COMM. MATH. SCI.
Vol. 1, No. 3, pp. 393–407
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ENTROPY SOLUTIONS OF A COMBUSTION MODEL*

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Abstract. We study weak solutions to a combustion model problem. An equivalent conservation law with discontinuous flux is derived. Definition of an entropy solution is given, and the existence and uniqueness of the entropy solutions is proved. The convergence of a projection method and an implicit finite difference scheme is also proved. Finally using this approach we prove the convergence of a random projection method.

Key words. combustion, finite difference method, detonation wave, stiff equation, conservation law

AMS subject classifications. 65M06 35L65 76M20 80A25

1. Introduction

The Euler system of equations with chemical reaction processes is usually stiff because the scale of reaction zones may be orders of magnitude smaller than the fluid dynamical space scales. There have been a number of works dealing with the numerical simulation of the problem. See [1][4][5][6][10][12].

There are also some mathematical studies on this subject. Especially the study for Majda's simplified combustion model[11],which is a 2×2 system, where a "lumped variable" is introduced to represent density, velocity and temperature. It was proved in [2] that if the approximate solution to a projection method tended to piecewise constants weak detonation waves, then the ignition temperature had to be less than a number of $u_{l*}-q$, where u_{l*} is the temperature behind a weak detonation wave which has the same speed as that of the desired strong detonation wave, and q is the binding energy. In [15] the convergence was studied rigorously. For the projection method and an upwind finite difference scheme some sufficient conditions were given to guarantee that the limit is a strong detonation wave solution or a weak detonation wave solution. We will continue our study in this paper.

We will make an assumption, which is "no cooling down", then deduce an equivalent conservation law with discontinuous flux. In [7] the authors studied a kind of conservation law, where the flux is discontinuous with respect to x. This is a different kind, where the flux is discontinuous with respect to the dependent variable. Using the concept of set-valued mapping we define weak solutions and entropy weak solutions for this conservation law. Then under one assumption, that for sufficient large $|x|$ the initial values are constants, different from the ignition temperature, existence and uniqueness of the entropy solutions are proved.

We will also introduce an implicit finite difference scheme for the conservation law and prove the convergence of the scheme. Furthermore it is shown that the limit is the entropy solution.

Without entropy condition we cannot expect uniqueness of the original Majda's model. Applying the above results we get the corresponding existence and uniqueness results for Majda's model. Especially for the Riemann problem we will show that different ignition temperatures yield two different kinds of profile of the unique solutions,

*Received: January 19, 2003; accepted (in revised version): March 12, 2003. Communicated by Shi Jin.
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strong detonation solutions and weak detonation solutions. This result explains some phenomena in the numerical simulation for combustion problems.

A random projection method was developed in the work by Bao and Jin [1]. The ignition temperature was assumed to be a random number with uniform distribution in the projection scheme. Strong detonation waves were obtained in one and two dimensional flows with chemical reaction. Using our approach we can prove the convergence of this method. We will study this method applied to Majda's model and prove that the limit is a strong detonation wave solution.

2. Majda's model

We consider the following Majda's model for combustion: [11]

$$\frac{\partial(u+qz)}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \tag{2.1}$$

$$\frac{\partial z}{\partial t}=-K\phi(u)z, \tag{2.2}$$

where u is a "lumped variable", representing density, velocity and temperature, $z \in [0,1]$ is a variable, representing the fraction of unburnt gas, q and K are positive constants, representing the binding energy and the rate of chemical reaction respectively, $f' > 0$, $f'' \geq a_0 > 0$,

$$\phi(u)=\begin{cases} 1, & u > U_i, \\ 0, & u < U_i, \end{cases}$$

and U_i is a constant, representing the ignition temperature.

We assume the initial condition,

$$u|_{t=0}=u_0(x), \qquad z|_{t=0}=z_0(x), \tag{2.3}$$

where naturally we require that $z_0(x)=0$ for a certain x if $u_0(x) > U_i$.

In the Chapman-Jouguet's model the rate of chemical reaction is assumed to be infinity. For this case the second equation in Majda's model is reduced to: $\frac{\partial z}{\partial t} \leq 0$, $\phi(u)z=0$, and if $u < U_i$ then $\frac{\partial z}{\partial t}=0$, therefore (2.2) is replaced by

$$z(x,t)=\begin{cases} 0, & \sup_{0\leq\tau\leq t} u(x,\tau) > U_i, \\ z_0(x), & \sup_{0\leq\tau\leq t} u(x,\tau) < U_i, \end{cases} \tag{2.4}$$

and

$$\frac{\partial z}{\partial t} \leq 0. \tag{2.5}$$

Global existence theorem for the problem (2.1) (2.4) (2.5) (2.3) was proved in [16].

The solutions to the above problem are not unique. For example the Riemann initial data are given as

$$u|_{t=0}=\begin{cases} u_l, & x<0, \\ u_r, & x>0, \end{cases} \qquad z|_{t=0}=\begin{cases} 0, & x<0, \\ 1, & x>0, \end{cases} \tag{2.6}$$

where $u_r < U_i \leq u_l$ and $u_l > u_r + q$. We assume that $u_l > u_{CJ}$, where u_{CJ} satisfies $f'(u_{CJ})=\frac{f(u_{CJ})-f(u_r)}{u_{CJ}-(u_r+q)}$, then there are two kinds of solutions satisfying the equations and the initial condition (2.6) (see [15]). The first one is

$$u(x,t)=\begin{cases} u_l, & x<st,\\ u_r, & x>st,\end{cases}\qquad z(x,t)=\begin{cases} 0, & x<st,\\ 1, & x>st,\end{cases} \tag{2.7}$$

where the constant s is the speed of the wave. By (2.1) s should satisfy the Rankine-Hugoniot condition,

$$s=\frac{f(u_l)-f(u_r)}{u_l-(u_r+q)}.$$

The second one is

$$u(x,t)=\begin{cases} u_l, & x<s_1t,\\ u_m, & s_1t<x<s_2t\\ u_r, & x>s_2t,\end{cases}\qquad z(x,t)=\begin{cases} 0, & x<s_2t,\\ 1, & x>s_2t,\end{cases} \tag{2.8}$$

where $u_l>u_m>u_r+q$, $u_m>U_i$, and

$$s_1=\frac{f(u_l)-f(u_m)}{u_l-u_m},\quad s_2=\frac{f(u_m)-f(u_r)}{u_m-(u_r+q)}.$$

The value of u_m is also not unique. The former one is called a strong detonation wave and the latter one is a solution with two waves, where a weak detonation wave is followed by a shock wave.

