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GLOBAL EXISTENCE OF SOLUTIONS FOR THE EINSTEIN-BOLTZMANN SYSTEM WITH COSMOLOGICAL CONSTANT IN THE ROBERTSON-WALKER SPACE-TIME*

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Abstract. We prove a global in time existence theorem for the initial value problem for the Einstein-Boltzmann system, with positive cosmological constant and arbitrarily large initial data, in the spatially homogeneous case, in a Robertson-Walker space-time.

Key words. Global existence in time, Einstein-Boltzmann, Cosmological constant

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1. Introduction

In the mathematical study of General Relativity, one of the main problems is to establish the existence and to give the properties of **global** solutions of the Einstein equations coupled to various field equations. The knowledge of the global dynamics of the relativistic kinetic matter is based on such results. In the case of **Collisionless** matter, the phenomena are governed by the Einstein-Vlasov system in the pure gravitational case, and by this system coupled to other fields equations, if other fields than the gravitational field are involved. In the collisionless case, several authors proved global results, see [1], [19] for reviews, [18], [23] and [3] for scalar matter fields, also see [21], [22] for the Einstein-Vlasov system with a cosmological constant. Now in the case of **Collisional** matter, the Einstein-Vlasov system is replaced by the Einstein-Boltzmann system, that seems to be the best approximation available and that describes the case of instantaneous, binary and elastic collisions. In contrast with the abundance of works in the collisionless case, the literature is very poor in the collisional case. If, due to its importance in collisional kinetic theory, several authors studied and proved global results for the single Boltzmann equation, see [5], [4] , [11] for the non-relativistic case, and [7], for the full relativistic case, very few authors studied the Einstein-Boltzmann system, see [2] for a local existence theorem. It then seems interesting for us, to extend to the collisional case, some global results obtained in the collisionless case. This was certainly the objective of the author in [13] and [14], in which he studied the existence of global solutions of the Einstein-Boltzmann system. Unfortunately, several points of the work are far from clear; such as, the use of a formulation which is valid only for the **non-relativistic** Boltzmann equation, or, concerning the Einstein equations, to abandon the evolution equations which are really relevant, and to concentrate only on the constraint equations, which, in the homogeneous case studied, reduce as we will see to a question of choice for the initial data.

In this paper, we study the collisional evolution of a kind of uncharged **massive** particles, under the only influence of their own gravitational field, which is a function of the position of the particles.

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The phenomenon is governed, as we said above, by the coupled Einstein-Boltzmann system we now introduce. The Einstein equations are the basic equations of the General Relativity. These equations express the fact that, the gravitational field is generated by the matter contents, acting as its sources. The gravitational field is represented, by a second order symmetric 2-tensor of Lorentzian type, called the metric tensor, we denote by g, whose components $g_{\alpha\beta}$, sometimes called "gravitational potentials", are subject to the Einstein equations, with sources represented by a second order symmetric 2-tensor we denote $T_{\alpha\beta}$, that summarizes all the matter contents and which is called the stress-matter tensor. Let us observe that, solving the Einstein equations is determining both the gravitational field and its sources. In our case, the only matter contents are the massive particles statistically described in terms of their **distribution function**, denoted f, and which is a non-negative real valued function, of both the position and the momentum of the particles, and that generates the gravitational field g, through the stress-matter tensor $T_{\alpha\beta}$. The scalar function f is physically interpreted as "probability of the presence density" of the particles during their collisional evolution, and is subject to the Boltzmann equation, defined by a non-linear operator Q called the "Collision Operator". In the binary and elastic scheme due to Lichnerowicz and Chernikov(1940), we adopt, at a given position, only 2 particles collide, in an instantaneous shock, without destroying each other, the collision affecting only their momenta which are not the same, before, and after the shock, only the sum of the 2 momenta being preserved.

We then study the coupled Einstein-Boltzmann system in (g,f). The system is **coupled** in the sense that f, which is subject to the Boltzmann equation generates the sources$T_{\alpha\beta}$ of the Einstein equations, whereas the metric g, which is subject to the Einstein equations, is in both the Collision operator, which is the r.h.s of the Boltzmann equation, and in the Lie derivative of f with respect to the vectors field tangent to the trajectories of the particles, and which is the l.h.s of the Boltzmann equation.

We now specify the geometric frame, i.e. the kind of space-time we are looking for. An important part of general relativity is the Cosmology, which is the study of the Universe on a large scale. A. Einstein and W. de Sitter introduced the cosmological models in 1917; A. Friedman and G. Lemaitre introduced the concept of the expanding Universe in 1920. Let us point out the fact that the Einstein equations are **overdetermined**, and physically meaning symmetry assumptions reduce the number of unknowns. A usual assumption is that the spatial geometry has constant curvature which is positive, zero or negative, respectively. Robertson and Walker showed in 1944 that "exact spherical symmetry about every point would imply that the universe is spatially homogeneous", see [10], p. 135. We look for a spatially homogeneous Friedman-Lemaitre-Robertson-Walker space-time, we will call a "Robertson-Walker space-time", which is, in Cosmology, the basic model for the study of the expanding Universe. The metric tensor g has only one unknown component we denote a, which is a **strictly positive** function called the Cosmological expansion factor; the spatial homogeneity means that a depends only on the time t and the distribution function f depends only on the time t and the 4-momentum p of the particles. The study of the Einstein-Boltzmann system then turns out to be the determination of the couple of scalar function (a, f).

In the present work, we consider the Einstein Equation with cosmological constant Λ. Our motivation is of a physical point of view. Recent measurements show that the case $\Lambda > 0$ is physically very interesting in the sense that one can prove, as we will see, that the expansion of the Universe is accelerating; in mathematical terms, this means that the mean curvature of the space-time tends to a constant at late times. For more details on the cosmological constant, see [20].

Our method preserves the physical nature of the problem that imposes to the distribution function f to be a non-negative function and nowhere, did we have to require a smallness assumption on the initial data, which can, consequently, be taken as arbitrarily large.

The paper is organized as follows:
In section 2, we introduce the Einstein and Boltzmann equations on a Robertson-Walker space-time.
In section 3, we study the Boltzmann equation in f.
In section 4 we study the Einstein equation in a.
In section 5, we prove the local existence theorem for the coupled Einstein-Boltzmann system.
In section 6, we prove the global existence theorem for the coupled Einstein-Boltzmann system.

2. The Boltzmann equation and the Einstein equations on a Robertson-Walker space-time

2.1. Notations and function spaces. A Greek index varies from 0 to 3 and a Latin index from 1 to 3, unless otherwise specified. We adopt the Einstein summation convention $a_\alpha b^\alpha = \sum_{\alpha=0}^{3} a_\alpha b^\alpha$. We consider the flat Robertson-Walker space-time denoted ($\mathbb{R}^4$, g) where, for
$x=(x^\alpha)=(x^0,x^i)\in\mathbb{R}^4$, $x^0=t$ denotes the time and $\bar{x}=(x^i)$ the space. g stands for the metric tensor with signature (-, +, +, +) that can be written:

$$g=-dt^2+a^2(t)[(dx^1)^2+(dx^2)^2+(dx^3)^2] \tag{2.1}$$

in which a is a strictly positive function of t, called the cosmological expansion factor. We consider the collisional evolution of a kind of **uncharged massive** relativistic particle in the time oriented space-time $(\mathbb{R}^4,g)$. The particles are statistically described by their **distribution function** we denote by f, which is a non-negative real-valued function of both the position (x^α) and the 4-momentum $p=(p^\alpha)$ of the particles, and that defines the coordinate of the tangent bundle $T(\mathbb{R}^4)$ i.e:

$$f:T(\mathbb{R}^4)\simeq\mathbb{R}^4\times\mathbb{R}^4\to\mathbb{R}^+, \quad (x^\alpha,p^\alpha)\mapsto f(x^\alpha,p^\alpha)\in\mathbb{R}^+.$$

For $\bar{p}=(p^i)$, $\bar{q}=(p^i)\in\mathbb{R}^3$, we set, as usual:

$$\bar{p}.\bar{q}=\sum_{i=1}^{3}p^iq^i; \quad |\bar{p}|=[\sum_{i=1}^{3}(p^i)^2]^{\frac{1}{2}}. \tag{2.2}$$

We suppose the rest mass $m>0$ of the particles normalized to unity, i.e
we take m = 1. The relativistic particles are then required to move on the future sheet of the mass-shell whose equation is $g(p,p)=-1$.
From this, we deduce, using (2.1) and (2.2):

$$p^0=\sqrt{1+a^2|\bar{p}|^2} \tag{2.3}$$

where the choice of $p^0>0$ symbolizes the fact that the particles eject towards the future . (2.3) shows that in fact, f is defined on the 7-dimensional subbundle of

$T(\mathbb{R}^4)$ with coordinates by $(x^\alpha),(p^i)$. Now, we consider the spatially homogeneous case which means that f depends only on t and $\bar{p}=(p^i)$. The framework we will refer to will be the subspace of $L^1(\mathbb{R}^3)$, denote $L^1_2(\mathbb{R}^3)$ and defined by :

$$L^1_2(\mathbb{R}^3)=\{f\in L^1(\mathbb{R}^3),\|f\|:=\int_{\mathbb{R}^3}\sqrt{1+|\bar{p}|^2}|f(\bar{p})|d\bar{p}<+\infty\} \tag{2.4}$$

where $|\bar{p}|$ is given by (2.2); $\|.\|$ is a norm on $L^1_2(\mathbb{R}^3)$ and $(L^1_2(\mathbb{R}^3),\|.\|)$ is a Banach space.
Let r be an arbitrary strictly positive real number. We set:

$$X_r=\{f\in L^1_2(\mathbb{R}^3), f\geq 0 \quad a.e, \|f\|\leq r\} \tag{2.5}$$

Endowed with the metric induced by the norm $\|.\|$, X_r is a complete and connected metric subspace of $(L^1_2(\mathbb{R}^3),\|.\|)$. Let I be a real interval. Set:

$$C[I;L^1_2(\mathbb{R}^3)]=\{f:I\to L^1_2(\mathbb{R}^3), \text{ f continuous and bounded }\}$$

endowed with the norm:

$$|||f|||=\underset{t\in I}{Sup}\|f(t)\| \tag{2.6}$$

$C[I;L^1_2(\mathbb{R}^3)]$ is a Banach space. X_r being defined by (2.5). We set:

$$C[I;X_r]=\{f\in C[I;L^1_2(\mathbb{R}^3)], f(t)\in X_r, \quad \forall t\in I\}. \tag{2.7}$$

Endowed with the metric induced by the norm $|||.|||$ defined by (2.6), $C[I;X_r]$ is a complete metric subspace of $(C[I;L^1_2(\mathbb{R}^3)],|||.|||)$.

