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DISPERSION AND UNIFORM L^1-STABILITY ESTIMATES OF THE VLASOV-POISSON SYSTEM IN A HALF SPACE*

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Abstract. We study explicit dispersion and uniform L^1-stability estimates to the Vlasov-Poisson system for a collisionless plasma in a half space, when the initial data is sufficiently small and decays fast enough in phase space. This extends the previous results on the dispersion and stability estimates for the whole space case.

Key words. Dispersion estimates, initial-boundary value problem, Vlasov-Poisson system, uniform L^1-stability.

AMS subject classifications. 35A05, 35B65, 78A35.

1. Introduction

In this paper, we are interested in the dispersive dynamics of a collisionless plasma consisting of several species of charged particles under the effect of an electromagnetic field confined in the half space $\Omega := \{x = (x_1, \dots, x_d) : x_1 \ge 0\}$. The dynamics of charged particles can be understood by the Vlasov-Maxwell system at the kinetic level. However when the speed of light is taken to be infinity and a magnetic field is ignored, we can use the Vlasov-Poisson (V-P) system with a self-consistent electrostatic field for the dynamic description of a collisionless plasma. Since the number of charged particles does not play any essential role in the analysis, we assume the plasma consists of only one species with mass m and charge q. Let $f = f(x,v,t)$ be a one-particle phase space density of one species plasma particles in phase space $(x,v) \in \Omega \times \mathbb{R}^d$ at time $t \in \mathbb{R}_+$. In this case, the V-P system with a self-consistent electric field $E = \nabla_x \varphi$ and normalized mass and charge reads

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_v f = 0, \quad (x,v) \in \Omega \times \mathbb{R}^d, \ t > 0, \\ &\Delta_x \varphi = d(d-2)\alpha(d)\rho, \quad \rho := \int_{\mathbb{R}^d} f dv, \end{aligned} \tag{1.1}$$

subject to initial and boundary conditions on the kinetic density and the electric potential:

$$f(x,v,0) = f_0(x,v), \quad (x,v) \in \Omega \times \mathbb{R}^d, \tag{1.2a}$$

$$f(x,v,t) = f(x, v - 2(v \cdot \nu_x)\nu_x, t), \qquad x \in \partial\Omega, \ \ v \cdot \nu_x < 0, \tag{1.2b}$$

$$\varphi(0,\bar{x},t) = 0, \ \ \bar{x} \in \mathbb{R}^{d-1}, \qquad \lim_{|x| \to \infty} |\varphi(x,t)| = 0, \qquad t > 0, \tag{1.2c}$$

where $\alpha(d)$ is the volume of the unit ball in $\mathbb{R}^d$ and $\nu_x = (-1, 0, \dots, 0) \in \mathbb{R}^d$ is the outward normal vector at $x \in \partial\Omega$.

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The Cauchy problem for the V-P system (1.1) is now well established. For example, we refer to [1, 2, 3, 10, 18, 19, 24, 25, 26, 28, 29, 30, 31, 32] for the global existence theory for smooth, weak, and renormalized solutions, [4, 5] for stability theory, and [9, 27] for a detailed survey of kinetic theory. In contrast, the initial and boundary value problem(IBVP) for the V-P system has not been much studied in previous literature, although it is much more complicated and interesting. So far, the existence theory of smooth solutions for IBVP to the V-P system can be categorized into two cases (*half space* and *bounded convex domain* problems) depending on the geometry of the physical domain under consideration. For the half space problem with a flat boundary [11, 12], it is well known that singularities in the distribution function can be formed from the boundary unless the electric field has a correct sign. In [11, 12], Guo first constructed global smooth solutions to the system (1.1) together with (1.2a) – (1.2b) and a Neumann condition for the electric field. Recently Hwang and Velazquez [21] successfully extended Guo's result [11, 12] to the case of a Dirichlet boundary condition (1.2c), where a new method had to be introduced to deal with the issues of the boundary value problem. On the other hand, in the case of a bounded convex domain, Hwang [20] establishes the global existence of smooth solutions for absorbing boundary conditions and for reflected boundary conditions with a symmetry assumption. Recently, the restriction of the symmetry assumption for the reflected boundary condition was completely removed in [22].

The purpose of this paper is twofold. First, we derive explicit dispersion estimates for small and smooth solutions in the context of IBVP setting, which correspond to the counterpart of the Bardos-Degond dispersion estimates [2] for whole space problem. Secondly, we again extend Chae-Ha's uniform L^1-stability estimate in [5] for the Bardos-Degond smooth solutions to the IBVP setting.

The rest of this paper is organized as follows. In section 2, we discuss the main assumptions for initial and boundary conditions, and present the main results regarding dispersion and stability estimates of the V-P system. In section 3, we provide basic a priori dispersion estimates which will be needed to establish the global existence of a smooth solution with a desirable decay property in dimension $d \geq 3$. Finally section 4 is devoted to the uniform L^1-stability estimate of small and decaying smooth solutions in high dimensions $d \geq 4$ as a direct application of dispersion estimates.

Notation. In this paper, C represents a generic constant which may depend on the initial data, but is independent of time t. We also employ simplified notations for global norms: for $1 \leq p,\ q \leq \infty,\ g = g(x,t), h = h(x,v,t)$,

$$||g(t)||_{L^p_x} := \Big(\int_\Omega |g(x,t)|^p dx \Big)^{\frac{1}{p}}, \qquad ||h(t)||_{L^p_x(L^q_v)} := \Big(\int_\Omega ||h(x,\cdot,t)||^p_{L^q(\mathbb{R}^d_v)} dx \Big)^{\frac{1}{p}}.$$

2. Description of main results

In this section, we discuss the main assumptions used for the initial and boundary conditions and summarize the main results on the dispersion estimates and uniform stability for small and smooth solutions.

2.1. Main assumptions. Since we are interested in classical C^1-solutions, the initial data should be sufficiently smooth and compatible with boundary conditions so that we can guarantee the non-existence of singularity in the phase space $\Omega \times \mathbb{R}^d$ at time $t = 0$. We now consider "*well-prepared*" initial data satisfying the following four conditions below, which are minimal conditions from the intersection of Bardos-Degond's framework [2] and Hwang-Velazquez's framework [21]:

- $\mathcal{C}1$ (Smoothness):

$$f_0 \in (C^2 \cap C^{1,\mu})(\Omega \times \mathbb{R}^d) \quad \text{for some } \mu \in (0,1).$$

- $\mathcal{C}2$ (Compact support, smallness and decay):

$$\mathrm{supp}_{(x,v)}(f_0) \text{ is bounded and}$$

$$\sum_{0\le i+j\le 2} \sup_{(x,v)} (1+|x|^2)^{\frac{\mu_1}{2}} (1+|v|^2)^{\frac{\mu_2}{2}} |\nabla_x^i \nabla_v^j f_0(x,v)| \le \eta,$$

$$\text{where} \quad \mu_1 > d+1, \qquad \mu_2 > d+2 \qquad \text{and} \quad 0 < \eta \ll 1.$$

- $\mathcal{C}3$ (Compatibility condition):

$$f_0(0,\bar{x},v_1,\bar{v}) = f_0(0,\bar{x},-v_1,\bar{v}), \quad \text{and}$$

$$v_1 \partial_{x_1} f_0(0,\bar{x},v_1,\bar{v}) + v_1 \partial_{x_1} f_0(0,\bar{x},-v_1,\bar{v}) + 2E_1 \partial_{v_1} f_0(0,\bar{x},v_1,\bar{v}) = 0.$$

  Here E_1 is the first component of the electric field E.
- $\mathcal{C}4$ (Vacuum condition near the singular set):
  $\exists$ positive constants C_1, C_2 independent of t such that

$$f_0(x,v) = 0, \quad \text{for} \quad \frac{1}{2}v_1^2 + C_2 \eta x_1 \le 2C_1 \eta.$$

Remark 2.1.

1. The conditions $\mathcal{C}1$ and $\mathcal{C}2$ are introduced to guarantee the existence of classical solutions with dispersion estimates.

2. The compatibility conditions $\mathcal{C}3$ can easily be derived from the specular reflection condition (1.2b), which can be rewritten as follows.

$$f(0,\bar{x};v_1,\bar{v};t) = f(0,\bar{x};-v_1,\bar{v};t). \tag{2.1}$$

We set $t=0$ to get the first compatibility condition. On the other hand, we differentiate (2.1) with respect to t to see

$$\partial_t f(0,\bar{x};v_1,\bar{v};t) = \partial_t f(0,\bar{x};-v_1,\bar{v};t).$$

Then we use equation (1.1) and set $t=0$ to get the second compatibility condition.

