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# NON MARKOVIAN BEHAVIOR OF THE BOLTZMANN-GRAD LIMIT OF LINEAR STOCHASTIC PARTICLE SYSTEMS

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**Abstract.** We will review some results which illustrate how the distribution of obstacles and the shape of the characteristic curves influence the convergence of the probability density of linear stochastic particle systems to the one particle probability density associated with a Markovian process in the Boltzmann–Grad asymptotics.

**Key words.** linear Boltzmann equation; Lorentz gas; lattice gas; force field; non Markovian.

**AMS subject classifications.** 82C40; 82C21; 60K35.

## 1. Introduction

The rigorous derivation of kinetic transport equations from particle systems has relevance in many problems, for applications and from a basic point of view. The easiest systems which we can analyze in this respect are the linear stochastic particle systems, namely stochastic particle systems where the particles under consideration (test particles) move among particles of different type (obstacles) interacting with the obstacles only (linear collisions). This kind of systems can be modelized through a Lorentz gas type model, where a test particle moves among obstacles which are stochastically distributed.

Linear particle systems can be described, under suitable conditions, by kinetic transport equations which are associated with a Markovian process. The Markovian behavior emerges in specific asymptotics, such as the Boltzmann–Grad limit, and corresponds to the fact that the obstacles are not too regularly distributed and that the probability of having recollisions (i.e. more than one collision with a given obstacle) for the test particle vanishes in this asymptotics.

In this paper we will give a general survey of results which describe how the non–Markovian character of a linear stochastic particle system may disappear in the Boltzmann–Grad asymptotics: in this case, the particle system is described by a kinetic equation of linear Boltzmann type. We will then discuss a relevant situation in which this may not happen; that is the case of the stochastic Lorentz gas in the presence of a force field. The present work is based on previously published results, therefore we will present only the main ideas and theorems, with references to the bibliography for more details on the subject.

In the first two sections we will review the derivation from particle systems of the linear Boltzmann equation from the Lorentz gas in the case of continuous and periodic distributions of obstacles; in the third section we will discuss the case of a system of absorbing obstacles in the presence of a force field.

## 2. Validity proof for the linear Boltzmann equation: obstacles with Poisson distributed positions

The dynamics of light particles moving in a medium can be described through the evolution of their single particle probability density in the phase space in a suitable asymptotics.

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We consider here one of the possible evolution equation for the probability density, the linear Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mu \int_{S_-^{d-1}} dn \Gamma(v,n)(f(t,x,v^{'}(v,n)) - f(t,x,v)) \\ f(0,x,v) = f_{in}(x,v) \in L^1(\mathbb{R}^d \times S^{d-1}), \end{cases} \tag{2.1}$$

which describes a cloud of light particles noninteracting among themselves and interacting with the medium through a collision process such that the kinetic energy of the light particles is preserved. In (2.1) $x \in \mathbb{R}^d$ and $v \in S^{d-1}$ are the position and the velocity variables, $t \in [0,T] \subset \mathbb{R}$ is the time variable, $f$ is the one particle probability density of the light particles in the phase space, $n \in S_-^{d-1} = \{\omega \in S^{d-1} | \omega \cdot v < 0\}$ is a unit vector, which we will call impact parameter, the integral operator on the right hand side represents the interaction of the particles with the medium, the positive function $\Gamma \in C(S^{d-1} \times S^{d-1})$ is the cross section associated with the collision process (we will consider collision processes described by a central potential), $v^{'} = v^{'}(v,n)$ is the post–collisional velocity (and $v^{'}(\cdot, n)$ is invertible).

We want to derive (2.1) from a particle system with minimal level of randomness, and in particular such that the interaction of the particles with the medium is deterministic.

A particle system which can be associated with (2.1) is the so called *Lorentz gas*, which is defined as follows.

We consider a test point particle, having initial position $(x,v) \in \mathbb{R}^d \times S^{d-1}$; the test particle moves among fixed obstacles, interacting with them through a positive, compactly supported potential $V_\varepsilon(z) = V(|z|/\varepsilon)$, $z \in \mathbb{R}^d$, of size of the support $r = \varepsilon$. $V$ is the potential associated with the cross section $\Gamma$ and we will not go into further details in this general introduction. The obstacles positions $\mathbf{c} = (c_1, \dots, c_n, \dots)$, $c_i \in \mathbb{R}^d$, are distributed according to a given probability distribution $P(d\mathbf{c})$, with mean density $\mu_\varepsilon$.