We make one assumption in the following. Let us assume that no cooling down happens in the process, that is, once $z=0$ and $u\ge U_i+q$ at time t_0 then they keep true for all $t\ge t_0$. Generally speaking it is not always the case, since it may happen that $u<U_i+q$ after combustion. However it is valid in the previous examples (2.7) and (2.8), and it should be always true in a neighborhood of detonation waves for general cases, hence it can be regarded as a condition of the local. This assumption will help us to study the behavior of solutions near detonation waves and to study the entropy condition.

Let $v=u+qz$, then we get the equation

$$\frac{\partial v}{\partial t}+\frac{\partial f_1(v)}{\partial x}=0, \tag{2.9}$$

where

$$f_1(v)=\begin{cases} f(v-q), & v\le U_i+q,\\ f(v), & v>U_i+q.\end{cases} \tag{2.10}$$

This is an equation with discontinuous flux f_1. The fractional steps projection finite difference scheme in [15] is indeed the finite difference scheme to the equation (2.9). We will study this equation in the next section, then we will study finite difference schemes for it, and then return to study the combustion model. Finally we will prove a result on the random projection method.

3. Single conservation law with discontinuous flux

We will study the equation (2.9) in general in this section. Let f_1 be a piecewise smooth discontinuous function. For simplicity we assume that there is one point of discontinuity, $v=u_0$. There is no significant difference if there are several points of discontinuity.

It is known that the Rankine-Hugoniot condition for (2.9) is

$$s = \frac{f_1(v_l) - f_1(v_r)}{v_l - v_r}, \tag{3.1}$$

where s is the slope of a discontinuity in a solution $v(x,t)$ and $v_l = v(x-0,t)$, $v_r = v(x+0,t)$. The wave speed is not bounded, because s can be arbitrary large if u_l and u_r are close to u_0. Also it known that the geometric entropy condition for admissible discontinuities is[14]

$$\begin{aligned} &f_1(v) - f_1(v_l) - s(v - v_l) \geq 0, v \in (v_l, v_r) \quad \text{for } v_r > v_l, \\ &f_1(v) - f_1(v_l) - s(v - v_l) \leq 0, v \in (v_r, v_l) \quad \text{for } v_r < v_l. \end{aligned} \tag{3.2}$$

To define a weak solution, the meaning of $f_1(u_0)$ should be clarified. We will consider f_1 as a set-valued mapping, that is, $f_1(u_0)$ is an interval $[f_1(u_0-0), f_1(u_0+0)]$ or $[f_1(u_0+0), f_1(u_0-0)]$.

Definition. $v \in L^\infty$ is a weak solution to (2.9), if there is a function $w \in L^\infty$, such that

$$\frac{\partial v}{\partial t} + \frac{\partial w}{\partial x} = 0, \tag{3.3}$$

in the sense of distributions, and $w \in f_1(v)$.

Let the initial condition be

$$v|_{t=0} = \varphi(x). \tag{3.4}$$

Before studying the existence and uniqueness problem of the initial value problem (2.9) (3.4), we will show two examples first. We consider the equation (2.9) with flux

$$f_1(v) = \begin{cases} v, & v < 0, \\ v+1, & v > 0. \end{cases}$$

Example 1.

$$\varphi(x) = \begin{cases} 0, & x < 0, \\ v_r, & x > 0, \end{cases} \qquad v_r > 0.$$

Then the solutions are

$$v = \begin{cases} 0, & x < st, \\ v_r, & x > st, \end{cases} \qquad s \in \frac{f_1(0) - f_1(v_r)}{-v_r}.$$

Example 2. $\varphi(x) = -x$. Then the solution is

$$v = \begin{cases} -x + t, & x < t \text{ or } x > t + \sqrt{2t}, \\ 0, & t < x < t + \sqrt{2t}. \end{cases}$$

From the first example we know that the solutions cannot be determined uniquely by the initial data and from the second example we know that there exist waves with infinite speed. Thus we will restrict ourselves to the following cases: We assume that there exists a positive constant X, such that $\varphi(x) \equiv \varphi(+\infty)$, for $x > X$, $\varphi(x) \equiv \varphi(-\infty)$, for $x < -X$, and $\varphi(+\infty) \neq u_0$, $\varphi(-\infty) \neq u_0$.

