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A RESULT OF L^2-WELL POSEDNESS CONCERNING THE SYSTEM OF LINEAR ELASTICITY IN 2D*

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Abstract. We give an L^2-well posedness result concerning an initial boundary value problem for the system of linear elasticity either in the half-plane or in a two dimensional bounded domain. Under the necessary uniform Kreiss Lopatinskii condition we construct here a dissipative Kreiss symmetrizer of our problem; actually, due to the characteristic boundary and the lack of a technical assumption given by T. Ohkubo, the main difficulty consists of building the dissipative symmetrizer near some special "boundary points".

Key words. Linear elasticity, initial boundary value problems, dissipative symmetrizers.

AMS subject classifications. 35L50, 74B05.

1. Introduction

We are concerned with the system of linear elasticity in two space dimension (2D). This system reads as follows

$$\begin{aligned}\partial_t F + \nabla z = 0,\\ \partial_t z + \mathrm{div} T = 0,\end{aligned} \tag{1.1}$$

where $F(x,t)\in \mathbf{M}_{2\times 2}(\mathbb{R})$, $z(x,t)\in\mathbb{R}^2$ (for $x=(x_1,x_2)\in\mathbb{R}^2$ and $t>0$) are the unknowns and we set

$$T := \lambda(F+F^T)+\mu(\mathrm{Tr}F)I_2, \tag{1.2}$$

with I_2 the identity matrix of order 2. The vector field z represents the opposite of the material velocity, while the stress tensor T is an isotropic function of the infinitesimal deformation tensor F and λ,μ are given positive constants (the so-called *Lamé coefficients*). A thorough analysis of the elasticity model can be found in the books of P. Ciarlet [2] and C. Dafermos [3]. Since in (1.1) the skew-symmetric part $F_{1,2}-F_{2,1}$ decouples from the rest, we may restrict to the system describing the evolution of z and the symmetric part of F. Since the system admits a quadratic energy

$$\frac{1}{2}|z|^2+\frac{\lambda}{4}|F+F^T|^2+\frac{\mu}{2}(\mathrm{Tr}F)^2, \tag{1.3}$$

it is Friedrichs symmetrizable. Setting $c_P := \sqrt{2\lambda+\mu}$ (the velocity of *pressure waves*), the choice of variables

$$u := (2\sqrt{\lambda(\lambda+\mu)}F_{1,1}, c_P\sqrt{\lambda}(F_{1,2}+F_{2,1}), c_P^2 F_{2,2}+\mu F_{1,1}, c_P z_1, c_P z_2)^T \tag{1.4}$$

puts the system (1.1) in the symmetric form

$$Lu := \frac{\partial u}{\partial t}+\sum_{\alpha=1}^{2} A^\alpha \frac{\partial u}{\partial x_\alpha} = 0, \tag{1.5}$$

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where the matrix $A(\xi) := \sum_{\alpha=1}^{2} A^\alpha \xi_\alpha$ has the form

$$A(\xi) = \begin{pmatrix} 0 & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) + \xi_2 a^2 \end{pmatrix}, \quad \xi = (\eta, \xi_2). \tag{1.6}$$

In particular, we have

$$a_2(\eta) + \xi_2 a^2 := \begin{pmatrix} 0_2 & b(\xi) \\ b(\xi)^T & 0_2 \end{pmatrix}, \qquad b(\xi) := \begin{pmatrix} \xi_2\sqrt{\lambda} & \eta\sqrt{\lambda} \\ \frac{\eta\mu}{c_P} & \xi_2 c_P \end{pmatrix}, \tag{1.7}$$

where 0_2 is the zero squared matrix of order 2 and $a_{2,1}(\eta)^T = (0, 0, 2\eta\sqrt{\lambda(\lambda+\mu)}/c_P, 0)$. We are interested in the well posedness of an initial boundary value problem (ibvp) such as

$$\begin{array}{ll} Lu(x_1,x_2,t) = f(x_1,x_2,t), & x_1 \in \mathbb{R}, \quad x_2, t > 0, \\ Bu(x_1,0,t) = g(x_1,t), & x_1 \in \mathbb{R}, \quad t > 0, \\ u(x_1,x_2,0) = a(x_1,x_2), & x_1 \in \mathbb{R}, \quad x_2 > 0. \end{array} \tag{1.8}$$