The flow $T^t_{\mathbf{c}}(x,v) = ((T^t_{\mathbf{c}}(x,v))_1, (T^t_{\mathbf{c}}(x,v))_2) = (x_{\mathbf{c}}(t), v_{\mathbf{c}}(t))$ associated with the equation of motion of the test particle

$$\begin{cases} \dot{x}_{\mathbf{c}} = v_{\mathbf{c}} \\ \dot{v}_{\mathbf{c}} = F_{\mathbf{c}}(t, x_{\mathbf{c}}, v_{\mathbf{c}}) \end{cases} \tag{2.2}$$

($F_{\mathbf{c}}$ is the force which acts on the particle during the interaction with the obstacles and we will consider the flow to be well–defined outside a set of vanishing measure w.r.t. $P(d\mathbf{c})$. This is the case in all situations we will consider.) induces an evolution on probability densities through the following definition on test functions $\phi$:

$$\int dx dv f_\varepsilon(t,x,v)\phi(x,v) = \int dx dv f_0(x,v) \mathbb{E}^c[\phi(T^t_{\mathbf{c}}(x,v))] \tag{2.3}$$

where $f_0$ is an initial probability density and $\mathbb{E}^c$ denotes the expectation with respect to $P(d\mathbf{c})$.

We say that equation (2.1) can be derived from the Lorentz gas dynamics in a suitable topology if $\lim_{\varepsilon \to 0} f_\varepsilon = f$ in that topology, where $f$ is the unique solution of (2.1).

The asymptotics in which this derivation is allowed, under suitable conditions on $P(d\mathbf{c})$, is the so called Boltzmann–Grad limit, a low density asymptotics such that $\varepsilon \to 0$, $\mu_\varepsilon \to \infty$, $\varepsilon^{d-1}\mu_\varepsilon \to \mu > 0$, where $\mu$ is a constant proportional to the inverse of the mean free path. Since (2.1) describes the evolution in time of the one particle probability density of a Markov process ([11]), this derivation is possible only if in the limit the process associated with the Lorentz gas dynamics (2.2) loose the non–Markovian character of the process for finite $\varepsilon$. In order to describe how this may happen, we start by illustrating the general procedure to validate (2.1) from particle systems for obstacles which are distributed according to a Poisson repartition, which has been performed first in [9] for hard–spheres obstacles. A derivation for more general potentials and distributions can be found in [13], while convergence for typical configurations is proved in [1]. In what follows, we will denote by $B_x(r)$ the sphere centered in $x$ and with radius $r$ and by $\chi(A)$ the characteristic function of the set $A$; we will call trajectory the trajectory in the position space associated with (2.2).

Let us then consider a Lorentz gas in which the scatterers are distributed according to a Poisson law with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-(d-1)}$, namely such that the probability distribution of finding exactly $N$ obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^d$ is given by:

$$P(d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|}\frac{\mu_\varepsilon^N}{N!}dc_1\dots dc_N, \tag{2.4}$$

Since the collision process (and the intercollisional dynamics) preserves $|v|$, the trajectory of the test particle is entirely contained in a ball of radius $R(T) = |v|T = T$. Moreover, equation (2.2) is reversible, so that we can express the evolved probability density as

$$f_\varepsilon(x,v,t) = \mathbb{E}^c[f_{in}(T_{\mathbf{c}}^{-t}(x,v))] = \sum_{N\geq 0} e^{-\mu_\varepsilon\,|B_0(R(T))|}\frac{\mu_\varepsilon^N}{N!}$$
$$\times\int_{B_0(R(T))}dc_1..\int_{B_0(R(T))}dc_N\chi(c_1\notin B_x(\varepsilon),\dots,c_N\notin B_x(\varepsilon))f_{in}(T_{\mathbf{c}}^{-t}(x,v))$$

or, defining[1]

$$\theta_\varepsilon^{\mathbf{c}}(t,x,v) = \{y\in\mathbb{R}^d, \exists s\in[0,t], |y-(T_{\mathbf{c}}^{-s}(x,v))_1|\leq\varepsilon\} \tag{2.5}$$

the tube of width $\varepsilon$ around the trajectory, and denoting by $C_1(x;\varepsilon) = \{c_1\notin B_x(\varepsilon),\dots,c_n\notin B_x(\varepsilon)\}$ and $C_2(t,x,v;\varepsilon) = \{c_1\in\theta_\varepsilon^{\mathbf{c}},\dots,c_n\in\theta_\varepsilon^{\mathbf{c}}\}$

$$f_\varepsilon(t,x,v) = \sum_{N\geq 0}\int_{(B_0(R(T)))^N}dc_1..dc_N e^{-\mu_\varepsilon\,|\theta_\varepsilon^{\mathbf{c}}|}\frac{\mu_\varepsilon^N}{N!}\chi(C_1)\chi(C_2)\,f_{in}(T_{\mathbf{c}}^{-t}(x,v)) \tag{2.6}$$

