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INSTABILITY OF VARIABLE MEDIA TO LONG WAVES WITH ODD DISPERSION RELATIONS*

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Abstract. The instability of variable media to a broad class of long waves having dispersion relations that are an odd function of wavenumber is examined. For Hamiltonian media, new necessary conditions for the existence and structure of global modes are obtained. For non-Hamiltonian media, an analysis of the complex WKB branch points yields explicit expressions for the frequency and structure of the global modes, which manifest as spatially oscillatory wave packets or smooth envelope structures. These distinct modes and their locations within the media can be predicted by simply examining the local convergence or divergence of the group velocity in the long wave limit.

Key words. Linear instability, Hamiltonian dynamics, variable media, long waves

MSC subject classifications. 76E09, 76E15, 76E20, 76E25

1. Introduction

Linear instabilities are often the crucial first stage in the evolution to fully finite-amplitude states and the transition to turbulence [1]. In variable media this first stage may be carried out by self-sustained oscillations resulting from the local resonant tuning by the media. This local tuning serves to partition the media into regions defined by their local stability properties, which are identified by either absolute instability (AI) or convective growth (CG). A medium is said to exhibit AI if a local wave source produces perturbations that grow in time at the source, even after the source is turned off. However, if AI is absent, a medium is said to exhibit CG if a local wave source produces perturbations that locally amplify due to the variations in the medium; but if the wave source is turned off, the medium returns to its original state. Therefore, the existence of a local region of AI is a necessary condition for the media to produce a self-sustained oscillation - termed global mode [2].

The linearized complex Ginzburg-Landau equation (LCGL) has been widely used as a model equation to study AI and CG in spatially developing media (e.g., [2], [3], [4]). In a fluid dynamical context, the LCGL equation represents a simple system that describes the local growth of perturbations in spatially developing media whose spatially uniform far field is *marginally* stable with respect to the classic Rayleigh stability criterion. The appeal of the LCGL equation rests largely on its ability to describe qualitatively, and often quantitatively, the linear stability properties of a vast array of wave phenomena whose dispersion relations are second order in wave number. Here we introduce another model equation which, like the LCGL, describes the local growth of perturbations in spatially developing media; but in contrast to

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the LCGL, the spatially uniform far field is *deeply* stable with respect to the Rayleigh criterion. Moreover, the model equation also describes the dynamics of a vast array of distinctly different wave phenomena, all of which share an underlying commonality: their dispersion relations are an odd function of wavenumber in the *long wave* limit. Examples of such wave phenomena include magnetoacoustic waves in the solar wind [5], Rossby waves in geophysical fluids [6], and buoyancy waves in water [7].

The main objective of this study is to derive new stability criteria for long waves with odd dispersion relations in spatially developing media and to examine the CG that occurs in response to a local wave source. In addition, we will carefully examine the mode structures and their relationship to the properties of the medium.

The paper is organized as follows. In Section 2 we develop, via a heuristic approach, a model wave equation that we postulate governs the dynamics of long waves in variable media. As we point out below, this model equation can also be rigorously derived using multiple space and time scales. In Section 3 we obtain several conservation laws which, for variable media characterized by Hamiltonian structure, yield necessary conditions for the existence and structure of the global mode instabilities that can exist in the system. Section 4 presents a WKB analysis, valid for non-Hamiltonian media, that details the structural characteristics of the two distinct types of global mode instabilities that can emerge. In Section 5 we close with a brief summary of our most important results.

2. Long wave equation

We begin by considering linear waves of amplitude A that have the following properties: $\partial A/\partial t = -i\omega A$ and $\partial A/\partial x = ikA$. These waves propagate in the streamwise (x) direction in a medium translating at constant velocity U, with well-defined phase frequency $\omega(k;U)$ and wavenumber k. We expand the frequency in a Taylor series about long-waves ($k = 0$) to obtain

$$\omega = \omega^0 + c_g^0 k + \frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}k^3 ..., \tag{2.1}$$

where we have used the fact that for odd dispersion relations $(dc_g/dk)|_{k=0} = 0$. In (2.1) $\omega^0 = \omega(k=0;U)$ and $c_g^0 = U + \hat{c}_g^0$ is the group velocity in the limit $k \to 0$, where $\hat{c}_g^0$ is the intrinsic group velocity in the same limit. Multiplying (2.1) by iA yields

$$\frac{\partial A}{\partial t} - \frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}\frac{\partial^3 A}{\partial x^3} + c_g^0\frac{\partial A}{\partial x} + i\omega^0 A = 0. \tag{2.2}$$