2.2. The Boltzmann equation in ($\mathbb{R}^4$, g). The Boltzmann equation on the curved space-time ($\mathbb{R}^4$, g) can be written:

$$p^\alpha\frac{\partial f}{\partial x^\alpha}-\Gamma^\alpha_{\mu\nu}p^\mu p^\nu\frac{\partial f}{\partial p^\alpha}=Q(f,f), \tag{2.8}$$

in which $\Gamma^\alpha_{\lambda\mu}$ are the Christoffel symbols of g; Q is a non-linear integral operator called the "Collision Operator". We specify this operator in detail in the next section. Now, since f depends only on t and (p^i), (2.8) can be written:

$$p^0\frac{\partial f}{\partial t}-\Gamma^i_{\mu\nu}p^\mu p^\nu\frac{\partial f}{\partial p^i}=Q(f,f). \tag{2.9}$$

We now express the Christoffel symbols which are defined by:

$$\Gamma^\lambda_{\alpha\beta}=\frac{1}{2}g^{\lambda\mu}[\partial_\alpha g_{\mu\beta}+\partial_\beta g_{\alpha\mu}-\partial_\mu g_{\alpha\beta}] \tag{2.10}$$

in which, the metric g is defined by (2.1) and $g^{\lambda\mu}$ denotes the inverse matrix of $g_{\lambda\mu}$; (2.1) gives:

$$g^{00}=g_{00}=-1;\quad g_{ii}=a^2;\quad g^{ii}=a^{-2};\quad g_{0i}=g^{0i}=0;\quad g_{ij}=g^{ij}=0 \quad for \quad i\neq j. \tag{2.11}$$

A direct computation, using (2.10) and (2.11) then gives, with $\dot{a}=\frac{da}{dt}$:

$$\Gamma^0_{ii}=\dot{a}a;\quad \Gamma^i_{i0}=\Gamma^i_{0i}=\frac{\dot{a}}{a};\quad \Gamma^0_{00}=0;\quad \Gamma^0_{\alpha\beta}=0 \quad for \quad \alpha\neq\beta;\quad \Gamma^k_{ij}=0. \tag{2.12}$$

The Boltzmann equation (2.9) then writes, using (2.12):

$$\frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a}\sum_{i=1}^{3} p^i \frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(f,f) \tag{2.13}$$

in which p^0 is given by (2.3). (2.13) is a non-linear p.d.e. in f which we study in the next section.

2.3. The Einstein equations. We consider the Einstein equations with a **cosmological constant** Λ and that can be written:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{2.14}$$

in which: $R_{\alpha\beta}$ is the Ricci tensor of g, contracted of the curvature tensor of g; $R = g^{\alpha\beta} R_{\alpha\beta} = R^{\alpha}_{\alpha}$ is the scalar curvature; $T_{\alpha\beta}$ is the stress-matter tensor that represents the matter contents, and that is generated by the distribution function f of the particles by:

$$T^{\alpha\beta} = \int_{\mathbb{R}^3} \frac{p^\alpha p^\beta f(t,\bar{p}) |g|^{\frac{1}{2}}}{p^0} dp^1 dp^2 dp^3 \tag{2.15}$$

in which $|g|$ is the determinant of g. We have, using (2.1), $|g|^{\frac{1}{2}} = a^3$. Recall that f is a function of t and $\bar{p} = (p^i)$; then $T^{\alpha\beta}$ is a function of t.
Notice that both sides of (2.14) are divergence free (see [6] and [12]). Now if $R^{\lambda}_{\alpha,\mu\beta}$ are the components of the curvature tensor of g, we have:

$$\begin{cases} R_{\alpha\beta} = R^{\lambda}_{\alpha,\lambda\beta} \\ \text{where} \\ R^{\lambda}_{\mu,\alpha\beta} = \partial_\alpha \Gamma^{\lambda}_{\mu\beta} - \partial_\beta \Gamma^{\lambda}_{\mu\alpha} + \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\lambda}_{\nu\beta}\Gamma^{\nu}_{\mu\alpha} \end{cases} \tag{2.16}$$

in which $\Gamma^{\lambda}_{\mu\beta}$ is defined by (2.11).

We now express the l.h.s of (2.14) in terms of a. We have to compute the Ricci tensor $R_{\alpha\beta}$ given by (2.16). The expression (2.12) of $\Gamma^{\lambda}_{\alpha\beta}$ in terms of a shows that the only non-zero components of the Ricci tensor are the $R_{\alpha\alpha}$ and that $R_{11} = R_{22} = R_{33}$. Then, it will be enough to compute $R_{00} = R^{\lambda}_{0,\lambda 0}$ and $R_{11} = R^{\lambda}_{1,\lambda 1}$. The expression (2.12) of $\Gamma^{\lambda}_{\alpha\beta}$ and formulae (2.16) give:

$$R^{0}_{0,00} = 0; \quad R^{1}_{0,10} = R^{2}_{0,20} = R^{3}_{0,30} = -\frac{\ddot{a}}{a};$$

$$R^{0}_{1,01} = a\ddot{a}; \quad R^{1}_{1,11} = 0; \quad R^{2}_{1,21} = R^{3}_{1,31} = (\dot{a})^2.$$

We then deduce that:

$$R_{00} = -3\frac{\ddot{a}}{a} \quad \textit{and} \quad R_{11} = a\ddot{a} + 2(\dot{a})^2.$$

We can then compute the scalar curvature R to be:

$$R = R^{\alpha}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].$$

The Einstein equations (2.14) then write, in terms of a, and using expression (2.1) of g:

$$\begin{cases} 3(\frac{\dot a}{a})^2 - \Lambda &= 8\pi T_{00} \\ -(\dot a)^2 - 2a\ddot a + a^2\Lambda &= 8\pi T_{11}. \end{cases}$$

2.4. The Einstein-Boltzmann system. In the present paper, we study the Einstein-Boltzmann system in (a,f), which can be written, using the above results:

$$\frac{\partial f}{\partial t} - 2\frac{\dot a}{a}\sum_{i=1}^{3} p^i \frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(f,f) \tag{2.17}$$

$$3(\frac{\dot a}{a})^2 - \Lambda = 8\pi T_{00} \tag{2.18}$$

$$-(\dot a)^2 - 2a\ddot a + a^2\Lambda = 8\pi T_{11} \tag{2.19}$$

in which $T_{\alpha\beta}$ is given in terms of a and f by (2.15).

3. Existence theorem for the Boltzmann equation

In this section, we suppose that the cosmological expansion factor a is given, and we prove an existence theorem for the initial value problem for the Boltzmann equation (2.13) or (2.17), on every bounded interval $I = [t_0, t_0 + T]$ with $t_0 \in \mathbb{R}_+$, $T \in \mathbb{R}_+$. We begin by specifying the collision operator Q in (2.13).

3.1. The Collision Operator. In the instanteneous, binary and elastic scheme due to Lichnerowicz and Chernikov, we consider, at a given position (t, x), only 2 particles colliding instanteneously without destroying each other, the collision affecting only the momenta of the 2 particles that change after the collision, only the sum of the 2 momenta being preserved, following the scheme:

$$\begin{array}{ccc} p & & p' \\ \searrow & & \nearrow \\ & (t,x) & \\ q\nearrow & & \searrow q' \end{array}$$

$$p + q = p' + q'.$$

The collision operator Q is then defined, using functions f, g on $\mathbb{R}^3$ by:

$$Q(f,g) = Q^+(f,g) - Q^-(f,g) \tag{3.1}$$

where

$$Q^+(f,g)(\bar p) = \int_{\mathbb{R}^3} \frac{a^3 d\bar q}{q^0} \int_{S^2} f(\bar p')g(\bar q')A(a,\bar p,\bar q,\bar p',\bar q')d\omega \tag{3.2}$$

$$Q^-(f,g)(\bar p) = \int_{\mathbb{R}^3} .\frac{a^3 d\bar q}{q^0} \int_{S^2} f(\bar p)g(\bar q)A(a,\bar p,\bar q,\bar p',\bar q')d\omega \tag{3.3}$$

whose elements we now introduce step by step, specifying properties and hypotheses:

1) S^2 is the unit sphere of $\mathbb{R}^3$ whose volume element is denoted dw.

2) A is a non-negative real-valued regular function of all its arguments, called the **collision kernel** or the **cross-section** of the collisions, on which we require the following boundedness, symmetry and Lipschitz continuity assumptions:

$$0 \leq A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \leq C_1 \tag{3.4}$$

$$A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = A(a, \bar{q}, \bar{p}, \bar{q}', \bar{p}') \tag{3.5}$$

$$A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = A(a, \bar{p}', \bar{q}', \bar{p}, \bar{q}) \tag{3.6}$$

$$|A(a_1, \bar{p}, \bar{q}, \bar{p}', \bar{q}') - A(a_2, \bar{p}, \bar{q}, \bar{p}', \bar{q}')| \leq \gamma |a_1 - a_2| \tag{3.7}$$

where C_1 and γ are strictly positive constants.