THEOREM 3.1. *We assume that $\varphi \in BV$, and $|\varphi(x)| \le M$. Then the problem (2.9) (3.4) admits a weak solution.*

Proof. Assuming $\delta > 0$, we define a smooth function $f_\delta(v)$ such that $f_\delta(v) = f_1(v)$ for $|v-u_0| \ge \delta$, and $\min_{v\in(u_0-\delta,u_0+\delta)} f_1(v)-\delta_1 < f_\delta(v) < \max_{v\in(u_0-\delta,u_0+\delta)} f_1(v)+\delta_1$ for $|v-u_0| < \delta$ with $\delta_1 \to 0$ as $\delta \to 0$. Then the equation

$$\frac{\partial v}{\partial t} + \frac{\partial f_\delta(v)}{\partial x} = 0, \tag{3.5}$$

with (3.4) admits a unique entropy solution v_δ[13][8], which satisfies

$$|v_\delta| \le M, \qquad \mathrm{var}_x v_\delta(\cdot,t) \le \mathrm{var}\varphi,$$

and

$$\|v_\delta(\cdot,t) - v_\delta(\cdot,\tau)\|_{L^1} \le C|t-\tau|.$$

Here and hereafter we will denote by C a generic positive constant. We extract a sequence of δ such that $\delta \to 0$, $v_\delta \to v$ a.e., and $f_\delta(v_\delta)$ converges to w, weak * in L^∞. Then v and w satisfy (3.3). If $v(x,t) \ne u_0$ at a point (x,t), then $w(x,t) = f_1(v(x,t))$ because of the continuity of f_1. If $v(x,t) = u_0$, then either $\min_{v\in(u_0-\delta,u_0+\delta)} f_1(v) - \delta_1 < f_\delta(v_\delta(x,t)) < \max_{v\in(u_0-\delta,u_0+\delta)} f_1(v) + \delta_1$ or $f_\delta(v_\delta(x,t)) = f_1(v_\delta(x,t))$. Passing to the limit we get $w(x,t) \in f_1(u_0)$. Consequently v is the weak solution to the equation (2.9). □

We start to study uniqueness. To begin with, we define entropy and entropy flux. Let $U \in C^2$ with $U'' \ge 0$ be a convex entropy. The entropy flux F is defined by

$$F(v) = \int_{u_1}^{v} U'(v) f_1'(v)\, dv + a, \tag{3.6}$$

where $u_1 < u_0$, and a is an arbitrary constant. F is discontinuous at $v = u_0$, hence we divide U into two parts: $U = U_1 + U_2$, where $U_1(v) = U'(u_0)(v-u_0)$. The corresponding entropy flux is

$$F_1(v) = U'(u_0)(f_1(v) - f_1(u_1)).$$

Then $F_2 = F - F_1 - a$ is continuous:

$$\begin{aligned} F_2(v) &= \int_{u_1}^{v} (U'(v) - U'(u_0)) f_1'(v)\, dv \\ &= (U'(v) - U'(u_0)) f_1(v) - (U'(u_1) - U'(u_0)) f_1(u_1) - \int_{u_1}^{v} U''(v) f_1(v)\, dv. \end{aligned}$$

Definition. A weak solution v to (2.9) is an entropy solution if it satisfies

$$\frac{\partial U(v)}{\partial t} + \frac{\partial W}{\partial x} \le 0, \tag{3.7}$$

in the sense of distributions, where $W = U'(u_0)(w - f_1(u_1)) + F_2(v) + a$.

THEOREM 3.2. *The weak solution in Theorem 3.1 is an entropy solution.*

Proof. We set

$$F_\delta(v) = \int_{u_1}^{v} U'(v) f_\delta'(v)\, dv + a,$$

where f_δ is given in the proof of Theorem 3.1. v_δ is an entropy solution to the equation (3.5), hence

$$\frac{\partial U(v_\delta)}{\partial t} + \frac{\partial F_\delta(v_\delta)}{\partial x} \leq 0. \tag{3.8}$$

Let $F_{1\delta}$, $F_{2\delta}$ be defined as F_1, F_2, then as $\delta \to 0$, $F_{1\delta}(v_\delta) = U'(u_0)(f_\delta(v_\delta) - f_\delta(u_1))$ converges to $U'(u_0)(w - f_1(u_1))$ weakly *, and

$$\begin{aligned} F_{2\delta}(v) &= \int_{u_1}^{v} (U'(v) - U'(u_0)) f'_\delta(v)\, dv \\ &= (U'(v) - U'(u_0)) f_\delta(v) - (U'(u_1) - U'(u_0)) f_\delta(u_1) - \int_{u_1}^{v} U''(v) f_\delta(v)\, dv \end{aligned}$$

converges to F_2 uniformly. Passing to the limit in (3.8) we get (3.7). ∎

Before we state our result on uniqueness, we will prove the following lemma first.

LEMMA 3.1. *There exists a positive constant X_1, depending on T, such that if v is the solution obtained in Theorem 3.1, then $v(x,t) \equiv \varphi(+\infty)$ for $x > X_1$, $t \leq T$, and $v(x,t) \equiv \varphi(-\infty)$ for $x < -X_1$, $t \leq T$, where T is an arbitrary positive constant.*

Proof. Because $|f'_\delta| < \infty$ in (3.5), $v_\delta(x,t) \equiv \varphi(+\infty)$ for large x. Therefore $\inf\{\xi; v_\delta(x,t) = \varphi(+\infty), \forall x > \xi\}$ is a curve satisfying the Lipschitz condition, and the Rankine-Hugoniot condition holds on the curve almost everywhere. The curve is denoted by $\xi = l(t)$. Let $v_+ = v(\xi + 0, t)$ and $v_- = v(\xi - 0, t)$. Then $v_+ = \varphi(+\infty)$. Since $\varphi(+\infty) \neq u_0$, there is a neighborhood of $\varphi(+\infty)$, $(\varphi(+\infty) - a, \varphi(+\infty) + a)$, such that u_0 does not belong to this neighborhood. If $v_- \in (\varphi(+\infty) - a, \varphi(+\infty) + a)$, then by the Rankine-Hugoniot condition $|l'| \leq \max|f'| \leq C$. If $v_- \notin (\varphi(+\infty) - a, \varphi(+\infty) + a)$, then $|v_- - v_+| \geq a$, we also get $|l'| \leq C$ by the Rankine-Hugoniot condition. Since $l(0) \leq X$, $|l(t)| \leq C$. There exists $X_1 > 0$, independent of δ, such that $v_\delta \equiv \varphi(+\infty)$ for $x > X_1$. Passing to the limit, we get that v possesses the same property. ∎