We want to compare (2.6) with the semi–explicit form of the solution of (2.1)

$$f(t,x,v) = \sum_{N\geq 0}\int_0^t dt_1\dots\int_0^{t_{N-1}}dt_N\int_{S_-^{d-1}}dn_1\dots\int_{S_-^{d-1}}dn_N\prod_{k=1}^N\Gamma(v_{k-1},n_k)$$
$$\times e^{-\mu\sum_{k=1}^{N+1}(t_{k-1}-t_k)\int_{S_-^{d-1}}dn_k\Gamma(v_{k-1},n_k)}\mu^N\,f_{in}(T_{\mathbf{t},\mathbf{n}}^{-t}(x,v)) \tag{2.7}$$

where $\mathbf{t} = (t_1,\dots,t_N)$, $t_i < t_{i-1}$, $i = 1,\dots,N+1$, $\mathbf{n} = (n_1,\dots,n_N)$, $\mathbf{v} = (v_1,\dots,v_N)$ are resp. sequences of collision times, impact parameters and (pre–collisional) velocities (recall that $v_n = v_n(n_n, v_{n-1})$), $t_0 = t$, $t_{N+1} = 0$, $v_0 = v$. Here $T_{\mathbf{t},\mathbf{n}}^{-t}(x,v) = (x-$

[1] The notation $(T_{\mathbf{c}}^{-s}(x,v))_1$ designates the first component (i.e. the position) in the couple $T_{\mathbf{c}}^{-s}(x,v)$.

$v(t-t_1)-v_1(t_1-t_2)-\ldots-v_N t_N, v_N)$ is the backward flow associated with the Boltzmann process described by (2.1). We need therefore to express the expectation value in (2.6), or part of it, in the same variables used in (2.7), i.e., collision times and impact parameters. Notice that $\|f_\varepsilon\|_{L^1(\mathbb{R}^d\times S^{d-1})} = \|f\|_{L^1(\mathbb{R}^d\times S^{d-1})} = \|f_{in}\|_{L^1(\mathbb{R}^d\times S^{d-1})} = 1$.

We observe that one of the reasons why (2.6) cannot be associated in general to a Markov process is that the measure of the flow tube $|\theta_\varepsilon^{\mathbf{c}}|$ does depend on the configuration of obstacles in a "non–factorizable" way; namely, assuming that $\mathbf{c}$ is ordered in such a way that $c_1$ is the first obstacle hit by the test particle, $c_2$ the second one, $c_3$ is the third (without counting recollisions with $c_1$), and so on, $|\theta_\varepsilon^{\mathbf{c}}|$ is not the sum of contributions coming from segments of the trajectory connecting $c_i$ with $c_{i+1}$. The main source of correlations comes from recollisions with a given obstacle and it is related, besides the mean density of $P(d\mathbf{c})$, to the topology of the trajectory in the space variable; less relevant effects are those due to the finite size of the obstacles (for instance, given that collisions are stochastic only because of the stochastic positions of the obstacles, at finite $\varepsilon$ two consecutive intercollisional laps of the trajectory are not independently distributed). These factors do affect the measure of the tube $|\theta_\varepsilon^{\mathbf{c}}|$.

Having in mind to compare (2.6) with (2.7) we proceed in the following way. First of all we notice that, on the set of configurations $C(t,x,v;\varepsilon)$ of non–superimposing obstacles $\mathbf{c}$, $c_i \in \theta_\varepsilon^{\mathbf{c}}$, $i=1,\ldots,N$, such that the test particle collides just once with each obstacle of the configuration, it is possible to define in a unique way, for each obstacle $c_i$, the entrance time $t_i$ of the test particle in the ball $B_{c_i}(\varepsilon)$ and its impact parameter $n_i$, as the unit exterior normal vector to $\partial B_{c_i}(\varepsilon)$ at the collision point. We can therefore define on this set the application ([9], [6], [7])

$$c_1,\ldots,c_N \longleftrightarrow t_1,n_1,\ldots,t_N,n_N, \tag{2.8}$$

($\mathbf{c}=\mathbf{c}(\mathbf{t},\mathbf{n})$) where $c_i$ is the $i$-th obstacle hit by the trajectory, with jacobian $J = (\varepsilon^{d-1})^N \prod_{k=1}^N \Gamma(v_{k-1},n_k)$, and express $f_\varepsilon$ as

$$f_\varepsilon = f_\varepsilon^M + f_\varepsilon^{NM}$$

where

$$\begin{aligned} f_\varepsilon^M(x,v,t) &= \sum_{N\ge 0}\int_{(B_0(R(T)))^N} dc_1\ldots dc_N e^{-\mu_\varepsilon |\theta_\varepsilon^{\mathbf{c}}|}\frac{\mu_\varepsilon^N}{N!}\,\chi(C_1\cap C_2\cap C) f_{in}(T_{\mathbf{c}}^{-t}(x,v)) \\ &= \sum_{N\ge 0}\int_0^t dt_1\ldots\int_0^{t_{N-1}} dt_N \int_{S_-^{d-1}} dn_1\ldots\int_{S_-^{d-1}} dn_N \prod_{k=1}^N \Gamma(v_{k-1},n_k)(\varepsilon^{d-1}\mu_\varepsilon)^N \\ &\times e^{-\mu_\varepsilon |\theta_\varepsilon^{\mathbf{c}(\mathbf{t},\mathbf{n})}|}\,\chi(\{(\mathbf{t},\mathbf{n})\in \mathbf{c}^{-1}(C_1\cap C_2\cap C)\}) f_{in}(T_{\mathbf{t},\mathbf{n}}^{-t}(x,v)) \end{aligned}$$

(and $f_\varepsilon^{NM} = f_\varepsilon - f_\varepsilon^M$). Since it is $f_\varepsilon^M$ which, in suitable conditions, can eventually be associated with a Markov process in the Boltzmann–Grad asymptotics, we will denote it in what follows as the Markovian part of $f_\varepsilon$ (in opposition to the non–Markovian part $f_\varepsilon^{NM}$: the denomination has no a priori relationship with the mathematical properties of the two components as densities associated with a stochastic process).