3) The conservation law $p + q = p' + q'$ splits into:

$$p^0 + q^0 = p'^0 + q'^0 \tag{3.8}$$

$$\bar{p} + \bar{q} = \bar{p}' + \bar{q}' \tag{3.9}$$

and (3.8) shows, using (2.3), the conservation of the quantity:

$$e = \sqrt{1 + a^2 |\bar{p}|^2} + \sqrt{1 + a^2 |\bar{q}|^2} \tag{3.10}$$

called elementary energy of the unit rest mass partcles; we can interpret (3.9) by setting, following Glassey, R, T., in [7]:

$$\begin{cases} \bar{p}' = \bar{p} + b(\bar{p}, \bar{q}, \omega)\omega \\ \bar{q}' = \bar{q} - b(\bar{p}, \bar{q}, \omega)\omega; \qquad \omega \in S^2 \end{cases} \tag{3.11}$$

in which $b(\bar{p}, \bar{q}, \omega)$ is a real-valued function. We prove, by a direct calculation, using (2.3) to express $\bar{p}'^0$, $\bar{q}'^0$ in terms of $\bar{p}'$, $\bar{p}'$, and now (3.11) to express $\bar{p}'$, $\bar{q}'$ in terms of $\bar{p}$, $\bar{q}$, that equation (3.8) leads to a quadratic equation in b, that solves to give:

$$b(\bar{p}, \bar{q}, \omega) = \frac{2 p^o q^o e a^2 \omega.(\hat{\bar{q}} - \hat{\bar{p}})}{e^2 - a^4 (\omega.(\bar{p} + \bar{q}))^2} \tag{3.12}$$

in which $\hat{\bar{p}} = \frac{\bar{p}}{p^0}$, $\hat{\bar{q}} = \frac{\bar{q}}{q^0}$, and e is given by (3.10). Another direct computation shows, using the classical properties of the determinants, that the Jacobian of the change of variables $(\bar{p}, \bar{q}) \to (\bar{p}', \bar{q}')$ in $\mathbb{R}^3 \times \mathbb{R}^3$, defined by (3.11) is given by:

$$\frac{\partial(\bar{p}', \bar{q}')}{\partial(\bar{p}, \bar{q})} = -\frac{p'^o q'^o}{p^o q^o}. \tag{3.13}$$

(3.13) shows, using once more (2.3) and the implicit function theorem, that the change of variable (3.11) is invertible and also allows us to compute $\bar{p}$, $\bar{q}$ in terms of $\bar{p}'$, $\bar{q}'$.

Finally, formulae (2.3) and (3.11) show that the functions to integrate in (3.2) and (3.3) completely express in terms of $\bar{p},\bar{q},\omega$; the integration with respect to $\bar{q}$ and ω leave functions $Q^{+}(f,g)$ and $Q^{-}(f,g)$ of the single variable $\bar{p}$. In practice, we will consider functions f on $\mathbb{R}\times\mathbb{R}^3$, that induce, for t fixed in $\mathbb{R}$, functions f(t) on $\mathbb{R}^3$, defined by $f(t)(\bar{p})=f(t,\bar{p})$.

Remark 3.1. *1) Formulae (3.12) and (3.13) are generalizations to the case of the Robertson-Walker space-time, of analogous formulae established by Glassey, R.T, in [7], in the case of the Minkowski space-time, to which the Robertson-Walker space-time reduces, when we take $a(t)=1$ in (2.1).*

2)The expression $b(\bar{p},\bar{q},\omega)=\omega.(\bar{p}-\bar{q})$ used by the author in [13] and [14] is valid only in the non-relativistic case.

3) $A=ke^{-a^2-|\bar{p}|^2-|\bar{q}|^2-|\bar{p}'|^2-|\bar{q}'|^2}$, $k>0$, is a simple example of functions satisfying assumptions (3.4), (3.5), (3.6) and (3.7).

3.2. Resolution of the Boltzmann equation. Our method follows that of Mucha in [13], [14] correcting some points we will specifiy.

We consider the Boltzmann equation on $[t_0,t_0+T]$ with $t_0\in\mathbb{R}_+$, $T\in\mathbb{R}_+^*$ and a is supposed to be given and defined on $[t_0,t_0+T]$.

The Boltzmann equation (2.13) is a first order p.d.e and its resolution is equivalent to the resolution of the associated characteristic system, which can be written, taking t as parameter:

$$\frac{dp^i}{dt}=-2\,\frac{\dot{a}}{a}p^i;\qquad \frac{df}{dt}=\frac{1}{p^0}Q(f,f). \tag{3.14}$$

We solve the initial value problem on $I=[t_0,t_0+T]$ with initial data:

$$p^i(t_0)=y^i;\qquad f(t_0)=f_{t_0}. \tag{3.15}$$

The equation in $\bar{p}=(p^i)$ solves directly to give, setting $y=(y^i)\in\mathbb{R}^3$;

$$\bar{p}(t_0+t,y)=\frac{a^2(t_0)}{a^2(t_0+t)}y,\quad t\in[0,T]. \tag{3.16}$$

The initial value problem for f is equivalent to the following integral equation in f, in which $\bar{p}$ stands this time for any independent variable in $\mathbb{R}^3$:

$$f(t_0+t,\bar{p})=f_{t_0}(\bar{p})+\int_{t_0}^{t_0+t}\frac{1}{p^0}Q(f,f)(s,\bar{p})ds\quad t\in[0,T]. \tag{3.17}$$

Finally, solving the Boltzmann equation (2.13) is equivalent to solving the integral equation (3.17). We prove:

Theorem 3.1. *Let a be a strictly positive continuous function such that $a(t)\geq\frac{3}{2}$ whenever a is defined . Let $f_{t_0}\in L^1_2(\mathbb{R}^3)$, $f_{t_0}\geq 0$, a.e, $r\in\mathbb{R}_0^*$ such that $r>\|f_{t_0}\|$. Then, the initial value problem for the Boltzmann equation on $[t_0,t_0+T]$, with initial data f_{t_0}, has a unique solution $f\in C[[t_0,t_0+T];X_r]$. Moreover, f satisfies the estimation:*

$$\underset{t\in[t_0,t_0+T]}{Sup}\|f(t)\|\leq\|f_{t_0}\|. \tag{3.18}$$

Theorem 3.1 is a direct consequence of the following result:

PROPOSITION 3.2. *Assume hypotheses of theorem 3.1 on: a, f_{t_0} and r.*

1) There exists an integer $n_0(r)$ such that, for every integer $n \geq n_0(r)$ and for every $v \in X_r$, the equation

$$\sqrt{n}u - \frac{1}{p^0}Q(u,u) = v \tag{3.19}$$

has a unique solution $u_n \in X_r$.

2) Let $n \in \mathbb{N}$, $n \geq n_0(r)$

i) For every $u \in X_r$, define R(n, Q)u to be the unique element of X_r such that:

$$\sqrt{n}R(n,Q)u - \frac{1}{p^0}Q\left[R(n,Q)u, R(n,Q)u\right] = u. \tag{3.20}$$

ii) Define operator Q_n on X_r by:

$$Q_n(u,u) = n\sqrt{n}R(n,Q)u - nu. \tag{3.21}$$

Then

a) The integral equation

$$f(t_0+t,\bar{p}) = f_{t_0}(\bar{p}) + \int_{t_0}^{t_0+t} Q_n(f,f)(s,\bar{p})ds \qquad t \in [0,T] \tag{3.22}$$

has a unique solution $f_n \in C[[t_0,t_0+T];X_r]$. Moreover, f_n satisfies the estimation:

$$\underset{t\in[t_0,t_0+T]}{Sup} \|f_n(t)\| \leq \|f_{t_0}\| \tag{3.23}$$

b)The sequence (f_n) converges in $C[[t_0,t_0+T];X_r]$to an element $f \in C[[t_0,t_0+T];X_r]$, which is the unique solution of the integral equation (3.17). The solution f satisfies the estimation (3.18).

The proof follows the same lines as the proof of theorem 4.1 in [16]. We will emphasize only on points where differences arise with the present case and we show how we proceed in such cases.

The proof of point 1) of Prop. 3.2 will use:

LEMMA 3.3. *Let $f,g \in L^1_2(\mathbb{R}^3)$. then $\frac{1}{p^0}Q^+(f,g)$, $\frac{1}{p^0}Q^-(f,g) \in L^1_2(\mathbb{R}^3)$ and*

$$\left\|\frac{1}{p^0}Q^+(f,g)\right\| \leq C(t)\,\|f\|\|g\|, \quad \left\|\frac{1}{p^0}Q^-(f,g)\right\| \leq C(t)\,\|f\|\|g\| \tag{3.24}$$

$$\begin{cases} \left\|\frac{1}{p^0}Q^+(f,f) - \frac{1}{p^0}Q^+(g,g)\right\| \leq C(t)(\|f\| + \|g\|)\,\|f-g\| \\ \\ \left\|\frac{1}{p^0}Q^-(f,f) - \frac{1}{p^0}Q^-(g,g)\right\| \leq C(t)(\|f\| + \|g\|)\,\|f-g\| \end{cases} \tag{3.25}$$

$$\left\|\frac{1}{p^0}Q(f,f) - \frac{1}{p^0}Q(g,g)\right\| \leq C(t)(\|f\| + \|g\|)\,\|f-g\| \tag{3.26}$$

where

$$C(t) = 32\pi C_1 a^3(t). \tag{3.27}$$

Proof. We deduce from (3.8) and $a>1$ that:

$$\sqrt{1+|\bar{p}|^2} \leq \sqrt{1+\frac{1}{a^2}}p^0 \leq \sqrt{1+\frac{1}{a^2}}(p^0+q^0) = \sqrt{1+\frac{1}{a^2}}(p'^0+q'^0) \leq 2(p'^0+q'^0)$$

Expression (3.2) of $Q^+(f,g)$ then gives, using (3.4):

$$\begin{aligned}\left\|\frac{1}{p^0}Q^+(f,g)\right\| &= \int_{\mathbb{R}^3}\left|\frac{\sqrt{1+|\bar{p}|^2}}{p^0}Q^+(f,g)\right|d\bar{p}\\ &\leq 2a^3(t)C_1\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\frac{p'^0+q'^0}{p^0q^0}|f(\bar{p}')||g(\bar{q}')|d\bar{p}d\bar{q}d\omega.\end{aligned}$$

We then deduce, using the change of variables (3.11) and (3.13) that gives $d\bar{p}d\bar{q} = \frac{p^0q^0}{p'^0q'^0}d\bar{p}'d\bar{q}'$ and the fact that, by (2.3) $\frac{p'^0+q'^0}{p'^0q'^0} = \frac{1}{q'^0}+\frac{1}{q'^0} \leq 2$

$$\left\|\frac{1}{p^0}Q^+(f,g)\right\| \leq 8\pi a^3(t)C_1\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|f(\bar{p}')||g(\bar{q}')|d\bar{p}'d\bar{q}' \leq C(t)\|f\|\|g\|.$$

The estimation of $\left\|\frac{1}{p^0}Q^-(f,g)\right\|$ follows the same way without change of variables and (3.24) follows. The inequalities (3.25) are consequences of (3.24) and the bilinearity of Q^+ and Q^-, that allows us to write, P standing for $\frac{1}{p^0}Q^+$ or $\frac{1}{p^0}Q^-$:

$P(f,f)-P(g,g) = P(f,f-g)+P(f-g,g)$.