The domain of the problem is the half-plane $\mathbb{R}^2_+ := \{x = (x_1,x_2) \in \mathbb{R}^2; x_2 > 0\}$ (so that the boundary $\partial(\mathbb{R}^2_+) = \{(x_1,0); x_1 \in \mathbb{R}\}$ will be identified with $\mathbb{R}$); the data f, g, a are given smooth functions and B is a given matrix in $\mathbf{M}_{2\times 5}(\mathbb{R})$ with $\mathrm{rank} B = 2$. Hereafter, for every integer $n \geq 2$, we will write $\mathbf{0}_n$ for the zero $n \times 1$ matrix. Since

$$A^2 = \begin{pmatrix} 0 & \mathbf{0}_4^T \\ \mathbf{0}_4 & a^2 \end{pmatrix}, \qquad a^2 := \begin{pmatrix} 0 & 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & 0 & c_P \\ \sqrt{\lambda} & 0 & 0 & 0 \\ 0 & c_P & 0 & 0 \end{pmatrix},$$

we immediately compute $\mathrm{rank} A^2 = 4$ so that the ibvp (1.8) is characteristic. In [5], A. Majda and S. Osher develop a general theory for uniformly characteristic ibvps including the foregoing 2D system (1.8). However, our work differs from Majda-Osher's paper for we provide explicitly an everywhere smooth symmetrizer of (1.8), whereas the symbolic symmetrizer constructed in [5] may display some singularities (cf. [5], Part II, Section 6). It must be even pointed out that the approach presented here turns out to be workable for the three dimensional (3D) linear elasticity as well, although some further technical difficulties might occur in this case. Applying our method to the 3D linear elasticity system will be rather interesting, since the latter system is no longer covered by Majda-Osher's theory; indeed the 3D counterpart of the symmetric system (1.5)-(1.7) does not obey a technical hypothesis needed in Majda-Osher's analysis (cf. Assumption 1.1. of [5]). The 3D linear elasticity will be studied in a next paper, by adapting the same procedure explained here. To carry out the announced symmetrizer's construction, we require that $\mathrm{Ker} A^2 = \mathbb{R} \times \{\mathbf{0}_4\} \subset \mathrm{Ker} B$, which yields $B = (\mathbf{0}_2, B_2)$ for $B_2 \in \mathbf{M}_{2\times 4}(\mathbb{R})$. This last assumption, called *reflexivity* in [7], is natural for characteristic ibvps, since for L^2 solutions u the best control of boundary terms that we expect is that of $A^2 u$; as a matter of fact, this restriction is also justified by Majda and Osher in [5]. It is well-known that the ibvp (1.8) is *strongly L^2-well posed* (see [1] for the notion of strong well posedness in a general framework) when the boundary condition $Bu = g$ is *maximal strictly dissipative* for the operator L. This means that the quadratic form $w \mapsto w^* A^2 w$ is non positive on $\mathrm{Ker} B$ and its restriction to $\mathrm{Ker} B$ vanishes only on $\mathrm{Ker} A^2$; moreover $\mathrm{Ker} B$ must be maximal with respect to the aforesaid property. We are interested here in the strong L^2-well posedness of (1.8) with general boundary condition $Bu=g$, satisfying the weaker but necessary *uniform Kreiss-Lopatinskii condition* (UKL). Recall that the characteristic ibvp (1.8) is said to fulfill the (UKL) condition provided that there exists a positive constant C for which the estimate below

$$|A^2V|\leq C|BV|,\quad V\in\mathcal{E}_-(\tau,\eta)$$

holds true for all pairs $(\tau,\eta)\in\mathbb{C}\times\mathbb{R}$ with $\Re\tau>0$; for any (τ,η) as before, we mean by $\mathcal{E}_-(\tau,\eta)$ the *stable subspace* of the system

$$(\tau I_5+iA^1\eta)V+A^2\frac{dV}{dx_2}=0,$$

obtained by taking the Fourier-Laplace transform of (1.5) with respect to (x_1,t). The reader is referred to [4] (see also [8] Chapter 14) for an exhaustive presentation of the (UKL) condition. According to notations (1.4)-(1.7), the main result of the paper may be stated as follows.

