$$

Due to the non-grazing hypothesis, (2.15) and the compatibility condition resulting from order 1 of the eikonal equation (2.6) provide $\partial_x^2\psi_k(T_k,x_{k-1}^{T_k})$. To solve the Riccati equation on $\partial_x^2\psi_k(t,x_k^t)$ with its given value at $t = T_k$, we need to study the imaginary part of $\partial_x^2\psi_k(T_k,x_{k-1}^{T_k})$. For $k' = k-1,k$, one has

$$\partial_t\partial_{\hat{v}} \, {}^\sigma\psi_{k'}(t,\hat{v},0) = D\sigma_{l_0}(\hat{v})^T \partial_t\partial_x\psi_{k'}(t,x_{k'}^t),$$

and

$$\begin{aligned}\partial_{\hat{v}}^2 \, {}^\sigma\psi_{k'}(t,\hat{v},0) = {} & D^2\sigma_{l_0}(\hat{v})\left(\partial_x\psi_{k'}(t,x_{k'}^t)\right) \\ & + D\sigma_{l_0}(\hat{v})^T \partial_x^2\psi_{k'}(t,x_{k'}^t) D\sigma_{l_0}(\hat{v}).\end{aligned}$$

Differentiating (P_{k-1}.a) and (P_k.a) yields

$$\mathrm{Im}\,\partial_t\partial_x\psi_{k'}(t,x_{k'}^t) = -\mathrm{Im}\,\partial_x^2\psi_{k'}(t,x_{k'}^t)\,\dot{x}_{k'}^t$$

and

$$\mathrm{Im}\,\partial_t^2\psi_{k'}(t,x_{k'}^t) = \dot{x}_{k'}^t \cdot \mathrm{Im}\,\partial_x^2\psi_{k'}(t,x_{k'}^t)\,\dot{x}_{k'}^t.$$

Denote

$$M_k = \partial_{t,\hat{v}}^2 \, {}^\sigma\psi_{k-1}(T_k,\hat{z}_k,0) = \partial_{t,\hat{v}}^2 \, {}^\sigma\psi_k(T_k,\hat{z}_k,0). \tag{2.16}$$

One therefore has

$$\operatorname{Im} M_k = \big(-\dot{x}_{k'}^{T_k},\, D\sigma_{l_0}(\hat{z}_k)\big)^T \operatorname{Im}\partial_x^2\psi_{k'}(T_k, x_{k-1}^{T_k})\big(-\dot{x}_{k'}^{T_k},\, D\sigma_{l_0}(\hat{z}_k)\big).$$

The non-grazing hypothesis ensures that the matrices $\big(-\dot{x}_{k'}^{T_k},\, D\sigma_{l_0}(\hat{z}_k)\big)$ are non singular. Since $\operatorname{Im}\partial_x^2\psi_{k-1}(T_k, x_{k-1}^{T_k})$ is positive definite by $(P_{k-1}.\mathrm{c})$, it follows that the same property holds true for $\operatorname{Im} M_k$ and consequently for $\operatorname{Im}\partial_x^2\psi_k(T_k, x_{k-1}^{T_k})$. Hence, the matrix $\partial_x^2\psi_k(t, x_k^t)$ solution of a Riccati equation with its given value at $t = T_k$ satisfies

$$\operatorname{Im}\partial_x^2\psi_k(t, x_k^t) \text{ is positive definite.} \qquad (P_k.\mathrm{c})$$

Higher order derivatives of the reflected phase on the associated ray are determined recursively. For $3 \le r \le R$, $\partial_x^r\psi_k(t, x_k^t)$ satisfies linear ODEs with a given value at $t = T_k$.

The second step is to prescribe that $d_{-m_B+j}^{k-1} + d_{-m_B+j}^{k}$ vanish up to the order $R-2j-2$ at $(T_k, x_{k-1}^{T_k})$. These requirements provide the derivatives of a_j^k up to the order $R-2j-2$ at $(T_k, x_{k-1}^{T_k})$. Hence, for $0 \le r \le R-2j-2$, $\partial_x^r a_j^k(t, x_k^t)$ satisfy linear systems of ODEs with initial conditions given at $t = T_k$.

It follows from this construction that the choice of the (truncated up to fixed orders) Taylor series of the phase and the amplitudes of the incident beam on the starting point of the ray recursively determines the (truncated up to fixed orders) Taylor series of successively reflected beams' phases and amplitudes.

Finally, the amplitudes a_j^k are multiplied by a cutoff equal to 1 near x_k^t. The reflected phases and amplitudes have the same forms as the incident ones

$$\psi_k(t,x) = \sum_{|\alpha| \le R} \frac{1}{\alpha!}(x - x_k^t)^\alpha \partial_x^\alpha \psi_k(t, x_k^t),$$

and

$$a_j^k(t,x) = \chi_d(x - x_k^t) \sum_{|\alpha| \le R-2j-2} \frac{1}{\alpha!}(x - x_k^t)^\alpha \partial_x^\alpha a_j^k(t, x_k^t),\ j = 1, \ldots, N.$$

2.2.2. Construction of beams associated to p_-. For the symbol p_-, the same construction applies for the associated incident and reflected beams.

An incident beam for p_- is a beam concentrated on the ray (t, x_0^{-t}), so it is simply $w_\varepsilon^0(-t,x)$. In fact, denoting $Pw_\varepsilon^0 = \sum_{j=0}^{N+2} \varepsilon^{j-2} c_j^0 e^{i\psi_0/\varepsilon}$, one can notice that $P[w_\varepsilon^0(-t,x)] = [Pw_\varepsilon^0](-t,x)$ and the amplitudes $c_j^0(-t,x)$ vanish on $x = x_0^{-t}$ up to the required orders.

Reflected beams for p_- are obtained by reflecting φ_0^t backwards. For $(y,\eta) \in \overset{o}{T^*\Omega}$, let $T_{-1}(y,\eta) < 0$ be the instant s.t. $x_0^{T_{-1}(y,\eta)}(y,\eta)$ strikes the boundary $\partial\Omega$. Denote φ_{-1}^t to be the Hamiltonian flow for h_+ determined by the condition (see figure 2.1)

$$\varphi_{-1}^{T_{-1}} = \mathcal{R} o \varphi_0^{T_{-1}}.$$

For $k > 1$, one can recursively define the instants of reflections T_{-k} and the Hamiltonians flows φ_{-k}^t for h_+ as follows:

$T_{-k}(y,\eta)$ is the instant s.t. $x_{-k+1}^{T_{-k}(y,\eta)}(y,\eta) \in \partial\Omega$ and $T_{-k}(y,\eta) < T_{-k+1}(y,\eta)$,
$\varphi_{-k}^{T_{-k}} = \mathcal{R} o \varphi_{-k+1}^{T_{-k}}$.

Assumption B3 implies that $T_k(y,\eta)\to-\infty$ when k goes to $-\infty$, and thus ensures a finite number $q_-(y,\eta)$ of reflections in $[-T,0]$.

Then we build Gaussian beams w_ε^{-k} for p_- after $1\le k\le q_-$ backwards reflections, by imposing $\|B(w_\varepsilon^0+\dots+w_\varepsilon^{-q_-})\|_{H^s([-T,0]\times\partial\Omega)}=O(\varepsilon^{m'})$ for some $m'>0$ and $s\ge 0$. We write these beams as

$$w_\varepsilon^{-k}=e^{i\psi_{-k}/\varepsilon}(a_0^{-k}+\dots+\varepsilon^N a_N^{-k}).$$

In particular, for $1\le k\le q_-$, the phase ψ_{-k} satisfies the following properties:

$$\partial_t\psi_{-k}(t,x_{-k}^t)=-h_+(x_{-k}^t,\xi_{-k}^t),\ \partial_x\psi_{-k}(t,x_{-k}^t)=\xi_{-k}^t, \qquad (P_{-k}.\mathrm{a})$$

$$\psi_{-k}(t,x_{-k}^t)\text{ is real}, \qquad (P_{-k}.\mathrm{b})$$

$$\operatorname{Im}\partial_x^2\psi_{-k}(t,x_{-k}^t)\text{ is positive definite}. \qquad (P_{-k}.\mathrm{c})$$

Noting that (t,x_{-k}^t), $k=1,\dots,q_-$, are successively reflected rays for p_-, the reflected beam of p_- after k reflections is simply $w_\varepsilon^{-k}(-t,x)$.

2.2.3. Error estimates for individual Gaussian beams. We fix $(y,\eta)\in \overset{o}{T^*}\Omega$ and choose d sufficiently small s.t. for $k=0,\dots,q_\pm$, $t\in[0,T]$ and $|x-x_{\pm k}^{\pm t}|\le d$,

$$\operatorname{Im}\psi_{\pm k}(\pm t,x)\ge \mathrm{const}(x-x_{\pm k}^{\pm t})^2. \qquad (2.17)$$

One can see that this choice is always possible by the properties $(P_k.\mathrm{a})$–$(P_k.\mathrm{b})$–$(P_k.\mathrm{c})$ of each phase ψ_k, $-q_-\le k\le q_+$.

For $t\in[0,T]$ and $x\in\mathbb{R}^n$, let

$$\underline{w}_\varepsilon^+(t,x)=\sum_{k=0}^{q_+}w_\varepsilon^k(t,x)\text{ and }\underline{w}_\varepsilon^-(t,x)=\sum_{k=0}^{q_-}w_\varepsilon^{-k}(-t,x). \qquad (2.18)$$

Then we have the following estimates on these constructed beams

Lemma 2.2.

1. $\|\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm(t,.)\|_{H^1(\Omega)}\lesssim 1$ *and* $\|\varepsilon^{-\frac{n}{4}+1}\partial_t\underline{w}_\varepsilon^\pm(t,.)\|_{L^2(\Omega)}\lesssim 1$ *uniformly w.r.t.* $t\in[0,T]$,
2. $\|P\left(\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm\right)(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}$ *uniformly w.r.t.* $t\in[0,T]$,
3. $\|B\left(\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm\right)\|_{H^s([0,T]\times\partial\Omega)}\lesssim\varepsilon^{-m_B-s+\frac{R+1}{2}}$, $s\ge 0$.

The proof of this Lemma and other results rely on this standard estimate for $p\in\mathbb{N}$

$$|x|^p e^{-x^2/\varepsilon}dx\lesssim\varepsilon^{\frac{p}{2}}e^{-x^2/(2\varepsilon)},\forall x\in\mathbb{R}^n. \qquad (2.19)$$

For more details, we refer the interested reader to [36] or [30].

2.3. Gaussian beams summation. The constructed functions $\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm$ are approximate solutions for the IBVP of the wave equation with initial data

$$\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm|_{t=0}=\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0|_{t=0}e^{i\psi_0|_{t=0}/\varepsilon}+\varepsilon^{-\frac{n}{4}+1}\sum_{k=1}^{q_\pm}w_\varepsilon^{\pm k}|_{t=0},$$

and

$$\partial_t\left(\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm\right)|_{t=0}=\pm\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j f_j^0 e^{i\psi_0|_{t=0}/\varepsilon}\pm\varepsilon^{-\frac{n}{4}+1}\sum_{k=1}^{q_\pm}\partial_t w_\varepsilon^{\pm k}|_{t=0},$$

where the f_j^0 are related to the phase and amplitudes of w_ε^0. One can show that the Assumptions B1–B2 imply that $x_k^0 \notin \bar{\Omega}$ for $k \neq 0$. The exponential decrease of the phases away from their associated rays leads to

$$\|w_\varepsilon^k|_{t=0}\|_{H^1(\Omega)} \lesssim \varepsilon^\infty \text{ and } \|\partial_t w_\varepsilon^k|_{t=0}\|_{L^2(\Omega)} \lesssim \varepsilon^\infty, k \neq 0.$$

Modulo infinitely small remainders, the initial conditions of $\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm$ are then

$$\left(\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0|_{t=0}e^{i\psi_0|_{t=0}/\varepsilon}, \pm\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j f_j^0 e^{i\psi_0|_{t=0}/\varepsilon}\right).$$

We wish to consider the IBVP (1.1) with general initial conditions $(u_\varepsilon^I, v_\varepsilon^I)$ in $H^1(\Omega)\times L^2(\Omega)$. Note that $\psi_0|_{t=0}$ has properties similar to ϕ_0, where $c_n\varepsilon^{-\frac{3n}{4}}e^{i\phi_0(x,y,\eta)/\varepsilon}$ denotes the kernel of T_ε^*; see formula (1.4) in the introduction. The first step is to build, for a fixed point $(y,\eta)\in \overset{o}{T^*}\Omega$, asymptotic solutions with initial conditions close to $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(.,y,\eta)/\varepsilon}, 0)$ and $(0, \varepsilon^{-\frac{n}{4}}e^{i\phi_0(.,y,\eta)/\varepsilon})$ in $H^1(\Omega)\times L^2(\Omega)$. Then one expects to fulfill more general initial data $(u_\varepsilon^I, v_\varepsilon^I)$ by decomposing u_ε^I on the family $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0/\varepsilon})_{(y,\eta)\in \overset{o}{T^*}\Omega}$ and v_ε^I on the family $(\varepsilon^{-\frac{n}{4}}e^{i\phi_0/\varepsilon})_{(y,\eta)\in \overset{o}{T^*}\Omega}$, indexed by (y,η).