Finally, (3.26) is a consequence of (3.25) and $Q = Q^+ - Q^-$. This completes the proof of the lemma 3.3. □

Now the continuous and strictly positive function $t \to a^3(t)$ is bounded from above, on the line segment $[t_0, t_0+T]$, and C(t) given by (3.27) is bounded from above by a constant $C(t_0,T)>0$. Hence, if we replace C(t) by $C(t_0,T)$ in the inequalities in lemma 3.3, we obtain the same inequalities with an absolute constant as the inequalities in proposition 3.1 in [16]. The proof of the point 1) of prop. 3.2 is then exactly the same as the proof of prop. 3.2 in [16].

The proof of the point 2) of Prop. 3.2 will use, $n_0(r)$ being the integer introduced in point 1:

Lemma 3.4. *We have, for every integer $n \geq n_0(r)$ and for every $u \in X_r$*

$$\| \sqrt{n}R(n,Q)u \| = \| u \| . \tag{3.28}$$

Proof. (3.28) is a consequence of:

$$\int_{\mathbb{R}^3} Q(f,f)(\bar{p})d\bar{p} = 0, \qquad \forall f \in L^1_2(\mathbb{R}^3) \tag{3.29}$$

that can be proved using the symmetry assumptions (3.5), (3.6) on A, and (3.8).

Now let us prove (3.28).

We have, multiplying equation (3.20) by $p^0 = \sqrt{1+a^2|\bar{p}|^2}$, integrating over $\mathbb{R}^3$, and using (3.29):

$$\sqrt{n}\int_{\mathbb{R}^3}\sqrt{1+a^2|\bar{p}|^2}R(n,Q)u(\bar{p})d\bar{p} = \int_{\mathbb{R}^3}\sqrt{1+a^2|\bar{p}|^2}u(\bar{p})d\bar{p}. \tag{3.30}$$

If we make in each side of (3.30) the change of variables $\bar{q}=B\bar{p}$ where $B=\mathrm{Diag}(a,a,a)$, then $|\bar{q}|^2=a^2|\bar{p}|^2$; $d\bar{p}=\frac{1}{a^3}d\bar{q}$ and (3.30) gives, using definition 2.4 of $\|.\|$:

$$\sqrt{n}\|R(n,Q)uoB^{-1}\|=\|uoB^{-1}\|. \tag{3.31}$$

But, if we compute $\|uoB\|$, using the above change of variable, we have: $\|uoB\|\leq\frac{1}{a^3}\left(1+\frac{1}{a}\right)\|u\|$. The assumption $a\geq\frac{3}{2}$ implies $\frac{1}{a^3}\left(1+\frac{1}{a}\right)\leq 1$, so that $\|uoB\|\leq\|u\|$, this implies that $uoB\in X_r$ if $u\in X_r$. Now since (3.31) holds for every $u\in X_r$, we have, replacing in (3.31) u by uoB, $\|\sqrt{n}R(n,Q)u\|=\|u\|$. We then have (3.28) and lemma 3.4 is proved.
Now (3.28) is exactly equality (3.10) in proposition 3.3 in [16]. We then prove exactly as for prop. 3.3 in [16], that all the other relations of that proposition hold in the present case. Using this result, the proof of point 2)a) of proposition 3.2 is the same as the proof of prop. 4.1 in [16] and the proof of point 2) b) of prop. 3.2 is the same as the proof of theorem 4.1 in [16], just replacing, $[0,+\infty[$ by $[t_0,t_0+T]$. This completes the proof of prop. 3.2 which gives directly theorem 3.1. ∎

Remark 3.2.

1) Our approximative equation (3.19) differs from that of Mucha in [13] and, contrary to his equation, it leads us to the approximative operator Q_n defined by (3.21) which has a clear domain.

2) Notice that the symmetry assumptions (3.5), (3.6), also specified by Bancel in [2], are indispensable to prove formula (3.29) also used by Mucha, without specifying these assumptions.

4. Existence theorem for the Einstein equations

4.1. The problem. In this paragrph, we study the Einstein equations (2.17)-(2.18) in a, when f is given. (2.17)-(2.18) can be written, using $T^{\alpha\beta}=g^{\alpha\lambda}g^{\beta\mu}T_{\lambda\mu}$:

$$(\frac{\dot{a}}{a})^2=\frac{8\pi}{3}T^{00}+\frac{\Lambda}{3} \tag{4.1}$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}(T^{00}+3a^2T^{11})+\frac{\Lambda}{3}. \tag{4.2}$$

In this paragraph, we suppose f fixed in $C[[0,T];X_r]$, with $f(0)=f_0\in L^1_2(\mathbb{R}^3)$, $f_0\geq 0$ a.e. and $r>\|f_0\|$, and we study the initial value problem in a.

4.2. Compatibility. The relations $R_{0i}=0$, $R_{ij}=0$ if $i\neq j$, $R_{11}=R_{22}=R_{33}$, imply for the Einstein tensor $S_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R$, that:

$$T_{11}=T_{22}=T_{33},\quad T_{0i}=0,\quad T_{ij}=0\quad for\quad i\neq j. \tag{4.3}$$

But the stress-matter tensor $T_{\alpha\beta}$ is defined by (2.15) in terms of the distribution function f. So, the relation (4.3) is in fact a condition to impose on f. We prove:

Proposition 4.1. *Let f_{t_0} and $r>0$ be defined as in theorem 3.1. Assume that, in addition, f_{t_0} is invariant by S_{O_3} and that, the collision kernel A satisfies*

$$A(a(t),M\bar{p},M\bar{q},M\bar{p}',M\bar{q}')=A(a(t),\bar{p},\bar{q},\bar{p}',\bar{q}'),\qquad\forall M\in S_{O_3}. \tag{4.4}$$

Then

1) The solution f of the integral equation (3.17) satisfies:

$$f(t_0+t, M\bar{p}) = f(t_0+t, \bar{p}), \quad \forall t \in [0,T], \quad \forall \bar{p} \in \mathbb{R}^3, \quad \forall M \in S_{O_3}. \tag{4.5}$$

2) The stress-matter tensor $T_{\alpha\beta}$ satisfies the conditions (4.3).

Proof. (4.5) is a consequence of the invariance of p^0 by S_{O_3} and the uniqueness theorem; (4.3) is obtained by choosing adequate elements of S_{O_3}. ◻

In all what follows, we assume that f_{t_0} is invariant by S_{O_3} and that the collision kernel A satisfies assumption (4.4). Notice that A defined in Remark 3.1 is an example of such a kernel.

4.3. The constraint equation. We study the Cauchy problem for the system (4.1)-(4.2) on $[0,T]$ with initial data:

$$a(0) = a_0; \quad \dot{a}(0) = \dot{a}_0. \tag{4.6}$$

Equation (4.1) is called a Hamiltonian constraint. It is proved (see for instance [12] p.29) that this equation is satisfied in the whole existence domain of the solution a of the initial value problem on [0, T], once it is satisfied for $t=0$. So, it will be the case if the initial data a_0, $\dot{a}_0$, f_0 satisfy, using expression (2.15) of T^{00}, the initial constraint:

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{8\pi a_0^3}{3}\int_{\mathbb{R}^3}\sqrt{1+a_0^2|\bar{p}|^2}\, f_0(\bar{p})d\bar{p} + \frac{\Lambda}{3}. \tag{4.7}$$

(4.7) gives two possible choices of $\dot{a}_0$, when a_0 and f_0 are given. We will choose, taking also into account the hypothesis on $a(t)$ in theorem 3.1:

$$a_0 \geq \frac{3}{2}; \quad f_0 \in L_2^1(\mathbb{R}^3); \quad f_0 \geq 0\, a.e \quad \dot{a}_0 > 0. \tag{4.8}$$

We now concentrate on (4.2) which is the evolution equation.

4.4. The evolution equation. We set $\theta = 3\frac{\dot{a}}{a}$, then $\dot{\theta} = 3[\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2]$ and (4.2) gives:

$$\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(T^{00} + 3a^2T^{11}) + \Lambda. \tag{4.9}$$

(4.9) is the Raychaudhuri equation in θ. We prove:

PROPOSITION 4.2. *There can exist no global regular solution for the coupled Einstein-Boltzmann system in the case $\Lambda < 0$.*

For the proof, we use the following result, proved in [9]:

LEMMA 4.3. *Let u and θ be 2 differentiable functions of t satisfying:*

$$\dot{\theta} < -\frac{\theta^2}{3}; \quad \dot{u} = -\frac{u^2}{3}; \quad \theta(t_1) = u(t_1) \tag{4.10}$$

for a given value t_1 of t. Then $\theta(t) \leq u(t)$ for $t \geq t_1$.