Theorem 1.1. *Let us consider the ibvp* (1.8)*; let the boundary matrix* $B\in\mathbf{M}_{2\times5}(\mathbb{R})$ *satisfy the (UKL) condition. Then for every data* $f\in L^2(\mathbb{R}^2_+\times(0,T))$, $g\in L^2(\mathbb{R}\times(0,T))$ *and* $a\in L^2(\mathbb{R}^2_+)$*, with arbitrary* $T>0$*, there exists one, and only one, solution* $u\in L^2(\mathbb{R}^2_+\times(0,T))$ *of* (1.8) *such that:*

a. $u\in C([0,T];L^2(\mathbb{R}^2_+))$*;*

b. A^2u *admits a trace* γ_0A^2u *on the boundary of* $\mathbb{R}^2_+$ *of class* $L^2(\mathbb{R}\times(0,T))$.

Finally, for every positive number γ*, the following a priori estimate holds true:*

$$e^{-2\gamma T}\|u(T)\|^2_{L^2}+\|u\|^2_{\gamma,T}\leq C\left(\|a\|^2_{L^2}+\int_0^T e^{-2\gamma t}\left(\frac{1}{\gamma}\|f(t)\|^2_{L^2}+\|g(t)\|^2_{L^2}\right)dt\right),\quad (1.9)$$

where the constant $C>0$ *does not depend on* f,g,a *and* γ,T*. In* (1.9) $\|.\|_{L^2}$ *denotes the norm in either* $L^2(\mathbb{R}^2_+)$ *or* $L^2(\mathbb{R})$*; moreover we have set*

$$\|u\|^2_{\gamma,T}:=\int_0^T\int_{\mathbb{R}}e^{-2\gamma t}|(\gamma_0A^2u)(x_1,t)|^2dx_1dt+\gamma\int_0^T\int_{\mathbb{R}^2_+}e^{-2\gamma t}|u(x_1,x_2,t)|^2dx_1dx_2dt.\quad (1.10)$$

In order to prove Theorem 1.1, we look for the existence of a *dissipative Kreiss symmetrizer* of (1.8) (cf. [1], [4], [6]).

Let us recall that a dissipative symmetrizer consists of a matrix-valued C^∞ bounded function $(\tau,\eta)\mapsto K(\tau,\eta)\in\mathbf{M}_{5\times5}(\mathbb{C})$ defined on $\{(\tau,\eta)\in\mathbb{C}\times\mathbb{R};\Re\tau\geq0,|\tau|+|\eta|\neq0\}$, fulfilling the following assumptions:

i. $\Sigma(\tau,\eta):=K(\tau,\eta)A^2$ is Hermitian;

ii. $\Sigma(\tau,\eta)$ must be non positive on Ker$B$ and its restriction to Ker$B$ vanishes only on KerA^2, uniformly in (τ,η);

iii. For $P(\tau,\eta):=K(\tau,\eta)(\tau I_5+iA(\eta,0))$, there exists a positive number c_0 such that

$$\Re P\geq c_0(\Re\tau)I_5,\qquad\forall(\tau,\eta):\Re\tau\geq0,|\tau|+|\eta|\neq0.\quad (1.11)$$

As in the case of a non-characteristic ibvp, a dissipative symmetrizer $K(\tau,\eta)$ turns out to be a fundamental tool in order to investigate the property of the well posedness. The existence of such a symmetrizer for a general characteristic Friedrichs symmetric ibvp fulfilling the (UKL) condition has been recently proved by D. Serre (see [1], §6.2), under an auxiliary assumption due to T. Ohkubo. Namely Ohkubo considers in [7] a Friedrichs symmetric system (1.5) for which the matrix $a_2(\eta)$ involved in (1.6) is identically zero. In fact the assumption made by Ohkubo is slightly more general and it is satisfied by many relevant physical examples such as the curl operator, Maxwell system and the shallow water equations. However the system of linear elasticity does not fall into the Ohkubo's case; indeed from (1.7) we derive the nontrivial $a_2(\eta)$

$$a_2(\eta)=\begin{pmatrix} 0 & 0 & 0 & \eta\sqrt{\lambda} \\ 0 & 0 & \eta\frac{\mu}{c_P} & 0 \\ 0 & \eta\frac{\mu}{c_P} & 0 & 0 \\ \eta\sqrt{\lambda} & 0 & 0 & 0 \end{pmatrix}.$$