Equation (2.2) describes the propagation and dispersion of long waves in a medium moving at constant velocity. Our goal, however, is to develop an equation that describes the propagation, dispersion, and local instability of long waves in spatially and temporally developing media, i.e., media for which $U = U(x,t)$. To account for such media and the instabilities that may ensue, we define the phase frequency to have an imaginary part at $k = 0$, i.e., $\omega^0 = i\sigma(x,t)$ in (2.1). In addition, we will assume that $\hat{c}_g^0$ is independent of $U(x,t)$. This assumption is valid for waves whose restoring force is largely independent of the translation speed of the medium. Therefore, paralleling (2.2), we assert the following heuristic model equation for long waves in spatially and temporally varying media:

$$\frac{\partial A}{\partial t} - \frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}\frac{\partial^3 A}{\partial x^3} + c_g^0(x,t)\frac{\partial A}{\partial x} - \sigma(x,t)A = 0. \tag{2.3}$$

Equation (2.3) is simply a linearized form of the Korteweg-deVries equation with variable coefficients. *It is important to emphasize that (2.3) can also be formally derived using the method of multiple space and time scales* (e.g., [6],[8]). However, such an approach tends to obscure the relationship of the coefficients to the dispersion relation, particularly the group velocity. Thus (2.3) highlights the fundamental physics of the wave dynamics in a way that is not only conceptually appealing, but, as shown below, also makes explicit the connection between the variations of the media and the wave dynamics.

3. Stability criteria for Hamiltonian media: conservation laws

Before proceeding to the detailed solutions of (2.3), it is instructive to obtain general stability criteria for media characterized by Hamiltonian structure. By defining the Hamiltonian as

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left[c_g^0 A^2 + \frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}\left(\frac{\partial A}{\partial x}\right)^2\right]dx \tag{3.1}$$

and writing its functional derivative as $\delta H/\delta A$, we can write (2.3) as the non-canonical Hamiltonian system

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H}{\delta A} = 0, \tag{3.2}$$

provided $\sigma = -\partial c_g^0/\partial x$, a condition that relates the local growth to the convergence of the group velocity in the long wave limit. This condition is consistent with systems that conserve wave action [9].

We exploit the symmetry properties of the Hamiltonian evolution equation to obtain, via Noether's theorem [10], conserved quantities that yield necessary conditions for the global instability of variable media. If the Hamiltonian (3.1) possesses translational symmetry in α, a functional Θ that satisfies

$$\frac{\partial A}{\partial \alpha} - \frac{\partial}{\partial x}\frac{\delta \Theta}{\delta A} = 0 \tag{3.3}$$

is time invariant. For example, if the Hamiltonian is invariant to translations in space ($\alpha = x$), we obtain the pseudomomentum, P, where

$$\Theta = P \equiv \frac{1}{2}\int_{-\infty}^{\infty} A^2 dx \tag{3.4}$$

is conserved for any dispersion relation for which c_g^0 varies only with time. Because the pseudomomentum is positive definite, a time varying, spatially uniform medium is stable. *Thus a necessary condition for global instability is a spatially developing medium.* With this stability condition as a basis, we hereafter focus attention on steady, spatially developing media.

If the Hamiltonian is invariant to translations in time ($\alpha = t$), for which c_g^0 varies only in space, we obtain conservation of pseudoenergy E, where $\Theta = E \equiv -H$. Conservation of E provides a constraint on the type of dispersion relations that can support linear global mode instabilities: *a necessary condition for a global mode in Hamiltonian media is*

$$c_g^0 \frac{d^2\hat{c}_g^0}{dk^2} < 0. \tag{3.5}$$

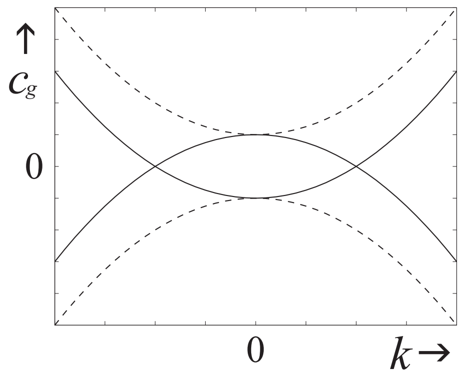

FIG. 3.1. *Possible configurations for the local group velocity. Dashed lines represent stable configurations, whereas solid lines represent configurations that may lead to instability.*

Thus instability may occur when there exists at least one point in the medium where the group velocity and its curvature at $k = 0$ have opposite signs (see Fig. 3.1). This constraint requires the existence of a wavenumber for which the group velocity vanishes. As we show below, the local vanishing of the group velocity is also a requirement for global mode instability in non-Hamiltonian media.