THEOREM 3.3. *If v_1 and v_2 are two entropy solutions to the initial value problem (2.9) (3.4), $v_1(x,t) = v_2(x,t) = \varphi(+\infty)$ for $x > X_1$, $t \leq T$, and $v_1(x,t) = v_2(x,t) = \varphi(-\infty)$ for $x < -X_1$, $t \leq T$, then $v_1 = v_2$, a.e. for $t \leq T$.*

Proof. Let $U(v) = |v - c|$ and $c \neq u_0$. We derive an entropy inequality with respect to this entropy. Let U_n be a C^2 convex entropy with $U_n(v) = U(v)$ for $|v - c| > 1/n$, then we replace U by U_n in (3.6) and get F_n. We also have F_{n1} and F_{n2}. It is easy to see the convergence of U_n and F_{n1} as $n \to \infty$. To see the convergence of F_{n2} we notice that $(U'_n(v) - U'_n(u_0)) f'_1(v)$ is continuous provided $\frac{1}{n} < |u_0 - c|$, and converges as $n \to \infty$. Let us drive the limit of $W_n = U'_n(u_0)(w - f_1(u_1)) + F_{n2}(v) + a$. We have the following expressions of the limit:

$$\lim_{n\to\infty} U'_n(u_0)(w - f_1(u_1)) = \begin{cases} w - f_1(u_1), & u_0 > c \\ f_1(u_1) - w, & u_0 < c \end{cases}.$$

We may assume that $u_1 < c$, then the limit of $F_{n2}(v)$ is

$$F_2(v) = \begin{cases} -2(f_1(v) - f_1(u_1)), & u_0 > c, v < c, \\ 0, & u_0 < c, v < c, \\ -2(f_1(c) - f_1(u_1)), & u_0 > c, v \geq c, \\ 2(f_1(v) - f_1(c)), & u_0 < c, v \geq c. \end{cases}$$

The limit of W_n is

$$W = \begin{cases} (w - f_1(c)) + (f_1(u_1) - f_1(c)) + a, & v > c \\ (f_1(c) - w) + (f_1(u_1) - f_1(c)) + a, & v \le c \end{cases} .$$

We take $a = f_1(c) - f_1(u_1)$ then

$$W = \begin{cases} w - f_1(c), & v > c, \\ f_1(c) - w, & v \le c \end{cases} , \tag{3.9}$$

and (3.7) holds.

If $c = u_0$, that is $U(v) = |v - u_0|$. Letting $c \to u_0 + 0$ in (3.9), we get the limit

$$W = \begin{cases} f_1(v) - f_1(u_0 + 0), & v > u_0, \\ f_1(u_0 + 0) - w, & v = u_0, \\ f_1(u_0 + 0) - f_1(v), & v < u_0. \end{cases}$$

Let $c \to u_0 + 0$ in (3.7), then it still holds for this critical case. Being the same, let $c \to u_0 - 0$, then we get

$$W = \begin{cases} f_1(v) - f_1(u_0 - 0), & v > u_0, \\ w - f_1(u_0 - 0), & v = u_0, \\ f_1(u_0 - 0) - f_1(v), & v < u_0, \end{cases}$$

and (3.7). We take $\eta \in [0, 1]$ and derive the following linear interpolation:

$$W = \begin{cases} f_1(v) - \eta f_1(u_0 + 0) - (1 - \eta) f_1(u_0 - 0), & v > u_0, \\ (1 - 2\eta) w + \eta f_1(u_0 + 0) - (1 - \eta) f_1(u_0 - 0), & v = u_0, \\ \eta f_1(u_0 + 0) + (1 - \eta) f_1(u_0 - 0) - f_1(v), & v < u_0. \end{cases} \tag{3.10}$$

(3.7) still holds for this W and $U = |v - u_0|$.

We turn now to study uniqueness. Let $x \in (-\infty, +\infty)$, $t \in (0, T]$, and $\xi \in (-\infty, +\infty)$, $\tau \in (0, T]$. We have

$$\frac{\partial v_1(x,t)}{\partial t} + \frac{\partial w_1(x,t)}{\partial x} = 0,$$

$$\frac{\partial v_2(\xi,\tau)}{\partial \tau} + \frac{\partial w_2(\xi,\tau)}{\partial \xi} = 0,$$

where $w_1 \in f_1(v_1)$, and $w_2 \in f_1(v_2)$. The solution v_1 satisfies the following entropy inequalities

$$\frac{\partial |v_1 - c_1|}{\partial t} + \frac{\partial W_1}{\partial x} \le 0,$$

where W_1 is given by (3.9) for $c_1 = c \ne u_0$, or by (3.10) for $c_1 = c = u_0$. We take $c_1 = v_2(\xi, \tau)$ in (3.9), and let $\eta f_1(u_0 + 0) + (1 - \eta) f_1(u_0 - 0)$ in (3.10) be $w_2(\xi, \tau)$. Being the same we can get the entropy inequalities for v_2,

$$\frac{\partial |v_2 - c_2|}{\partial \tau} + \frac{\partial W_2}{\partial \xi} \le 0,$$

where the constants c_2, $\eta f_1(u_0+0)+(1-\eta)f_1(u_0-0)$ assume the values $v_1(x,t)$ and $w_1(x,t)$. We notice that if $v_1(x,t)\neq u_0$ or $v_2(\xi,\tau)\neq u_0$, then $W_1=W_2$, regarded as the functions of x,t,ξ,τ. it remains to check the case of $v_1(x,t)=v_2(\xi,\tau)=u_0$. Let

$$w_1(x,t)=\eta_1 f_1(u_0+0)+(1-\eta_1)f_1(u_0-0),$$

and

$$w_2(\xi,\tau)=\eta_2 f_1(u_0+0)+(1-\eta_2)f_1(u_0-0),$$

then

$$W_1=(1-2\eta_2)(\eta_1 f_1(u_0+0)+(1-\eta_1)f_1(u_0-0))+\eta_2 f_1(u_0+0)-(1-\eta_2)f_1(u_0-0),$$

$$W_2=(1-2\eta_1)(\eta_2 f_1(u_0+0)+(1-\eta_2)f_1(u_0-0))+\eta_1 f_1(u_0+0)-(1-\eta_1)f_1(u_0-0).$$

They are equal. We conclude that the entropy and entropy flux for v_1 and v_2 are symmetric.