3. In [12] the existence of a unique classical solution f to (1.1)with $f \in C^{1,\mu'}$ for some $0<\mu'<\mu$ has been established under the correct sign for the Neumann data. The vacuum condition $\mathcal{C}4$ is stronger than the flatness assumption employed in [12], which is

$$\mathcal{C}4': \quad \textit{There exist } \delta_0 > 0 \textit{ small such that } f_0 = \textit{constant} \quad \textit{for } \ x_1 + v_1^2 \le \delta_0.$$

This vacuum condition $\mathcal{C}4$ will be crucially used to guarantee the existence of a uniform lower bound for v_1^i in the back-time m-cycle $\{(x^i,v^i,t^i)\}_{0\le i\le m}$ (see Lemma 3.3), which again leads to the control of the number of bounces at the boundary for large time (Lemma 3.5).

2.2. Discussion of main results. In this part, we discuss the main results presented in this paper. The dispersion estimate for the V-P system (1.1) – (1.2a) was first noticed by Bardos and Degond in [2], and was employed effectively for the construction of global smooth small amplitude solutions in three physical dimensions. Recently Bardos-Degond's dispersion estimates were employed in the study of the uniform L^1-stability estimate in higher dimensions [6], and was further refined in [23].

In this paper, we extend Bardos-Degond's dispersion estimates [2] for the whole space problem to the case of a half space. Thus this allows some physical situations ([8, 17] and references therein) possessing physical boundaries such as plasma sheaths, electron guns, and diodes. However, this extension requires careful treatment of difficulties coming from repeated bounces of charged particles off the boundary along their back-time trajectories (see Definition 3.1). Bounces at the boundary destroy some of fine structures of particle trajectories due to the jumps of normal velocities at the boundary, and hence it is necessary to refine the trajectory analysis due to the complex bounces in order to obtain time-decay estimates (see Lemma 3.1 and Lemma 3.2 for details). This issue of the repeated bounces and the corresponding difficulty were already raised in the problem of the global-in-time construction of classical solutions to the Vlasov equation in a half space with general initial data, and careful analysis along particle paths was carried out in the context of global existence [11, 12, 20, 21, 22]. In [12] the flatness condition $\mathcal{C}4'$ near the singular set is made to deal with the issue by showing that trajectories stay away from the boundary. In our case, we show that trajectories are away from the boundary uniformly in time due to the time-decay of the electric field and its derivatives in Lemma 3.3. We replace the flatness condition $\mathcal{C}4'$ with the stronger vacuum condition $\mathcal{C}4$ near the singular set to establish the dispersion estimates.

Notice that the direction of the electric field at the boundary plays an important role; even in the linear problem with a given field, smooth solutions are not guaranteed unless the field directs outwards at the boundary, which is the case in the plasma physics. More precisely, under the incorrect sign of the field, singularities of particles moving tangentially at the boundary may enter inside of the domain along the particle paths, which result in the singularities of the solution. It means that we are forced to restrict ourselves to the plasma physics case with one kind of species to establish dispersion estimates in a half space. This is a noticeable difference compared to the full space problem [2], where the dispersion estimates hold for the gravitational case as well as long as the initial data is sufficiently small.

The dispersion estimates for the macroscopic variables ρ and E are determined by bootstrapping arguments, i.e., assuming that the electric field and its spatial derivatives decay at certain geometric rates, we show that the V-P system is dominated by the pure transport equation, which yields the dispersion estimates, and hence a self-consistent electric field also satisfies the desired decay estimate even in a half space to close the loop in the bootstrapping arguments. We next briefly explain how the pure transport equation yields the desired dispersion estimate. Consider the pure transport equation:

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f = 0, \quad (x,v)\in\Omega\times\mathbb{R}^d, \quad t\in\mathbb{R}_+,\\ &\qquad f(x,v,0) = f_0(x,v). \end{aligned} \tag{2.2}$$

In this case, the kinetic density f can be represented explicitly in terms of the initial datum f_0 by tracking the backward particle trajectory issued from (x,v,t) (see Figure 1). We denote $\bar{x}$ and $\bar{v}$ by the projection points of x and v onto $\partial\Omega = \left\{x\in\mathbb{R}^d \ : \ x_1 = 0\right\}$,

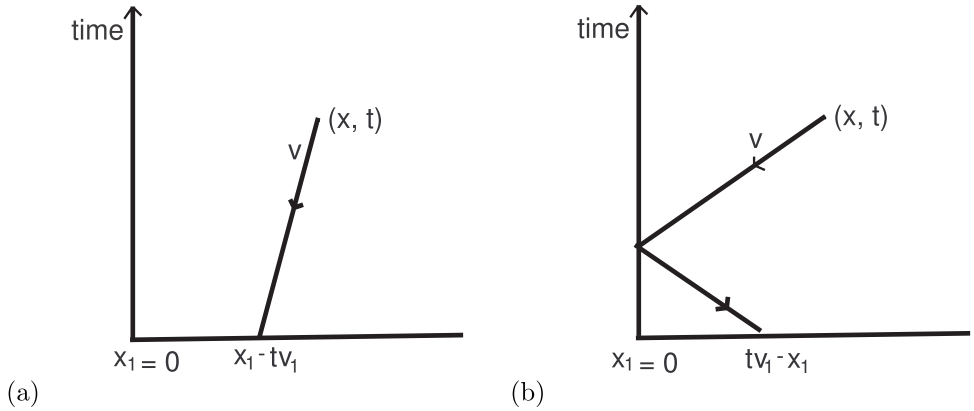

FIG. 2.1. *Schematic diagram for the first component of backward particle trajectories issued from (x,v) at time t: (a) $v_1 < \frac{x_1}{t}$, and (b) $v_1 \geq \frac{x_1}{t}$.*

and we assume that $f_0(x,v)$ is bounded by some integrable function $\zeta = \zeta(x)$:

$$\sup_{v \in \mathbb{R}^d} |f_0(x,v)| \leq \zeta(x), \qquad \int_\Omega \zeta(x)\,dx < \infty.$$

Then by a change of variables, we have

$$\begin{aligned}
\rho(x,t) &= \int_{\mathbb{R}^{d-1}} \Big(\int_{-\infty}^{\infty} f(x,v,t)\,dv_1 \Big) d\bar{v} \\
&= \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\frac{x_1}{t}} f_0(x - tv, v)\,dv_1 d\bar{v} \\
&\quad + \int_{\mathbb{R}^{d-1}} \int_{\frac{x_1}{t}}^{\infty} f_0(tv_1 - x_1, \bar{x} - t\bar{v}, -v_1 \bar{v})\,dv_1 d\bar{v} \\
&\leq C t^{-d} \int_\Omega \zeta(x)\,dx \\
&\leq C t^{-d}, \quad t \gg 1.
\end{aligned}$$

This time-decay of ρ also implies the time-decay of electric potential and fields (see Lemma 3.5). We next turn to the discussion of uniform L^1-stability of the Vlasov-Poisson system. Unlike the Boltzmann equation, the nonlinear functional approach in [13, 14] incorporating the collision potential $\mathcal{D}(f)$ cannot be applied directly to the L^1-stability estimates of (1.1), (1.2a)–(1.2c). Hence we instead derive a Gronwall type estimate (see section 4):

$$||f(t) - \bar{f}(t)||_{L^1} \leq ||f_0 - \bar{f}_0||_{L^1} + C \int_0^t (1+s)^{-(d-2)} ||f(s) - \bar{f}(s)||_{L^1} ds.$$

Here we used the simplified notation for L^1-norm:

$$||f(t) - \bar{f}(t)||_{L^1} := ||f(\cdot,\cdot,t) - \bar{f}(\cdot,\cdot,t)||_{L^1(\Omega \times \mathbb{R}^d)}.$$

Since

$$\int_0^t (1+s)^{-(d-2)} < C \qquad \text{for} \quad d \geq 4,$$

the standard Gronwall's Lemma yields the uniform L^1-stability for physical space dimensions $d \geq 4$. We next summarize the main results of this paper.