Let us recover the notation of the beams referring to the starting points of the incident flow. We fix $(y,\eta)\in \overset{o}{T^*}\Omega$ and consider the incident beam $w_\varepsilon^0(t,x,y,\eta)$ associated to the ray $(t, x_0^t(y,\eta))$ and the reflected beams $w_\varepsilon^{\pm k}(t,x,y,\eta)$, $k=1,\dots,q_\pm$. Taylor formulae (2.12) yields at $t=0$

$$\psi_0(0,x,y,\eta) = \sum_{|\alpha|\le R}\frac{1}{\alpha!}(x-y)^\alpha \partial_x^\alpha \psi_0(0,y,y,\eta).$$

If one chooses the following initial spatial derivatives on the ray for the incident beam's phase

$$\psi_0(0,y,y,\eta) = 0, \partial_x^2\psi_0(0,y,y,\eta) = iId \text{ and } \partial_x^\alpha\psi_0(0,y,y,\eta) = 0, 3\le|\alpha|\le R,$$

then $(P_0$.a) implies

$$\psi_0(0,x,y,\eta) = \eta\cdot(x-y) + i(x-y)^2/2 = \phi_0(x,y,\eta). \tag{2.20}$$

We assume henceforth that the incident beam's phase satisfies (2.20). Consider an approximate solution

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^+ + \underline{w}_\varepsilon^-).$$

Its initial data is

$$\left(\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0|_{t=0}e^{i\phi_0/\varepsilon}, 0\right),$$

with a remainder of order ε^∞ in $H^1(\Omega)\times L^2(\Omega)$. To get the form $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0/\varepsilon}, 0)$, one has to make a suitable choice for the amplitudes. The expansion (2.13) at $t=0$ yields

$$a_j^0(0,x,y,\eta) = \chi_d(x-y)\sum_{|\alpha|\le R-2j-2}\frac{1}{\alpha!}(x-y)^\alpha\partial_x^\alpha a_j^0(0,y,y,\eta), j=0,\dots,N,$$

and one has full choice for the initial spatial derivatives of a_j^0 on the ray up to the order $R-2j-2$. Under the assumptions

$$a_0^0(0,y,y,\eta)=1, \partial_x^\alpha a_0^0(0,y,y,\eta)=0 \text{ for } 1\leq|\alpha|\leq R-2,$$
$$\partial_x^\alpha a_j^0(0,y,y,\eta)=0 \text{ for } |\alpha|\leq R-2j-2, 1\leq j\leq N,$$

one obtains

$$\sum_{j=0}^{N}\varepsilon^j a_j^0(0,x,y,\eta)=\chi_d(x-y). \tag{2.21}$$

Taking advantage of the exponential decrease of $e^{i\phi_0(x,y,\eta)/\varepsilon}$ for $|x-y|\geq d/2$, one deduces that

$$\|\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0(0,.,y,\eta)e^{i\phi_0(.,y,\eta)/\varepsilon}-\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(.,y,\eta)/\varepsilon}\|_{H^1(\Omega)}\lesssim\varepsilon^\infty.$$

We keep the notations a_j^0 and w_ε^0 to denote the amplitudes satisfying (2.21) and the associated incident beam. For $1\leq k\leq q_\pm$, we denote $w_\varepsilon^{\pm k}$ to be the corresponding reflected beams and $\underline{w}_\varepsilon^\pm$ to be the sum of the incident and reflected beams for $p_\pm$.

Next, we shift to the initial condition on the time derivative, for which we construct a new incident beam $w_\varepsilon^{0'}$ with amplitudes $a_j^{0'}$. Indeed, an approximate solution

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+'}-\underline{w}_\varepsilon^{-'}),$$

has initial data

$$\left(0,\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0'}+\partial_t a_{j-1}^{0}{}'\right)|_{t=0}e^{i\phi_0/\varepsilon}\right),$$

modulo a remainder of order ε^∞ in $H^1(\Omega)\times L^2(\Omega)$, with $a_{-1}^0{}'=a_{N+1}^0{}'=0$. In order to approach the form $(0,\varepsilon^{-\frac{n}{4}}e^{i\phi_0/\varepsilon})$, we derive new initial Taylor series for the incident beam's amplitudes. As $\partial_t\psi_0(0,y,y,\eta)=-c(y)|\eta|$, we impose

$$a_0^{0'}(0,y,y,\eta)=i\left(c(y)|\eta|\right)^{-1}, \partial_x^\alpha\left(\partial_t\psi_0 a_0^{0'}\right)(0,y,y,\eta)=0 \text{ for } 1\leq|\alpha|\leq R-2,$$
$$\partial_x^\alpha\left(i\partial_t\psi_0 a_j^{0'}+\partial_t a_{j-1}^{0}{}'\right)(0,y,y,\eta)=0 \text{ for } |\alpha|\leq R-2j-2, 1\leq j\leq N.$$

One obtains

$$\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0'}+\partial_t a_{j-1}^{0}{}'\right)(0,x,y,\eta)=1+\sum_{j=0}^{N}\varepsilon^j\sum_{|\alpha|=R-2j-1}(x-y)^\alpha z_\alpha(x,y,\eta)$$
$$+\varepsilon^{N+1}\partial_t a_N^0{}'(0,x,y,\eta), \tag{2.22}$$

where z_α are smooth remainders that vanish for $|x-y|\geq d$. Making use of (2.10) and (2.19), one can show that

$$\|\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0'}+\partial_t a_{j-1}^{0}{}'\right)(0,.,y,\eta)e^{i\phi_0(.,y,\eta)/\varepsilon}-\varepsilon^{-\frac{n}{4}}e^{i\phi_0(.,y,\eta)/\varepsilon}\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

Let $w_\varepsilon^{\pm k'}$, $1\le k\le q_\pm$, be the reflected beams associated to $w_\varepsilon^{0'}$ and denote $\underline{w}_\varepsilon^{\pm'}$ to be the sum of the so obtained incident and reflected beams for $p_\pm$. Hence, the approximate solutions

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+}+\underline{w}_\varepsilon^{-})(t,x,y,\eta) \text{ and } \frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+\prime}-\underline{w}_\varepsilon^{-\prime})(t,x,y,\eta),$$

have the required initial data

$$\left(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(x,y,\eta)/\varepsilon},0\right) \text{ and } \left(0,\varepsilon^{-\frac{n}{4}}e^{i\phi_0(x,y,\eta)/\varepsilon}\right),$$

modulo remainders of respective orders ε^∞ and $\varepsilon^{\frac{R-1}{2}}$ in $H^1(\Omega)\times L^2(\Omega)$.

To fulfill general initial conditions $(u_\varepsilon^I,v_\varepsilon^I)$, the previous computations together with the identity $T_\varepsilon^*T_\varepsilon=Id$, suggest that we look for an approximate solution such as

$$\frac{c_n}{2}\varepsilon^{-\frac{3n}{4}}\int_{\overset{o}{T^*}\Omega}T_\varepsilon u_\varepsilon^I(y,\eta)\left(\underline{w}_\varepsilon^{+}(t,x,y,\eta)+\underline{w}_\varepsilon^{-}(t,x,y,\eta)\right)dyd\eta$$
$$+\frac{c_n}{2}\varepsilon^{-\frac{3n}{4}}\int_{\overset{o}{T^*}\Omega}\varepsilon T_\varepsilon v_\varepsilon^I\left(\underline{w}_\varepsilon^{+\prime}(t,x,y,\eta)-\underline{w}_\varepsilon^{-\prime}(t,x,y,\eta)\right)dyd\eta.$$

Let us notice that it is not clear that the previous integral is well defined.

Firstly, the construction of $\underline{w}_\varepsilon^{\pm^{(\prime)}}(t,x,y,\eta)$ breaks down when y approaches the boundary $\partial\Omega$ because the numbers of reflections in $[0,\pm T]$ become infinitely large. Next we need to tackle the problem of integration for large η.

One way to overcome these two problems is to require that the initial FBI transforms are compactly supported modulo small remainders. This requirement is in the spirit of considering only compactly supported symbols in the study of the FIOs of [24]. Nevertheless, this restriction was recently removed by Rousse and Swart in [40]. In particular, a general boundedness result of FIOs with complex phases for subquadratic Hamiltonians was established therein. The proof is rather subtle and relies in particular on Cotlar-Stein estimate. The same arguments can be used for the constant coefficient wave equation but do not seem to work for the general wave equation. In fact, in this case, the second derivatives of the Hamiltonian are not bounded and thus the proof of [40] needs to be adapted.

A last problem related to the wave equation is the integration for small η.

In view of all these difficulties, this explains why we made the Assumptions A2 and A3 on the initial data in the introduction, which we recall

$$u_\varepsilon^I \text{ and } v_\varepsilon^I \text{ are supported in a fixed compact } K\subset\Omega,$$
$$\|T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R_\eta^c)}=O(\varepsilon^\infty) \text{ and } \|T_\varepsilon v_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R_\eta^c)}=O(\varepsilon^\infty),$$

where $R_\eta=\{\eta\in\mathbb{R}^n, r_0\le|\eta|\le r_\infty\}$, $0<r_0<r_\infty$. These assumptions are satisfied for instance by a large class of WKB functions $ae^{i\Phi/\varepsilon}$, $a\in\mathcal{C}_0^\infty(\Omega)$. Indeed the non-stationary phase lemma implies that the FBI transform of such a function is of order $O(\varepsilon^\infty)$ outside the compact set

$$\mathcal{A}\times\mathcal{B}=\{y\in\mathbb{R}^n,\operatorname{dist}(y,\operatorname{supp}a)\le c\}\times\{\eta\in\mathbb{R}^n,\operatorname{dist}(\eta,\partial_x\Phi(\mathcal{A}))\le c\}, c>0;$$

see Lemmas 4.2 and 4.3 of [39]. Thus $ae^{i\Phi/\varepsilon}$ satisfies Assumption A3, provided that $\partial_x\Phi$ does not vanish on $\operatorname{supp}a$.

Remark 2.3. Another strategy can be used to match initial conditions of WKB form in a Gaussian beams summation [28, 44]. It consists of integrating the beams associated to rays that start from $y\in\operatorname{supp}a$ with the direction $\eta=\partial_x\Phi(y)$. The accuracy of such obtained solutions faces a damage caused by caustics, namely an extra factor $\varepsilon^{\frac{1-n}{4}}$ appears in the error estimate. This loss originates from the restriction to rays $x^{\pm t}_{\pm k}(y,\partial_x\Phi(y))$ $(k=0,\dots,N_\pm)$, which technically leads to considering the deformation matrices $\partial_y[x^{\pm t}_{\pm k}(y,\partial_x\Phi(y))]$ singular at caustics (see [28, Lemma 5.1]). The summation over rays starting with general directions η independent of y uses the symplectic maps $\varphi^{\pm t}_{\pm k}$ and thus provides a phase space description of the solution that unfolds the caustics.

Let ρ be a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n,[0,1])$ supported in a compact $K_y\subset\Omega$ and satisfying

$$\rho(y)=1 \text{ if } \operatorname{dist}(y,K)<\Delta \text{ for a small } \Delta>0, \tag{2.23}$$

and ϕ a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n,[0,1])$ supported in a compact $K_\eta\subset\mathbb{R}^n\backslash\{0\}$ s.t. $\phi=1$ on R_η.