Analogously to the non-characteristic case, we are led to find a Kreiss symmetrizer $K(\tau,\eta)$ which is a homogeneous function of degree zero with respect to (τ,η); so it will be enough to build $K(\tau,\eta)$ in the unit hemi-sphere defined as the set of pairs (τ,η) such that $\Re\tau\geq 0$ and $|\tau|^2+\eta^2=1$. By a compactness argument, we still reduce to define the matrix $K(\tau,\eta)$, with properties i.-iii., locally in a neighborhood of each point of the unit hemi-sphere. An inspection of the proof given in [1] for the Ohkubo's case shows that the same arguments, applied there, may be straightforwardly repeated to build a dissipative symmetrizer of the elasticity system near the "interior points" (τ,η), with $\Re\tau>0$, and the "boundary points" (τ,η), with $\Re\tau=0$ and $\tau\neq 0$. However, the main difficulty is to make a dissipative symmetrizer of (1.8) in a neighborhood of the "central points" $(0,\eta)$ for $\eta\neq 0$; indeed it is near these central points that the Ohkubo's assumption $a_2(\eta)\equiv 0_4$ plays a fundamental role in the construction of such a symmetrizer shown in [1]. In the next section 2, we will construct a dissipative symmetrizer of (1.8) in the vicinity of the critical points $(0,\eta)$, $\eta\neq 0$, by following an approach which does not involve the Ohkubo's hypothesis; then the proof of Theorem 1.1 will be achieved by making use of standard a priori estimates. In section 3, we study the ibvp (1.8) in an open bounded domain. An analogy of Theorem 1.1 will be proved by extending the analysis of section 2; the key point consists of reducing the original problem into a finite family of variable coefficient problems in the half-plane, by introducing a smooth partition of unity and local changes of coordinates.

2. Construction of a Kreiss symmetyrizer at points $(0,\eta)$, $\eta\in\mathbb{R}\setminus\{0\}$

Throughout this section, we will assume that the boundary condition $Bu=g$ in the ibvp (1.8) satisfies the (UKL) assumption. Let us recall that we are looking for a C^∞ map $(\tau,\eta)\mapsto K(\tau,\eta)\in\mathbf{M}_{5\times 5}(\mathbb{C})$, defined in an open neighborhood of each point $(0,\eta_0)$, $\eta_0\neq 0$, displaying assumptions i.-iii. in the previous section. As we already noticed, by a homogeneity argument, we may restrict our construction near the points $(0,\pm 1)$ on the unit hemi-sphere defined by the equation $|\tau|^2+\eta^2=1$ with $\Re\tau\geq 0$. Let us recall also that the explicit expressions of the matrices A^2 and $\tau I_5+iA(\eta,0)$ are respectively

$$A^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & 0 & 0 & c_P \\ 0 & \sqrt{\lambda} & 0 & 0 & 0 \\ 0 & 0 & c_P & 0 & 0 \end{pmatrix} \tag{2.1}$$

and

$$\tau I_5+iA(\eta,0)=\begin{pmatrix}\tau & 0 & 0 & i\Theta\eta & 0\\ 0 & \tau & 0 & 0 & i\sqrt{\lambda}\eta\\ 0 & 0 & \tau & i\frac{\mu}{c_P}\eta & 0\\ i\Theta\eta & 0 & i\frac{\mu}{c_P}\eta & \tau & 0\\ 0 & i\sqrt{\lambda}\eta & 0 & 0 & \tau\end{pmatrix} \tag{2.2}$$

where $\Theta:=2\frac{\sqrt{\lambda(\lambda+\mu)}}{c_P}$. Requiring the matrix $\Sigma=KA^2$ is Hermitian immediately yields for $K(\tau,\eta)$ the following expression

$$K=\begin{pmatrix}k_{1,1} & \mathbf{0}_4^T\\ \mathbf{k}_1 & K_2\end{pmatrix}, \tag{2.3}$$