The function $f_\varepsilon^M$ reflects the effect of recollisions because of the forbidden volume associated with the characteristic function which defines the set where the change of variables (2.8) is one–to–one: in order to obtain a Markov process, we would like this forbidden volume to vanish in the Boltzmann–Grad limit. In the case of obstacles with Poisson distribution, this is in fact what happens, as a direct consequence of the structure of $P(d\mathbf{c})$ and of the trivial intercollisional dynamics. The convergence of $f^M$ follows, through Lebesgue's dominated convergence theorem, from $\mu_\varepsilon|\theta^{\mathbf{c}}_\varepsilon| \approx \mu \sum_{k=1}^{N+1} \int_{t_k}^{t_{k-1}} d\tau \int_{S_-^{d-1}} dn_k \Gamma(v_{k-1}, n_k)$, on $\chi(\mathbf{c}^{-1}(C_1 \cap C_2 \cap C)) = 1$, and from $\lim_{\varepsilon\to 0} \mathbb{E}^c[\chi(\mathbf{c}^{-1}(C_1 \cap C_2 \cap C))] = 1$. Since $\|f^M_\varepsilon\|_{L^1(\mathbb{R}^d\times S^{d-1}))} \to \|f_{in}\|_{L^1(\mathbb{R}^d\times S^{d-1})}$, the validity of (2.1) is proved (see [9] for the hard–sphere case).

The Markovian behavior emerging in the Boltzmann–Grad asymptotics can be explained as follows: since the intercollisional trajectories of the test particle are straight lines, the test particle can recollide with a given obstacle only after having suffered a collision with a different obstacle in the medium. The probability of this kind of event vanishes in the Boltzmann–Grad limit for the repartition $P(d\mathbf{c})$ given by (2.4) and therefore the correlations due to recollisions disappear when $\varepsilon \to 0$.

*Remark* In this particularly simple case, since the $L^1$-norm of (2.6) and (2.7) is preserved, proving $f^M_\varepsilon \xrightarrow{L^1} f$ implies $f^{NM}_\varepsilon \xrightarrow{L^1} 0$, so that no explicit evaluation of the non–Markovian part is required. This is not the case when in (2.1) the linear operator on the right-hand side is such that no semi–explicit form for the solution for the corresponding equation is available, like in the case of long range potentials ([6]) or of Fokker–Planck's type equations ([7]): in such cases, an explicit bound on $f^{NM}$ is required.

**3. Proof of validity for the linear Boltzmann equation: obstacles on a lattice**

When we consider a probability distribution of obstacles such that the centers of the obstacles $c_i$ lie on a lattice, the situation changes. In this case there is one more source of non–Markovianity, namely the fact that we are working on a discrete space: even though we would consider random collision laws in the absence of recollisions, the particle process at finite $\varepsilon$ cannot be Markovian, as a process in continuous time. Indeed,time intervals between collision times are now correlated, due to the constraint that the centers of the obstacles have to lie on the lattice. These correlations can remain in the Boltzmann–Grad asymptotics, as a memory effect due to the regularity of the distribution of obstacles.

In order to understand what may happen for periodic distributions of obstacles, we will analyze the following particle system in $\mathbb{R}^2$ (*lattice gas*, see [5], [12]).

Let consider the two-dimensional lattice $\mathbb{Z}^2_\lambda = \{(j_1\lambda, j_2\lambda)| j_i \in \mathbb{Z}, \quad i = 1,2\}$ with cells of size $\lambda = \varepsilon^{1/(2-\delta)}$, $0 < \delta \le 1$, and $\mathcal{C} = \{((j_1 + 1/2)\lambda, (j_2 + 1/2)\lambda)| j_i \in \mathbb{Z}, \quad i = 1,2\}$, the lattice formed by the centers. On each site of the lattice we define a random variable $n_c$, the occupation number, taking the value 1 or 0 with probability $p \equiv \varepsilon^{\delta/(2-\delta)}$ and $1-p$ respectively, independently for all $c \in \mathcal{C}$; in each $c$ s.t. $n_c = 1$ we place a disk of radius $\varepsilon$. For a given scatterer configuration $\{n_c\}_{c\in\mathcal{C}}$, we denote by $\Lambda_{\mathbf{c}} = \bigcup_{n_c=1} B_c(\varepsilon)$ the region occupied by the set of scatterers.

The relationship among the parameters ($p$, $\lambda$, $r$) corresponds to the Boltzmann–Grad scaling.