One can establish that the Assumptions A2 and A3 imply

$$\|(1-\rho(y)\phi(\eta))T_\varepsilon u^I_\varepsilon\|_{L^2_{y,\eta}}\lesssim\varepsilon^\infty \text{ and } \|(1-\rho(y)\phi(\eta))T_\varepsilon v^I_\varepsilon\|_{L^2_{y,\eta}}\lesssim\varepsilon^\infty.$$

In fact, viewing the FBI transform as the Fourier Transform of some auxiliary function yields by the Parseval equality the following result

Lemma 2.4. *Let a be a positive real and G a measurable subset of $\mathbb{R}^n$ s.t.* $\operatorname{dist}(G,K)\geq a$. *If $u\in L^2(\mathbb{R}^n_w)$ is supported in K then*

$$\|\mathbf{1}_G(y)T_\varepsilon u\|_{L^2_{y,\eta}}=c_n\varepsilon^{-\frac{n}{4}}\|\mathbf{1}_G(y)u(w)e^{-(w-y)^2/(2\varepsilon)}\|_{L^2_{w,y}}\lesssim e^{-a^2/(4\varepsilon)}\|u\|_{L^2_w}.$$

On the other hand, if (y,η) varies in $K_y\times K_\eta$, then $q_+(y,\eta)$ is uniformly bounded. In fact, for $j\geq 1$, the T_j are positive, depend continuously on (y,η), and property (2.14) ensures that $T_j\nearrow+\infty$ when $j\to+\infty$. Thus they uniformly go to $+\infty$ on the compact $K_y\times K_\eta$, by Dini's theorem on the sequence $(1/T_j)_{j\geq1}$. As $T_{q_+}\leq T$, it follows that $\sup\limits_{K_y\times K_\eta} q_+<+\infty$. The same result holds true for q_-. Denote $N_\pm=\sup\limits_{K_y\times K_\eta} q_\pm$. The construction of the reflected beams in section 2.2 may be continued up to $N_\pm$ reflections.

The final result of the discussion above is an approximate solution proposed as

$$\begin{aligned} u^R_\varepsilon(t,x)=\frac{1}{2}\varepsilon^{-\frac{3n}{4}}c_n\int_{\mathbb{R}^{2n}}\rho(y)\phi(\eta)\Big[&\varepsilon T_\varepsilon v^I_\varepsilon(y,\eta)\Big(\sum_{k=0}^{N_+}w^{k\prime}_\varepsilon(t,x,y,\eta)-\sum_{k=0}^{N_-}w^{-k\prime}_\varepsilon(-t,x,y,\eta)\Big)\\ &+T_\varepsilon u^I_\varepsilon(y,\eta)\Big(\sum_{k=0}^{N_+}w^k_\varepsilon(t,x,y,\eta)+\sum_{k=0}^{N_-}w^{-k}_\varepsilon(-t,x,y,\eta)\Big)\Big]dyd\eta. \end{aligned} \tag{2.24}$$

This approximate solution is indexed by R, the order of vanishing of the eikonal equation (2.6) on the ray. The incident beams' phase fulfills the initial conditions (2.20) and their amplitudes satisfy respectively (2.21) for w^0_ε and (2.22) for $w^{0\prime}_\varepsilon$ for every $(y,\eta)\in\operatorname{supp}\rho\otimes\phi$. The size $d\in]0,1]$ of the support of the cut-offs multiplying the amplitudes no longer depends on (y,η) and would be chosen sufficiently small to satisfy various constraints we set out in the following section.

In the sequel, we prove that this family of functions (u_ε^R) indeed allows to approach the exact solution of the IBVP problem (1.1) to any arbitrary power of ε by choosing the order R. The difference between the asymptotic solutions and the exact one is investigated in $\mathcal{C}([0,T],H^1(\Omega))\times\mathcal{C}^1([0,T],L^2(\Omega))$ by means of error estimates in the interior equation, the boundary condition, and the initial conditions. The only assumptions needed on the initial data are A1, A2 and A3.

3. Justification of the asymptotics

We aim to estimate $\|u_\varepsilon^R(t,.)-u_\varepsilon(t,.)\|_{H^1(\Omega)}$ and $\|\partial_t u_\varepsilon^R(t,.)-\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)}$ for $t\in[0,T]$.

It follows from standard results [7] that the IBVP for the wave equation is well-posed, and furthermore one has the energy estimate (as a consequence of [25, p.185] for the Dirichlet problem and of [3, p.224] for the Neumann problem)

$$\begin{aligned}&\underset{t\in[0,T]}{Sup}\|u_\varepsilon^R(t,.)-u_\varepsilon(t,.)\|_{H^1(\Omega)}+\underset{t\in[0,T]}{Sup}\|\partial_t u_\varepsilon^R(t,.)-\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)}\\ &\lesssim \underset{t\in[0,T]}{Sup}\|Pu_\varepsilon^R\|_{L^2(\Omega)}+\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)}\\ &\quad+\|u_\varepsilon^R(0,.)-u_\varepsilon^I\|_{H^1(\Omega)}+\|\partial_t u_\varepsilon^R(0,.)-v_\varepsilon^I\|_{L^2(\Omega)},\end{aligned}\tag{3.1}$$

where $s=1$ for Dirichlet and $s=\frac{1}{2}$ for Neumann.

The asymptotics will be proven by estimating each term of the right hand side of this energy estimate.

Since the error estimates in the interior and near the boundary use similar computations, a unified framework will be used by considering the more general problem of estimates linked with a suitable family of approximation operators O^α in section 3.1. Then in section 3.2 we use these estimates for the interior term $\|Pu_\varepsilon^R\|_{L^2(\Omega)}$ in 3.2.1, the boundary term $\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)}$ in 3.2.2, and the initial conditions errors in 3.2.3. All these estimates are gathered in section 3.3 to prove Theorem 1.1.

3.1. Approximation operators.

Let $K_{z,\theta}$ be a compact of $\mathbb{R}^{2n}$ and

$$E_r=\{(x,z,\theta)\in\mathbb{R}^n\times K_{z,\theta},|x-z|\le r\},r>0.$$

Consider a complex phase function Φ smooth on an open set containing E_{r_0} for some $r_0\in]0,1]$. We assume, for $(z,\theta)\in K_{z,\theta}$, that

$$\begin{aligned}&\partial_x\Phi(z,z,\theta)=\theta,\\ &\Phi(z,z,\theta)\text{ is real,}\\ &\partial_x^2\Phi(z,z,\theta)\text{ has a positive definite imaginary part.}\end{aligned}\tag{Q1}$$

The Taylor expansion of Φ together with Assumptions (Q1) imply the existence of some constant $r[\Phi]\in]0,r_0]$ s.t. for $(x,z,\theta)\in E_{r[\phi]}$

$$\operatorname{Im}\Phi(x,z,\theta)\ge\text{const}(x-z)^2.$$

Consider a sequence $l_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^{2n}_{z,\theta},\mathbb{C})$. We assume that

$$\begin{aligned}&l_\varepsilon(x,z,\theta)=0\text{ if }(x,z,\theta)\notin E_{r[\Phi]},\\ &l_\varepsilon\text{ is uniformly bounded in }L^\infty(\mathbb{R}^{3n}).\end{aligned}\tag{Q2}$$

For a given multi-index α, let the operators $O^0(l_\varepsilon,\Phi/\varepsilon)$ and $O^\alpha(l_\varepsilon,\Phi/\varepsilon)$ be given by

$$\left[O^0(l_\varepsilon,\Phi/\varepsilon)h\right](x)=\int_{\mathbb{R}^{2n}} h(z,\theta)l_\varepsilon(x,z,\theta)e^{i\Phi(x,z,\theta)/\varepsilon}dzd\theta, h\in L^2(\mathbb{R}^{2n}),$$

and

$$[O^\alpha(l_\varepsilon,\Phi/\varepsilon)h](x)=\int_{\mathbb{R}^{2n}} h(z,\theta)l_\varepsilon(x,z,\theta)(x-z)^\alpha e^{i\Phi(x,z,\theta)/\varepsilon}dzd\theta, h\in L^2(\mathbb{R}^{2n}),$$

with $x\in\mathbb{R}^n$.

Let us show that these are operators from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$. For $x\in\mathbb{R}^n$ we have

$$\int |l_\varepsilon e^{i\Phi/\varepsilon}|dzd\theta \lesssim \int_{(z,\theta)\in K_{z,\theta}} e^{-\mathrm{const}(x-z)^2/\varepsilon}dzd\theta,$$

and thus

$$\int |l_\varepsilon e^{i\Phi/\varepsilon}|dzd\theta \lesssim \varepsilon^{\frac{n}{2}}.$$

Similarly, for $(z,\theta)\in K_{z,\theta}$

$$\int |l_\varepsilon e^{i\Phi/\varepsilon}|dx \lesssim \varepsilon^{\frac{n}{2}}.$$

It is then immediate by Schur's lemma that

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)} \lesssim \varepsilon^{\frac{n}{2}}.$$

Similar arguments lead to the estimate

$$\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)} \lesssim \varepsilon^{\frac{n}{2}+\frac{|\alpha|}{2}}.$$

However, the use of the module inside the previous integrals makes one lose the highly oscillatory character of $e^{i\Phi/\varepsilon}$, that is the contribution of $e^{i\theta\cdot(x-z)/\varepsilon}$. In fact, a better estimate on the norms of these operators is available if a precise control on l_ε is assumed. This is stated in the following lemma

LEMMA 3.1. *Assume that $\varepsilon^{\frac{k}{2}}\partial^k_{x_b}l_\varepsilon$ ($b=1,\dots,n$) is uniformly bounded in $L^\infty(\mathbb{R}^{3n})$, at any order $k\in\mathbb{N}$. Then, one has*

1. $\|O^0(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)} \lesssim \varepsilon^{\frac{3n}{4}}$,
2. $\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)} \lesssim \varepsilon^{\frac{3n}{4}+\frac{|\alpha|}{2}}$.

Proof. 1. Let $h\in L^2(\mathbb{R}^{2n})$. We shall use the notations $f(x)$ for $f(x,z,\theta)$ and $f'(x)$ for $f(x,z',\theta')$. First of all, we make explicit the L^2 norm of $O^0(l_\varepsilon,\Phi/\varepsilon)h$ as

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)h\|^2_{L^2}=\int_{\mathbb{R}^{4n}} h\overline{h}'e^{i\Phi(z)/\varepsilon-i\Phi'(z')/\varepsilon}e^{i(\theta'.z'-\theta.z)/\varepsilon}$$
$$\Big[\int_{\mathbb{R}^n} l_\varepsilon(x)\overline{l}'_\varepsilon(x)e^{i(\theta-\theta').x/\varepsilon}e^{i\Theta(x,z,\theta,z',\theta')/\varepsilon}dx\Big]dzdz'd\theta d\theta', \tag{3.2}$$

where

$$\Theta(x,z,\theta,z',\theta') = \sum_{|\alpha|=2}(x-z)^\alpha \int_0^1 \tfrac{2}{\alpha!}(1-s)\partial_x^\alpha\Phi(z+s(x-z),z,\theta)ds - \sum_{|\alpha|=2}(x-z')^\alpha \int_0^1 \tfrac{2}{\alpha!}(1-s)\partial_x^\alpha\overline{\Phi}(z'+s(x-z'),z',\theta')ds.$$

Let I_ε denote the integral inside the brackets, which we begin to estimate. For $1\leq b\leq n$ and $K\in\mathbb{N}$, successive integrations by parts give

$$I_\varepsilon(z,z',\theta,\theta')i^K\varepsilon^{-K}(\theta_b-\theta_b')^K = (-1)^K\sum_{N+N'=K}\binom{K}{N}\int_{\mathbb{R}^n} e^{i(\theta-\theta').x/\varepsilon}\partial_{x_b}^N[e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon\bar{l}'_\varepsilon]dx,$$

where $\binom{K}{N}$ denotes the standard binomial coefficient. To estimate $\partial_{x_b}^N[e^{i\Theta/\varepsilon}]$, $N\in\mathbb{N}$, we use the following result, of which proof is postponed to the end of this section

Lemma 3.2. *Let $p\in\mathbb{N}^*$ and consider a complex phase function F_p of the form*

$$F_p(x,z)=\sum_{|\alpha|=p}(x-z)^\alpha f_\alpha(x,z),$$

with f_α smooth on some open set of $\mathbb{R}^{2n}$ containing a subset S and $\partial_x^k f_\alpha$ bounded on S for any $k\geq 0$.
Then for $(x,z)\in S$, $|x-z|\leq 1$, small ε, $N\in\mathbb{N}$, and $b=1,\dots,n$, one has

$$|\partial_{x_b}^N[e^{iF_p/\varepsilon}]|\leq \max_{\substack{|\alpha|=p\\0\leq s\leq N\\1\leq k\leq N}}\big(\sup_S|\partial_{x_b}^s f_\alpha|\big)^k\Big(\sum_{\frac{N}{p}\leq k\leq N}\varepsilon^{-k}|x-z|^{kp-N}+\sum_{1\leq k<\frac{N}{p}}\varepsilon^{-N/p}\Big)|e^{iF_p/\varepsilon}|.$$