THEOREM 2.1. *Suppose the main hypotheses $\mathcal{C}1$ - $\mathcal{C}4$ with $d \geq 3$ hold. Let (f,ϕ) be the smooth solution to* (1.1)*,* (1.2a) – (1.2c) *with initial data f_0. Then f satisfies the following dispersion estimates:*

(i) $\|\rho(t)\|_{L^\infty_x} + \|\nabla_x \rho(t)\|_{L^\infty_x} + \left\|\nabla_x^2 \rho(t)\right\|_{L^\infty_x} \leq \dfrac{C\eta}{(1+t)^d}.$

(ii) $\|\rho(t)\|_{L^1_x} + \|\nabla_x \rho(t)\|_{L^1_x} \leq C\eta.$

(iii) $\|\nabla_v f(t)\|_{L^\infty_x(L^1_v)} \leq \dfrac{C\eta}{(1+t)^{d-1}}, \quad \|\nabla_v f(t)\|_{L^1_x(L^1_v)} \leq C\eta(1+t).$

THEOREM 2.2. *Suppose the main hypotheses $\mathcal{C}1 - \mathcal{C}4$ with $d \geq 4$ hold. Let f and $\bar{f}$ be smooth solutions to* (1.1)*,* (1.2a) – (1.2c) *with initial data f_0 and $\bar{f}_0$ respectively. Then smooth solutions are uniformly L^1-stable with respect to initial data:*

$$||f(t) - \bar{f}(t)||_{L^1} \leq G||f_0 - \bar{f}_0||_{L^1},$$

where G is a positive constant independent of time t.

REMARK 2.2. The uniform stability estimates of (1.1) in three dimensions $d=3$ is still an interesting open problem. However for the regularized systems of the V-P system such as the Vlasov-Yukawa and the Vlasov-Poisson-Fokkeer-Planck systems, the above uniform L^1-stability estimates hold for the physically interesting dimension $d=3$ (see [15, 16]).

3. Dispersion estimates in a half space

In this section, we study the dynamic behavior of particle trajectories near the singular set, and provide the dispersion property for small decaying solutions to the V-P system in a half space Ω. Throughout this section we always assume the electric field E is a C^1 in space and continuous in time, namely $E \in C^{1,0}_{x,t}(\Omega \times \mathbb{R}_+)$. For such a given force field E, we consider the linear Vlasov equation in a half space with the specular reflection boundary condition;

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \qquad (x,v) \in \Omega \times \mathbb{R}^d, \quad t \in \mathbb{R}_+, \\
&f(x,v,0) = f_0(x,v) \geq 0, \quad (x,v) \in \Omega \times \mathbb{R}^d, \\
&f(x,v,0) = f(x, v - 2(v\cdot\nu_x)\nu_x, t), \quad x \in \partial\Omega, \quad v \cdot \nu_x < 0.
\end{aligned} \tag{3.1}$$

3.1. Generalized particle trajectories. Define the generalized particle trajectory as a piecewise smooth solution satisfying the following ODE system: For a given phase-space position $(x,v) \in \Omega \times \mathbb{R}^d$ at time t,

$$\begin{aligned}
&\dot{X}(s;t,x,v) = V(s;t,x,v), \quad X(t;t,x,v) = x, \quad s \in [0,t), \\
&\dot{V}(s;t,x,v) = E(X(s;t,x,v), s), \quad V(t;t,x,v) = v,
\end{aligned} \tag{3.2}$$

with the specular reflection condition imposed on the boundary of Ω. Here $\dot{X}$ and $\dot{V}$ represent $\frac{dX}{ds}$ and $\frac{dV}{ds}$ respectively. This can be formulated as follows.

Case 1 ($X(s;t,x,v)$ stays in Ω for all $s \in [0,t)$): In this case, the generalized particle trajectory is simply the solution of the ODE system:

$$\begin{aligned}
&\dot{X}(s;t,x,v) = V(s;t,x,v), \qquad X(t;t,x,v) = x, \\
&\dot{V}(s;t,x,v) = E(X(s;t,x,v), s), \; V(t;t,x,v) = v.
\end{aligned}$$

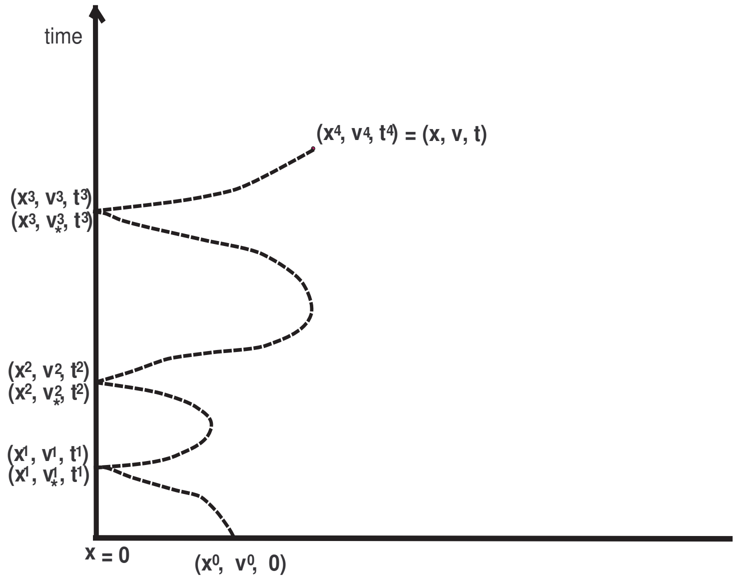

Fig. 3.1. *Schematic diagram with $l=4$.*

Case 2 ($X(s;t,x,v)$ hits the boundary of Ω for some $s\in[0,t)$). Suppose $s=\tau$ is the last hitting time to $\partial\Omega$ before $s=t$, i.e.,

$$X_1(s;t,x,v)\to 0, \quad \text{as } s\to\tau^+,$$

where $s\to\tau^+$ means that s approaches τ from the right. Then

$$\lim_{s\to\tau^+} V_1(s;t,x,v)\geq 0.$$

Define the generalized particle trajectory for $s\in(\tau,t)$ as the solution of the following ODE system:

$$\begin{aligned}
&\dot{X}(s;t,x,v)=V(s;t,x,v), && X_1(\tau;t,x,v)=0,\\
&\dot{V}(s;t,x,v)=E(X(s;t,x,v),s), && \lim_{s\to\tau^+}V_1(s;t,x,v)=-\lim_{s\to\tau^-}V_1(s;t,x,v).
\end{aligned}$$

For the time interval $s\in(0,\tau)$, we repeat the above procedure between successive hitting time intervals. Below, we review the concept "*back-time cycles*" which consist of the points where the particles hit the boundary (see figure 3.1). Below we denote by v_* the specularly reflected velocity of v, i.e.,

$$v_*:=v-2(v\cdot\nu_x)\nu_x.$$

Definition 3.1. [12] *Given a C^1-field E, we call a "back-time l-cycle connecting (x,v,t) and $(x^0,v^0,0)$" by the trajectories in $\bar{\Omega}\times\mathbb{R}^d\times\mathbb{R}_+$ which connect*

$(x,v,t)=(x^l,v^l,t^l)$ *with* $(x^{l-1},v^{l-1},t^{l-1})$, $(x^{l-1},v_*^{l-1},t^{l-1})$ *with* $(x^{l-2},v^{l-2},t^{l-1}),\ldots,$ (x^i,v_*^i,t^i) *with* $(x^{i-1},v^{i-1},t^{i-1}),\ldots,$ (x^1,v_*^1,t^1) *with* $(x^0,v^0,0),$

where $t^i > t^{i-1}$, $x^i \in \partial\Omega$ *for* $1 \le i \le l-1$, $v^i \cdot \nu_x \ge 0$, $1 \le i \le l$ *and* $t^0 = 0$.