In steady, streamwise uniform media, global mode instability requires that pseudoenergy and pseudomomentum both vanish. However, in steady, spatially developing media, only the pseudoenergy must vanish for instability. In this case, the pseudomomentum is non-zero and modulated by the spatial variations in the medium, i.e.,

$$\frac{dP}{dt} = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{dc_g^0}{dx} A^2 dx. \tag{3.6}$$

Thus in regions where the long wave limit of the group velocity is convergent (divergent), the long wave amplitude increases (decreases). This means that an unstable global mode must be anchored to regions for which $dc_g^0/dx < 0$. If the pseudoenergy does not vanish, the medium is globally stable. In this case, variations in the pseudomomentum state that traveling long waves exhibit local convective growth in regions where $dc_g^0/dx < 0$.

In addition to the conservation of pseudomomentum and pseudoenergy, other conservation laws can be obtained by identifying the Casimir invariants C that satisfy $\partial/\partial x[\delta C/\delta A] = 0$. Here we only consider the Casimir invariant mass,

$$M = \int_{-\infty}^{\infty} A dx. \tag{3.7}$$

Because M is conserved, the integrated amplitude must vanish. *Thus, global instability in Hamiltonian media implies oscillatory wave structure.* As we show in the following section, along with these oscillatory modes, another class of modes exist in general media that are characterized by a smooth, envelope structure.

4. Stability criteria for general media

4.1. WKB solutions. The general stability criteria obtained for Hamiltonian media will now be extended to general media. In so doing we will use WKB methods to obtain explicit expressions for the global mode growth rates and structures. Our formulation hinges on assuming that the medium is slowly varying, which pivots on introducing $X=\delta x$, where $\delta \ll 1$ measures the ratio of the long wave variation to that of the medium. We seek solutions of the form $A(X,t)=a(X)exp[-i(\omega_0+\delta\omega_1+...)t]+c.c.$, where the ω_j are the contributions to the complex frequency and c.c. denotes the complex conjugate of the preceding term. We expand the amplitude $a(X)$ in standard WKB form [11], $a(X)=exp[S_0(X;\omega_0)/\delta+S_1(X;\omega_1)+...]$, where the local wavenumber, $k_0(X,\omega_0)$, is defined by $dS_0/dX=ik_0$. Insertion of $a(X)$ into (2.3) yields the following dispersion relation

$$D(k_0,\omega_0;X)\equiv \frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}k_0^3+c_g^0k_0-\omega_0+i\sigma=0. \tag{4.1}$$

The leading order (complex) frequency of the global mode is determined by examining the branch points in the complex X plane and subsequently identifying the location of the complex saddle point, X_0, where $\partial\omega_0/\partial x=0$ [2]. Consider the cubic dispersion relation (4.1), where $k_0,\omega_0,X\in\mathbb{C}$. A point X_0 that is both a square root branch point and a complex saddle point is defined by

$$D(k_0,\omega_0;X_0)=0, \qquad \left.\frac{\partial D}{\partial k_0}\right|_{X=X_0}=0, \qquad \left.\frac{\partial D}{\partial X}\right|_{X=X_0}=0. \tag{4.2}$$

From these expressions we obtain the following equation whose roots are the branch points:

$$F_{br}(X_0)=c_g^0\left(\frac{dU}{dX}\right)^2-\frac{1}{2}\frac{d^2\hat{c}_g^0}{dk^2}\left(\frac{d\sigma}{dX}\right)^2=0. \tag{4.3}$$

At the branch point the local wavenumber is

$$k_0(X_0,\omega_0)=-i\frac{d\sigma/dX}{dU/dX}, \tag{4.4}$$

and the leading order approximation to the global mode frequency is

$$\omega_0(X_0)=i\left(\sigma+\frac{1}{6}\frac{d^2\hat{c}_g^0}{dk^2}\left[\frac{d\sigma/dX}{dU/dX}\right]^3-c_g^0\frac{d\sigma/dX}{dU/dX}\right). \tag{4.5}$$