We write $\Theta=F_2-\bar{F}_2'$ with

$$F_2(x,z,\theta)=\sum_{|\alpha|=2}(x-z)^\alpha\int_0^1\frac{2}{\alpha!}(1-s)\partial_x^\alpha\Phi(z+s(x-z),z,\theta)ds,$$

for $(x,z,\theta)\in E_{r[\Phi]}$. By Leibnitz formula, $\partial_{x_b}^N[e^{i\Theta/\varepsilon}]$ is a sum of terms of the form

$$\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]\partial_{x_b}^{N_2}[e^{-i\bar{F}_2'/\varepsilon}], 0\leq N_1,N_2\leq N, N_1+N_2=N.$$

Note that $\operatorname{Im}F_2=\operatorname{Im}\Phi$. Lemma 3.2 yields for $N_1\in\mathbb{N}$ and $(x,z,\theta)\in E_{r[\Phi]}$

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]|\lesssim\left(\sum_{\frac{N_1}{2}\leq k\leq N_1}\varepsilon^{-k}|x-z|^{2k-N_1}+\varepsilon^{-N_1/2}\right)e^{-\text{const}(x-z)^2/\varepsilon}.$$

Hence

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]|\lesssim\varepsilon^{-\frac{N_1}{2}}e^{-\text{const}(x-z)^2/\varepsilon}.$$

A similar estimate may be obtained for $|\partial_{x_b}^{N_2}[e^{i\bar{F}_2'/\varepsilon}]|$ when $(x,z',\theta')\in E_{r[\Phi]}$. It follows, for $(x,z,\theta),(x,z',\theta')\in E_{r[\Phi]}$, that

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]\partial_{x_b}^{N_2}[e^{-i\bar{F}_2'/\varepsilon}]|\lesssim\varepsilon^{-\frac{N_1+N_2}{2}}e^{-\text{const}(x-z)^2/\varepsilon}e^{-\text{const}(x-z')^2/\varepsilon},$$

and thus

$$|\partial_{x_b}^N[e^{i\Theta/\varepsilon}]| \lesssim \varepsilon^{-\frac{N}{2}} e^{-\mathrm{const}(2x-z-z')^2/\varepsilon} e^{-\mathrm{const}(z-z')^2/\varepsilon}, N \in \mathbb{N}.$$

Since $\varepsilon^{\frac{N'}{2}} \partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]$, $N' \in \mathbb{N}$, is uniformly bounded,

$$|\partial_{x_b}^N[e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]| \lesssim \varepsilon^{-\frac{N+N'}{2}} e^{-\mathrm{const}(2x-z-z')^2/\varepsilon} e^{-\mathrm{const}(z-z')^2/\varepsilon},$$

and we deduce that

$$|I_\varepsilon(z,z',\theta,\theta')\left(\frac{\theta_b-\theta'_b}{\sqrt{\varepsilon}}\right)^K| \lesssim \varepsilon^{\frac{n}{2}} e^{-\mathrm{const}(z-z')^2/\varepsilon},$$

for $b=1,\dots,n$ and $K \in \mathbb{N}$.

Choosing $K>n$ and coming back to (3.2) gives

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{n}{2}} \int_{\mathbb{R}^{4n}} |h||h'| e^{-\mathrm{const}(z-z')^2/\varepsilon} dz dz' (1+(\theta-\theta')^2/\varepsilon)^{-\frac{K}{2}} d\theta d\theta'.$$

Upon using the change of variables:

$$(z,z') = (u+\sqrt{\varepsilon}v, u-\sqrt{\varepsilon}v) \text{ and } (\theta,\theta') = (\sigma+\sqrt{\varepsilon}\delta, \sigma-\sqrt{\varepsilon}\delta),$$

we have

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{3n}{2}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} |h(u+\sqrt{\varepsilon}v,\sigma+\sqrt{\varepsilon}\delta)||h(u-\sqrt{\varepsilon}v,\sigma-\sqrt{\varepsilon}\delta)| du d\sigma \\ e^{-\mathrm{const}\, v^2}(1+4\delta^2)^{-\frac{K}{2}} dv d\delta,$$

from which, using the Cauchy-Schwartz inequality for the first integral, we obtain

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{3n}{2}} \|h\|_{L^2}^2.$$

2. The arguments are similar to the previous case. For a multi-index α, we have

$$\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)h\|_{L^2}^2 = \int_{\mathbb{R}^{4n}} h\bar{h}' e^{i\Phi(z)/\varepsilon - i\Phi'(z')/\varepsilon} e^{i(\theta'.z'-\theta.z)/\varepsilon} \\ I_\varepsilon^\alpha(z,z',\theta,\theta') dz dz' d\theta d\theta',$$

where, for $b=1,\dots,n$ and $K \in \mathbb{N}$

$$I_\varepsilon^\alpha(z,z',\theta,\theta') i^K \varepsilon^{-K}(\theta_b-\theta'_b)^K = (-1)^K \sum_{N+N'=K} \binom{K}{N} \int_{\mathbb{R}^n} e^{i(\theta-\theta').x/\varepsilon} \\ \partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}] \partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon] dx.$$

We note that $\partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]$ is a finite sum of terms of the form

$$(x-z)^{\alpha-ke^b}(x-z')^{\alpha-le^b} \partial_{x_b}^m[e^{i\Theta/\varepsilon}],$$

where $k,l \le \alpha_b$, $k+l+m=N$ and e^b denotes the vector of $\mathbb{R}^n$ s.t. $e_a^b = \delta_{ab}$.
For $(x,z,\theta),(x,z',\theta') \in E_{r[\Phi]}$, it follows that

$$|\partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]| \lesssim \varepsilon^{|\alpha|-\frac{N}{2}} e^{-\mathrm{const}(2x-z-z')^2/\varepsilon} e^{-\mathrm{const}(z-z')^2/\varepsilon}.$$

Since $\varepsilon^{\frac{N'}{2}}\partial_{x_b}^{N'}[l_\varepsilon l'_\varepsilon]$ is uniformly bounded,

$$|\partial_{x_b}^{N}[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]|\lesssim \varepsilon^{|\alpha|-\frac{N+N'}{2}}e^{-\mathrm{const}(2x-z-z')^2/\varepsilon}e^{-\mathrm{const}(z-z')^2/\varepsilon},$$

and thus

$$|I_\varepsilon^\alpha(z,z',\theta,\theta')\left(\frac{\theta_b-\theta'_b}{\sqrt{\varepsilon}}\right)^K|\lesssim \varepsilon^{\frac{n}{2}+|\alpha|}e^{-\mathrm{const}(z-z')^2/\varepsilon},$$

and finally

$$\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)h\|_{L^2}^2\lesssim \varepsilon^{\frac{3n}{2}+|\alpha|}\|h\|_{L^2}^2.$$

□

Similar computations can be carried out for a phase Φ and a sequence of amplitudes l_ε that depend on a parameter $m\in[0,M]$. In this case, we consider for $m\in[0,M]$ a compact $K_{z,\theta}(m)\subset\mathbb{R}^{2n}$ and denote for $r>0$

$$E_r=\{(m,x,z,\theta)\in[0,M]\times\mathbb{R}^{3n},(z,\theta)\in K_{z,\theta}(m),|x-z|\le r\}.$$

We are interested in a phase function Φ smooth on an open set containing E_{r_0} for some $r_0\in]0,1]$. We make the further assumption

$$E_{r_0} \text{ is compact,}$$

which is obviously fulfilled when no parameter m interferes. Assuming, for $m\in[0,M]$ and $(z,\theta)\in K_{z,\theta}(m)$, that

$$\begin{aligned}&\partial_x\Phi(m,z,z,\theta)=\theta,\\&\Phi(m,z,z,\theta) \text{ is real,}\\&\partial_x^2\Phi(m,z,z,\theta) \text{ has a positive definite imaginary part,}\end{aligned}\tag{Q1'}$$

one can find $r[\Phi]\in]0,r_0]$ s.t. for $(m,x,z,\theta)\in E_{r[\phi]}$

$$\mathrm{Im}\,\Phi(m,x,z,\theta)\ge \mathrm{const}(x-z)^2.$$

Similarly, the sequence l_ε will be assumed to belong to $\mathcal{C}^\infty([0,M]\times\mathbb{R}^n_x\times\mathbb{R}^{2n}_{z,\theta},\mathbb{C})$ and to satisfy

$$\begin{aligned}&\text{for } m\in[0,M], l_\varepsilon(m,x,z,\theta)=0 \text{ if } (m,x,z,\theta)\notin E_{r[\Phi]},\\&l_\varepsilon \text{ is uniformly bounded in } L^\infty([0,M]\times\mathbb{R}^{3n}).\end{aligned}\tag{Q2'}$$

One can then define, for every given $m\in[0,M]$ and α multiindex ($|\alpha|\ge 0$), the operators $O^\alpha(l_\varepsilon(m,.),\Phi(m,.)/\varepsilon)$, for which the following estimate may be established

Lemma 3.3. *Assume that $\varepsilon^{\frac{k}{2}}\partial_{x_b}^k l_\varepsilon$ ($b=1,\dots,n$) is uniformly bounded in $L^\infty([0,M]\times\mathbb{R}^{3n})$, at any order $k\in\mathbb{N}$. Then, one has*

$$\|O^\alpha(l_\varepsilon(m,.),\Phi(m,.)/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim \varepsilon^{\frac{3n}{4}+\frac{|\alpha|}{2}},\ \textit{uniformly w.r.t. } m\in[0,M].$$

In fact, all the estimates used in the proof of Lemma 3.1 hold true with a parameter $m\in[0,M]$, since $E_{r[\phi]}$ is still compact, owing to the compactness of E_{r_0}.

We now give the proof of Lemma 3.2. Using the formula of composite functions' high derivatives (see, e.g., [11] p.161), the N^{th} partial derivative of $e^{iF_p/\varepsilon}$ is

$$\partial_{x_b}^N[e^{iF_p/\varepsilon}]=\sum_{k=1}^N\left(\frac{i}{\varepsilon}\right)^k\prod_{\substack{j^1+\cdots+j^k=N\\ j^1,\dots,j^k\geq 1}}\frac{N!}{k!j^1!\dots j^k!}\partial_{x_b}^{j^1}F_p\dots\partial_{x_b}^{j^k}F_p e^{iF_p/\varepsilon},N\in\mathbb{N}^*.$$

Each derivative $\partial_{x_b}^j F_p$ is a linear combination of

$$(x-z)^{\alpha+(s-j)e^b}\partial_{x_b}^s f_\alpha,|\alpha|=p,0\leq s\leq j\text{ and }\alpha_b\geq j-s.$$

The product $\partial_{x_b}^{j^1}F_p\dots\partial_{x_b}^{j^k}F_p$ is then a linear combination of

$$(x-z)^{\alpha^1+(s^1-j^1)e^b+\cdots+\alpha^k+(s^k-j^k)e^b}\partial_{x_b}^{s^1}f_{\alpha^1}\dots\partial_{x_b}^{s^k}f_{\alpha^k},$$

where for $i=1,\dots,k$, $|\alpha^i|=p$, $0\leq s^i\leq j^i$, and $\alpha_b^i\geq j^i-s^i$. As $j^1+\cdots+j^k=N$, then for $N/p\leq k\leq N$ and $|x-z|\leq 1$ one has

$$|(x-z)^{\alpha^1+(s^1-j^1)e^b+\cdots+\alpha^k+(s^k-j^k)e^b}|\leq|x-z|^{kp-N}.$$

Thus for $N\in\mathbb{N}^*$, $(x,z)\in S$, $|x-z|\leq 1$, and small ε,

$$|\partial_{x_b}^N[e^{iF_p/\varepsilon}]|\leq\max_{\substack{|\alpha|=p\\0\leq s\leq N\\1\leq k\leq N}}\big(\sup_S|\partial_{x_b}^s f_\alpha|\big)^k\Big(\sum_{\frac{N}{p}\leq k\leq N}\varepsilon^{-k}|x-z|^{kp-N}+\sum_{1\leq k<\frac{N}{p}}\varepsilon^{-N/p}\Big)|e^{iF_p/\varepsilon}|,$$

which of course is also valid for $N=0$.