It is routine to obtain that

$$\int_{-X_1}^{X_1}|v_1-v_2|\,dx\leq\int_{-X_1}^{X_1}|v_1(x,0)-v_2(x,0)|\,dx,$$

which implies uniqueness. □

4. An implicit finite difference scheme

The system (2.1) (2.2) is stiff, in the sense K is very large. Therefore the splitting method is a natural approach. It is a kind of fractional step method and the computation processes in two steps in each time interval with length Δt:

First step. Solving

$$\frac{\partial(u+qz)}{\partial t}+\frac{\partial f(u)}{\partial x}=0,$$

$$\frac{\partial z}{\partial t}=0.$$

Second step. Solving

$$\frac{\partial(u+qz)}{\partial t}=0,$$

$$\frac{\partial z}{\partial t}=-K\phi(u)z.$$

For very large K the second step is in fact a projection, where $K=\infty$. The difference scheme for this fractional steps scheme is just the finite difference scheme for (2.9).

To solve the problem (2.9) (3.4) we cannot expect the CFL condition to be satisfied for an explicit scheme, since the wave speed may be infinity. However if $f_1'\geq 0$ in the sense of distributions, the following implicit finite difference scheme can be applied. Let $v_i^j = v(i\Delta x, j\Delta t)$, then the scheme reads

$$\frac{v_i^{j+1} - v_i^j}{\Delta t} + \frac{f_1(v_i^{j+1}) - f_1(v_{i-1}^{j+1})}{\Delta x} = 0. \tag{4.1}$$

The equation can be solved step by step from the left to the right, because it can be rewritten as

$$v_i^{j+1} + rf_1(v_i^{j+1}) = v_i^j + rf_1(v_{i-1}^{j+1}), \tag{4.2}$$

where $r = \Delta t/\Delta x$ is assumed to be a fixed number. The derivatives of the left hand side with respect to v_i^{j+1} is positive, so there is a unique v_i^{j+1}, and even if $v_i^{j+1} = u_0$, $w \in f_1(v_i^{j+1})$ is still fixed.

THEOREM 4.1. *If $\varphi(x) \in [m, M]$, $\varphi \in BV$, and $f_1(\varphi) \in BV$ then $\lim_{i\to\infty} v_i^j = \varphi(+\infty)$, and the following estimates hold:*

$$m \le v_i^j \le M. \tag{4.3}$$

$$\sum_i |v_i^j - v_{i-1}^j| \le \mathrm{var}\varphi. \tag{4.4}$$

$$\sum_i |z_i^j| \le r\mathrm{var} f_1(\varphi), \tag{4.5}$$

with $z_i^j = v_i^j - v_i^{j-1}$.

$$\sum_i |f_1(v_i^{j+1}) - f_1(v_{i-1}^{j+1})| \le \frac{1}{r}\sum_i |z_i^{j+1}|. \tag{4.6}$$

Proof. (4.3) is proved by induction.

To prove $v_i^j \to \varphi(+\infty)$ we consider graph $G = \{(v, w); w \in f_1(v)\}$. There is an one to one correspondence between a parameter $p = v + rw$ and the point (v, w) on G, and (v, w) depends continuously on p. Let $p_i^j = v_i^j + rf_1(v_i^j)$ and $p_\infty = \varphi(+\infty) + rf_1(\varphi(+\infty))$, then because $\varphi(+\infty) \ne u_0$, $v_i^j \to \varphi(+\infty)$ is equivalent to $p_i^j \to p_\infty$. For definiteness we assume that $\varphi(+\infty) - u_0 = a > 0$.

We prove by induction that $|v_i^j - \varphi(+\infty)| < C\alpha^i$, and $|p_i^j - p_\infty| < C\alpha^i$ for large i, where $\alpha \in (0, 1)$, depending on j. We suppose that these inequalities are valid for a certain j. By equation (4.2)

$$p_{i-1}^{j+1} = v_{i-1}^{j+1} + rf_1(v_{i-1}^{j+1}) = v_i^j + rf_1(v_{i-1}^{j+1}) + (v_{i-1}^{j+1} - v_i^j) = p_i^{j+1} + (v_{i-1}^{j+1} - v_i^j). \tag{4.7}$$

If $v_{i-1}^{j+1} > u_0$, then since f_1 is differentiable, there exists a constant C_1, such that

$$|p_{i-1}^{j+1} - p_\infty| \le C_1|v_{i-1}^{j+1} - \varphi(+\infty)|.$$

By (4.7) we have

$$p_{i-1}^{j+1} - p_\infty = p_i^{j+1} - p_\infty + (v_{i-1}^{j+1} - \varphi(+\infty)) + (\varphi(+\infty) - v_i^j).$$

We note that $p_{i-1}^{j+1} - p_\infty$ and $v_{i-1}^{j+1} - \varphi(+\infty)$ have the same sign, thus

$$|p_i^{j+1} - p_\infty| \leq \left(1 - \frac{1}{C_1}\right) |p_{i-1}^{j+1} - p_\infty| + C\alpha^i, \quad \forall i. \tag{4.8}$$

If $v_{i-1}^{j+1} \leq u_0$, then $v_{i-1}^{j+1} - v_i^j \leq -a + C\alpha^i$. Then by (4.7) we get $p_i^{j+1} - p_\infty \geq p_{i-1}^{j+1} - p_\infty + a - C\alpha^i$. $p_{i-1}^{j+1} - p_\infty$ is bounded, hence if i is large enough, so that $C\alpha^i \leq a/2$, then there is a constant C_1, such that $|p_{i-1}^{j+1} - p_\infty| \leq C_1|a - C\alpha^i|$. Then

$$p_i^{j+1} - p_\infty \geq (p_{i-1}^{j+1} - p_\infty) + \frac{1}{C_1}|p_{i-1}^{j+1} - p_\infty| = \left(1 - \frac{1}{C_1}\right)(p_{i-1}^{j+1} - p_\infty).$$

Moreover, the scheme (4.1) is monotone, hence $p_i^{j+1} \leq \max(p_{i-1}^{j+1}, p_i^j) \leq p_\infty + C\alpha^i$. Therefore (4.8) also holds.