We consider then a test point particle with initial position $x \in \mathbb{R}^2 \setminus \partial\Lambda_{\mathbf{c}}$, and with an initial velocity $v \in S^1$ (as a matter of convenience, we allow particles to start inside a scatterer, since, in the limit as $\varepsilon$ goes to zero, the fractional volume of the scatterers vanishes). The particle moves with velocity $v$ until the time $t = \min\{\tau > 0 \mid x + v\tau \in \partial\Lambda_{\mathbf{c}},\ v\cdot\omega \le 0\}$ at which it encounters an obstacle and interacts with it through an hard–sphere interaction (elastic collision). The post–collisional velocity is given by $v' = v - 2(v \cdot n)n$, where $n$ is the unit exterior normal to the surface of the obstacle at the point met by the test particle. The procedure is then iterated. We denote by $z_\varepsilon(t) = T_\varepsilon^t(x, v)$ the flow constructed in this way.

As usual, we want to compare the evolved probability density for the particle system, $f_\varepsilon(x, v, t)$, defined through the flow $T_\varepsilon^t$ for a given $f_{in}$ by (2.3), with the semi–explicit solution of (2.1) given by (2.7), but in this case no expression similar to (2.6) is available. We therefore proceed in the following way: we first define a process $\tilde{z}_\varepsilon$ which is Markovian as a discrete time process; we prove the convergence of the probability density associated with this Markovian process, $\tilde{f}_\varepsilon$, to the solution of (2.1); we prove finally the equivalence, in the Boltzmann–Grad limit, between $\tilde{z}_\varepsilon$ and the original process $z_\varepsilon$.

The Markovian process $\tilde{z}_\varepsilon$ is defined considering again the lattice $\mathcal{C}$ and assuming now that the occupation number $n_c = 1$, $\forall c \in \mathcal{C}$ (in each site of the lattice we have an obstacle of radius $\varepsilon$ with center on the site); a test particle moves freely among the obstacles and, each time it encounters an obstacle, it performs an elastic collision with probability $p = \varepsilon^{\delta/(2-\delta)}$ or goes ahead with probability $1 - p$. We will denote by $\tilde{T}_\varepsilon^t(x, v)$ the corresponding flow. The probability of realization of a given trajectory is $p^k(1 - p)^h$, where $k$ is the number of collisions and $h$ the number of times that the trajectory crosses an obstacle. The process $\tilde{z}_\varepsilon$ is not reversible, but the collision law is, so that it is possible to write $\tilde{f}_\varepsilon(x, v, t) = \mathbb{E}[(Rf_{in})(\tilde{T}_\varepsilon^t(x, -v))]$, where $(Rf)(x, v) = f(x, -v)$ and $\mathbb{E}$ denotes the expectation value with respect to $\tilde{z}_\varepsilon$. Since for the process $\tilde{z}_\varepsilon$ recollisions are absent (each collision is independent from the others), the only relevant source of non–Markovian behavior (as a continuous time process) is now the discretization of the space. Moreover, the evolution induced on functions by $\tilde{z}_\varepsilon$, as in the case of the evolution associated with (2.1), is described by a linear semi–group. More precisely, we define the sets $S_i$ of possible times for the $i$-th scattering for a trajectory starting at $(x, v)$ as

$$
\begin{aligned}
S_1(t) &= \left\{\tau \in (0, t) \mid x + v\tau \in \partial\Lambda_{\mathbf{c}},\ v \cdot \omega \le 0\right\}, \\
&\qquad\qquad\qquad\vdots \\
S_n(t, t_1 \dots t_{n-1}) &= \big\{\tau \in (t_{n-1}, t) \mid t_i \in S_i,\ i = 1 \dots n-1, \\
&\qquad x + t_1 v \dots + (\tau - t_{n-1}) v_{n-1} \in \partial\Lambda_{\mathbf{c}},\ v_n \cdot \omega \le 0\big\} \\
S_{n+1}(t, t_1 \dots t_n) &= \big\{\tau \in (t_n, t) \mid t_i \in S_i,\ i = 1 \dots n, \\
&\qquad x + t_1 v \dots + (\tau - t_n) v_n \in \partial\Lambda_{\mathbf{c}},\ v_{n+1} \cdot \omega \le 0\big\},
\end{aligned}
$$

where $v_0 = v$ and $v_i = v'_{i-1}$ is the postcollisional velocity with incoming velocity $v_{i-1}$. We denote then by $s_n := |S_n(t, t_1 \dots t_{n-1})|$ the cardinality of the set $S_n$, and by $k^{(n)} = |S_{n+1}(t, t_1 \dots t_n)| + \sum_{j=1}^n |S_j(t_i, t_1 \dots t_{j-1})|$ the number of encounters with an obstacle which do not correspond to a scattering event, given that scattering did occur at $t_1 ... t_n$. With these definitions, we have, for a function $g$,

$$
g_\varepsilon(x, v, t) = \sum_{n \ge 0} (V_\varepsilon^t g)_n(x, v, t) = \sum_{n \ge 0} g_{\varepsilon,n}(x, v, t)
$$

where $g_{\varepsilon,0}(x, v, t) = (1 - \varepsilon)^{s_1} g(x + vt, v)$ and, for $n > 1$,

$$
\begin{aligned}
g_{\varepsilon,n}(x, v, t) = \sum_{t_1 \in S_1(t)} \cdots \sum_{t_n \in S_n(t, t_1 \dots t_{n-1})} & p^n (1 - p)^{k^{(n)}} \\
& g(x + t_1 v + (t_2 - t_1) v_1 + \dots + (t - t_{n-1}) v_n, v_n)\,.
\end{aligned}
$$