3.2. Error estimates. The different terms of the energy estimate (3.1) will be estimated separately. Our main interest is to prove that the interior and boundary errors given for individual beams in Lemma 2.2 hold true for an infinite sum of beams, when the starting points of the incident flow vary in the compact $K_y\times K_\eta$. The control we have is that we can make the Gaussian beams vanish outside the very neighborhood of their associated rays by making the parameter d as small as needed.

3.2.1. The interior estimate of Pu_ε^R. In this section, we will prove that

$$\underset{t\in[0,T]}{Sup}\|Pu_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

For $0\leq k\leq N_+$, one has by construction

$$Pw_\varepsilon^k=\sum_{j=0}^{N+2}\varepsilon^{j-2}c_j^k e^{i\psi_k/\varepsilon},$$

where c_j^k is null on (t,x_k^t), up to the order $R-2j$, for $j=0,\dots,N+1$. One may write

$$Pw_\varepsilon^k(t,x)=\sum_{j=0}^{N+1}\varepsilon^{j-2}\Big(\sum_{|\alpha|=R-2j+1}(x-x_k^t)^\alpha r_\alpha^k(t,x)e^{i\psi_k(t,x)/\varepsilon}\Big)+\varepsilon^N c_{N+2}^k(t,x)e^{i\psi_k(t,x)/\varepsilon},$$

where r_α^k denotes the remainder in the Taylor formulae of c_j^k near x_k^t. Applying P to (2.24) then gives terms of the form

$$p_\varepsilon^{j,k}(t,x) = \varepsilon^{-\frac{3n}{4}-1+j} \sum_{|\alpha|=R-2j+1} \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)h_\varepsilon(y,\eta) \\ (x-x_k^t)^\alpha r_\alpha^k(t,x,y,\eta)e^{i\psi_k(t,x,y,\eta)/\varepsilon}dyd\eta,$$

with $j=0,\dots,N+1$, and

$$p_\varepsilon^{N+2,k}(t,x) = \varepsilon^{-\frac{3n}{4}+N+1}\int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)h_\varepsilon(y,\eta)c_{N+2}^k(t,x,y,\eta)e^{i\psi_k(t,x,y,\eta)/\varepsilon}dyd\eta,$$

where h_ε is either $\varepsilon^{-1}T_\varepsilon u_\varepsilon^I$ or $T_\varepsilon v_\varepsilon^I$ and $0\le k\le N_+$. Other terms of the same form come from $Pw_\varepsilon^{k'}$, $0\le k\le N_+$, and $P[w_\varepsilon^{-k(')}(-t,.)]$, $0\le k\le N_-$.

Let $\tilde f(t,x,z,\theta) = f(t,x,\{\varphi_k^t\}^{-1}(z,\theta))$. Using the volume preserving change of variables $(z,\theta)=\varphi_k^t(y,\eta)$ in the definition of $p_\varepsilon^{j,k}(t,x)$, $0\le j\le N+1$, writes it as a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}-1+j}\int_{\mathbb{R}^{2n}} \widetilde{\rho\otimes\phi}(t,z,\theta)\tilde h_\varepsilon(t,z,\theta)(x-z)^\alpha \tilde r_\alpha^k(t,x,z,\theta)e^{i\tilde\psi_k(t,x,z,\theta)/\varepsilon}dzd\theta,$$

with $|\alpha|=R-2j+1$. Similarly, $p_\varepsilon^{N+2,k}(t,x)$ is a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}+N+1}\int_{\mathbb{R}^{2n}} \widetilde{\rho\otimes\phi}(t,z,\theta)\tilde h_\varepsilon(t,z,\theta)\tilde c_{N+2}^k(t,x,z,\theta)e^{i\tilde\psi_k(t,x,z,\theta)/\varepsilon}dzd\theta.$$

We want to estimate these integrals with the help of the operators O^α applied to $\mathbf{1}_{\operatorname{supp}\widetilde{\rho\otimes\phi}(t,.)}\tilde h_\varepsilon$. Clearly $\mathbf{1}_{\operatorname{supp}\widetilde{\rho\otimes\phi}(t,.)}\widetilde{T_\varepsilon v_\varepsilon^I}(t,.)$ is uniformly bounded (w.r.t. ε and t) in $L^2(\mathbb{R}^{2n})$. But more work is needed for estimating $\varepsilon^{-1}\mathbf{1}_{\operatorname{supp}\widetilde{\rho\otimes\phi}(t,.)}\widetilde{T_\varepsilon u_\varepsilon^I}(t,.)$, which is given in the following result.

Lemma 3.4. $\|\varepsilon^{-1}T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})}\lesssim 1$.

Proof. Differentiating (1.3) w.r.t. y_b, $1\le b\le n$, yields

$$\varepsilon^{\frac12}\partial_{y_b}(T_\varepsilon u_\varepsilon^I) = i\eta_b\varepsilon^{-\frac12}T_\varepsilon u_\varepsilon^I - c_n\varepsilon^{-\frac{3n}{4}}\int_{\mathbb{R}^n} u_\varepsilon^I(w)\varepsilon^{-\frac12}(y_b-w_b)e^{i\eta.(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)}dw.$$

The left hand side is bounded in $L^2_{y,\eta}$ because $\partial_{y_b}(T_\varepsilon u_\varepsilon^I)=T_\varepsilon(\partial_{w_b}u_\varepsilon^I)$. The second term of the right hand side is the Fourier transform of a bounded function in L^2_w, thus it can be estimated using the Parseval equality. One obtains

$$\|\varepsilon^{-\frac{3n}{4}}\int_{\mathbb{R}^n} u_\varepsilon^I(w)\varepsilon^{-\frac12}(y_b-w_b)e^{i\eta.(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)}dw\|_{L^2_{y,\eta}}\lesssim \|u_\varepsilon^I\|_{L^2_w}.$$

Thus $\|\varepsilon^{-\frac12}\eta_bT_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}}\lesssim 1$ and consequently $\|\varepsilon^{-\frac12}\phi(\eta)T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}}\lesssim 1$. Assumption $A3$ yields

$$\|\varepsilon^{-\frac12}T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}}\lesssim 1.$$

Hence $\|u_\varepsilon^I\|_{L^2}\lesssim\sqrt{\varepsilon}$. Reproducing the same arguments on the following equality

$$\partial_{y_b}(T_\varepsilon u_\varepsilon^I) = i\eta_b\varepsilon^{-1}T_\varepsilon u_\varepsilon^I - c_n\varepsilon^{-\frac{3n}{4}}\int_{\mathbb{R}^n}\left(\varepsilon^{-\frac12}u_\varepsilon^I\right)(w)\varepsilon^{-\frac12}(y_b-w_b)e^{\frac{i}{\varepsilon}\eta.(y-w)-\frac{1}{2\varepsilon}(y-w)^2}dw,$$

leads to $\|u_\varepsilon^I\|_{L^2} \lesssim \varepsilon$. □

Let us now check if a family of operators O^α may be used. First, each phase $\tilde{\psi}_k$ is smooth on an open set containing

$$E_1 = \{(t,x,z,\theta) \in [0,T] \times \mathbb{R}^{3n}, (z,\theta) \in \varphi_k^t(K_y \times K_\eta), |x-z| \le 1\}.$$

E_1 is compact, since the map $(t,y,\eta) \mapsto (t, \varphi_k^t(y,\eta))$ is continuous. For $t \in [0,T]$ and $(z,\theta) \in \varphi_k^t(K_y \times K_\eta)$, one has by ($P_k$.a), ($P_k$.b), and ($P_k$.c),

$$\begin{aligned} &\partial_x \tilde{\psi}_k(t,z,z,\theta) = \tilde{\xi}_k^t(z,\theta) = \theta, \\ &\tilde{\psi}_k(t,z,z,\theta) \text{ is real,} \\ &\partial_x^2 \tilde{\psi}_k(t,z,z,\theta) \text{ has a positive definite imaginary part.} \end{aligned}$$

Hence $\tilde{\psi}_k$ satisfies properties (Q1'). We fix some $r[\tilde{\psi}_k] \in]0,1]$ so that

$$\operatorname{Im} \tilde{\psi}_k(t,x,z,\theta) \ge \text{const}(x-z)^2 \text{ for every } (t,x,z,\theta) \in E_{r[\tilde{\psi}_k]}. \tag{3.3}$$

Next, for $R - 2N - 1 \le |\alpha| \le R + 1$, let

$$l^{\alpha,k}(t,x,z,\theta) = \widetilde{\rho \otimes \phi}(t,z,\theta) \tilde{r}_\alpha^k(t,x,z,\theta), t \in [0,T],$$

and

$$l^{0,k}(t,x,z,\theta) = \widetilde{\rho \otimes \phi}(t,z,\theta) \tilde{c}_{N+2}^k(t,x,z,\theta), t \in [0,T].$$

Then the $l^{\alpha,k}$, $|\alpha| = R - 2N - 1, \dots, R + 1$, and $l^{0,k}$ are smooth w.r.t. all their variables. Assume that

$$d \le r[\tilde{\psi}_k], k = 0, \dots, N_+. \tag{3.4}$$

Because of the cut-offs χ_d in the beams' amplitudes, it follows that $\tilde{c}_{N+2}^k(t,x,z,\theta) = \tilde{r}_\alpha^k(t,x,z,\theta) = 0$ if $|x-z| \ge r[\tilde{\psi}_k]$. Furthermore, $\widetilde{\rho \otimes \phi}(t,z,\theta) = 0$ for $(z,\theta) \notin \varphi_k^t(K_y \times K_\eta)$. The $l^{\alpha,k}$ therefore satisfy Assumptions (Q2').

It follows that the operators O^α can be used to obtain for $t \in [0,T]$ and $x \in \mathbb{R}^n$

$$p_\varepsilon^{j,k}(t,x) = \varepsilon^{-\frac{3n}{4} - 1 + j} \sum_{|\alpha| = R - 2j + 1} \left[O^\alpha \left(l^{\alpha,k}(t,.), \tilde{\psi}_k(t,.)/\varepsilon \right) \mathbf{1}_{\operatorname{supp} \widetilde{\rho \otimes \phi}(t,.)} \tilde{h}_\varepsilon(t,.) \right](x),$$

with $j = 0, \dots N + 1$, and

$$p_\varepsilon^{N+2,k}(t,x) = \varepsilon^{-\frac{3n}{4} + N + 1} \left[O^0 \left(l^{0,k}(t,.), \tilde{\psi}_k(t,.)/\varepsilon \right) \mathbf{1}_{\operatorname{supp} \widetilde{\rho \otimes \phi}(t,.)} \tilde{h}_\varepsilon(t,.) \right](x).$$

Applying Lemma 3.3 and making use of (2.10) yields

$$\|p_\varepsilon^{j,k}(t,.)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}, \text{ uniformly w.r.t. } t \in [0,T], \text{ for } j = 0, \dots, N+2.$$

3.2.2. The boundary estimate of Bu_ε^R. We will now estimate $Bu_\varepsilon^R|_{\partial\Omega}$, $B = \mathbf{D}$ or $\mathbf{N}$ standing for Dirichlet and Neumann operators respectively. We shall prove that

$$\|\mathbf{D} u_\varepsilon^R\|_{H^1([0,T] \times \partial\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}} \text{ and } \|\mathbf{N} u_\varepsilon^R\|_{H^{1/2}([0,T] \times \partial\Omega)} \lesssim \varepsilon^{\frac{R-2}{2}}. \tag{3.5}$$

To this end, we note that the boundary operator B applied to (2.24) is a sum of terms arising from Bw_ε^k, $0\le k\le N_+$ such as

$$b_\varepsilon^j(t,x)=\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\rho(y)\phi(\eta)h_\varepsilon(y,\eta)\sum_{k=0}^{N_+}d^k_{-m_B+j}(t,x',y,\eta)e^{i\psi_k(t,x',y,\eta)/\varepsilon}dyd\eta, \tag{3.6}$$

with $j=0,\dots,N+m_B$, and others with the same form arising from $Bw_\varepsilon^{k'}$, $0\le k\le N_+$, and $B[w_\varepsilon^{-k(')}(-t,.)]$, $0\le k\le N_-$.

Above and as in the previous section, h_ε is either $\varepsilon^{-1}T_\varepsilon u_\varepsilon^I$ or $T_\varepsilon v_\varepsilon^I$ and thus is uniformly bounded in L^2.

We first study the support of the amplitudes. Next we use local boundary coordinates to expand the boundary phases and introduce a change of variables on (y,η) that makes the obtained phases satisfy properties (Q1). The previous results on the approximation operators O^α are then used to estimate the boundary norms.