We obtain by induction that

$$|p_i^{j+1} - p_\infty| \leq Ci\left(\max\left(1 - \frac{1}{C_1}, \alpha\right)\right)^i.$$

Taking $\alpha' > \max\left(1 - \frac{1}{C_1}, \alpha\right)$, we get

$$|p_i^{j+1} - p_\infty| \leq C{\alpha'}^i,$$

thus the induction is complete.

Let $w_i^j = f_1(v_i^j)$, $\nu_i^j = v_i^j - v_{i-1}^j$, and $\omega_i^j = w_i^j - w_{i-1}^j$, then the equation (4.2) is

$$v_i^{j+1} + rw_i^{j+1} = v_i^j + rw_{i-1}^{j+1}.$$

By subtracting with respect to i and $i-1$, we get

$$\nu_i^{j+1} + r\omega_i^{j+1} = \nu_i^j + r\omega_{i-1}^{j+1}.$$

We note that if $\nu_i^{j+1} \neq 0$, then owing to $f_1' \geq 0$, ω_i^{j+1} possesses the same sign with ν_i^{j+1}. Therefore

$$|\nu_i^{j+1}| + r|\omega_i^{j+1}| = |\nu_i^{j+1} + r\omega_i^{j+1}| \leq |\nu_i^j| + r|\omega_{i-1}^{j+1}|.$$

We also note that ν_i^j and ω_i^j vanish for sufficiently small i, hence

$$\sum_{i=-\infty}^{I} |\nu_i^{j+1}| + r|\omega_I^{j+1}| \leq \sum_{i=-\infty}^{I} |\nu_i^j|.$$

Passing to the limit as $I \to \infty$ yields (4.4). Using the same approach we can get $\sum_1 |z_i^j| \leq \sum_i |z_i^0|$, where v_i^{-1} is given by the equation (4.1) in an opposite direction. Then (4.5) follows from the equation (4.1) . Finally the equation (4.1) gives $\omega_i^{j+1} = \frac{1}{r} z_i^{j+1}$, which implies (4.6). ∎

Applying Theorem 4.1 we can extract a sequence of approximate solutions converging to a weak solution $v(x,t)$. Moreover we can prove that it is an entropy solution.

THEOREM 4.2. *$v(x,t)$ is an entropy solution.*

Proof. Let $U \in C^2$, $U'' \geq 0$, and F is given by (3.6). F is discontinuous at $v = u_0$, As before we set $F(v_i^j) = U'(u_0)(f_1(v_i^j) - f_1(u_1)) + F_2(v_i^j) + a$.

We are going to prove a discrete entropy inequality. Let $\theta \in [0,1]$ we define

$$v(\theta) = \theta v_{i-1}^{j+1} + (1-\theta)v_i^{j+1},$$

$$u(\theta) = v_i^{j+1} + r(f_1(v_i^{j+1}) - f_1(v(\theta))),$$

and

$$H(\theta) = U(v_i^{j+1}) - U(u(\theta)) + r(F(v_i^{j+1}) - F(v(\theta))).$$

The functions u and H may be discontinuous. We consider the continuous case first, so we assume that $v(\theta) \neq u_0$.

We have

$$\begin{aligned} H'(\theta) &= -U'(u(\theta))u'(\theta) - rF'(v(\theta))v'(\theta) \\ &= r(U'(u(\theta)) - U'(v(\theta)))f_1'(v(\theta))v'(\theta) \\ &= rU''(u(\theta) - v(\theta))f_1'(v(\theta))v'(\theta) \\ &= rU''(v_i^{j+1} - v(\theta) + r(f_1(v_i^{j+1}) - f_1(v(\theta))))f_1'(v(\theta))(v_{i-1}^{j+1} - v_i^{j+1}). \end{aligned}$$

We see that $v_i^{j+1} - v(\theta) = \theta(v_i^{j+1} - v_{i-1}^{j+1})$, so $v_i^{j+1} - v(\theta)$, $f_1(v_i^{j+1}) - f_1(v(\theta))$, and $v_i^{j+1} - v_{i-1}^{j+1}$ possess the same sign. Consequently $H' \leq 0$. Because $H(0) = 0$, $H(1) \leq 0$, which gives the discrete entropy inequality

$$U(v_i^{j+1}) - U(v_i^j) + r(F(v_i^{j+1}) - F(v_{i-1}^{j+1})) \leq 0. \tag{4.9}$$

If the functions are discontinuous, there are some cases: The point of discontinuity is a single point $\theta_0 \in [0,1]$, where $v(\theta_0) = u_0$, or $v(\theta) \equiv u_0$. We consider one case, $\theta_0 \in (0,1)$. The consideration for the others are similar. Let $f_+ = f_1 \circ v(\theta_0 + 0)$, $f_- = f_1 \circ v(\theta_0 - 0)$, and analogously F_+ and F_-. We define

$$f(\eta) = \eta f_+ + (1-\eta)f_-, \quad F(\eta) = \eta F_+ + (1-\eta)F_-,$$

$$u(\eta) = v_i^{j+1} + r(f_1(v_i^{j+1}) - f(\eta)),$$

and

$$H(\eta) = U(v_i^{j+1}) - U(u(\eta)) + r(F(v_i^{j+1}) - F(\eta)),$$

with $\eta \in [0,1]$. Then

$$\begin{aligned} H'(\eta) &= r(U'(u(\eta)) - U'(u_0))(f_+ - f_-) \\ &= rU''(u(\eta) - u_0)(f_+ - f_-) \\ &= rU''(v_i^{j+1} - u_0 + r(f_1(v_i^{j+1}) - f(\eta)))(f_+ - f_-). \end{aligned}$$