Proof of Proposition 4.2: Suppose $\Lambda < 0$. Then the Raychaudhuri equation (4.9) gives $\dot{\theta} < -\frac{\theta^2}{3}$. We have, integrating the equation in u on $[t_1, t]$ when $u \neq 0$ and since $\theta(t_1) = u(t_1)$:

$$u(t) = \frac{3\theta(t_1)}{3 + \theta(t_1)(t - t_1)}; \quad t \geq t_1. \tag{4.11}$$

Derive both sides of (4.1) and use (3.29) to conclude that, necessarily $\dot{a} < 0$, which implies $\theta = 3\frac{\dot{a}}{a} < 0$, so that in (4.10) we have $\theta(t_1) < 0$. Now (4.11) shows that, since by (4.10) u is a decreasing function:

$$u(t) \to -\infty \quad when \quad 3 + \theta(t_1)(t - t_1) \underset{>}{\to} 0, \quad and \quad t_1 \leq t < t_1 - \frac{3}{\theta(t_1)} := t^*. \tag{4.12}$$

By Lemma 4.3, (4.12) implies that $\theta(t) = 3\frac{\dot{a}}{a}(t) \to -\infty$ when $t \underset{<}{\to} t^*$, then:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \to +\infty \quad when \quad t \underset{<}{\to} t^*. \tag{4.13}$$

Now, since $\dot{a} < 0$, a is a decreasing function on $[t_1, t^*]$, and then, $a(t) \leq a(t_1)$, $\forall t \in [t_1, t^*]$; and (4.1)then implies, since $\Lambda < 0$:

$$(\dot{a})^2 \leq \frac{8\pi}{3} a^5(t_1)\sqrt{1 + a^2(t_1)}|||f||| := C^2(t_1, f), \quad \forall t \in [t_1, t^*]$$

so that:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \leq \frac{C^2(t_1, f)}{a^2(t)} \quad \forall t \in [t_1, t^*].$$

(4.13) then implies that $\frac{C^2(t_1,f)}{a^2(t)} \to +\infty$ when $t \underset{<}{\to} t^*$ and this can happen only if: $a(t) \to 0$ when $t \underset{<}{\to} t^*$. So the cosmological expansion factor a tends to zero in a finite time and such a solution (a, f) cannot be global towards the future. This completes the proof of proposition 4.2.

We now study the case $\Lambda > 0$. Notice that since $T^{00} \geq 0$, $T^{11} \geq 0$ (4.1)-(4.2) gives by subtraction $\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 < 0$; this implies $\frac{d}{dt}(\frac{\dot{a}}{a}) < 0$, which shows that, in all the cases, $\theta = 3\frac{\dot{a}}{a}$ is a decreasing function. In the case $\Lambda > 0$, the Hamiltonian constraint (4.1) gives: $\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \geq \frac{\Lambda}{3}$ i.e $\left[\frac{\dot{a}(t)}{a(t)} - \sqrt{\frac{\Lambda}{3}}\right]\left[\frac{\dot{a}(t)}{a(t)} + \sqrt{\frac{\Lambda}{3}})\right] \geq 0$, which is equivalent to:

$$\frac{\dot{a}(t)}{a(t)} \geq \sqrt{\frac{\Lambda}{3}} \tag{4.14}$$

or

$$\frac{\dot{a}(t)}{a(t)} \leq -\sqrt{\frac{\Lambda}{3}}. \tag{4.15}$$

The continuity of $t \to \frac{\dot{a}(t)}{a(t)}$ implies that we have to choose between (4.14) and (4.15). Since $a > 0$, (4.15) implies $\dot{a} < 0$ and a is decreasing; (4.14) implies that $\dot{a} > 0$, then a is increasing and since $\frac{\dot{a}}{a}$ is decreasing, this gives on $[t_0, t]$:

$$a(t) \geq a(t_0); \quad \sqrt{\frac{\Lambda}{3}} \leq \frac{\dot{a}(t)}{a(t)} \leq \frac{\dot{a}(t_0)}{a(t_0)}, \quad t \geq t_0. \tag{4.16}$$

Recall that our aim is to study the coupled Einstein-Boltzmann system and we had to require, for the study of the Boltzmann equation, that a, which is positive be bounded away from zero. This problem is solved by choosing (4.14). Another important consequence of (4.14) is that it implies on $[t_0,t]$

$$a(t)\geq a(t_0)e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}, \quad t\geq t_0.$$

This shows that the cosmological expansion factor has an exponential growth; but, it also shows that, an eventual global solution a will be unbounded, and this is why, in order to use standard results, we make the change of variable:

$$e=\frac{1}{a} \tag{4.17}$$

which gives:

$$\dot{e}=-\frac{\dot{a}}{a^2}. \tag{4.18}$$

Then, using (2.15), T^{00} and T^{11} in the r.h.s of (4.2) express in terms of e and f, and we set:

$$\rho=T^{00}=\frac{1}{e^3}\int_{\mathbb{R}^3}\sqrt{1+\frac{1}{e^2}|\bar{p}|^2}f(t,\bar{p})d\bar{p} \tag{4.19}$$

$$P=a^2T^{11}=\frac{1}{e^5}\int_{\mathbb{R}^3}\frac{(p^1)^2f(t,\bar{p})}{\sqrt{1+\frac{1}{e^2}|\bar{p}|^2}}d\bar{p}. \tag{4.20}$$

ρ stands for the density and P for the pressure. One verifies that:

$$P\leq\rho. \tag{4.21}$$

Recall that $r>0$ is such that $r>\|f_0\|$. If we set:

$$d_0=3\sqrt{\frac{\Lambda}{3}+\frac{16\pi}{3}ra_0^4} \tag{4.22}$$

one checks easily, using $a\geq a_0\geq\frac{3}{2}$, (4.16) with $t_0=0$ (4.8) (4.7) and (4.22), that

$$e=\frac{1}{a}\in[0,\frac{2}{3}]; \quad et \quad \theta=3\frac{\dot{a}}{a}\in[\sqrt{3\Lambda},d_0]. \tag{4.23}$$

A direct calculation shows, using (4.17), (4.18), (4.19), (4.20) , and (4.9), that the Einstein evolution equation (4.2) is equivalent to the following first order system in (e,θ):

$$\dot{e}=-\frac{\theta}{3}\times e \tag{4.24}$$

$$\dot{\theta}=-\frac{\theta^2}{3}-4\pi(\rho+3P)+\Lambda \tag{4.25}$$

with $\rho=\rho(e,f)$ and $P=P(e,f)$ given by (4.19) and (4.20).
We will study the initial value problem for the system (4.24)-(4.25) with initial data (e_0,θ_0) and $t=0$. By virtue of the change of variables (4.23), we will deduce a solution for the Einstein evolution equation (4.2) by setting:

$$e(0)=\frac{1}{a_0} \quad \theta(0)=3\frac{\dot{a}_0}{a_0} \tag{4.26}$$

in which a_0, $\dot{a}_0$ satisfy the constraint (4.8)-(4.7)with f_0 given. By virtue of (4.23) it will be enough for the study of the initial value problem for (4.24)-(4.25) to take e_0, θ_0 such that

$$0<e_0\leq\frac{2}{3}, \quad 0<\theta_0\leq d_0. \tag{4.27}$$

Remark 4.1. *The equivalence of the evolution equation (4.2) and the system (4.24)-(4.25) requires that any solution (e,θ) of (4.24)-(4.25)-(4.27) satisfies $e>0$ and $\theta>0$.*

For the global existence theorem, we will need the following a priori estimation.

Proposition 4.4. *Let $\delta>0$ and $t_0\in\mathbb{R}_+$ be given; suppose that in (4.19) and (4.20), $f\in C[t_0,t_0+\delta;L^1_2(\mathbb{R}^3)]$ is given. Let (e,θ) be any solution of the system (4.24)-(4.25) on $[t_0,t_0+\delta]$. Then θ and $a=\frac{1}{e}$ satisfy the inequalities:*

$$\theta(t_0+t)\leq\theta(t_0)+\Lambda, \quad t\in[0,\delta] \tag{4.28}$$

$$a(t_0+t)\leq a(t_0)e^{(\frac{\theta(t_0)}{3}+\frac{\Lambda}{3})(t_0+t+1)^2}, \quad t\in[0,\delta]. \tag{4.29}$$

Proof. Since $\rho\geq0$, $P\geq0$, (4.25) implies: $\dot{\theta}\leq\Lambda$. Integrating this inequality on $[t_0,t_0+t]$ where $t\in[0,\delta]$ yields (4.28).
Next, since $e>0$, integrating (4.24) that writes $-\frac{\dot{e}}{e}=\frac{\theta}{3}$ over $[t_0,t_0+t]$, $t\in[0,\delta]$ gives:

$$a(t_0+t)\leq a(t_0)e^{\int_{t_0}^{t_0+t}\frac{\theta(s)}{3}ds}, \quad t\in[0,\delta]. \tag{4.30}$$

Now setting in (4.28) $s=t_0+t\in[t_0,t_0+\delta]$,and integrating on $[t_0,t_0+t]$ yields:

$$\int_{t_0}^{t_0+t}\frac{\theta(s)}{3}ds\leq(\frac{\theta(t_0)}{3}+\frac{\Lambda}{3})(t_0+t+1)^2, \quad t\in[0,\delta] \tag{4.31}$$

(4.29) then follows from (a) and (b). □

We deduce:

Proposition 4.5. *Let $T>0$ and $f\in[[0,T];X_r]$ be given. Suppose that the initial value problem (4.24)-(4.25)-(4.26) with initial data (4.26) at $t=0$ satisfying the constraints (4.8)-(4.7) has a solution $(\Xi=\frac{1}{\Omega},\Theta)$ on $[0,t_0]$ with $0\leq t_0<T$. Then, any solution $(e=\frac{1}{a},\theta)$ of the initial value problem for the system (4.24)-(4.25) on $[t_0,t_0+\delta]$, $\delta>0$, with initial data $(e,\theta)(t_0)=(\Xi,\Theta)(t_0)$ at $t=t_0$, satisfy the inequalities:*

$$a(t_0+t)\leq C_2e^{C_3(t_0+t+1)^2}, \quad t\in[0,\delta] \tag{4.32}$$

$$\theta(t_0+t)\leq 3\gamma_1+\Lambda(T+t), \quad t\in[0,\delta] \tag{4.33}$$

where:

$$C_2=a_0e^{\gamma_1}; \quad C_3=\gamma_1+\frac{\Lambda}{3}T; \quad \gamma_1=\gamma_1(a_0,r,T)=(\frac{\Lambda}{3}+\sqrt{\frac{\Lambda}{3}+3ra_0^4})(T+1)^2. \tag{4.34}$$

Proof. We apply proposition 4.4 to the solution (e,θ) of (4.24)-(4.25) on $[t_0,t_0+\delta]$; (4.29) gives, since $a(t_0)=\Xi(t_0)$, $\theta(t_0)=\Theta(t_0)$:

$$a(t_0+t)\leq\Xi(t_0)e^{(\frac{\Theta(t_0)}{3}+\frac{\Lambda}{3})(t_0+t+1)^2}, \quad t\in[0,\delta]. \tag{a}$$

Now apply (4.29) to the solution (Ξ,Θ) of (4.24)-(4.25) on $[0,t_0]$, at the point t_0. We obtain by setting in (4.29), $t_0=0$, $t=t_0$ and since $a(t_0)=\Xi(t_0)$, $\Xi(0)=a_0$, $\Theta(0)=3\frac{\dot a_0}{a_0}$, $0\leq t_0<T$:

$$\Xi(t_0)\leq a_0e^{(\frac{\dot a_0}{a_0}+\frac{\Lambda}{3})(t_0+1)^2}\leq a_0e^{(\frac{\dot a_0}{a_0}+\frac{\Lambda}{3})(T+1)^2}, \quad t\in[0,\delta]. \tag{b}$$

Now since the initial data at $t=0$ satisfy (4.8), (4.7), this implies, using $\dot a_0>0$, $a_0>1$, $\|f_0\|<r$:

$$\frac{\dot a_0}{a_0}\leq\sqrt{\frac{\Lambda}{3}+\frac{8\pi}{3}ra_0^4} \tag{c}$$

and (b) gives:

$$\Xi(t_0)\leq a_0e^{\gamma_1} \tag{d}$$

with

$$\gamma_1=\left(\frac{\Lambda}{3}+\sqrt{\frac{\Lambda}{3}+\frac{8\pi}{3}ra_0^4}\right)(T+1)^2. \tag{e}$$

Now, we apply (4.28) to the solution (Ξ,Θ) of (4.24)-(4.25)-(4.26), on $[0,t_0]$ at point t_0, $t=t_0$ and since $\Xi(0)=3\frac{\dot a_0}{a_0}$, $0\leq t_0<T$:

$$\Xi(t_0)\leq\Xi(0)+\Lambda t_0\leq 3\frac{\dot a_0}{a_0}+\Lambda T. \tag{f}$$

We obtain, using (c) and (f):

$$\frac{\Xi(t_0)}{3}+\frac{\Lambda}{3}\leq\left(\frac{\Lambda}{3}+\sqrt{\frac{\Lambda}{3}+\frac{8\pi}{3}ra_0^4}\right)+\frac{\Lambda}{3}T.$$

This gives, using the definition (e) of γ_1:

$$\frac{\Xi(t_0)}{3}+\frac{\Lambda}{3}\leq\gamma_1+\frac{\Lambda}{3}T \tag{g}$$

and (4.32) follows from (a), (d) and (g).
The inequality (4.33) follows from (4.28), $\theta(t_0)=\Xi(t_0)$, (f), (c) and (e). This completes the proof of prop. 4.5 □

We can now prove:

PROPOSITION 4.6. *Let $T>0$ and $f\in C[[0,T];X_r]$ be given. Then the initial value problem for the system (4.24)-(4.25) with initial data (e_0,θ_0) at $t=0$ satisfying (4.27), has an unique solution (e,θ) on $[0,T]$.*

The proof of proposition 4.6 will use the following result:

LEMMA 4.7. *Let $e_1,e_2\in]0,\frac{2}{3}]$; then we have, C being a constant:*

$$|\rho(e_1,f)-\rho(e_2,f)|\leq \frac{C}{e_1^4e_2^6}\|f(t)\||e_1-e_2| \tag{4.35}$$

$$|P(e_1,f)-P(e_2,f)|\leq \frac{C}{e_1^4e_2^7}\|f(t)\||e_1-e_2|. \tag{4.36}$$

Proof. We can write, using (4.19)

$$\begin{aligned}|\rho(e_1,f)-\rho(e_2,f)|\leq |\frac{1}{e_1^3}-\frac{1}{e_2^3}|\int_{\mathbb{R}^3}\sqrt{1+\frac{1}{e_1^2}|\bar p|^2}f(t,\bar p)d\bar p\\ +\frac{1}{e_2^3}\int_{\mathbb{R}^3}|\sqrt{1+\frac{1}{e_1^2}|\bar p|^2}-\sqrt{1+\frac{1}{e_2^2}|\bar p|^2}|f(t,\bar p)d\bar p.\end{aligned} \tag{a}$$

But we have, by usual factorization, using $e_1,e_2\in]0,1]$:

$$|\frac{1}{e_1^3}-\frac{1}{e_2^3}|\leq \frac{|e_1-e_2|}{e_1^3e_2^3};\quad |\sqrt{1+\frac{1}{e_1^2}|\bar p|^2}-\sqrt{1+\frac{1}{e_2^2}|\bar p|^2}|\leq \frac{3\sqrt{1+|\bar p|^2}|e_1-e_2|}{e_1^3e_2^3}. \tag{b}$$

Then (4.35) follows from (a) and (b); (4.36) is obtained by a similar way. ∎

Proof of Proposition 4.6: Define, using (4.24), (4.25), (4.27), the function F on $]0,\frac{2}{3}]\times]0,d_0]$ by:

$$F(e,\theta)=\left[-\frac{\theta e}{3},\quad -\frac{\theta^2}{3}-4\pi(\rho+3P)+\Lambda\right]. \tag{4.37}$$

Then by usual factorization and using (4.37), (4.38), it appears that F is locally Lipschitzian with respect the $\mathbb{R}^2$-norm, and continuous in t. Now, by (4.32) with $a=\frac{1}{e}$, F is uniformly bounded. Hence, prop. 4.6 follows from the standard theorem of differential systems.

We deduce a result that will be useful to prove the global existence for the coupled Einstein-Boltzmann system. We will use the number D_0 defined by:

$$D_0=3\gamma_1+\Lambda(T+1) \tag{4.38}$$

where γ_1 is defined by (4.34). We prove:

PROPOSITION 4.8. *Let $T>0$ and $f\in C[[0,T];X_r]$ be given. Let $(\Xi=\frac{1}{\Omega},\Theta)$ be the solution of the initial value problem for the system (4.24)-(4.25) with initial data (e_0,θ_0) at $t=0$ given by (4.26) with a_0, $\dot a_0$ satisfying the constraints (4.8)-(4.7). Let*

$t_0 \in [0,T]$. *Then the initial values problem for the system (4.24)-(4.25), with the initial data* $(\Xi = \frac{1}{\Omega}, \Theta)(t_0)$ *at* $t = t_0$ *has a unique solution* $(e = \frac{1}{a}, \theta)$ *on* $[t_0, t_0+\delta]$, *where* $\delta > 0$ *is independent of* t_0. *The solution* $(e = \frac{1}{a}, \theta)$ *satisfies the inequalities:*

$$\frac{3}{2} \leq a(t_0+t) \leq C_2 e^{C_3(t_0+t+1)^2}, \quad t \in [0,\delta] \tag{4.39}$$

$$\sqrt{3\Lambda} \leq \theta(t_0+t) \leq 3\gamma_1 + \Lambda(T+t), \quad t \in [0,\delta] \tag{4.40}$$

where C_2, C_3 *and* γ_1 *are defined by (4.34)*

Proof. We have $\theta_0 = 3\frac{\dot{a}_0}{a_0}$ and (4.8) implies $\theta_0 \geq \sqrt{3\Lambda}$, so, the proof given for prop. 4.6 with the function F given by (4.37) and defined this time on $]0, \frac{2}{3}] \times [\sqrt{3\Lambda}, d_0]$ leads to a solution (Ξ, Θ) satisfying $\Theta \geq \sqrt{3\Lambda}$, $\Xi \leq \frac{2}{3}$, and the existence of (e, θ). Now (4.39) follows from (4.32) and (4.40) from (4.33). ∎

Theorem 4.9. *Let* $T > 0$ *and* $f \in C[[0,T]; X_r]$ *be given. Then the initial value problem for the Einstein equation (4.1)-(4.2)-(4.6) with initial data* a_0, $\dot{a}_0$ *satisfying the constraints (4.8)-(4.7) has an unique solution* a *on* $[0,T]$.