Support of the amplitudes

Due to Assumptions B1–B2–B3, the rays stay away from the boundary except for times near the instants of reflections. For $(y,\eta)\in K_y\times K_\eta$ and $t\in[0,T]$ near some $T_k(y,\eta)$, $0\le k\le N_+$, only $x_k^t(y,\eta)$ and, if $k\neq 0$, $x_{k-1}^t(y,\eta)$ approach the boundary. This suggests that the meaningful contributions to the boundary norm of b_ε^j are the quantities $d^{k-1}_{-m_B+j}(.,y,\eta)e^{i\psi_{k-1}(.,y,\eta)/\varepsilon}+d^k_{-m_B+j}(.,y,\eta)e^{i\psi_k(.,y,\eta)/\varepsilon}$ near $T_k(y,\eta)$, $k=1,\dots,N_+$. Furthermore, for t in the neighborhood of $T_k(y,\eta)$ and $x'\in\partial\Omega$, one expects $d^{k-1}_{-m_B+j}(t,x',y,\eta)$ and $d^k_{-m_B+j}(t,x',y,\eta)$ to vanish away from $x_{k-1}^{T_k(y,\eta)}(y,\eta)$, because of the cut-offs in the amplitudes. In the remainder, we show that these two intuitive points are true. The key argument is that (t,y,η) vary in a compact set.

The first point is rather easy to see. For $(y,\eta)\in K_y\times K_\eta$, let us consider a period smaller than any lapse of time between two successive reflections, say $\beta(y,\eta)=\min_{0\le k\le N_+}(T_k(y,\eta)-T_{k-1}(y,\eta))/3$, $(T_0=0)$, and define the intervals

$$I_0(y,\eta)=\emptyset, I_k(y,\eta)=[T_k(y,\eta)-\beta(y,\eta),T_k(y,\eta)+\beta(y,\eta)] \text{ for } k=1,\dots,N_+, \quad \text{and } I_{N_++1}(y,\eta)=\emptyset.$$

For each $k=0,\dots,N_+$, let

$$A_k=\{(t,y,\eta)\in[0,T]\times K_y\times K_\eta, t\notin \overset{o}{I}_k(y,\eta)\cup \overset{o}{I}_{k+1}(y,\eta)\}.$$

For $(t,y,\eta)\in A_k$, $\operatorname{dist}(x_k^t(y,\eta),\partial\Omega)>0$ and then has a positive lower bound by continuity on the compact A_k. One has by (3.3) and (3.4)

$$\psi_k(t,x,y,\eta)\ge \text{const}(x-x_k^t(y,\eta))^2,$$

for $(t,x,y,\eta)\in[0,T]\times\mathbb{R}^n\times K_y\times K_\eta$ s.t. $|x-x_k^t(y,\eta)|\le d$. Thus

$$|d^k_{-m_B+j}(t,x',y,\eta)e^{i\psi_k(t,x',y,\eta)/\varepsilon}|\le e^{-\text{const}/\varepsilon} \text{ for } (t,y,\eta)\in A_k \text{ and } x'\in\partial\Omega.$$

All we have to care about then is the contribution to the norm at the boundary of $d^{k-1}_{-m_B+j}e^{i\psi_{k-1}/\varepsilon}$ and $d^k_{-m_B+j}e^{i\psi_k/\varepsilon}$ at times in the interval I_k, $k=1,\dots,N_+$. Let

$$q_\varepsilon^{j,k}=\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\rho\otimes\phi h_\varepsilon\mathbf{1}_{I_k}(t)(d^{k-1}_{-m_B+j}e^{i\psi_{k-1}/\varepsilon} + d^k_{-m_B+j}e^{i\psi_k/\varepsilon})dyd\eta.$$

Summing over $k=1,\dots,N_+$ yields

$$\|b_\varepsilon^j\|_{L^2([0,T]\times\partial\Omega)} \lesssim \sum_{k=1}^{N_+} \|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} + \varepsilon^\infty. \tag{3.7}$$

For the second point, we partition the set of starting points (y,η) according to the part of the boundary the rays $x_{k-1}^t(y,\eta)$ reach at $t=T_k(y,\eta)$. Let (u_l) be a $\mathcal{C}^\infty$ partition of unity associated to the covering $(\mathcal{U}_l)$ introduced in subsection 2.2.1 and $\pi_l(y,\eta)=\rho(y)\phi(\eta)u_l(x_{k-1}^{T_k}(y,\eta))$. Then

$$\|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} \lesssim \sum_{l=1}^{L} \|m_\varepsilon^{j,k,l}\|_{L^2([0,T]\times\partial\Omega)},$$

where

$$m_\varepsilon^{j,k,l} = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \int_{\mathbb{R}^{2n}} h_\varepsilon \pi_l \mathbf{1}_{I_k}(t)(d_{-m_b+j}^{k-1} e^{i\psi_{k-1}/\varepsilon} + d_{-m_b+j}^{k} e^{i\psi_k/\varepsilon}) dy d\eta. \tag{3.8}$$

We fix $1\le l\le L$ and $1\le k\le N_+$. For $0<\delta<\min_{K_y\times K_\eta}\beta$, let

$$B_\delta = \{(t,y,\eta)\in[0,T]\times\operatorname{supp}\pi_l,\, t\in I_k(y,\eta)\setminus]T_k(y,\eta)-\delta, T_k(y,\eta)+\delta[\}.$$

If (t,y,η) is in the compact set B_δ, then $\operatorname{dist}(x_k^t(y,\eta),\partial\Omega)>0$. Let $d(\delta)\in]0,\delta]$ s.t. $d(\delta)<\min_{(t,y,\eta)\in B_\delta}\operatorname{dist}(x_k^t(y,\eta),\partial\Omega)$ and consider the set

$$S_\delta = \{(t,x',y,\eta)\in[0,T]\times\partial\Omega\times\operatorname{supp}\pi_l, t\in I_k(y,\eta) \text{ and } |x'-x_k^t(y,\eta)|\le d(\delta)\}.$$

If $(t,x',y,\eta)\in S_\delta$ then $t\in]T_k(y,\eta)-\delta,T_k(y,\eta)+\delta[$ and consequently

$$\begin{aligned}|x'-x_{k-1}^{T_k(y,\eta)}(y,\eta)| &\le |x'-x_{k-1}^t(y,\eta)| + |t-T_k(y,\eta)| \sup_{s\in[t,T_k(y,\eta)]} |\dot{x}_{k-1}^s(y,\eta)| \\ &\le (1+\|c\|_\infty)\delta,\end{aligned}$$

which implies that $x'\in\mathcal{U}_l$ for sufficiently small δ, since $x_{k-1}^{T_k(y,\eta)}(y,\eta)$ varies in a compact set of $\mathcal{U}_l$. Assume that $d\le d(\delta)$. Thus, $\operatorname{supp}\big(\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^k(t,x',y,\eta)\big)$ is included in S_δ. On the other hand, as σ_l is a diffeomorphism between $\mathcal{N}_l$ and $\mathcal{U}_l$, one has

$$|\sigma_l(\hat{v})-\sigma_l(\hat{v}')|\ge \text{const}\,|\hat{v}-\hat{v}'| \text{ for every } \hat{v},\hat{v}'\in\mathcal{N}_l.$$

Therefore, there exists $\kappa>0$ s.t.

$$\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^k(t,\sigma_l(\hat{v}),y,\eta)=0 \text{ if } |t-T_k(y,\eta)|\ge\delta \text{ or } |\hat{v}-\hat{z}_k(y,\eta)|\ge\kappa\delta,$$

where $\sigma_l(\hat{z}_k(y,\eta))=x_{k-1}^{T_k(y,\eta)}(y,\eta)$.

The same result holds true for $\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^{k-1}(t,\sigma_l(\hat{v}),y,\eta)$, assuming that $d\le d'(\delta)$ with $d'(\delta)\in]0,\delta]$ and $d'(\delta)<\min_{(t,y,\eta)\in B_\delta}\operatorname{dist}(x_{k-1}^t(y,\eta),\partial\Omega)$. Furthermore

$$m_\varepsilon^{j,k,l}(t,x')=0 \text{ if } x'\notin\mathcal{U}_l.$$

Expansion of the boundary phases For simplicity of notation, we shall drop the exponents and indexes l. We expand the phase ${}^{\sigma}\psi_{k-1}$ on $[0,T]\times\mathcal{N}\times\{0\}$ near $(T_k,\hat{z}_k)$

$$\begin{aligned}{}^{\sigma}\psi_{k-1}(t,\hat{v},0) &= \psi_{k-1}(t,\sigma(\hat{v})+v_n\nu(\sigma(\hat{v})))|_{v_n=0}\\ &= {}^{\sigma}\psi_{k-1}(T_k,\hat{z}_k,0)+(t-T_k,\hat{v}-\hat{z}_k)\cdot(\tau,\hat{\theta}_k)\\ &\quad+\frac{1}{2}(t-T_k,\hat{v}-\hat{z}_k)\cdot M_k(t-T_k,\hat{v}-\hat{z}_k)\\ &\quad+\sum_{|\alpha|=3}(t-T_k,\hat{v}-\hat{z}_k)^{\alpha}\\ &\qquad\int_0^1\frac{3}{\alpha!}(1-s)^2\partial^{\alpha}_{t,\hat{v}}\,{}^{\sigma}\psi_{k-1}(T_k+s(t-T_k),\hat{z}_k+s(\hat{v}-\hat{z}_k),0)ds,\end{aligned}$$

where $\hat{\theta}_k=D\sigma(\hat{z}_k)^T\xi^{T_k}_{k-1}$ and the matrix M_k defined in (2.16) has a positive definite imaginary part. Remember that all the quantities of the previous formulae depend on $(y,\eta)\in(x^{T_k}_{k-1})^{<-1>}(\mathcal{U})$. For the purpose of obtaining a phase satistfying (Q1), the form of ${}^{\sigma}\psi_{k-1}|_{v_n=0}$ suggests the change of variables $(C):(z,\theta)=\vartheta(y,\eta)$, with

$$\vartheta:(y,\eta)\in(x^{T_k}_{k-1})^{<-1>}(\mathcal{U})\mapsto(T_k,\hat{z}_k,\tau,\hat{\theta}_k).$$

Because tangential rays are avoided, the function $T_k\in\mathcal{C}^{\infty}((x^{T_k}_{k-1})^{<-1>}(\mathcal{U}))$ so ϑ is $\mathcal{C}^{\infty}$. Note that $\xi^{T_k}_{k-1}=\Sigma(\hat{z}_k)\hat{\theta}_k+(\nu(\sigma(\hat{z}_k))\cdot\xi^{T_k}_{k-1})\nu(\sigma(\hat{z}_k))$ with $\Sigma=D\sigma\left(D\sigma^TD\sigma\right)^{-1}$. Hence ϑ is bijective and its inverse is given by

$$\begin{aligned}\vartheta^{-1}:(T_k,\hat{z}_k,\tau,\hat{\theta}_k)&\in\vartheta((x^{T_k}_{k-1})^{<-1>}(\mathcal{U}))\\ &\mapsto\{\varphi^{T_k}_{k-1}\}^{-1}(\sigma(\hat{z}_k),\Sigma(\hat{z}_k)\hat{\theta}_k+(\tau^2/c^2(\sigma(\hat{z}_k))-|\Sigma(\hat{z}_k)\hat{\theta}_k|^2)^{\frac{1}{2}}\nu(\sigma(\hat{z}_k))).\end{aligned}$$

ϑ^{-1} is $\mathcal{C}^{\infty}$ on $\vartheta\left((x^{T_k}_{k-1})^{<-1>}(\mathcal{U})\right)$ because the square root in the previous expression never vanishes. Consequently, ϑ is a $\mathcal{C}^{\infty}$ diffeomorphism.