Remark 3.1. In the following, for the simplicity of notation, we use the simplified notation

$$X(s) := X(s;t,x,v), \quad V(s) := V(s;t,x,v) \quad \textit{and} \quad E(s) := E(X(s;t,x,v),s).$$

Note that if $\{(x^i,v^i,t^i)\}_{0\le i\le m}$ is the back-time m-cycle with $t^0=0$ and $(x^m,v^m,t^m) = (x,v,t)$, then we have

$$v_1^1 = v_1^0 + \int_0^{t^1} E_1(\tau)\,d\tau,$$
$$v_1^{k+1} = -v_1^k + \int_{t^k}^{t^{k+1}} E_1(\tau)\,d\tau, \ 1 \le k \le m.$$

Recall that Green's function [7] for Laplace's equation in the half space Ω is given by

$$G(x,y) = \frac{1}{d(d-2)\alpha(d)}\left[\frac{1}{|x-y|^{d-2}} - \frac{1}{|x-y^*|^{d-2}}\right],$$

where $y^* = (-y_1, y_2, \ldots, y_d)$ is the reflection point of y with respect to $\partial\Omega$. Then the electric potential and fields, which are solutions to the Poisson equation in (1.1) and (1.2c) are given by the following representations:

$$\phi(x,t) = d(d-2)\alpha(d)\int_\Omega G(x,y)\rho(y,t)dy \quad \text{and}$$
$$E(x,t) = -d(d-2)\alpha(d)\int_\Omega \nabla_x G(x,y)\rho(y,t)dy. \tag{3.3}$$

Below, we show that if the electric field and its spatial derivatives decay fast enough, the particle trajectories stay bounded away from a singular set for all time. For this purpose, we consider a forward particle trajectory $(\hat{X},\hat{V})$ issued from (x,v) at time $t=0$:

$$\hat{X}(s) := X(s;0,x,v), \quad \hat{V}(s) := V(s;0,x,v) \quad \text{and} \quad \hat{E}(s) := E(X(s;0,x,v),s), \quad s > 0,$$

and introduce a quantity Γ which can detect the singular set at the boundary:

$$\Gamma(s) := \frac{1}{2}\hat{V}_1^2(s) - E_1\left(0, \bar{\hat{X}}(s), s\right)\hat{X}_1(s). \tag{3.4}$$

Lemma 3.2. *Consider equation* (3.1) *for* $d \ge 3$ *with the well-prepared initial data* f_0. *Suppose the force field* E *satisfies*

$$\|E(t)\|_{L_x^\infty} \le \frac{\eta}{(1+t)^{d-1}} \quad \textit{and} \quad \|\nabla_x E(t)\|_{L_x^\infty} \le \frac{\eta}{(1+t)^\lambda} \quad \textit{for} \quad \lambda > 2.$$

Then we have the following two estimates:

1. *The* v*-support is uniformly bounded in time, i.e., there exists a positive constant* C *which may depend on the size of* x *and* v*-supports of* f_0 *and* η *such that*

$$\left|\hat{V}(t)\right| \le C.$$

2. There exists a positive constant C_1 satisfying

$$|\Gamma(t)-\Gamma(0)|\le C_1\eta.$$

Proof.

(i) For given $(x,v)\in\Omega\times\mathbb{R}^d$, we use the time-decay of E and compact support condition $\mathcal{C}2$ in section 2 to get

$$\begin{aligned}\left|\hat{V}(t)\right| &\le |v|+\int_0^t ||\hat{E}(\tau)||_{L_x^\infty}d\tau\\ &\le C+\eta\int_0^t \frac{d\tau}{(1+\tau)^{d-1}}\le C.\end{aligned}$$

(ii) (Step 1): We first claim

$$||\partial_t E(t)||_{L_x^\infty}\le \frac{C\eta}{(1+t)^\lambda}, \qquad \lambda>2. \tag{3.5}$$

The proof of claim: We use the continuity equation resulting from (1.1)

$$\partial_t\rho(x,t)+\nabla_x\cdot j(x,t)=0, \qquad j(x,t):=\int_{\mathbb{R}^N} vf(x,v,t)\,dv,$$

and the representation formula (3.3) for $E(x,t)$ to see that

$$\begin{aligned}\partial_t E(x,t) &= d(d-2)\alpha(d)\int_\Omega \nabla_y G(x,y)\,\partial_t\rho(y,t)\,dy\\ &= -d(d-2)\alpha(d)\int_\Omega \nabla_y G(x,y)\,\nabla_y\cdot j(y,t)\,dy\\ &= d(d-2)\alpha(d)\Big[\int_\Omega [\nabla_y\cdot\nabla_y G(x,y)]j(y,t)\,dy\\ &\quad -\int_{\partial\Omega}\nabla_y G(x,y)\left(\int_{\mathbb{R}^d}\nu_y\cdot vf(y,v,t)\,dv\right)dS_y\Big]\\ &= d(d-2)\alpha(d)\int_\Omega \Big[\nabla_y\cdot\nabla_y G(x,y)\Big]j(y,t)\,dy,\end{aligned} \tag{3.6}$$

where ν_y is the outward normal vector at $y\in\partial\Omega$, and dS_y is the surface volume element on $\partial\Omega$, while we used the specular reflection condition for f to take care of the boundary contribution. For $y\in\partial\Omega$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\nu_y\cdot vf(y,v,t)\,dv\\ &\quad=-\int_{\mathbb{R}^{d-1}}\Big(\int_{-\infty}^{\infty} v_1 f(y,v,t)dv_1\Big)d\bar{v}\\ &\quad=-\int_{\mathbb{R}^{d-1}}\Big(\int_0^{\infty} v_1 f(y,v,t)dv_1+\int_{-\infty}^0 v_1 f(y,v,t)dv_1\Big)d\bar{v}\\ &\quad=-\int_{\mathbb{R}^{d-1}}\Big(\int_0^{\infty} v_1 f(y,v_1,\bar{v},t)dv_1+\int_{-\infty}^0 v^1 f(y,-v_1,\bar{v},t)dv_1\Big)d\bar{v}\\ &\quad=0.\end{aligned}$$

Here we used a change of variable $-v_1\leftrightarrow\tilde{v}_1$ and the specular reflection condition:

$$f(y,v_1,\bar{v},t)=f(y,-v_1,\bar{v},t).$$

We use (3.3) and decay in L^∞ for $\nabla_x E$ to deduce that

$$\begin{aligned}\|\partial_t E(t)\|_{L^\infty_x} &\le C\left\|\int_\Omega \nabla_y\cdot\nabla_y G(\cdot,y)\,j(y,t)\,dy\right\|_{L^\infty_x}\\ &\le C\|\nabla_x E(t)\|_{L^\infty_x}\\ &\le \frac{C\eta}{(1+t)^\lambda}.\end{aligned}$$

This completes the proof of the claim.

(Step 2): Note that the uniform boundedness of $V(t)$ implies that for $s\in[0,t]$,

$$\left|\hat{X}(s)\right|\le C(1+s), \tag{3.7}$$

where C depends only on the x- and v-support of f_0 and η. We now return to estimating the time change of the quantity $\Gamma(s)$. We differentiate the functional (3.4) in time to yield

$$\begin{aligned}\frac{d}{ds}\Gamma(s) = \hat{V}_1(s)\hat{E}_1\left(\hat{X}(s),s\right) &- \left[\partial_t\hat{E}_1\left(0,\bar{\hat{X}}(s),s\right)+\partial_{\bar{x}}E\left(0,\bar{\hat{X}}(s),s\right)\cdot\bar{\hat{V}}\right]\hat{X}_1(s)\\ &-\hat{E}_1\left(0,\bar{\hat{X}}(s),s\right)\hat{V}_1(s).\end{aligned}$$

We use the result of (i) and (3.5) – (3.7) to obtain

$$\left|\frac{d}{ds}\Gamma(s)\right|\le C\eta\left[\frac{1}{(1+s)^{d-1}}+\frac{1}{(1+s)^{\lambda-1}}\right].$$

Note that the right hand side of the above inequality is integrable in time for $d\ge 3,\lambda>2$. ☐

A singularity may be formed in the C^1 norm as in [12] and thus we need to control particle trajectories starting from the support of the first derivatives of f_0. This is made possible in the repulsive case of the field together with the flatness assumption $\mathcal{C}4'$. We present a similar estimate given in [12] on the control of the trajectories away from the singularities. Due to the time decay of the electric field and its derivatives, we have the uniform-in-time control of the trajectories here. Note that in the plasma physics case with one sign of charge, we apply the maximum principle to the Poisson equation appearing in (3.1) to get

$$E_1\left(0,\bar{\hat{X}}(s),s\right)<0.$$

Lemma 3.3. *Consider equation* (3.1) *for $d\ge 3$ with the well-prepared initial data f_0. Suppose the force field E satisfies*

$$\begin{aligned}&\|E(t)\|_{L^\infty_x}\le\frac{\eta}{(1+t)^{d-1}}\quad\text{and}\quad\|\nabla_x E(t)\|_{L^\infty_x}\le\frac{\eta}{(1+t)^\lambda}\quad\text{for}\quad\lambda>2,\\ &E_1(0,\bar{x},0)\le -C_2\eta,\quad\text{for } (0,\bar{x})\in x\text{-support of } f_0.\end{aligned}$$

Let $\{(x^i,v^i,t^i)\}_{0\le i\le m}$ be the back-time m-cycle with $t^0=0$ satisfying

$$(x^m,v^m,t^m)=(x,v,t),\qquad \frac{1}{2}\left(v_1^0\right)^2+C_2\eta x_1^0\ge 2C_1\eta.$$

Then there exists a uniform lower bound of $|v_1^i|$ *for* $1 \le i \le m-1$*:*

$$|v_1^i| \ge \sqrt{2C_1\eta},$$

where C_1 *is a positive constant appearing in Lemma 3.1.*

Proof. Note that although the definition for Γ in (3.4) was defined for the forward particle trajectory $(\hat{X},\hat{V})$, the functional Γ can be defined for the backward particle trajectory (X,V) in the same way.