Proof. It is a consequence of prop. 4.6 and the equivalence of the system (4.1)-(4.2) and (4.24)-(4.25). ∎

5. Local existence of solutions for the coupled Einstein-Boltzmann system

5.1. Equations and functional framework. Given the equivalence of the initial value problems (4.1)-(4.2)-(4.6) with constraint (4.8)-(4.7) and (4.24)-(4.25)-(4.26), and following the study of the Boltzmann equation, paragraph 3, by the characteristic method that leads to (3.14), we study the initial value problem for the first order system:

$$\frac{df}{dt} = \frac{1}{p^0} Q(f,f) \tag{5.1}$$

$$\dot{e} = -\frac{\theta}{3} e \tag{5.2}$$

$$\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) \tag{5.3}$$

with, in (5.3) $\rho = \rho(e,f)$ and $P = P(e,f)$ defined by (4.19) and (4.20). The initial data at $t = 0$ are denoted f_0; e_0; θ_0 i.e.

$$f(0,\bar{p}) = f_0(\bar{p}); \quad e(0) = e_0; \quad \theta(0) = \theta_0 \tag{5.4}$$

where we take

$$f_0 \in L^1_2(\mathbb{R}^3); \quad f_0 \geq 0 \quad a.e; e_0 = \frac{1}{a_0}; \quad \theta_0 = 3\frac{\dot{a}_0}{a_0}, \tag{5.5}$$

with, following (4.8)-(4.7), a_0, $\dot{a}_0$, f_0 subject to the constraints:

$$a_0 \geq \frac{3}{2}, \qquad \frac{\dot{a}_0}{a_0} = \sqrt{\frac{8\pi}{3}\int_{\mathbb{R}^3} a_0^3 \sqrt{1 + a_0^2|p|^2} f_0(\bar{p}) d\bar{p} + \frac{\Lambda}{3}}. \tag{5.6}$$

We are going to prove the existence of the solution (f,e,θ) of the above initial value problem, on an interval $[0,l]$, $l>0$. Given the study in paragraphs 3 and 4, the functional framework will be the Banach space $E=L^1_2(\mathbb{R}^3)\times\mathbb{R}\times\mathbb{R}$, endowed with the norm

$$\|(f,e,\theta)\|_E=\|f\|+|e|+|\theta|$$

where $\|.\|$ is defined by (2.4) and $|.|$ is the absolute value in $\mathbb{R}$.

5.2. The local existence theorem.

PROPOSITION 5.1. *There exists an interval* $[0,l]$, $l>0$ *such that, the initial value problem for the system (5.1)-(5.2)-(5.3) with initial data* $(f_0,e_0,\theta_0)\in L^1_2(\mathbb{R}^3)\times\mathbb{R}\times\mathbb{R}$ *has a unique solution* (f,e,θ) *on* $[0,l]$.

The proof of Prop. 5.1 will use the following result in which $C>0$ is a constant.

LEMMA 5.2. *We have for* $f,f_1,f_2\in L^1_2(\mathbb{R}^3)$, $e_1,e_2\in]0,\frac{3}{2}]$:

$$\left\|\left(\frac{1}{p^0(e_1)}-\frac{1}{p^0(e_2)}\right)Q(f,f)(e_2)\right\|\le\frac{C}{e_1^2e_2^5}\|f\||e_1-e_2| \tag{5.7}$$

$$\left\|\frac{Q(f_1,f_1)(e_2)-Q(f_2,f_2)(e_2)}{p^0(e_2)}\right\|\le\frac{C}{e_2^3}(\|f_1\|+\|f_2\|)\|f_2-f_1\| \tag{5.8}$$

$$\left\|\frac{1}{p^0(e_1)}[Q(f,f)(e_1)-Q(f,f)(e_2)]\right\|\le\frac{C\|f\|^2}{e_1^5e_2^3}|e_1-e_2| \tag{5.9}$$

$$\left\|\frac{Q(f_1,f_1)(e_1)}{p^0(e_1)}-\frac{Q(f_2,f_2)(e_2)}{p^0(e_2)}\right\|\le\frac{C(\|f_1\|+\|f_1\|^2+\|f_2\|)(|e_1-e_2|+\|f_1-f_2\|)}{e_1^5e_2^5}. \tag{5.10}$$

Proof. We have by usual factorization, using $p^0=\sqrt{1+\frac{1}{e^2}|\bar{p}|^2}$, $0<e_i<1$:

$$\left\|\left(\frac{1}{p^0(e_1)}-\frac{1}{p^0(e_2)}\right)Q(f,f)(e_2)\right\|\le\frac{2}{e_1^2e_2^5}\frac{Q(f,f)(e_2)}{p^0(e_2)}|e_1-e_2|.$$

(5.9) then follows from (3.26) with $g=0$ and $a=\frac{1}{e_2}$ in (3.27). (5.10) follows from (3.26) and (3.27). To obtain (5.11), we use $Q=Q^- -Q^-$. We prove by usual factorization, using expression (3.2) of Q^+ and proceeding as for (3.24):

$$\left\|\frac{Q(f_1,f_1)(e_1)-Q(f_1,f_1)(e_2)}{p^0(e_1)}\right\|\le\frac{C\|f\|^2|e_1-e_2|}{e_1^5e_2^3} \tag{a}$$

and an analogous estimation for Q^-, using (3.3). Then (5.11) follows. (5.12) is a consequence of (5.9), (5.10) and (5.11) adding and subtracting adequate terms. □

Proof. **Proof of Prop. 5.1:** By (4.19), (4.20), ρ and P are linear; so, we deduce from (4.35), (4.36) that:

$$|\rho(e_1,f_1)-\rho(e_2,f_2)|\le\frac{C(\|f_1\|+1)}{e_1^4e_2^6}(|e_2-e_1|+\|f_1-f_2\|) \tag{5.11}$$

$$|P(e_1,f_1)-P(e_2,f_2)|\leq \frac{C(\|f_1\|+1)}{e_1^4 e_2^7}(|e_2-e_1|+\|f_1-f_2\|). \tag{5.12}$$

Now, by (5.12), (5.13) and (5.14), we can conclude that the function F defined by the r.h.s of (5.1), (5.2), (5.3) is locally Lipschitzian in (f,e,θ) with respect to the norm of E. Proposition 5.1 then follows from the standard theorem on first order differential system for functions with values in a Banach space. Notice the e,θ are continuous and $f\in C[0,l;L^1_2(\mathbb{R}^3)]$. □

From proposition 5.1, we deduce the following theorem:

THEOREM 5.3. *Let $f_0\in L^1_2(\mathbb{R}^3)$, $f_0\geq 0$ a.e, $a_0\geq \frac{3}{2}$ and let a strictly positive cosmological constant Λ, be given. Define $\dot{a}_0$ by the relation (5.6).*
Let $r>\|f_0\|$. Then, there exists a number $l>0$ such that the initial value problem for the coupled Einstein-Boltzmann system (2.13)-(4.1)-(4.2) with the initial data $(f_0,a_0,\dot{a}_0)$, has an unique solution (f, a) on $[0,l]$. The solution (f, a) has the following properties:

$$(i)\ a \text{ is an increasing function.} \quad (ii)\ f\in C[[0,l],X_r] \quad (iii)\ \||f\||\leq \|f_0\|. \tag{5.13}$$

Proof. Choose in proposition 5.1, f_0,e_0 and θ_0 as in (5.5), then theorem 5.3 is a consequence of prop. 5.1, the equivalence between (2.13) and (5.1), and between (5.2)-(5.3) and (4.1)-(4.2). (i) is given by (4.14), (ii) by the uniqueness theorem 3.1, and (iii) by (3.18). □

6. Global existence theorem for the coupled Einstein-Boltzmann system

6.1. The method. Here, we prove that the local solution obtained in §5 is, in fact, a global solution. Let us sketch the method we adopt: Denote $[0,T]$, where $T>0$, the **maximal** existence domain of the solution of (5.1)-(5.2)-(5.3), with initial data (f_0,e_0,θ_0) defined by (5.4)-(5.5)-(5.6); here we denote this solution $(\tilde{f},\tilde{e},\tilde{\theta})$, in other words we have, on $[0,T]$

$$\dot{\tilde{f}}=\frac{1}{p^0(\tilde{e})}Q(\tilde{f},\tilde{f}) \tag{6.1}$$

$$\dot{\tilde{e}}=-\frac{\tilde{\theta}}{3}\tilde{e} \tag{6.2}$$

$$\dot{\tilde{\theta}}=-\frac{\tilde{\theta}^2}{3}-4\pi(\tilde{\rho}+3\tilde{P})+\Lambda \tag{6.3}$$

$$\tilde{f}(0)=f_0\in X_r, \qquad \tilde{e}(0)=e_0=\frac{1}{a_0}, \qquad \tilde{\theta}(0)=3\frac{\dot{a}_0}{a_0} \tag{6.4}$$

with in (6.3) $\tilde{\rho}=\rho(\tilde{f},\tilde{e})$: $\tilde{P}=P(\tilde{f},\tilde{e})$ and in (6.4) f_0, a_0, θ_0 subject to the constraints (5.6). $r>0$ is given such that $r>\|f_0\|$.
If $T=+\infty$, the problem is solved. We are going to show that, if we suppose that $T<+\infty$, then the solution $(\tilde{f},\tilde{e},\tilde{\theta})$ can be extended beyond T, which contradicts the maximality of T. Suppose $0<T<+\infty$ and let $t_0\in[0,T]$. We will show that, there

exists a strictly positive number $\delta>0$, **independent of** t_0, such that the following system in (e,θ) on $[t_0,t_0+\delta]$, in which $\tilde{a}=\frac{1}{\tilde{e}}$:

$$\dot{f}=\frac{1}{p^0(e)}Q(f,f) \tag{6.5}$$

$$\dot{e}=-\frac{\theta}{3}e \tag{6.6}$$

$$\dot{\theta}=-\frac{\theta^2}{3}-4\pi(\rho+3P)+\Lambda \tag{6.7}$$

$$f(t_0)=\tilde{f}(t_0), \quad e(t_0)=\tilde{e}(t_0), \quad \theta(t_0)=\tilde{\theta}(t_0)=3\frac{\dot{\tilde{a}}(t_0)}{\tilde{a}(t_0)} \tag{6.8}$$

has a solution (f,e,θ) on $[t_0,t_0+\delta]$. Then, by taking t_0 sufficiently close to T, for example, to such that $0<T-t_0<\frac{\delta}{2}$, hence $T<t_0+\frac{\delta}{2}$, we can extend the solution $(\tilde{f},\tilde{e},\tilde{\theta})$ to $[0,t_0+\frac{\delta}{2}]$ that contains strictly $[0,T]$, and this will contradict the maximality of T. We need some preliminaries results.
In what follows, we suppose $0<T<+\infty$ and $t_0\in[0,T]$.