Let $v=(t,\hat{v})$, $z=(T_k,\hat{z}_k)$, and $\theta=(\tau,\hat{\theta}_k)$ and denote $\tilde{f}(v,z,\theta)=f(v,\vartheta^{-1}(z,\theta))$. We may write ${}^{\sigma}\tilde{\psi}_{k-1}|_{v_n=0}$ as

$$\begin{aligned}{}^{\sigma}\tilde{\psi}_{k-1}(v,0,z,\theta) = &\ {}^{\sigma}\tilde{\psi}_{k-1}(z,0,z,\theta)+\theta\cdot(v-z)+\tfrac{1}{2}(v-z)\tilde{M}_k(z,\theta)(v-z)\\ &+\sum_{3\le|\alpha|\le R}\tfrac{1}{\alpha!}(v-z)^{\alpha}\partial^{\alpha}_{t,\hat{v}}\,{}^{\sigma}\tilde{\psi}_{k-1}(z,0,z,\theta)+\tilde{r}_{k-1}(v,z,\theta)\\ :=&\ \tilde{\lambda}(v,z,\theta)+\tilde{r}_{k-1}(v,z,\theta).\end{aligned}$$

Since ${}^{\sigma}\psi_k$ and ${}^{\sigma}\psi_{k-1}$ have by construction the same derivatives w.r.t. v up to the order R at $(z,0)$, the expansion of ${}^{\sigma}\tilde{\psi}_k|_{v_n=0}$ involves the same derivatives up to the order R and a remainder $\tilde{r}_k$

$${}^{\sigma}\tilde{\psi}_k(v,0,z,\theta)=\tilde{\lambda}(v,z,\theta)+\tilde{r}_k(v,z,\theta).$$

With the change of variables (C), ${}^{\sigma}m^{j,k}_{\varepsilon}$ may be written on $[0,T]\times\mathcal{N}\times\{0\}$ as

$$\begin{aligned}{}^{\sigma}m^{j,k}_{\varepsilon}=\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_{\varepsilon}\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t)({}^{\sigma}\tilde{d}^{k-1}_{-m_B+j}e^{i(\tilde{\lambda}+\tilde{r}_{k-1})/\varepsilon}\\ +{}^{\sigma}\tilde{d}^{k}_{-m_B+j}e^{i(\tilde{\lambda}+\tilde{r}_k)/\varepsilon})|\det\vartheta|dzd\theta,\end{aligned}$$

where $\tilde{I}_k$ denotes $[\tilde{T}_k-\tilde{\beta},\tilde{T}_k+\tilde{\beta}]$. We split the previous integral into two integrals which can be estimated using the operators O^α

$$\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_\varepsilon\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t)({}^\sigma\tilde{d}^{k-1}_{-m_B+j}+{}^\sigma\tilde{d}^k_{-m_B+j})e^{i(\tilde{\lambda}+\tilde{r}_{k-1})/\varepsilon}|\det\vartheta|dzd\theta := ①,$$

$$\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_\varepsilon\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t){}^\sigma\tilde{d}^k_{-m_B+j}e^{i\tilde{\lambda}/\varepsilon}(e^{i\tilde{r}_k/\varepsilon}-e^{i\tilde{r}_{k-1}/\varepsilon})|\det\vartheta|dzd\theta := ②.$$

Estimate of ①: The phase $\tilde{\lambda}+\tilde{r}_{k-1}$ is smooth on an open set containing $E_{r_0}=\{(v,z,\theta)\in\mathbb{R}^n\times\operatorname{supp}\tilde{\pi},|v-z|\le r_0\}$ for some $r_0\in]0,1]$. Furthermore, $\tilde{\lambda}+\tilde{r}_{k-1}$ satisfies the required properties (Q1). We fix $r[\tilde{\lambda}+\tilde{r}_{k-1}]\in]0,r_0]$.

Since ${}^\sigma d^{k-1}_{-m_B+j}+{}^\sigma d^k_{-m_B+j}$ is zero at $v=z$ up to the order $R-2j-2$ by construction, one has

$$\left({}^\sigma\tilde{d}^{k-1}_{-m_B+j}+{}^\sigma\tilde{d}^k_{-m_B+j}\right)(v,z,\theta)=\sum_{|\alpha|=R-2j-1}(v-z)^\alpha\tilde{s}^k_\alpha(v,z,\theta),$$

where $\tilde{s}^k_\alpha$ are smooth remainders. Let

$$a^{\alpha,k}(v,z,\theta)=\tilde{\pi}(z,\theta)\mathbf{1}_{\tilde{I}_k}(t)\tilde{s}^k_\alpha(v,z,\theta)|\det\vartheta(z,\theta)|.$$

The $a^{\alpha,k}$ are smooth and $a^{\alpha,k}(t,\hat{v},T_k,\hat{z}_k,\theta)=0$ if $|t-T_k|\ge\delta$ or $|\hat{v}-\hat{z}_k|\ge\kappa\delta$ or $(z,\theta)\notin\operatorname{supp}(\tilde{\pi})$. Then the $a^{\alpha,k}$ satisfy the properties (Q2), assuming δ is small enough to ensure $|(\delta,\kappa\delta)|\le r[\tilde{\lambda}+\tilde{r}_{k-1}]$.

Therefore

$$①=\varepsilon^{-\frac{3n}{4}+1-m_B+j}\sum_{|\alpha|=R-2j-1}O^\alpha\left(a^{\alpha,k},(\tilde{\lambda}+\tilde{r}_{k-1})/\varepsilon\right)\mathbf{1}_{\operatorname{supp}\tilde{\pi}}\tilde{h}_\varepsilon.$$

One deduces

$$\|①\|_{L^2([0,T]\times\mathcal{N})}\lesssim\varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}. \tag{3.9}$$

Estimate of ②: This is the term for which Lemma 3.1 is fully used. We write $\tilde{\lambda}$ as $\tilde{\lambda}=\beta+2\gamma$ where

$$\gamma=\frac{1}{4}(v-z)\tilde{M}_k(z,\theta)(v-z)\text{ and }\beta=\tilde{\lambda}-\frac{1}{2}(v-z)\tilde{M}_k(z,\theta)(v-z).$$

The part $\beta+\gamma$ will play the role of the phase for the operators O^α, while $e^{i\gamma/\varepsilon}$ will be enclosed in the amplitude to give it a good behavior. The phase $\beta+\gamma$ is smooth on an open set containing E_{r_0} and satisfies the properties (Q1). We associate to this phase some constant $r[\beta+\gamma]$ and impose on δ to satisfy $|(\delta,\kappa\delta)|\le r[\beta+\gamma]$.

Let

$$c^{j,k}_\varepsilon=\varepsilon^{-\frac{R-1}{2}}\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t){}^\sigma\tilde{d}^k_{-m_B+j}e^{i\gamma/\varepsilon}(e^{i\tilde{r}_k/\varepsilon}-e^{i\tilde{r}_{k-1}/\varepsilon})|\det\vartheta|.$$

One has

$$|c^{j,k}_\varepsilon|\lesssim\varepsilon^{-\frac{R-1}{2}}e^{-\operatorname{const}(v-z)^2/\varepsilon}|e^{i\tilde{r}_k/\varepsilon}-e^{i\tilde{r}_{k-1}/\varepsilon}|.$$

If δ is small enough,

$$e^{-\operatorname{const}(v-z)^2/\varepsilon}|e^{i\tilde{r}_k/\varepsilon}-e^{i\tilde{r}_{k-1}/\varepsilon}|\lesssim\varepsilon^{-1}|v-z|^{R+1}e^{-\operatorname{const}(v-z)^2/(2\varepsilon)},$$

so that

$$|c_\varepsilon^{j,k}| \lesssim 1.$$

Hence $c_\varepsilon^{j,k}$ is smooth and satisfies the properties (Q2):

$$c_\varepsilon^{j,k}(v,z,\theta)=0 \text{ if } |v-z|\geq r[\beta+\gamma] \text{ or } (z,\theta)\notin \operatorname{supp}(\tilde\pi),$$
$$c_\varepsilon^{j,k} \text{ is uniformly bounded in } L^\infty(\mathbb{R}^{3n}).$$

To make use of the estimates of Lemma 3.1, we aim to show that for $N\in\mathbb{N}$, $\varepsilon^{\frac{N}{2}}\partial_{v_b}^N c_\varepsilon^{j,k}$ $(b=1,\dots,n)$ is uniformly bounded in $L^\infty(\mathbb{R}^{3n})$. For this purpose, we write $\partial_{v_b}^N[e^{i\gamma/\varepsilon}(e^{i\tilde r_k/\varepsilon}-e^{i\tilde r_{k-1}/\varepsilon})]$ as a sum of terms of the form

$$\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]\partial_{v_b}^{N_2}[e^{i\tilde r_k/\varepsilon}-e^{i\tilde r_{k-1}/\varepsilon}], 0\leq N_1,N_2\leq N, N_1+N_2=N.$$

As the remainders $\tilde r_k$ and $\tilde r_{k-1}$ are of order $R+1$, Lemma 3.2 yields for $N_1,N_2\in\mathbb{N}$, $(z,\theta)\in\operatorname{supp}\tilde\pi$, $|v-z|\leq|(\delta,\kappa\delta)|$ and δ sufficiently small

$$|\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]|\lesssim \varepsilon^{-\frac{N_1}{2}}e^{-\mathrm{const}(v-z)^2/\varepsilon},$$
$$|\partial_{v_b}^{N_2}[e^{i\tilde r_k/\varepsilon}-e^{i\tilde r_{k-1}/\varepsilon}]|\lesssim\Big(\sum_{\frac{N_2}{R+1}\leq k\leq N_2}\varepsilon^{-k}|v-z|^{k(R+1)-N_2}+\sum_{1\leq k<\frac{N_2}{R+1}}\varepsilon^{-\frac{N_2}{R+1}}\Big)$$
$$\Big(|e^{i\tilde r_k/\varepsilon}|+|e^{i\tilde r_{k-1}/\varepsilon}|\Big).$$

The second sum in the last inequality is zero when $N_2/(R+1)\leq 1$. Remember that $R\geq 2$. If $N_2/(R+1)>1$ then $N_2(R-1)/(2(R+1))>(R-1)/2$ and consequently $-N_2/(R+1)>-N_2/2+(R-1)/2$. Thus

$$|\partial_{v_b}^{N_2}[e^{i\tilde r_k/\varepsilon}-e^{i\tilde r_{k-1}/\varepsilon}]|\lesssim\Big(\sum_{\frac{N_2}{R+1}\leq k\leq N_2}\varepsilon^{-k}|v-z|^{k(R+1)-N_2}+\varepsilon^{-\frac{N_2}{2}+\frac{R-1}{2}}\Big)$$
$$\Big(|e^{i\tilde r_k/\varepsilon}|+|e^{i\tilde r_{k-1}/\varepsilon}|\Big).$$

Hence, for $(z,\theta)\in\operatorname{supp}\tilde\pi$ and $|v-z|\leq|(\delta,\kappa\delta)|$

$$|\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]\partial_{v_b}^{N_2}[e^{i\tilde r_k/\varepsilon}-e^{i\tilde r_{k-1}/\varepsilon}]|\lesssim\varepsilon^{-\frac{N_1}{2}-\frac{N_2}{2}+\frac{R-1}{2}}.$$

It follows that

$$|\partial_{v_b}^N c_\varepsilon^{j,k}|\lesssim\varepsilon^{-\frac{N}{2}}.$$

One can use the operator O^0 to write

$$② = \varepsilon^{-\frac{3n}{4}+1-m_B+j}\varepsilon^{\frac{R-1}{2}}O^0\left(c_\varepsilon^{j,k},(\beta+\gamma)/\varepsilon\right)\mathbf{1}_{\operatorname{supp}\tilde\pi}\tilde h_\varepsilon,$$

and thus

$$\|②\|_{L^2([0,T]\times\mathcal{N})}\lesssim\varepsilon^{\frac{R+1}{2}-m_B+j}\|h_\varepsilon\|_{L^2}. \tag{3.10}$$

Using (3.9) and (3.10) yields

$$\|m_\varepsilon^{j,k,l}\|_{L^2([0,T]\times\partial\Omega)}\lesssim\varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}.$$

One has a similar bound for $q_\varepsilon^{j,k}$ by summing over $l=1,\dots,L$,

$$\|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}.$$

Plugging this into (3.7) gives

$$\|b_\varepsilon^j\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}.$$

All in all, we have shown that

$$\|Bu_\varepsilon^R\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}.$$

This result can be adapted to the integer Sobolev spaces as follows:

$$\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B-s}, s\in\mathbb{N}.$$

An interpolation argument ([27, p.49]) enables the same estimate for non integer Sobolev spaces $H^s([0,T]\times\partial\Omega)$, $s>0$. This proves (3.5).

3.2.3. The initial conditions. In this section we estimate the difference between $(u_\varepsilon^R|_{t=0},\partial_t u_\varepsilon^R|_{t=0})$ and $(u_\varepsilon^I,v_\varepsilon^I)$ in $H^1(\Omega)\times L^2(\Omega)$.