We use $x_1^i = 0$, the assumption on $E_1(0,\bar{x},0)$ and the definition of Γ in (3.4) to see that

$$\Gamma(t^i) = \frac{1}{2}(v_1^i)^2 - E_1(0,\bar{x}(t^i),t^i)x_1^i(t^i) = \frac{1}{2}(v_1^i)^2, \quad \text{and} \tag{3.8}$$

$$\Gamma(0) = \frac{1}{2}(v_1^0)^2 - E_1(0,\bar{x}^0,0)x_1^0 \ge \frac{1}{2}(v_1^0)^2 + C_2\eta x_1^0 \ge 2C_1\eta. \tag{3.9}$$

On the other hand, it follows from Lemma 3.1 that

$$|\Gamma(t^i) - \Gamma(0)| \le C_1\eta. \tag{3.10}$$

Hence we combine the estimates (3.8), (3.9), and (3.10) to conclude that

$$\Gamma(t^i) = \frac{1}{2}(v_1^i)^2 \ge C_1\eta.$$

□

3.2. Dispersion estimates. In this part, we present several dispersion estimates. We first present a series of Lemmas.

LEMMA 3.4. *Consider equation* (3.1) *for* $d \ge 3$ *with the well-prepared initial data* f_0*. Suppose that for* $0 < \eta \ll 1, \lambda > 2$*,*

$$\|E(t)\|_{L^\infty} \le \frac{\eta}{(1+t)^{d-1}}, \quad \|\nabla_x E(t)\|_{L^\infty} + \left\|\nabla_x^2 E(t)\right\|_{L^\infty} \le \frac{\eta}{(1+t)^{\lambda}}, \tag{3.11}$$

and let $(X(s),V(s)) = X(s;t,x,v)$*,* $V(s;t,x,v)$ *be the generalized trajectories passing through* (x,v) *at time* t*. Then there exists a constant* $C = C(\eta)$ *satisfying*

$$\left|\frac{\partial X}{\partial x}\right| + \left|\frac{\partial V}{\partial x}\right| + \left|\frac{\partial^2 X}{\partial x^2}\right| + \left|\frac{\partial^2 V}{\partial x^2}\right| + \frac{1}{t}\left|\frac{\partial X}{\partial v}\right| + \frac{1}{t}\left|\frac{\partial V}{\partial v}\right| \le C.$$

Here derivatives of (X,V) are considered piecewise sense, and $|A|$ denotes any matrix norm for a matrix A.

Proof. Let (x,v,t) be given in $\Omega \times \mathbb{R}^d \times \mathbb{R}_+$.

Case 1: (x,v,t) and (x^b,v^b,t^b), $x_1^b = 0, v_1^b > 0$ are connected through a trajectory.

In this case, we have

$$\begin{aligned} v^b &= v + \int_t^{t^b} E(X(\tau),\tau)\,d\tau, \\ x^b &= x - v\left(t - t^b\right) + \int_t^{t^b}\int_t^s E(X(\tau),\tau)\,d\tau ds. \end{aligned} \tag{3.12}$$

We differentiate the second equation of (3.12) to yield

$$\partial_{v_j} x_i^b = -\delta_{ij}\left(t-t^b\right) + v_i \partial_{v_j} t^b + \int_t^{t^b}\int_t^s \partial_x E_i \cdot \partial_{v_j} X(\tau)\, d\tau ds, \quad i=1,\ldots,d.$$

For $i=1$, we use $\partial_{v_j} x_1^b = 0, \;\; j=1,\ldots,d$ to find

$$\partial_{v_j} t^b = \frac{1}{v_1^b}\left[\delta_{1j}\left(t-t^b\right) - \int_t^{t^b}\int_t^s \partial_x E_1 \cdot \partial_{v_j} X(\tau)\, d\tau ds\right] \quad \text{for } \;\; v_1^b > 0.$$

Thus Lemma 3.2 for the estimate v_1^b and (3.11) yield

$$\left|\partial_{v_j} x_i^b\right| \le C\left(t-t^b\right) + \eta \int_{t^b}^t \int_s^t \frac{|\partial_v X(\tau)|}{(1+\tau)^\lambda} d\tau ds,$$

i.e., we have

$$|\partial_v X| \le C\left(t-t^b\right) + \eta \int_{t^b}^t \int_s^t \frac{|\partial_v X(\tau)|}{(1+\tau)^\lambda} d\tau ds.$$

Since $\lambda > 2$, the Gronwall inequality yields

$$\left|\partial_v x^b\right| \le C\left(t-t^b\right). \tag{3.13}$$

In a similar manner, we obtain

$$\begin{aligned} &\left|\partial_v v^b\right| \le C + C\left(t-t^b\right),\; \left|\partial_v t^b\right| \le C\left(t-t^b\right), \\ &\left|\partial_x x^b\right| \le C,\; \left|\partial_x v^b\right| \le C,\; \left|\partial_x t^b\right| \le C. \end{aligned} \tag{3.14}$$

Case 2: Let $w^k = \{\left(x^k, v^k, t^k\right)\}_{0\le k\le m}$ be the back-time m-cycle of (x,v,t), where $t^0 = 0, t^m = t$ and $x_1^k = 0$ for $1 \le k \le m-1$.

It follows from the chain rule that

$$\partial_v X(0;t,x,v) = \frac{\partial x^0}{\partial x^1}\frac{\partial x^1}{\partial x^2}\cdots\frac{\partial x^{m-2}}{\partial x^{m-1}}\frac{\partial x^{m-1}}{\partial v}.$$

By (3.13), we have

$$\frac{\partial x^{m-1}}{\partial v} \le C\left(t-t^{m-1}\right).$$

Using (3.12), we have

$$\begin{aligned} \partial_{x_j^{k+1}} x_i^k &= \delta_{ij} + v_i^k \partial_{x_j^{k+1}} t^k + \int_{t^{k+1}}^{t^k}\int_{t^{k+1}}^s \partial_x E_i \cdot \partial_{x_j} X(\tau)\, d\tau ds, \\ \partial_{x_j^{k+1}} t^k &= -\frac{1}{v_1^k}\left[\delta_{1j} + \int_{t^{k+1}}^{t^k}\int_{t^{k+1}}^s \partial_x E_i \cdot \partial_{x_j} X(\tau) d\tau ds\right]. \end{aligned}$$