where $k_{1,1}=k_{1,1}(\tau,\eta)\in\mathbb{C},\mathbf{k}_1=\mathbf{k}_1(\tau,\eta)\in\mathbf{M}_{4\times1}(\mathbb{C})$ and $K_2=K_2(\tau,\eta)\in\mathbf{M}_{4\times4}(\mathbb{C})$ are smooth functions; consequently, Σ reduces to

$$\Sigma=\begin{pmatrix}0 & \mathbf{0}_4^T\\ \mathbf{0}_4 & \Sigma_2\end{pmatrix}, \tag{2.4}$$

where $\Sigma_2:=K_2a^2$ must be Hermitian too. From this last condition, the next equalities involving the terms $k_{i,j}=k_{i,j}(\tau,\eta)$ $(i,j=2,\dots,5)$ of K_2 are plainly derived.

$$\begin{array}{ll} k_{2,4},k_{3,5},k_{4,2},k_{5,3}\in\mathbb{R}, & \\ \sqrt{\lambda}k_{3,4}=c_P\bar{k}_{2,5}, & c_Pk_{4,5}=\sqrt{\lambda}\bar{k}_{3,2},\\ \sqrt{\lambda}k_{5,2}=c_P\bar{k}_{4,3}, & k_{5,5}=\bar{k}_{3,3},\\ k_{4,4}=\bar{k}_{2,2}, & \sqrt{\lambda}k_{5,4}=c_P\bar{k}_{2,3}.\end{array} \tag{2.5}$$

On the other hand, in view of $\mathrm{Ker}A^2\subset\mathrm{Ker}B$, the assumption ii. about Σ translates into the existence of a positive constant ϵ_0 such that

$$\Sigma_{2|\mathrm{Ker}B_2}\leq-\epsilon_0I_4, \tag{2.6}$$

for all (τ,η), with $\Re\tau\geq0$, in a neighborhood of $(0,\pm1)$ on the hemi-sphere. According to (2.5) and setting also $\mathbf{k}_1=(k_{2,1},k_{3,1},k_{4,1},k_{5,1})^T$ we find

$$\begin{aligned}(z')^*\Sigma_2z'=&\sqrt{\lambda}k_{2,4}|z_2|^2+c_Pk_{3,5}|z_3|^2+\sqrt{\lambda}k_{4,2}|z_4|^2+c_Pk_{5,3}|z_5|^2\\ &+2\Re(c_Pk_{2,5}\bar{z}_2z_3)+2\Re(\sqrt{\lambda}k_{2,2}\bar{z}_2z_4)+2\Re(c_Pk_{2,3}\bar{z}_2z_5)\\ &+2\Re(\sqrt{\lambda}k_{3,2}\bar{z}_3z_4)+2\Re(c_Pk_{3,3}\bar{z}_3z_5)+2\Re(c_Pk_{4,3}\bar{z}_4z_5),\end{aligned} \tag{2.7}$$