By construction,

$$\begin{aligned} u_\varepsilon^R(0,x) = \frac{1}{2}\varepsilon^{-\frac{3n}{4}}c_n\int_{\mathbb{R}^{2n}} &\rho(y)\phi(\eta)\varepsilon T_\varepsilon v_\varepsilon^I(y,\eta)\Big[\sum_{k=0}^{N_+} w_\varepsilon^{k'}(0,x,y,\eta) - \sum_{k=0}^{N_-} w_\varepsilon^{-k'}(0,x,y,\eta)\Big] \\ &+\rho(y)\phi(\eta)T_\varepsilon u_\varepsilon^I(y,\eta)\Big[\sum_{k=0}^{N_+} w_\varepsilon^{k}(0,x,y,\eta) + \sum_{k=0}^{N_-} w_\varepsilon^{-k}(0,x,y,\eta)\Big]dyd\eta. \end{aligned}$$

As $\operatorname{dist}(x^0_{\pm k}(y,\eta),\bar\Omega)>0$ for $(y,\eta)\in K_y\times K_\eta$, $k=1,\dots,N_\pm$, $w_\varepsilon^{\pm k^{(\prime)}}(0,x,y,\eta)$ are uniformly exponentially decreasing for $x\in\Omega$ and $(y,\eta)\in K_y\times K_\eta$. Thus, only the incident beams contribute to $u_\varepsilon^R(0,x)$ in Ω and

$$u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}}c_n\int_{\mathbb{R}^{2n}}\rho(y)\phi(\eta)T_\varepsilon u_\varepsilon^I(y,\eta)w_\varepsilon^0(0,x,y,\eta)dyd\eta + O(\varepsilon^\infty)$$

uniformly w.r.t. $x\in\Omega$. The initial values for the phase and the amplitudes of w_ε^0 have been fixed in (2.20) and (2.21). Hence

$$u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}}c_n\int_{\mathbb{R}^{2n}}\rho(y)\phi(\eta)T_\varepsilon u_\varepsilon^I(y,\eta)\chi_d(x-y)e^{i\phi_0(x,y,\eta)/\varepsilon}dyd\eta + O(\varepsilon^\infty),$$

uniformly w.r.t. $x\in\Omega$. It follows, uniformly for $x\in\Omega$, that

$$\begin{aligned} u_\varepsilon^R(0,x) &= T_\varepsilon^*\rho\otimes\phi T_\varepsilon u_\varepsilon^I(x) \\ &\quad + \varepsilon^{-\frac{3n}{4}}c_n\int_{\mathbb{R}^{2n}}\rho(y)\phi(\eta)T_\varepsilon u_\varepsilon^I(y,\eta)(\chi_d(x-y)-1)e^{i\phi_0(x,y,\eta)/\varepsilon}dyd\eta + O(\varepsilon^\infty). \end{aligned}$$

One wants to get rid of the second integral by making use of the exponential decrease of $e^{i\phi_0(x,y,\eta)/\varepsilon}$ for $|x-y|\ge d/2$. The following estimate is immediate by the Cauchy-Schwartz inequality:

LEMMA 3.5. *Let a be a positive real number and $h \in L^2(\mathbb{R}^{2n}_{y,\eta})$. Then*

$$\Big\| \int_{|x-y|\geq a} h \mathbf{1}_{K_y\times K_\eta}(y,\eta) e^{-(x-y)^2/(2\varepsilon)} dy d\eta \Big\|_{L^2_x} \lesssim \|h\|_{L^2_{y,\eta}} e^{-a^2/(4\varepsilon)}.$$

The previous Lemma leads to

$$\|u^R_\varepsilon|_{t=0} - T^*_\varepsilon \rho\otimes\phi T_\varepsilon u^I_\varepsilon\|_{L^2(\Omega)} \lesssim \varepsilon^\infty,$$

by using the boundedness of T^*_ε from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ (this result follows, e.g., from [31, p.97]). On the other hand, $\rho\otimes\phi T_\varepsilon u^I_\varepsilon$ approaches $T_\varepsilon u^I_\varepsilon$ up to a small remainder. In fact, as $\rho(y) = 1$ if $\mathrm{dist}(y,K) < \Delta$, one has by Lemma 2.4 and Assumption A3

$$\|T_\varepsilon u^I_\varepsilon - \rho\otimes\phi T_\varepsilon u^I_\varepsilon\|_{L^2_{y,\eta}} \lesssim \varepsilon^\infty,$$

and consequently

$$\|u^R_\varepsilon|_{t=0} - u^I_\varepsilon\|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Moving to the spatial derivatives of u^R_ε, one has

$$\partial_{x_b} u^R_\varepsilon(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u^I_\varepsilon(y,\eta) \sum_{j=0}^{N} \varepsilon^j \partial_{x_b}\left[a^0_j(0,x,y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon}\right] dy d\eta$$
$$+ O(\varepsilon^\infty), \text{ uniformly w.r.t. } x\in\Omega.$$

Plugging the initial condition (2.21) for the incident amplitudes into the previous equation yields a simpler expression

$$\partial_{x_b} u^R_\varepsilon(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u^I_\varepsilon(y,\eta) \partial_{x_b}\big(\chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon}\big) dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x\in\Omega.$$

Since $\partial_{x_b}\big(\chi(x-y)e^{i\phi_0/\varepsilon}\big) = -\partial_{y_b}\big(\chi(x-y)e^{i\phi_0/\varepsilon}\big)$, integration by parts leads to

$$\partial_{x_b} u^R_\varepsilon(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \partial_{y_b}\big(\rho T_\varepsilon u^I_\varepsilon\big) \phi \chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x\in\Omega.$$

Application of Lemma 3.5 and then Lemma 2.4 shows that the term involving $\partial_{y_b}\rho$ has an exponentially decreasing contribution in $L^2(\Omega)$. On the other hand, the y derivative of the FBI transform is the FBI transform of the derivative. Thus

$$\|\partial_{x_b} u^R_\varepsilon|_{t=0} - \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho\otimes\phi T_\varepsilon\left(\partial_{x_b} u^I_\varepsilon\right) \chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta\|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Again, Lemmas 3.5–2.4 and Assumption A3 imply

$$\|\partial_{x_b} u^R_\varepsilon|_{t=0} - \partial_{x_b} u^I_\varepsilon\|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Time differentiation of u^R_ε is somewhat different. The contribution of reflected beams is still uniformly exponentially decreasing for $x\in\Omega$

$$\partial_t u^R_\varepsilon|_{t=0}(x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon v^I_\varepsilon(y,\eta) \varepsilon \partial_t w^{0\prime}_\varepsilon(0,x,y,\eta) dy d\eta + O(\varepsilon^\infty),$$

with

$$\varepsilon\partial_t w_\varepsilon^{0\prime} = \sum_{j=0}^{N+1} \varepsilon^j \left(i\partial_t \psi_0 a_j^{0\prime} + \partial_t a_{j-1}^{0}{}' \right) e^{i\psi_0/\varepsilon}.$$

The initial values (2.20) and (2.22) for the phase and amplitudes of $w_\varepsilon^{0\prime}$ yield

$$\varepsilon\partial_t w_\varepsilon^{0\prime}(0,x,y,\eta) = e^{i\phi_0(x,y,\eta)/\varepsilon} + \sum_{j=0}^{N} \varepsilon^j \sum_{|\alpha|=R-2j-1} (x-y)^\alpha z_\alpha(x,y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon}$$
$$+\varepsilon^{N+1}\partial_t a_N^{0}{}'(0,x,y,\eta)e^{i\phi_0(x,y,\eta)/\varepsilon},$$

where z_α are smooth remainders that vanish for $|x-y| \geq d$. We can use the operators O^α to estimate the contribution of the terms $(x-y)^\alpha z_\alpha$ to the norm of $u_\varepsilon^R|_{t=0}$

$$\|\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} \rho\otimes\phi T_\varepsilon v_\varepsilon^I \varepsilon^j (x-y)^\alpha z_\alpha e^{i\phi_0/\varepsilon} dy d\eta\|_{L_x^2} = \varepsilon^{-\frac{3n}{4}+j} \|O^\alpha(\rho\otimes\phi z_\alpha, \phi_0/\varepsilon) T_\varepsilon v_\varepsilon^I\|_{L_x^2}$$
$$\lesssim \varepsilon^{\frac{R-1}{2}}, \text{ for } j=0,\dots,N.$$

We also have

$$\|\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} \rho\otimes\phi T_\varepsilon v_\varepsilon^I \varepsilon^{N+1} \partial_t a_N^{0}{}'|_{t=0} e^{i\phi_0/\varepsilon} dy d\eta\|_{L_x^2}$$
$$= \varepsilon^{-\frac{3n}{4}+N+1} \|O^0 \left(\rho\otimes\phi \partial_t a_N^{0}{}'|_{t=0}, \phi_0/\varepsilon \right) T_\varepsilon v_\varepsilon^I\|_{L_x^2}$$
$$\lesssim \varepsilon^{N+1}.$$

It follows, with the help of (2.10), that

$$\|\partial_t u_\varepsilon^R|_{t=0} - T_\varepsilon^* \rho\otimes\phi T_\varepsilon v_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}},$$

and finally, from Lemma 2.4 and Assumption A3,

$$\|\partial_t u_\varepsilon^R|_{t=0} - v_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}.$$

Hence

$$\|\partial_t u_\varepsilon^R|_{t=0} - v_\varepsilon^I\|_{L^2(\Omega)} + \|u_\varepsilon^R|_{t=0} - u_\varepsilon^I\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}.$$

3.3. Proof of the main theorem. Now we may collect the previous estimates in order to bound the difference between u_ε the exact solution for (1.1) and u_ε^R the approximate solution of order R.

For the Dirichlet case, the errors in the interior, at the boundary, and in the initial conditions exhibit the same scale of ε, and the energy estimate leads to

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon(t,.) - u_\varepsilon^R(t,.)\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}, \ \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon(t,.) - \partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}. \tag{3.11}$$

For the Neumann case, one loses an order $\sqrt{\varepsilon}$ in the boundary estimate, and thus the energy estimate yields

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon(t,.) - u_\varepsilon^R(t,.)\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{R-2}{2}}, \ \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon(t,.) - \partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-2}{2}}.$$

However, when comparing the ansatz at order R and $R+1$ in the difference between u_ε^{R+1} and u_ε^R, we can make use of further powers of $((x-x_k^t)^\alpha)_{|\alpha|=R+1}$ between the phases and $((x-x_k^t)^\alpha)_{|\alpha|=R-2j-1}$ in the amplitudes. Using the approximation operators yields uniformly in time

$$\|u_\varepsilon^{R+1}(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}},\|\partial_t u_\varepsilon^{R+1}(t,.)-\partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

Hence one may improve the estimate for the Neumann case by using the approximate solution at the next order $R+1$

$$\begin{aligned}\|u_\varepsilon(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}&\lesssim\|u_\varepsilon(t,.)-u_\varepsilon^{R+1}(t,.)\|_{H^1(\Omega)}+\|u_\varepsilon^{R+1}(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\\&\lesssim\varepsilon^{\frac{R-1}{2}}.\end{aligned}$$

This leads to the same estimate (3.11) for the Neumann case.

Remark 3.6. The FBI transforms of u_ε^I and v_ε^I are uniformly locally infinitely small outside the frequency sets $Fs(u_\varepsilon^I)$ and $Fs(v_\varepsilon^I)$ respectively, as ε tends to 0 (see [31, p.98]). These sets may be phase space submanifolds of lower dimensions. For instance, for WKB initial data, $Fs(ae^{i\Phi/\varepsilon})=\{(y,\partial_x\Phi(y)),y\in\operatorname{supp}a\}$. For numerical computations, one therefore has to discretize neighborhoods of $(K_y\times K_\eta)\cap Fs(u_\varepsilon^I)$ and $(K_y\times K_\eta)\cap Fs(v_\varepsilon^I)$. Numerically studying the behaviour of FBI transforms in the associated computational domains could lead to interesting results on the optimal mesh size. Details on numerical FBI transforms are given in [26].

4. Conclusion

We have shown that Gaussian beams summation can be used to construct asymptotic solutions for the wave equation in a convex domain. Rigorous estimate of the difference between the obtained approximate solutions and the exact solution has been given. We have proven that the precision of a Gaussian beams superposition depends only on the accuracy to which the used individual beams satisfy the wave equation; no extra order of error is induced by the summation process. A large class of initial data, including the WKB form, is allowed. The boundary condition can be either of Dirichlet or Neumann type. The obtained solutions are global and thus well suited to numerical computations, which will be performed in a coming work [1].

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