4.2. Global mode instability. As shown below, the spatial variations in the media lead to two distinct types of global mode structures: oscillatory modes and envelope modes. To illustrate these two types of global mode structures, we first calculate the branch points and the corresponding global mode frequencies. To simplify the analysis, we assume Gaussian variation in the medium, which is perhaps the simplest way to represent local variations in a variety of media. To simplify the coefficients in (2.3), we choose: $d^2\hat{c}_g^0/dk_0^2=\pm 6, c_g^0=1-2e^{-X^2/9}$, and $\sigma=4\alpha X/9e^{-X^2/9}$, where the free parameter α will be set to ± 1. For this choice of coefficients, the medium is non-Hamiltonian, which implies $\sigma\neq -\partial c_g^0/\partial x$. We note that irrespective

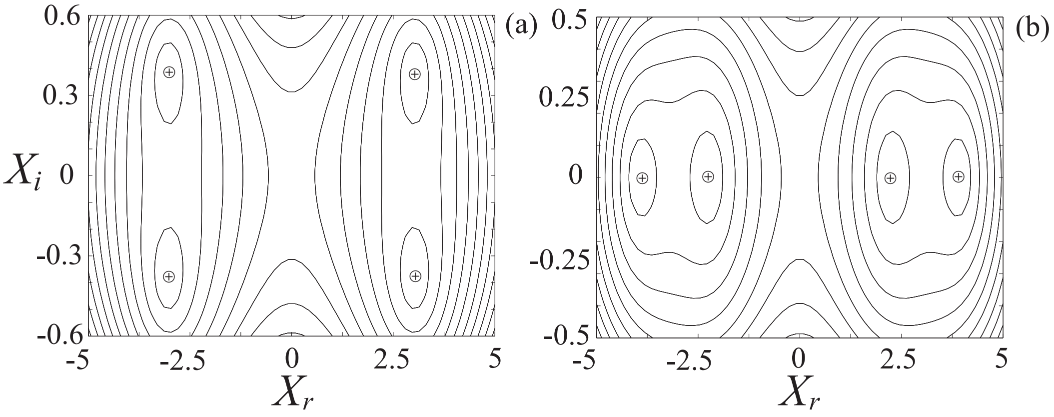

FIG. 4.1. *Branch points in the complex plane, which are denoted by the circled* $+$*. The contours are the function* F_{br}.

of the choice of media variation or model parameters, there are only two possible global mode structures − oscillatory and envelope.

Figure 4.1 shows the spatial configuration of the real and complex branch points found in the system. Figure 4.1a shows, for $d^2\hat{c}_g^0/dk_0^2 = 6$ and $\alpha = -1$, two pairs of branch points that are located symmetrically about the X_r axis. The branch points on the left are associated with equal growth rates $Im(\omega_0) = 1.3$ but opposite frequencies; $Re(\omega_0) = 0.067$ for the point where $X_i > 0$ and $Re(\omega_0) = -0.067$ for the point where $X_i < 0$. Because this wave-pair has opposite frequencies, their linear superposition produces a standing oscillation or stationary global mode. The waves associated with the branch points on the right are conjugates of those on the left; thus they exponentially decay with time and are physically irrelevant to the stability problem.

Figure 4.1b shows for, $d^2\hat{c}_g^0/dk_0^2 = -6$ and $\alpha = -1$, two pairs of real branch points situated symmetrically about the imaginary axis. For the left pair of branch points we have, from left to right, [$Re(\omega_0) = 0$, $Im(\omega_0) = 1.5$] and [$Re(\omega_0) = 0$, $Im(\omega_0) = 1.4$], respectively. Similar to Fig. 4.1a, the waves associated with the right pair of branch points are the conjugates of the left pair.

Figure 4.2 shows, for $d^2\hat{c}_g^0/dk_0^2 = 6$, that the leading order WKB solution yields two distinct types of global modes: oscillatory mode [$\alpha = -1$] and envelope mode [$\alpha = 1$]. For $d^2\hat{c}_g^0/dk_0^2 = 6$, (2.3) yields oscillatory and envelope modes on complex branch points. Other parameter choices lead to both global mode types on real branch points (see Fig. 4.1b). The oscillatory mode (Fig. 4.2a) is characterized by a slowly modulated oscillatory structure, which is anchored to the left-hand side of the Gaussian well, where the group velocity in the long wave limit is convergent. This is consistent with the predictions for Hamiltonian media derived in Section 3. The envelope mode is characterized by a broad, non-oscillatory structure (Fig. 4.2b), which, in contrast to the oscillatory mode, is anchored to the right-hand side of the Gaussian well, where the group velocity is divergent.