Notice that here the quantity $-k^{(n)}\log(1-p)$ is somehow analogous to $\mu_\varepsilon|\theta^c|$ for the continuous model in (2.6).

For this particle system it is possible to prove the following theorem ([12]).

THEOREM 3.1. *Let $f_{in}:\mathbb{R}^2\times S^1\to\mathbb{R}^+$ be the initial probability density. Then, for any $t>0$, $0<\delta\le 1$: $\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(\cdot,t)=f(\cdot,t)$ weakly in measure, where $f(\cdot,t)\in L^1(\mathbb{R}^2\times S^1)$ is the unique solution of the transport equation (2.1) with $\mu=1$, $\Gamma(v,n)=\frac{1}{2}|v\cdot n|$ and $f(x,v,0^+)=f_{in}(x,v)$.*

The proof of the theorem is obtained by estimating the number of crossings with the obstacles of a line segment of length $L$ starting from $(x,v)$, $s(x,v,L)$: when $\varepsilon\to 0$, we get $\varepsilon^{\delta/(2-\delta)}s(x,v,L)\approx L$ (with an error bounded in $L^1(S^1;L^\infty(\mathbb{R}^2))$; we then use this estimate to evaluate the distance between the two semi–groups associated respectively with the evolution of the Markovian particle system and with the evolution associated with (2.1). Since for both semi–groups the part of the two evolved functions associated with trajectories which have two or more collisions (up to the time $t$) is bounded by $t^2$ (see Prop. 3 in [12]), it is sufficient to prove that in the $\varepsilon\to 0$ limit $g_{\varepsilon,0}(x,v,t)\approx e^{-t}g(x+vt,v)$ and $g_{\varepsilon,1}(x,v,t)=\sum_{t_1\in S_1(t)}p(1-p)^{s_1+s_2}g(x+t_1v+(t-t_1)v_1,v_1)\approx\frac{e^{-t}}{2}\int_0^t dt_1\int_{S^1_-}g(x+t_1v+(t-t_1)v_1,v_1)|v\cdot n|\,dn$ in $L^1$, for small enough $t$, i.e. that the distribution of trajectories with at most one jump for the Markovian process and for the linear Boltzmann process are near when observed on small time intervals; then, using the semi–group property of the evolutions, we get an $L^1$ estimate of the distance between two evolved functions at a generic time $t$. The fact that the behavior of $s(x,v,L)$ is asymptotically $s(x,v,L)\approx Lp^{-1}$ is one of the key ingredient of the proof: it means that in the $\varepsilon\to 0$ limit the regularity of the lattice does not prevent the limit process to have an exponential distribution for the free path lengths (or equivalently for the free times).

Then, since $\tilde{z}_\varepsilon$ and $z_\varepsilon$ differ only on the set of trajectories s. t. there are obstacles which are met more than once, the equivalence between the two processes is proved by showing that the measure of this set (both with respect to $\tilde{z}_\varepsilon$ and $z_\varepsilon$) vanishes in the $\varepsilon\to 0$ limit (see sec. 5 in [12]). We get finally the convergence theorem for the original particle system [12]:

THEOREM 3.2. *Let $f_{in}:\mathbb{R}^2\times S^1\to\mathbb{R}^+$ be the initial probability density. Then, for any $t>0$, $0<\delta\le 1$: $\lim_{\varepsilon\to 0}f_\varepsilon(\cdot,t)=f(\cdot,t)$ in $\mathcal{D}'$, where $f(\cdot,t)$ is the unique solution of the transport equation (2.1) with $\mu=1$ and $\Gamma(v,n)=\frac{1}{2}|v\cdot n|$ and $f(x,v,0^+)=f_{in}(x,v)$.*

As in the case of the Poisson distributed obstacles, we see that the stochasticity in the positions of the obstacles is relevant to get a vanishing measure for the set of trajectories with recollisions. We notice, by the way, that the distribution of obstacles for the lattice gas is asymptotically equivalent to the Poisson distribution. In addition to that, we need to have an estimate for $s(x,v,L)$ of the kind we mentioned in the Boltzmann-Grad limit, namely not strongly depending on $v$ (and therefore on the structure of the lattice): this is trivial in the case $\delta=1$, where the lattice s.t. $n_c=1$ for all $c\in\mathcal{C}$ is a finite horizon billiard, but it is not in the case $0<\delta<1$, when the associated billiard has an infinite horizon.