$$\begin{aligned}z^*Pz=&\tau k_{11}|z_1|^2+(\tau k_{2,1}+i\Theta\eta k_{2,4})\bar{z}_2z_1+(\tau k_{3,1}+i\Theta\tfrac{c_P}{\sqrt{\lambda}}\eta\bar{k}_{2,5})\bar{z}_3z_1\\ &+(\tau k_{4,1}+i\Theta\eta\bar{k}_{2,2})\bar{z}_4z_1+(\tau k_{5,1}+i\Theta\tfrac{c_P}{\sqrt{\lambda}}\eta\bar{k}_{2,3})\bar{z}_5z_1+(\tau k_{2,2}+i\sqrt{\lambda}\eta k_{2,5})|z_2|^2\\ &+(\tau k_{3,2}+i\sqrt{\lambda}\eta k_{3,5})\bar{z}_3z_2+(\tau k_{4,2}+i\tfrac{\lambda}{c_P}\eta\bar{k}_{3,2})\bar{z}_4z_2+(\tau\tfrac{c_P}{\sqrt{\lambda}}\bar{k}_{4,3}+i\sqrt{\lambda}\eta\bar{k}_{3,3})\bar{z}_5z_2\\ &+(\tau k_{2,3}+i\tfrac{\mu}{c_P}\eta k_{2,4})\bar{z}_2z_3+(\tau k_{3,3}+i\tfrac{\mu}{\sqrt{\lambda}}\eta\bar{k}_{2,5})|z_3|^2+(\tau k_{4,3}+i\tfrac{\mu}{c_P}\eta\bar{k}_{2,2})\bar{z}_4z_3\\ &+(\tau k_{5,3}+i\tfrac{\mu}{\sqrt{\lambda}}\eta\bar{k}_{2,3})\bar{z}_5z_3+i\Theta\eta k_{1,1}\bar{z}_1z_4+(i\Theta\eta k_{2,1}+i\tfrac{\mu}{c_P}\eta k_{2,3}+\tau k_{2,4})\bar{z}_2z_4\\ &+(i\Theta\eta k_{3,1}+i\tfrac{\mu}{c_P}\eta k_{3,3}+\tau\tfrac{c_P}{\sqrt{\lambda}}\bar{k}_{2,5})\bar{z}_3z_4+(i\Theta\eta k_{4,1}+i\tfrac{\mu}{c_P}\eta k_{4,3}+\tau\bar{k}_{2,2})|z_4|^2\\ &+(i\Theta\eta k_{5,1}+i\tfrac{\mu}{c_P}\eta k_{5,3}+\tau\tfrac{c_P}{\sqrt{\lambda}}\bar{k}_{2,3})\bar{z}_5z_4+(i\sqrt{\lambda}\eta k_{2,2}+\tau k_{2,5})\bar{z}_2z_5\\ &+(i\sqrt{\lambda}\eta k_{3,2}+\tau k_{3,5})\bar{z}_3z_5+(i\sqrt{\lambda}\eta k_{4,2}+\tau\tfrac{\sqrt{\lambda}}{c_P}\bar{k}_{3,2})\bar{z}_4z_5+(ic_P\eta\bar{k}_{4,3}+\tau\bar{k}_{3,3})|z_5|^2,\end{aligned} \tag{2.8}$$

for any $z'=(z_2,z_3,z_4,z_5)^T\in\mathbb{C}^4$ and $z=(z_1,z')^T$. We now specialize the values of the elements $k_{i,j}$ $(i,j=1,\dots,5)$ of $K(\tau,\eta)$; precisely for any $\tau=\gamma+i\rho$, with $\gamma\geq 0$, and real η such that $|\tau|^2+\eta^2=1$, $|\tau|$ is sufficiently small and η ranges in a small neighborhood $\mathcal{V}$ of ±1, we set

$$\begin{aligned}
&k_{j,j}=h+\chi\bar{\tau}, \qquad j=1,2,3;\\
&k_{2,1}=k_{3,1}=k_{5,1}=\gamma\eta, \qquad k_{4,1}=-iM\eta;\\
&k_{2,3}=k_{2,5}=-\tfrac{\sqrt{\lambda}\rho\gamma}{\Theta c_P}, \qquad k_{2,4}=-\tfrac{\rho\gamma}{\Theta};\\
&k_{3,2}=0, \qquad k_{3,5}=-\tfrac{\mu\rho\gamma}{\sqrt{\lambda}\Theta c_P};\\
&k_{4,2}=-A, \qquad k_{4,3}=iN\eta, \qquad k_{5,3}=-\tfrac{\sqrt{\lambda}c_P}{\mu}A.
\end{aligned} \tag{2.9}$$

Here h,χ,A,M,N are positive constants that will be chosen later on in a suitable way. The elements $k_{i,j}$ which are not listed above will be determined by positions (2.9) themselves, according to (2.5). Replacing (2.9) into (2.7) gives