4.3. Convective growth. In globally stable media, waves originating in the far field may undergo local CG. The local CG rate is simply the local temporal growth rate obtained from (4.1), i.e., $\omega_i = \sigma$. As shown in [12], CG can be interpreted as either local temporal growth or as local spatial growth, which are related by $\omega_i \approx -k_{0i}c_{gr}$,

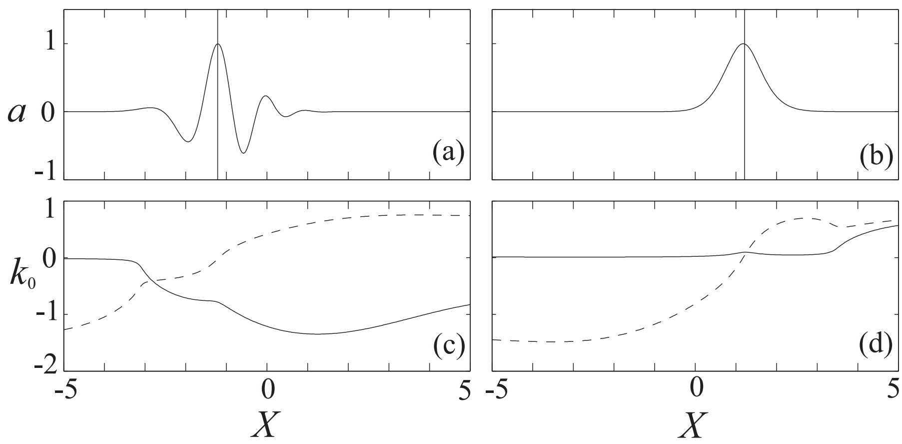

FIG. 4.2. *The lowest order WKB solution for the two types of global modes that arise due to spatial variations in the medium: (a) oscillatory mode and (b) envelope mode. Plotted in (c) and (d) are the lowest order wavenumbers corresponding to (a) and (b), respectively. The vertical line denotes the location of the projection of the branch point onto the real axis.*

where $c_{gr} = Re(c_g^0)$. Thus the net CG at position X for a wave train originating at infinity can be written as

$$G(X) = exp\left(-\frac{1}{\delta}\int_{-\infty}^{X} k_{0i}dX'\right) \approx exp\left(\frac{1}{\delta}\int_{-\infty}^{X} \frac{\sigma}{c_{gr}}dX'\right). \tag{4.6}$$

Equation (4.6) clearly shows that within the growth region ($\sigma/c_{gr} > 0$), the slowest moving waves experience the largest CG. In addition, if the growth rate distribution σ/c_{gr} is symmetric (anti-symmetric), the wave train will (will not) retain energy upon exiting the growth region. *Thus, a necessary condition for convective instability is*

$$\int_D \frac{\sigma}{c_{gr}}dx > 0, \tag{4.7}$$

where D denotes integration over the entire domain, either open or periodic. The net CG that occurs for symmetric growth rate distributions depends on whether radiation (decay) or periodic boundary conditions are imposed. For radiation boundary conditions, a finite amount of CG will occur as the wave makes a single traverse of the growth region, whereas for periodic boundary conditions there will be continual CG (i.e., convective instability) as the wave periodically enters and exits the growth region.

5. Conclusions

We have examined the instability of variable media to a broad class of long waves having dispersion relations that are an odd function of wavenumber. We have found that by examining the convergence or divergence of the group velocity in the long wave limit [i.e., dc_g^0/dx], we can predict the structures of the instabilities that emerge. If instability exists where the group velocity field is convergent, oscillatory wave packets emerge. If, on the other hand, instability exists where the group velocity field is divergent, smooth envelope modes emerge. Because these linear instabilities are the first stage in the evolution to finite amplitude states and the transition to turbulence, it will be of interest to see to what extent the group velocity field still serves as a predictor of the structure and location of the global modes as they evolve to weakly nonlinear or perhaps fully nonlinear regimes.

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