This convergence result for the lattice gas is no more valid when we consider the limit $\delta=0$, that is the deterministic particle system in which $n_c=1$ for each lattice site and $p=1$. This case has been studied in [3], [10], [4], for a periodic Lorentz gas with absorbing obstacles: for such a system, the first collision time is not exponentially distributed because the structure of the lattice allows very long free trajectories which do influence in a relevant way the free times distribution. Notice that for a lattice gas with absorbing obstacles, $z_\varepsilon = \tilde{z}_\varepsilon$.

## 4. Proof of validity for the linear Boltzmann equation: obstacles with Poisson distributed positions and force field

As we saw in the previous sections, the linear Boltzmann equation can be derived from linear particle systems such that the distribution of the positions of the obstacles is stochastic: this is sufficient in many cases to obtain a Markovian limit. This is no more the case when a force field is present. An example of deviation from the Markovian behavior is given in [2], where the authors analyze the motion in $d = 2$ of charged particles in a medium under the action of a constant magnetic field $B$ and derive heuristically in the Boltzmann-Grad asymptotics an equation which has a non–Markovian collision term. This is a consequence of the high probability of recollision with a given obstacle, which originates from the peculiar shape of the trajectory of the test particle in the dynamics defined by the Lorentz force.

In fact, when the trajectories are no more straight lines, recollisions may occur even in the presence of just one single obstacle (this is for instance the case in [2], where the intercollisional trajectories are arcs of a circle), so that a good stochasticity in the distribution of the positions of the obstacles may no more be enough to guarantee the Markovianity of the Boltzmann–Grad limit.

We will illustrate here how such a non–Markovian behavior of the Boltzmann–Grad limit can be rigorously derived for stochastic particle systems in which the random medium is modeled through fixed obstacles and under the action of a force field; we will show moreover how, adding some stochasticity in the distribution of the velocity of the obstacles, it is possible to recover a Markovian limit.

We consider a Lorentz gas with Poisson distributed, absorbing obstacles and a force field $F \equiv F(t,x)$, $F \in C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$ acting on the test particle: $P(dc)$ is given then by (2.4) and the test particle, with initial position $x \in \mathbb{R}^2$ and initial velocity $v \in \mathbb{R}^2$, moves according to the equation of motion

$$\frac{d}{dt}(T_1^t(x,v)) = T_2^t(x,v), \quad \frac{d}{dt}(T_2^t(x,v)) = F(t, T_1^t(x,v)), \tag{4.1}$$

up to the first time $\tau_{\mathbf{c}}(x,v)$ when the particle enters an obstacle; then the particle disappears. The flow $T^t$, which does not depend on the configuration of obstacles, is well–defined for all $t$ and for $t \in [0,T]$ the trajectory $T_1^{-t}(x,v)$ is included in some ball $B_0(R(T))$ depending on the initial datum $(x,v)$.

This particle system is particularly easy to study because recollisions are absent and $f_\varepsilon = f_\varepsilon^M$ can be calculated explicitly, being

$$f_\varepsilon(t,x,v) = \mathbb{E}^c[f_{in}(T^{-t}(x,v))\, 1_{\{t \le \tau_{\mathbf{c}}(x,v)\}}] = e^{-\mu_\varepsilon\, |\theta_\varepsilon(t,x,v)|}\, f_{in}(T^{-t}(x,v)) \tag{4.2}$$

for a given initial datum $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$. The behavior in the Boltzmann–Grad limit of such a particle system would be Markovian if the limit density would satisfy the following (generalized) linear Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -2\, \mu\, |v|\, f \tag{4.3}$$

whose solution is

$$f(t,x,v) = e^{-2\mu \int_0^t |T_2^{-s}(x,v)| ds}\, f_{in}(T^{-t}(x,v)). \tag{4.4}$$

In order to have $\lim_{\varepsilon\to 0} f_\varepsilon = f$, the asymptotic behavior for the tube (2.5) when $\varepsilon \to 0$ should be given by:

$$\varepsilon^{-1}|\theta_\varepsilon(t,x,v)| \approx 2\int_0^t |T_2^{-s}(x,v)|ds. \tag{4.5}$$

Yet the best we can hope for actually is to get:

$$\varepsilon^{-1}|\theta_\varepsilon(t,x,v)| \approx 2l(\gamma) \tag{4.6}$$

where $l(\gamma)$ is the 1-dimensional measure of the set $\gamma = \{y \in \mathbb{R}^2, y \in \cup_{\sigma\in[0,t[}\{T_1^{-\sigma}(x,v)\}\}$. In (4.6) we have the length of the trajectory of the particle up to time $t$, while in (4.5) it appears the distance travelled by the particle along the trajectory up to time $t$, and the two quantities in general do not coincide; whenever $l(\gamma) \neq \int_0^t |T_2^{-s}(x,v)|ds$, (4.2) does not converge in the $\varepsilon \to 0$ limit to (4.4), and this is for instance what happens in [2] for $t > 2\pi/constB$. In more pathological cases, $\varepsilon^{-1}|\theta_\varepsilon(t,x,v)|$ may not follow at all the behavior described by (4.6). These two situations correspond to the fact that after a time smaller than the considered time $t$, the particle comes sufficiently near to a space point it already visited in the past, and therefore it is possible to predict if it will suffer or not a collision, and this happens for a set of times whose closure has non-vanishing measure in the limit.