$f_1(v_i^{j+1}) - f(\eta)$ possesses the same sign with $v_i^{j+1} - u_0$, and $f_+ - f_-$ has the opposite sign, so $H'(\eta) \leq 0$. Taking this fact into account, the discrete entropy inequality (4.9) also holds.

Let $\Delta x \to 0$ and $\Delta t \to 0$, then we get the entropy inequality for the weak solution v. ∎

5. Application to the combustion model

The previous results can be applied to the combustion model (2.1) (2.4) (2.5) (2.3). If f_1 is given by (2.10), then the entropy solution exists. For piecewise smooth solution the geometric entropy condition (3.2) is equivalent to the entropy inequality (3.7)[9]. We apply the condition (3.2) to the weak solution (2.7) (2.8) to get the following:

THEOREM 5.1. *Among the weak solutions (2.7) (2.8) to the problem (2.1) (2.4) (2.5) (2.6) the entropy solution is*

$$u(x,t) = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \qquad z(x,t) = \begin{cases} 0, & x < st, \\ 1, & x > st, \end{cases}$$

if $U_i \in [u_{l*} - q, u_l - q)$, where s and $u_{l*} < u_{CJ}$ are given by

$$s = \frac{f(u_l) - f(u_r)}{u_l - (u_r + q)} = \frac{f(u_l) - f(u_{l*})}{u_l - u_{l*}},$$

and

$$u(x,t) = \begin{cases} u_l, & x < s_1 t, \\ U_i + q, & s_1 t < x < s_2 t \\ u_r, & x > s_2 t, \end{cases} \qquad z(x,t) = \begin{cases} 0, & x < s_2 t, \\ 1, & x > s_2 t, \end{cases}$$

if $U_i \in (u_r, u_{l*} - q)$, where

$$s_1 = \frac{f(u_l) - f(U_i + q)}{u_l - U_i - q}, \quad s_2 = \frac{f(U_i + q) - f(u_r)}{U_i - u_r}.$$

It is the unique solution in the sense of that it corresponds to an entropy solution to the equation (2.9) and $u \equiv u_l$ for $t \le T$, $x < X_1$, and $u \equiv u_r$ for $t \le T$, $x > X_1$, where T, X_1 are positive constants.

Proof. The corresponding solutions v to the equation (2.9) are

$$v(x,t) = \begin{cases} u_l, & x < st, \\ u_r + q, & x > st, \end{cases} \tag{5.1}$$

and

$$v(x,t) = \begin{cases} u_l, & x < s_1 t, \\ u_m, & s_1 t < x < s_2 t \\ u_r + q, & x > s_2 t. \end{cases} \tag{5.2}$$

We consider a set

$$\begin{aligned} J = &\{\tilde{g} \in C([u_r + q, u_l]); \tilde{g}(u_r + q) = f_1(u_r + q), \tilde{g}(u_l) = f_1(u_l), \\ &\tilde{g}''(v) \le 0, \tilde{g}(v) \ge f_1(v), \forall v \in [u_r + q, u_l]\}. \end{aligned}$$

The solutions (5.1) (5.2)in the (v, f_1) plane are a line segment and a broken line segment:

$$g(v) = f_1(u_l)\frac{v - u_r - q}{u_l - u_r - q} + f_1(u_r + q)\frac{u_l - v}{u_l - u_r - q}, \quad v \in [u_r + q, u_l], \tag{5.3}$$

$$g(v) = \begin{cases} f_1(u_l)\frac{v-u_m}{u_l-u_m} + f_1(u_m)\frac{u_l-v}{u_l-u_m}, & v \in [u_m, u_l], \\ f_1(u_m)\frac{v-u_r-q}{u_m-u_r-q} + f_1(u_r+q)\frac{u_m-v}{u_m-u_r-q}, & v \in [u_r+q, u_m]. \end{cases} \tag{5.4}$$

The geometric entropy condition (3.2) is equivalent to the following:

$$g(v) = \inf_{\tilde{g}\in J} \tilde{g}(v), \qquad v \in [u_r+q, u_l].$$

We consider (5.3) first. We consider a critical case, $g(U_i+q) = f_1(U_i+q+0)$, that is

$$s = \frac{f_1(u_l) - f_1(u_r+q)}{u_l - (u_r+q)} = \frac{f_1(u_l) - f_1(U_i+q+0)}{u_l - U_i - q}.$$

Therefore $U_i+q = u_{l*}$. If $U_i+q \geq u_{l*}$, (5.1) is an entropy solution, and if $U_i+q < u_{l*}$, (5.1) is not an entropy solution.

Next we consider (5.4). If $U_i+q > u_{l*}$ it is not a solution, because the function is not concave. If $U_i+q \leq u_{l*}$, $u_m = U_i+q$ is the only possibility.

Uniqueness follows directly from Theorem 3.3. □

We have proved in the previous section that the implicit scheme gives the entropy solutions, and we have given some sufficient conditions for the explicit scheme in [15], where the weak solutions are also entropy solutions.

6. A random projection method

To design a method for the numerical simulation to strong detonation waves, a random projection method is introduced in [1]. We are going to prove that if our implicit finite difference scheme is applied in the random projection method, then it yields the strong detonation wave solution (2.7) for Majda's model.