6.2. The functional framework. In all what follows, C_2, C_3 , D_0 are the absolute constant defined by (4.34) and (4.38). We set, for $\delta>0$:

$$E^{\delta}_{t_0}=\left\{e\in C[t_0,t_0+\delta], \frac{1}{C_2}e^{-C_3(t_0+t+1)^2}\leq e(t_0+t)\leq\frac{2}{3} \quad ,\forall t\in[0,\delta[\right\}$$

$$F^{\delta}_{t_0}=\left\{\theta\in C[t_0,t_0+\delta], \theta, \sqrt{3\Lambda}\leq\theta(t_0+t)\leq D_0 \quad \forall t\in[0,\delta[\right\}$$

where $C[t_0,t_0+\delta]$ is the space of continuous (and hence bounded) functions on $[t_0,t_0+\delta]$. One verifies easily that $E^{\delta}_{t_0}$ and $F^{\delta}_{t_0}$ are complete metric subspaces of the Banach space $(C[t_0,t_0+\delta],\|.\|_\infty)$ where $\|u\|_\infty=\underset{t\in[t_0,t_0+\delta]}{Sup}|u(t)|$.

6.3. The global existence theorem.

PROPOSITION 6.1. *There exists a strictly positive real number $\delta>0$ depending only on the absolute constants a_0, Λ , r and T such that the initial value problem (6.5)-(6.6)-(6.7)-(6.8) has a solution $(f,e=\frac{1}{a},\theta)\in C[[t_0,t_0+\delta];X_r]\times E^{\delta}_{t_0}\times F^{\delta}_{t_0}$.*

Proof. It will be enough, if we look for δ such that $0<\delta<1$. By theorem 3.1, we know that if we fix $\bar{e}\in E^{\delta}_{t_0}$ and if we set $\bar{a}=\frac{1}{\bar{e}}$, then (6.5) has an unique solution $f\in C[[t_0,t_0+\delta];X_r]$, such that, $f(t_0)=\tilde{f}(t_0)$, and, by (3.18) and (5.13):

$$\|f(t)\|\leq\|\tilde{f}(t_0)\|\leq\|f_0\|\leq r. \tag{6.9}$$

Next, by proposition 4.8 in which we set $\Xi=\tilde{e},\Theta=\tilde{\theta}$, we know that if $\bar{f}$ is given in $C[[t_0,t_0+\delta];X_r]$, then (6.6)-(6.7) has a unique solution (e,θ) on $[t_0,t_0+\delta]$ such that

$e(t_0) = \frac{1}{\tilde{a}(t_0)}$; $\theta(t_0) = 3\frac{\dot{\tilde{a}}(t_0)}{\tilde{a}(t_0)}$. Now (4.39) and (4.40) show that $(e = \frac{1}{a}, \theta) \in E^\delta_{t_0} \times F^\delta_{t_0}$. This allows us to define the application :

$$G : C[[t_0, t_0+\delta]; X_r] \times E^\delta_{t_0} \to C[[t_0, t_0+\delta]; X_r] \times (E^\delta_{t_0} \times F^\delta_{t_0}) \tag{6.10}$$

$$(\bar{f}, \bar{e}) \mapsto G(\bar{f}, \bar{e}) = [f, (e, \theta)]. \tag{6.11}$$

We are going to show that we can find $\delta > 0$ such that G defined by (6.10) induces a contracting map of the complete metric space $C[[t_0, t_0+\delta]; X_r] \times E^\delta_{t_0}$ into itself, that will hence, have a unique fixed point (f, e); this will allow us to find θ such that (f, e, θ) be the unique solution of (6.5)-(6.6)-(6.7)-(6.8) in $C[[t_0, t_0+\delta]; X_r] \times (E^\delta_{t_0} \times F^\delta_{t_0})$. So if we set in (6.5) $e = \bar{e} \in E^\delta_{t_0}$, in (6.7) $\rho = \bar{\rho} = \rho(e, \bar{f})$, $P = \bar{P} = P(e, \bar{f})$ where $\bar{f} \in C[[t_0, t_0+\delta]; X_r]$, we have a solution (f, e, θ) of that system, or, equivalently, a solution (f, e, θ) of the following integral system with $t \in [0, \delta]$:

$$f(t_0+t) = \tilde{f}(t_0) + \int_{t_0}^{t_0+t} \frac{1}{p^0(\bar{e})} Q(f, f)(\bar{e})(s) ds \tag{6.12}$$

$$e(t_0+t) = \tilde{e}(t_0) + \int_{t_0}^{t_0-t} \frac{\theta(s)e(s)}{3} ds \tag{6.13}$$

$$\theta(t_0+t) = \tilde{\theta}(t_0) + \int_{t_0}^{t_0+t} [-\frac{\theta^2}{3} - 4\pi(\bar{\rho} + 3\bar{P}) + \Lambda](s) ds \tag{6.14}$$

Let $\bar{e}_1, \bar{e}_2 \in E^\delta_{t_0}$; $\bar{f}_1, \bar{f}_2 \in C[[t_0, t_0+\delta]; X_r]$. To $\bar{e}_i$ (resp $\bar{f}_i$), i = 1, 2, corresponds by G, the solution f_i (resp (e_i, θ_i)) of (6.12), [resp (6.13)-(6.14)].
We now write each equation for i = 1, 2 and subtract. We apply (5.10) in which we set $e_1 = \bar{e}_1$, $e_2 = \bar{e}_2$ and (5.11)-(5.12), in which we set $f_1 = \bar{f}_1$, $f_2 = \bar{f}_2$. Since $\bar{e}_1, \bar{e}_2, e_1, e_2 \in E^\delta_{t_0}$, (4.39), shows that: $\frac{1}{e_i}, \frac{1}{\bar{e}_i} \leq C_2 e^{C_3(T+2)^2}$, since $t_0 < T$, $t \leq \delta < 1$. So we deduce from (6.12), (6.13) and (6.14), using (5.10), (5.11), (5.12), $|||\bar{f}_i||| < r$, $0 \leq t \leq \delta$, the definition of $E^\delta_{t_0}$, $F^\delta_{t_0}$ $(|\theta_i| \leq D_0(a_0, r, \Lambda, T))$, see (4.38), the definition (2.16) of the norm $|||.|||$ of function over $[t_0, t_0+\delta]$, that:

$$|||f_1 - f_2||| \leq \delta M_1(\|\bar{e}_1 - \bar{e}_2\|_\infty + |||f_1 - f_2|||) \tag{6.15}$$

$$\|e_1 - e_2\|_\infty \leq \delta M_2(\|e_1 - e_2\|_\infty + \|\theta_1 - \theta_2\|_\infty) \tag{6.16}$$

$$\|\theta_1 - \theta_2\|_\infty \leq \delta M_3(\|e_1 - e_2\|_\infty + \|\theta_1 - \theta_2\|_\infty + |||\bar{f}_1 - \bar{f}_2|||). \tag{6.17}$$

where M_1, M_2, M_3 are constants depending only on a_0, Λ, r and T. We have by addition of (6.16) and (6.17):

$$\|e_1 - e_2\|_\infty + \|\theta_1 - \theta_2\|_\infty) \leq 2\delta(M_2 + M_3)(\|e_1 - e_2\|_\infty + \|\theta_1 - \theta_2\|_\infty + |||\bar{f}_1 - \bar{f}_2|||). \tag{6.18}$$

Then, if we choose δ such that:

$$\delta = Inf\left[1, \frac{1}{8(M_1 + M_2 + M_3)}\right]. \tag{6.19}$$

We can arrange differences in $\bar{e}_i, \bar{f}_i$ in the r.h.s and those in e_i, f_i, θ_i in the l.h.s and obtain by addition:

$$|||f_1 - f_2||| + \|e_1 - e_2\|_\infty + \|\theta_1 - \theta_2\|_\infty \le \frac{1}{3}(|||\bar{f}_1 - \bar{f}_2||| + \|\bar{e}_1 - \bar{e}_2\|_\infty)$$

from which, we deduce:

$$|||f_1 - f_2||| + \|e_1 - e_2\|_\infty \le \frac{1}{3}(|||\bar{f}_1 - \bar{f}_2||| + \|\bar{e}_1 - \bar{e}_2\|_\infty). \tag{6.20}$$

(6.20) shows that the map $(\bar{f}, \bar{e}) \mapsto (f, e)$ is a contracting map from the complete metric space $C[[t_0, t_0 + \delta]; X_r] \times E^\delta_{t_0}$ into itself, for every δ satisfying (6.19), which shows that such a δ depends only on a_0, r, Λ and T. This map has an unique fixed point (f, e); since e is known, (6.6) determines θ by $\theta = -3\frac{\dot{e}}{e}$, and (f, e, θ) is a solution of (6.5)-(6.6)-(6.7)-(6.8) in $C[[t_0, t_0 + \delta]; X_r] \times E^\delta_{t_0} \times F^\delta_{t_0}$. This completes the proof of proposition 6.1. □

We can then state:

THEOREM 6.2. *The initial value problem for the Einstein-Boltzmann system with a strictly positive cosmological constant* Λ *on a Robertson-Walker space-time has a global solution (a, f) on* $[0, +\infty[$, *for arbitrarily large initial data* a_0 *and* $f_0 \in L^1_2(\mathbb{R}^3)$, $f_0 \ge 0$ *a.e.*

REMARK 6.1. *1) Nowhere in the proof did we have to restrict the size of the initial data* a_0, f_0, *which can then be taken as arbitrarily large.*

2) In [13], the author considered only the Hamiltonian constraint (4.1), in the case $\Lambda = 0$, *without studying the evolution equation (4.2) that is, as we saw, the main problem to solve, since the Hamiltonian constraint (4.1) is satisfied once it is the case for the initial data.*

3) We will prove in a future paper, that theorem 6.2 extends to the case $\Lambda = 0$.

4) In the future, we will try to relax hypotheses on the collision kernel A.

5) Details on the present paper can be found in [17].

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