We use (3.11) and the above equation to obtain

$$\left|\partial_{x^{k+1}} x^k\right| \le 1 + \frac{C\eta}{\left(1+t^k\right)^{\lambda-2}},$$

where we used $x_1^k = 0$ for $1 \le k \le m-1$. Let $\alpha = \lambda - 3 > 0$, then we have

$$\begin{aligned}
\left|\frac{\partial x^0}{\partial x^1}\right|\left|\frac{\partial x^1}{\partial x^2}\right|\cdots\left|\frac{\partial x^{m-2}}{\partial x^{m-1}}\right| &\le \Pi_{k=0}^{m-2}\left\{1+\frac{C\eta}{(1+t^k)^{\alpha+1}}\right\} \\
&\le (1+z)\,\Pi_{k=1}^{k=m-2}(1+a_k z) \\
&= (1+z)(1+A_1 z + A_2 z^2 + \ldots + A_{m-2} z^{m-2}),
\end{aligned}$$

where

$$z := C\eta, \quad a_k := \frac{1}{(1+t^k)^{\alpha+1}}.$$

We make the following basic observations:

$$\begin{aligned}
A_1 &= \sum_{k=1}^{m-2}\frac{1}{(1+t^k)^{\alpha+1}} \le \int_0^t (1+s)^{-\alpha-1}\,ds \le \frac{1}{\alpha}, \\
A_2 &= \sum_{k<l}\frac{1}{(1+t^k)^{\alpha+1}}\frac{1}{(1+t^l)^{\alpha+1}} \le \left(\sum_{k=1}^{m-2}\frac{1}{(1+t^k)^{\alpha+1}}\right)^2 \\
&= A_1^2 \le \frac{1}{\alpha^2}, \\
&\;\;\vdots \\
A_{m-2} &\le A_1^{m-2} \le \frac{1}{\alpha^{m-2}}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\left|\frac{\partial x^0}{\partial x^1}\right|\left|\frac{\partial x^1}{\partial x^2}\right|\cdots\left|\frac{\partial x^{m-2}}{\partial x^{m-1}}\right| &\le (1+z)\left[1+\frac{z}{\alpha}+\ldots+\left(\frac{z}{\alpha}\right)^{m-2}\right] \\
&\le C,
\end{aligned}$$

if η is small enough such that

$$\frac{z}{\alpha} = \frac{C\eta}{\lambda-3} < 1.$$

Therefore we have

$$|\partial_v X(0;t,x,v)| \le C\left(t - t^{m-1}\right) \le Ct.$$

The other estimates can be obtained similarly. □

LEMMA 3.5. *Suppose the force field E satisfies*

$$\|E(t)\|_{L_x^\infty} \le \frac{\eta}{(1+t)^{d-1}}, \quad d \ge 3,$$

and let (x,v,t) be emanated from the point $(x^0,v^0,0)$, which is in the singular set, i.e.,

$$\frac{1}{2}(v_1^0)^2 + C_2\eta x_1^0 \le 2C_1\eta.$$

Then there is at most one bounce in the time-interval $\left[\frac{t}{2}, t\right]$ *if* t *is large enough.*

Proof. Suppose a particle trajectory issued from (x^0, v^0) hits $\partial\Omega$ at $\left(x^j, v^j, t^j\right)$ and $\left(x^{j+1}, v^{j+1}, t^{j+1}\right)$ successively with $\frac{t}{2} \le t^j < t^{j+1} \le t$. Then we have

$$0 = -v_1^j\left(t^{j+1} - t^j\right) + \int_{t^j}^{t^{j+1}} \int_{\tau}^{t^{j+1}} E_1(s)\, ds d\tau.$$

Then the time spent for the bounce is bounded from below by

$$\Delta t = t^{j+1} - t^j \ge \frac{2\left|v_1^j\right|}{\sup_{t/2 \le s \le t} \|E(s)\|_{L^\infty}} \ge \frac{2\sqrt{2C_1\eta}\left(1 + \frac{t}{2}\right)^{d-1}}{\eta} > \frac{t}{2}, \quad t \gg 1.$$

This gives a contradiction. Hence there is at most one bounce between $\frac{t}{2}$ and t for $t \gg 1$. ∎

Proposition 3.6. *Consider equation* (3.1). *Suppose that main hypotheses* $\mathcal{C}1 - \mathcal{C}4$ *in section 2 hold, and the force field* E *satisfies*

$$\|E(t)\|_{L^\infty} \le \frac{\eta}{(1+t)^{d-1}}, \quad \|\nabla_x E(t)\|_{L^\infty} + \left\|\nabla_x^2 E(t)\right\|_{L^\infty} \le \frac{\eta}{(1+t)^{\lambda}}, \quad d \ge 3, \quad \lambda > 2. \tag{3.15}$$

Then we have the following a priori estimates

(i) $\|\rho(t)\|_{L_x^\infty(\Omega)} + \|\nabla_x \rho(t)\|_{L_x^\infty(\Omega)} + \left\|\nabla_x^2 \rho(t)\right\|_{L_x^\infty(\Omega)} \le \dfrac{C\eta}{(1+t)^d}$.

(ii) $\|\rho(t)\|_{L_x^1(\Omega)} + \|\nabla_x \rho(t)\|_{L_x^1(\Omega)} \le C\eta$.

(iii) $\|\nabla_v f(t)\|_{L_x^\infty(L_v^1)} \le \dfrac{C\eta}{(1+t)^{d-1}}, \qquad \|\nabla_v f(t)\|_{L_x^1(L_v^1)} \le C\eta(1+t)$.

Proof. We only consider the time-decay estimates of ρ. The other estimates can be treated similarly. We set $t^* \gg 1$, and we separate the estimate for ρ into two steps (small time and large time).

Case 1 ($t \in [0, t^*)$): Note that

$$|V(0;t,x,v)| \ge |v| - \int_0^t \|E(\tau)\|_{L^\infty} d\tau \ge |v| - \frac{\eta}{d-2}, \qquad |v| \gg 1,$$

where we used

$$\int_0^t ||E(\tau)||_{L_x^\infty} d\tau \le \frac{\eta}{d-2}.$$

We now use

$$f(x,v,t) = f_0(X(0;t,x,v), V(0;t,x,v))$$

to see that

$$|\rho(x,t)| = \int_{\mathbb{R}^d} f(x,v,t) dv$$

$$\begin{aligned}
&\le C\eta \int_{\mathbb{R}^d} \frac{dv}{(1+|V(0;t,x,v)|^2)^{\frac{\mu_2}{2}}} \\
&\le C\eta \Big[|\{v : |v| \le \frac{\eta}{d-2}\}| + \int_{|v|\ge \frac{\eta}{d-2}} \Big(1+(|v|-\eta/(d-2))^2\Big)^{-\mu_2/2} dv \Big].
\end{aligned}$$

Thus, for small η and for $0 \le t \le t^*$, we have

$$\|\rho(t)\|_{L^\infty} \le \frac{C\eta}{(1+t)^d}.$$

Case 2 ($t \ge t^*$): If the possible bounce occurs at time less than $\frac{3t}{4}$ along the trajectory, then we split the time interval $[0,t]$ into the two parts, namely $\left[0, \frac{3t}{4}\right]$ and $\left[\frac{3t}{4}, t\right]$. We set

$$X' := X\left(\frac{3t}{4};t,x,v\right),\ V' := V\left(\frac{3t}{4};t,x,v\right),$$

$$X := X(0;t,x,v),\ V := V(0;t,x,v).$$

Then we have no bounces over $\left[\frac{3t}{4}, t\right]$ and we have, as in [2],

$$\left|\det\left(\frac{\partial X'}{\partial v}\right)\right| \ge \frac{1}{2}\left(\frac{t}{4}\right)^d,$$

and by Lemma 3.4 forward in time,

$$\left|\det\left(\frac{\partial X'}{\partial X}\right)\right| \le C.$$

Thus we obtain

$$\begin{aligned}
\rho(x,t) &= \int_{\frac{1}{2}(V_1)^2 + C_2\eta X_1 + \ge 2C_1\eta} f_0(X(0;t,x,v), V(0;t,x,v))dv \\
&\le C\eta \int_{\frac{1}{2}(V_1)^2 + C_2\eta X_1 \ge 2C_1\eta} \frac{dv}{(1+|X|^2)^{\frac{\mu_1}{2}}} \\
&\le C\eta \int_{\frac{1}{2}(V_1)^2 + C_2\eta X_1 \ge 2C_1\eta} \frac{1}{(1+|X|^2)^{\frac{\mu_1}{2}}} \left|\det\left(\frac{\partial X'}{\partial X}\right)\right| \left|\det\left(\frac{\partial X'}{\partial v}\right)\right|^{-1} dX \\
&\le \frac{C\eta}{t^d} \\
&\le \frac{C\eta}{(1+t)^d}.
\end{aligned}$$