$$(z')^*\Sigma_2 z'=\Phi(z')-AQ(z')+R_\chi(z'), \tag{2.10}$$

where

$$\begin{aligned}
\Phi(z')=&-\tfrac{\sqrt{\lambda}\rho\gamma}{\Theta}|z_2|^2-\tfrac{\mu\rho\gamma}{\sqrt{\lambda}\Theta}|z_3|^2+2\Re\left(-\tfrac{\sqrt{\lambda}\rho\gamma}{\Theta}\bar{z}_2z_3\right)+2\Re(\sqrt{\lambda}h\bar{z}_2z_4)\\
&+2\Re\left(-\tfrac{\sqrt{\lambda}\rho\gamma}{\Theta}\bar{z}_2z_5\right)+2\Re(c_Ph\bar{z}_3z_5)+2\Re(ic_PN\eta\bar{z}_4z_5),
\end{aligned} \tag{2.11}$$

$$Q(z')=\sqrt{\lambda}|z_4|^2+\frac{\sqrt{\lambda}c_P^2}{\mu}|z_5|^2, \tag{2.12}$$

and

$$R_\chi(z')=2\Re(\chi\sqrt{\lambda}\bar{\tau}\bar{z}_2z_4)+2\Re(c_P\chi\bar{\tau}\bar{z}_3z_5). \tag{2.13}$$

As in the Ohkubo's case treated in [1], let us introduce for every $\eta\neq0$ the vector space

$$H(\eta):=\sum_{\substack{\xi\in\mathbb{R}(\eta,0)^T+\mathbb{R}\mathbf{e}_2^T\\ \xi\neq\mathbf{0}_2}}\operatorname{Ker}A(\xi),$$

where, hereafter, $\mathbf{e}_1=(1,0)$, $\mathbf{e}_2=(0,1)$. Concerning a 2D Friedrichs symmetric system like (1.5), we see that, for any $\eta\neq0$, $H(\eta)$ is an isotropic subspace of all matrices $A(\xi)$ and actually it does not depend on η (however these two properties would be false in space dimension larger than 2); since in particular $\operatorname{rank}A^2=4$, the dimension of $H(\eta)$ is not larger than 3. Since $H(\eta)$ contains $\operatorname{Ker}A^2=\mathbb{R}\times\{\mathbf{0}_4\}$, for any $\eta\neq0$ it can be split as $\mathbb{R}\times H_1(\eta)$, where $H_1(\eta)$ is an isotropic subspace for both a^2 and $a_2(\eta)$ of dimension not larger than 2. In the case of the linear elasticity system a direct computation shows that $H(\eta)=H(\mathbf{e}_1)=\mathbb{R}^3\times\{\mathbf{0}_2\}$, for $\eta\neq0$. Moreover we may check that for any $\eta\neq0$ there is no vector $U\in H(\eta)^\perp$ other than $U=\mathbf{0}_5$ such that

$$A((\eta,0)^T+\xi_2\mathbf{e}_2^T)U\in H(\eta)^\perp,$$

for non real $\xi_2=-i\sigma$. In view of the preceding properties, it can be shown that a boundary matrix $B=(\mathbf{0}_2,B_2)$ satisfies the (UKL) condition near the central points $(0,\eta)$, $\eta\neq0$, if, and only if,

$$\mathbb{C}^4=\operatorname{Ker}B_2\oplus H_1(\eta), \tag{2.14}$$

where $H_1(\eta)$ denotes both the real space and its complexification. We refer to [1] (see Proposition 6.6 there) for the proof of (2.14) in the framework of a general Friedrichs symmetric ibvp with characteristic boundary. Under the Ohkubo's assumption $a_2(\eta)\equiv 0_4$, we might also prove that $H_1(\eta)=\mathrm{Ker}(a_{2,1}(\eta)^T)$ as $\eta\neq 0$ (see again [1], §6.1.4 and Proposition 6.7); hence, because of $\mathrm{Ker}(a_{2,1}(\eta)^T)\oplus^{\perp}\mathrm{R}(a_{2,1}(\eta))=\mathbb{C}^4$, we should conclude that $\mathrm{Ker}B_2$ may be represented as the set $\mathrm{Ker}B_2=\{p+q; p=Dq, q\in\mathrm{R}(a_{2,1}(\eta))\}$, where $D=D(\eta)$ is a given linear operator from $\mathrm{R}(a_{2,1}(\eta))$ to $\mathrm{Ker}(a_{2,1}(\eta)^T)$, depending smoothly and boundedly on $\frac{\eta}{|\eta|}$. Nevertheless, when the Ohkubo's assumption is removed, generally $H_1(\eta)$ becomes a proper subspace of $\mathrm{Ker}(a_{2,1}(\eta)^T)$; when $\eta\neq 0$, for the elasticity system in the symmetric form (1.5) we easily compute