Let us denote the function defined in (2.10) by $f(v, U_i)$, then we fix a lower bound v_l and an upper bound v_u. We assume that U_i is random between v_l and v_u. In fact we assume that it satisfies the van der Corput rule: Let j be an integer $j = \sum_{k=0}^{n} i_k 2^k$, then $\theta_j = \sum_{k=0}^{n} i_k 2^{-(k+1)} \in [0,1]$. U_i is given by $U_i = v_l + \theta_j(v_u - v_l)$. The corresponding function $f(v, U_i)$ is denoted by $f^j(v)$. Replacing the function f_1 in (4.1) by f^j, we obtain a new difference scheme which is in fact equivalent to the random projection scheme to Majda's model (2.1) (2.4) (2.5) (2.3).

THEOREM 6.1. *Under the conditions of Theorem 4.1, the approximate solution of the random projection scheme tends to an entropy solution to the equation*

$$\frac{\partial v}{\partial t} + \frac{\partial \bar{f}(v)}{\partial x} = 0, \tag{6.1}$$

as $\Delta x \to 0$ *and* $\Delta t \to 0$ *with* $r = \Delta t/\Delta x$ *fixed, where*

$$\bar{f}(v) = \begin{cases} f(v-q), & v \leq v_l + q, \\ \frac{1}{v_u - v_l}\int_{v_l}^{v_u} f(v, U_i)\, dU_i, & v_l + q < v \leq v_u + q, \\ f(v), & v > v_u + q. \end{cases}$$

Proof. The estimates of Theorem 4.1 still holds for this scheme, hence as $\Delta x \to 0$ and $\Delta t \to 0$ with $r = \Delta t/\Delta x$ fixed, there is a convergent sequence. Let $v(x,t)$ be the limit. We are going to study the limit of flux. Letting m be an integer, we consider the average $\tilde{v}_i^j = \sum_{k=0}^{2^m-1} v_i^{j+k}/2^m$, which has the same property of convergence as v_i^j.

The estimate (4.5) implies Lipschitz continuous with respect to t, so the limit is the same, $\lim \tilde{v}_i^j = \lim v_i^j = v(x,t)$. We obtain the difference scheme

$$\frac{\tilde{v}_i^{j+1} - \tilde{v}_i^j}{\Delta t} + \frac{\sum_{k=0}^{2^m-1} \Big(f^{j+k}(v_i^{j+k+1}) - f^{j+k}(v_{i-1}^{j+k+1}) \Big)/2^m}{\Delta x} = 0.$$

By the property of van der Corput quasi-random series, if the interval $[v_l, v_u]$ is divided into 2^m equal subintervals, then there exists exactly one point of U_i, associated with f^{j+k}, in each subinterval. Thus $\sum_{k=0}^{2^m-1} f^{j+k}(v(x,t))/2^m$ is the Riemann sum of the integral $\frac{1}{v_u - v_l}\int_{v_l}^{v_u} f(v, U_i)\, dU_i$. Let $\varepsilon > 0$ be an arbitrary constant. We take m large enough so that

$$\left| \sum_{k=0}^{2^m-1} f^{j+k}(v(x,t))/2^m - \bar{f}(v(x,t)) \right| < \varepsilon.$$

For the given $v(x,t)$ there are at least $2^m - 2$ random number U_i satisfying $|v(x,t) - U_i - q| > (v_u - v_l)/2^m$, and we have that v_i^{j+k} converges to $v(x,t)$ almost everywhere, so

$$|f^{j+k}(v_i^{j+k}) - f^{j+k}(v(x,t))| \le C|v_i^{j+k} - v(x,t)| < \varepsilon,$$

for almost all f^{j+k} except 2, provided Δx, Δt are small enough. For the remaining two terms we note that f^{j+k} is bounded, therefore

$$\left| \sum_{k=0}^{2^m-1} \Big(f^{j+k}(v_i^{j+k+1}) - f^{j+k}(v(x,t)) \Big)/2^m \right| \le \varepsilon + \frac{4C}{2^m}.$$

We take m large enough to get $4C/2^m < \varepsilon$.

In conclusion we take ε first, then determine m, which is independent of (x,t), then take Δx, Δt small enough, and obtain

$$\left| \sum_{k=0}^{2^m-1} f^{j+k}(v_i^{j+k+1})/2^m - \bar{f}(v(x,t)) \right| \le C\varepsilon$$

pointwise for almost all (x,t). Passing to the limit, it holds that

$$\left| \int\int \left(v\frac{\partial \varphi}{\partial t} + \bar{f}(v)\frac{\partial \varphi}{\partial x} \right) dxdt \right| \le C\varepsilon \|\varphi\|_{C^1},$$

for any $\varphi \in C_0^1$. However ε is arbitrary, so we obtain

$$\int\int \left(v\frac{\partial \varphi}{\partial t} + \bar{f}(v)\frac{\partial \varphi}{\partial x} \right) dxdt = 0$$

finally. The entropy inequality can be proved in the same way.

It is known that the entropy solution to the initial value problem of the equation (6.1) is unique, consequently the approximate solution v_i^j tends to v as $\Delta x \to 0$, $\Delta t \to 0$. ∎

We notice that $\bar{f}$ is continuous, and by direct calculation we find that it is convex between $v_l + q$ and $v_u + q$. We also notice that $\bar{f}(v_l + q) = f(v_l)$, and $\bar{f}(v_u + q) = f(v_u + q)$. As a result, if we take $v_l = u_r$ and $v_u = u_l - q$, and consider the Riemann problem (2.1) (2.4) (2.5) (2.6), then the random projection method gives the strong detonation wave solution (5.1).

Acknowledgments. The author is very grateful to the referees for their constructive comments to improve the paper. This work was supported by the China State Major Key Project for Basic researches and the Research Fund for the Doctorial Program of Higher Education.

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