What actually can be proved for this particle system is the following theorem ([8]):

Theorem 4.1. *Let* $\mathbf{c}$ *be distributed according to the Poisson's repartition (2.4) on* $\mathbb{R}^2$ *and* $F \equiv F(t,x) \in C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$ *such that for a. e. initial data* $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $T_2^s(x,v) \neq 0$ *for* $s \in \mathbb{R}$, *where* $T^t$ *is the flow defined (for* $t \in \mathbb{R}$*) by (4.1). Then for* $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, *we have, when* $\varepsilon \to 0$, $f_\varepsilon \overset{L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}{\longrightarrow} f$ *for all* $T > 0$, *where* $f$ *is the unique solution in* $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ *of the equation*

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = -\,2\,\mu\,|v|\,f\,1_{\{x\neq T_1^{-s}(x,v),s\in]0,t[\}} \\ f(0,x,v) = f_{in}(x,v). \end{cases} \tag{4.7}$$

The proof of the theorem is based on a lemma which states the validity of (4.6) for the class of forces considered: for this class in fact the radius of curvature of the space trajectory is bounded from below, a condition which prevents accumulation points of self–crossing of the trajectory and allows to obtain estimates of the measure of the flow tube by using a convenient set of coordinates (see [8]).

Equation (4.7) differs from (4.3) whenever the trajectories cross themselves, for a set of times of nonzero measure, for a nonzero measure set of initial data, and in this case the behavior of the solution is non–Markovian. This may happen for very common force fields (for instance, we have such a kind of behavior for the harmonic oscillator, $F(x) = -x$), and of course for forces depending on the velocity, as the Lorentz force in [2], since it is a consequence of the topological properties of the trajectories associated with (2.2).

In order to recover a Markovian behavior, we need to impose some additional stochasticity.

We do it by considering a system of absorbing obstacles in a force field $F(t,x)$, with initial position $\mathbf{c}$ still distributed according to the Poisson law (2.4), but now moving with a (fixed) velocity $\mathbf{w} = (w_1,..,w_N)$ which is distributed according to a centered Gaussian law with variance 1. The velocities of the obstacles are independent from each other and independent of $\mathbf{c}$. We denote the expectation with respect to the measure we just described by $\mathbb{E}^{c,w}$ and the density associated with this particle system as $g_\varepsilon(t,x,v) = \mathbb{E}^{c,w}[g_{in}(T^{-t}(x,v))\, 1_{\{t\leq\tau_{\mathbf{c},\mathbf{w}}(x,v)\}}]$

Using a procedure analogous to the one used to prove theorem (4.1), we get the following theorem

Theorem 4.2. *Let $\mathbf{c},\mathbf{w}$ be given by a repartition as described above, and $F \equiv F(t,x)$ be a given force in $C(\mathbb{R};W^{1,+\infty}(\mathbb{R}^2))$. Then for a given $g_{in} \in L^1(\mathbb{R}^2\times\mathbb{R}^2)$, $g_\varepsilon \overset{L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}{\longrightarrow} g$ for all $T>0$, where $g$ is the unique solution in $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ of the equation*

$$\begin{cases} \partial_t g + v\cdot\nabla_x g + F\cdot\nabla_v g = -\,2\,\mu\, g \int_{w\in\mathbb{R}^2} |v-w|\, \frac{e^{-\frac{|w|^2}{2}}}{2\pi}\, dw \\ g(0,x,v) = g_{in}(x,v) \end{cases} \tag{4.8}$$

Equation (4.8) describes now a Markov process, and it is reasonable to think that the same result is valid for other kinds of stochasticy as the one we chose. Notice that in this case, the only hypothesis we make on the force field is that $F$ is Lipschitz; the theorem is therefore very general and gives a way to recover a Markovian limit for linear particle systems in a wide class of force fields. Moreover, it suggests that the non–Markovianity arising in the linear case may not occur in the non–linear case (Vlasov–Boltzmann equation).

In conclusion, we did review some results about the derivation from particle systems of the linear Boltzmann equation (2.1) and of its generalized form (4.3). We did show that, while for the free Lorentz gas a good stochasticity in the positions of the obstacles allows to derive in the Boltzmann–Grad limit a Markovian equation for the probability density, when a force field is present we need an additional stochasticity (in the velocity of the obstacles) in order to get a Markovian behavior in the limit. This is due to the high probability of the set of trajectories with recollisions which may derive from the topology of the trajectories associated with the equation of motion (4.1) even for nice probability distributions for the localization of obstacles in the medium.

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