If the bounce occurs at time $t' \in [\frac{3t}{4}, t]$ along the trajectory, then there are no bounces over $[\frac{t}{2}, \frac{3t}{4}]$ by Lemma 3.4, so we split $[0,t]$ into the three parts, $\left[0, \frac{t}{2}\right]$, $\left[\frac{t}{2}, \frac{3t}{4}\right]$, $\left[\frac{3t}{4}, t\right]$. We set

$$X'' := X\left(\frac{3t}{4};t,x,v\right), V'' := V\left(\frac{3t}{4};t,x,v\right),$$

$$X' := X\left(\frac{t}{2};t,x,v\right), V' := V\left(\frac{t}{2};t,x,v\right),$$

$$X := X(0;t,x,v), \quad V := V(0;t,x,v).$$

Then we have

$$\left|\det\left(\frac{\partial X'}{\partial V''}\right)\right| \geq \frac{1}{2}\left(\frac{t}{4}\right)^d, \quad \left|\det\left(\frac{\partial v}{\partial V''}\right)\right| \leq C, \ \left|\det\left(\frac{\partial X'}{\partial X}\right)\right| \leq C.$$

Thus we obtain

$$\begin{aligned}
|\rho(x,t)| &\leq C\eta \int_{\frac{1}{2}(V_1)^2 + C_2\eta X_1 \geq 2C_1\eta} \frac{dv}{(1+|X|^2)^{\mu_1/2}} \\
&\leq C\eta \int_{\frac{1}{2}(V_1)^2 + C_2\eta X_1 \geq 2C_1\eta} \frac{1}{(1+|X|^2)^{\mu_1/2}} \left|\det\left(\frac{\partial X'}{\partial X}\right)\right| \left|\det\left(\frac{\partial X'}{\partial V''}\right)\right|^{-1} \\
&\quad \times \left|\det\left(\frac{\partial v}{\partial V''}\right)\right| dX \\
&\leq \frac{C\eta}{t^d}.
\end{aligned}$$

In a similar manner, we can deduce the other estimates in the lemma. □

Remark 3.2. Recently the more refined time-decay estimates for $\nabla_x^k \rho$ was obtained in [23] in the framework of Bardos-Degond [2]:

$$||\nabla_x^k \rho(t)||_{L_x^\infty} \leq \frac{C}{(1+t)^{k+3}} \qquad k \geq 0, \quad t \geq 0.$$

In order to close the argument on the existence part, we give the following lemmas:

Lemma 3.7. [6] *Let $\rho = \rho(x)$ be a smooth function in $L^1(\Omega) \cap W^{1,\infty}(\Omega)$, and we set*

$$\mathcal{N}(\rho)(x) := \int_\Omega \frac{\rho(y)\,dy}{|x-y|^{d-2}}.$$

Then we have the following estimates (analogous to Lemma 1 in [2]:)

(i) $\|\mathcal{N}(\rho)\|_{L_x^\infty} \leq C(d) \|\rho\|_{L_x^1}^{\frac{2}{d}} \|\rho\|_{L_x^\infty}^{\frac{d-2}{d}}$,
(ii) $\|\nabla_x \mathcal{N}(\rho)\|_{L_x^\infty} \leq C(d) \|\rho\|_{L_x^1}^{\frac{1}{d}} \|\rho\|_{L_x^\infty}^{\frac{d-1}{d}}$,
(iii) $\left\|\nabla_x^2 \mathcal{N}(\rho)\right\|_{L_x^\infty} \leq C(d,\kappa) \|\rho\|_{L_x^1}^{\frac{\kappa}{\kappa+d}} \|\rho\|_{L_x^\infty}^{\frac{d(1-\kappa)}{\kappa+d}} \|\nabla_x \rho\|_{L_x^\infty}^{\frac{d\kappa}{\kappa+d}}$,

where $0 < \kappa < 1$.

As a Corollary of Lemma 3.5, we have decay estimates on E.

Corollary 3.8. *Suppose ρ satisfies the a priori estimates in Proposition 3.1. Then we have the following decay estimates for E through the formula (3.3):*

$$||E(t)||_{L_x^\infty} \leq \frac{C\eta}{(1+t)^{d-1}}, \quad \|\nabla_x E(t)\|_{L_x^\infty} + \left\|\nabla_x^2 E(t)\right\|_{L_x^\infty} \leq \frac{C\eta}{(1+t)^d}.$$

Proof. Recall that

$$E(x,t) = (d-2)\int_{\Omega}\Big[-\frac{x_i - y_i}{|x-y|^d} + \frac{x_i - y_i^*}{|x-y^*|^d}\Big]\rho(y,t)dy.$$

(i) We first consider the L_x^∞-estimates $E(x,t)$. We apply the result of Lemma 3.7 to see that

$$\begin{aligned} ||E(t)||_{L_x^\infty} &\le C||\nabla_x(\rho)(t)||_{L_x^\infty} \\ &\le C\,\|\rho(t)\|_{L_x^1}^{\frac{1}{d}}\,\|\rho(t)\|_{L_x^\infty}^{\frac{d-1}{d}} \\ &\le \frac{C\eta}{(1+t)^{d-1}}. \end{aligned}$$

(ii) For the estimate of $\nabla_x E$, we use Lemma 3.7 to find

$$\begin{aligned} ||\nabla_x E(t)||_{L_x^\infty} &\le C||\nabla_x^2(\rho)(t)||_{L_x^\infty} \\ &\le C\,\|\rho(t)\|_{L_x^1}^{\frac{\kappa}{\varepsilon+d}}\,\|\rho(t)\|_{L_x^\infty}^{\frac{d(1-\kappa)}{\kappa+d}}\,\|\nabla_x\rho(t)\|_{L_x^\infty}^{\frac{d\kappa}{\kappa+d}} \\ &\le \frac{C\eta}{(1+t)^d}. \end{aligned}$$

The term $||\nabla_x^2 E(t)||_{L_x^\infty}$ can be treated similarly. ∎

Remark 3.3. Proposition 3.1 and Corollary 3.1 give the consistency of the ansatz for decay rate of E.

We provide the proof of Theorem 2.1. We proceed exactly as in [12], only adding the fact that the dispersion estimate in Proposition 3.1 is synchronized well in every iterating step.

Proof of Theorem 2.1.

Proof. Let f^0 be a suitable smooth extension of f_0 to $\bar{\Omega}\times\mathbb{R}^3$ satisfying the corresponding compatibility condition specified in [12] (equation (5.4)) and also the dispersion estimates in Proposition 3.1. We define the iteration sequences as follows.

$$\begin{aligned} &\partial_t f^{n+1} + v\cdot\nabla_x f^{n+1} + \nabla\phi^n\cdot\nabla_v f^{n+1} = 0 \\ &f^{n+1}(x,v,0) = f_0(x,v) \ge 0, \\ &\Delta\phi^n = d(d-2)\alpha(d)\rho^n, \quad \text{with} \\ &f^{n+1}(0,\bar{x};v_1,\bar{v},t) = f^{n+1}(0,\bar{x};-v_1,\bar{v},t), \qquad \phi^n(0,\bar{x},t) = 0. \end{aligned}$$

Then according to Lemmas 3.1 – 3.5 in the previous section, the following estimates hold for every iterating step:

$$\begin{aligned} \|\rho^n(t)\|_{L_x^\infty} + \|\nabla_x\rho^n(t)\|_{L_x^\infty} + \left\|\nabla_x^2\rho^n(t)\right\|_{L_x^\infty} &\le \frac{C\eta}{(1+t)^d}, \\ \|\rho^n(t)\|_{L_x^1} + \|\nabla_x\rho^n(t)\|_{L_x^1} &\le C\eta, \end{aligned}$$

and so does for the solution (f,ϕ). For the details, we refer to [2]. ∎

4. Uniform L^1-stability estimate

In this section, we study the uniform L^1-stability of C^1-solutions to (1.1), (1.2a) – (1.2c). For the case of full-space problem, the uniform L^1-stability has been obtained in [6] by deriving a Grownall type estimate for the L^1-distance employing the dispersion estimates. Although the proof in our presentation is similar to that of the Cauchy problem, we present its proof for self-containedness. As in [6], We first need to estimate the following functional:

$$\mathcal{K}[f](x,t) := \max_{1\le i\le d}\int_{\Omega}\frac{||\partial_{v_i}f(y,t)||_{L^1_v}}{|x-y|^{d-1}}dy.$$

LEMMA 4.1. *Suppose the main hypotheses $\mathcal{C}1-\mathcal{C}4$ in section 2 hold, and let f be a smooth C^1-solution of* (1.1), (1.2a) – (1.2c) *with smooth initial data f_0. Then we have*

$$||\mathcal{K}[f](t)||_{L^\infty_x} \le \frac{C}{(1+t)^{d-2}}.$$

Proof. Let $1\le i\le d$ and recall Lemma 3.6:

$$||\partial_{v_i}f(t)||_{L^\infty_x(L^1_v)} \le \frac{C}{(1+t)^{d-1}} \quad \text{and} \quad ||\partial_{v_i}f(t)||_{L^1_x(L^1_v)} \le C(1+t). \tag{4.1}$$

Let r be a positive constant to be determined later, and for $x\in\Omega$ we set

$$A_r(x) := \Omega\cap\{y\in\mathbb{R}^d : |y-x|\le r\} \quad \text{and} \quad B_r(x) := \Omega - A_r(x).$$

Note that

$$\begin{aligned}\int_{\Omega}\frac{||\partial_{v_i}f(y,t)||_{L^1_v}}{|x-y|^{d-1}}dy &= \int_{A_r(x)}\frac{||\partial_{v_i}f(y,t)||_{L^1_v}}{|x-y|^{d-1}}dy + \int_{B_r(x)}\frac{||\partial_{v_i}f(y,t)||_{L^1_v}}{|x-y|^{d-1}}dy\\ &\le C(d)r||\partial_{v_i}f(t)||_{L^\infty_x(L^1_v)} + \frac{||\partial_{v_i}f(t)||_{L^1_x(L^1_v)}}{r^{d-1}},\end{aligned} \tag{4.2}$$

where $C(d)$ is a positive constant depending only on d. In order to minimize the right hand side of (4.2), we choose r such that

$$C(d)r||\partial_{v_i}f(t)||_{L^\infty_x(L^1_v)} = \frac{||\partial_{v_i}f(t)||_{L^1_x(L^1_v)}}{r^{d-1}},$$

i.e.,

$$r = \Big(\frac{||\partial_{v_i}f(t)||_{L^1_x(L^1_v)}}{||\partial_{v_i}f(t)||_{L^\infty_x(L^1_v)}C(d)}\Big)^{\frac{1}{d}}.$$

Hence for such r, we use (4.1) and (4.2) to see that

$$\begin{aligned}\mathcal{K}[f](x,t) &\le 2\max_{1\le i\le d}C(d)^{\frac{d-1}{d}}||\partial_{v_i}f(t)||^{1-\frac{1}{d}}_{L^\infty_x(L^1_v)}||\partial_{v_i}f(t)||^{\frac{1}{d}}_{L^1_x(L^1_v)}\\ &\le \mathcal{O}(1)(1+t)^{-(d-2)}.\end{aligned}$$

We take a supremum over x to get the desired result. ∎

REMARK 4.1. Note that $||\mathcal{K}[f](t)||_{L^\infty_x}$ is integrable in t for $d\ge 4$.

Based on the above estimate, we obtain the uniform L^1-stability estimate.

The proof of Theorem 2.2. Let f and $\bar{f}$ be smooth C^1-solutions of (1.1), (1.2a) – (1.2c) corresponding to initial data f_0 and $\bar{f}_0$ respectively:

$$\partial_t f + v\cdot\nabla_x f + E(f)\cdot\nabla_v f = 0, \tag{4.3}$$

$$\partial_t \bar{f} + v\cdot\nabla_x \bar{f} + E(\bar{f})\cdot\nabla_v \bar{f} = 0. \tag{4.4}$$

We subtract (4.4) from (4.3) to see that

$$\partial_t(f-\bar{f}) + v\cdot\nabla_x(f-\bar{f}) + E(f)\cdot\nabla_v(f-\bar{f}) = (E(\bar{f})-E(f))\cdot\nabla_v\bar{f}. \tag{4.5}$$

Let $(x,v,t)\in\Omega\times\mathbb{R}^d\times\mathbb{R}_+$ be fixed, and let $(X(s),V(s))$ be the trajectories of particles for f passing through the point (x,v) at time t, i.e.,

$$X(t)=x \qquad \text{and} \qquad V(t)=v.$$

We integrate (4.5) along the trajectory $(X(s),V(s))$ to obtain

$$\begin{aligned}(f-\bar{f})(x,v,t) &= (f_0-\bar{f}_0)(X(0),V(0))\\ &\quad + \int_0^t (E(\bar{f})-E(f))(X(s),s)\cdot\nabla_v\bar{f}(X(s),V(s),s)ds.\end{aligned} \tag{4.6}$$

Note that Green's function for a half space and the Dirichlet boundary condition on electric potential yield

$$\begin{aligned}E(f)(x,t) &= \int_\Omega \nabla_x\left[\frac{1}{|x-y|^{d-2}} - \frac{1}{|x-y^*|^{d-2}}\right]\rho(y,t)dy\\ &= -(d-2)\int_\Omega\left[\frac{x_i-y_i}{|x-y|^d} - \frac{x_i-y_i^*}{|x-y^*|^d}\right]\rho(y,t)dy.\end{aligned}$$

Here $y^*=(-y_1,y_2,\ldots,y_d)$ denotes the reflection point of $y\in\Omega$ with respect to the $\{x_1=0\}$ plane. We use $|x-y|\le|x-y^*|$ to estimate

$$\left|\frac{x_i-y_i}{|x-y|^d} - \frac{x_i-y_i^*}{|x-y^*|^d}\right| \le \frac{1}{|x-y|^{d-1}} + \frac{1}{|x-y^*|^{d-1}} \le \frac{2}{|x-y|^{d-1}}.$$

We now take an absolute value for (4.6), and integrate it over the phase space to see that

$$\begin{aligned}&||f(t)-\bar{f}(t)||_{L^1}\\ &\le ||f_0-\bar{f}_0||_{L^1} + \int_0^t \iint_{\Omega\times\mathbb{R}^d} |(E(\bar{f})-E(f))(X(s),s)||\nabla_v\bar{f}(X(s),V(s),s)|dvdxds\\ &\le ||f_0-\bar{f}_0||_{L^1} + C\int_0^t\int_\Omega |\rho(y,s)-\bar{\rho}(y,s)|\\ &\qquad \times\Big[\int_\Omega \frac{dX(s)}{|X(s)-y|^{d-1}}\Big(\int_{\mathbb{R}^d}|\nabla_v\bar{f}(X(s),V(s),t)dV(s)\Big)\Big]dyds\\ &\le ||f_0-\bar{f}_0||_{L^1} + C\int_0^t\iint_{\Omega\times\mathbb{R}^d}\mathcal{K}[f](y,s)\Big(|f(y,v_*,s)-\bar{f}(y,v_*,s)|\Big)dv_*dyds\\ &= ||f_0-\bar{f}_0||_{L^1} + C\int_0^t ||\mathcal{K}[f](s)||_{L^\infty_x}||f(s)-\bar{f}(s)||_{L^1}ds,\end{aligned} \tag{4.7}$$

where the Liouville principle $dxdv = dX(s)dV(s)$ was employed to obtain

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}^d} \frac{1}{|X(s)-y|^{d-1}} |\nabla_v \bar{f}(X(s),V(s),t)| dX(s)dV(s)\\ &= \iint_{\Omega\times\mathbb{R}^d} \frac{1}{|x-y|^{d-1}} |\nabla_v \bar{f}(x,v,t)| dvdx.\end{aligned}$$

We now use Lemma 4.1 and (4.7) to see

$$\begin{aligned}||f(t)-\bar{f}(t)||_{L^1} &\le ||f_0-\bar{f}_0||_{L^1} + C\int_0^t ||K[f](s)||_{L^\infty_x} ||f(s)-\bar{f}(s)||_{L^1} ds\\ &\le ||f_0-\bar{f}_0||_{L^1} + C\int_0^t (1+s)^{-(d-2)} ||f(s)-\bar{f}(s)||_{L^1} ds.\end{aligned}$$

Hence Gronwall's Lemma yields

$$\begin{aligned}||f(t)-\bar{f}(t)||_{L^1} &\le ||f_0-\bar{f}_0||_{L^1} \exp\Big(C\int_0^t (1+s)^{-(d-2)} ds\Big)\\ &= G||f_0-\bar{f}_0||_{L^1},\end{aligned}$$

where

$$G := \exp\Big(C\int_0^\infty (1+s)^{-(d-2)} ds\Big) < \infty \qquad \text{for } d \ge 4.$$

This completes the proof.

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