$$\mathrm{Ker}(a_{2,1}(\eta)^T)=\{(z_2,z_3,0,z_5)^T; z_j\in\mathbb{C}, j=2,3,5\},$$

while

$$H_1(\eta)=\{(z_2,z_3,0,0)^T; z_j\in\mathbb{C}, j=2,3\}.$$

Mimicking the analysis performed in [1] for the Ohkubo's case and denoting by $H_1(\eta)^{\perp}$ the subspace of $\mathrm{Ker}(a_{2,1}(\eta)^T)$ orthogonal to $H_1(\eta)$, we now provide the following decomposition.

$$\mathbb{C}^4=H_1(\eta)\oplus^{\perp}H_1(\eta)^{\perp}\oplus^{\perp}\mathrm{R}(a_{2,1}(\eta)).$$

Therefore, in view of (2.14), for a boundary matrix $B=(\mathbf{0}_2,B_2)$ satisfying the (UKL) condition we get

$$\mathrm{Ker}B_2=\{r+Dr; r\in H_1(\eta)^{\perp}\oplus^{\perp}\mathrm{R}(a_{2,1}(\eta))\},$$

where $D=D(\eta)$ is a given linear operator from $H_1(\eta)^{\perp}\oplus^{\perp}\mathrm{R}(a_{2,1}(\eta))$ to $H_1(\eta)$, depending smoothly and boundedly on $\frac{\eta}{|\eta|}$. Coming back to the elasticity case, we have $H_1^{\perp}(\eta)=\{(0,0,0,z_5); z_5\in\mathbb{C}\}$, so that $\mathrm{Ker}B_2$ may be characterized as the set of vectors of the form $r+Dr$, for $r=(0,0,z_4,z_5)^T$ spanning $\{\mathbf{0}_2\}\times\mathbb{C}^2$ and $D=D(\eta)$ taking its values in $\mathbb{C}^2\times\{\mathbf{0}_2\}=\{(z_2,z_3,0,0)^T, z_j\in\mathbb{C}, j=2,3\}$. This allows us to conclude that the quadratic form Q in (2.12) is positive definite in $\mathrm{Ker}B_2$; indeed $Q(z')\geq 0$ for every z', moreover for $z'=r+Dr$ with $r=(0,0,z_4,z_5)$, $Q(z')=0$ implies $r=0$ then $z'=r+Dr=0$. Thus there exists a positive constant ϵ such that

$$Q(z')\geq\epsilon|z'|^2,\quad z'\in\mathrm{Ker}B_2. \tag{2.15}$$

The constant ϵ may be chosen independent of η, for η belonging to a small neighborhood $\mathcal{V}$ of ± 1. On the other hand, the quadratic form Φ in (2.11) can be estimated from above as

$$\Phi(z')\leq c^*|z'|^2, \tag{2.16}$$

where $c^*>0$ depends only on the constants h and N involved in (2.9), when $|\tau|$ is sufficiently small and η ranges over $\mathcal{V}$. More explicitly, assuming without loss of generality $0\leq\gamma,|\rho|,|\eta|\leq 1$, we may choose $c^*=\max\{3\sqrt{\lambda}/\Theta+\sqrt{\lambda}h,(\mu+\lambda)/(\sqrt{\lambda}\Theta)+c_Ph,\sqrt{\lambda}h+c_PN,\sqrt{\lambda}/\Theta+c_Ph+c_PN\}$. Finally we compute that

$$R_\chi(z')\leq C_1\chi|\tau||z'|^2,$$

for $C_1 = \max\{c_P, \sqrt{\lambda}\}$. By adding the preceding estimates, we find that

$$(z')^*\Sigma_2 z' \le (c^* - \epsilon A + C_1\chi|\tau|)|z'|^2, \qquad z' \in \mathrm{Ker} B_2. \tag{2.17}$$

For given positive constants h, N and δ^*, we may always find $A>0$ sufficiently large so that $c^* - \epsilon A < -\delta^*$; it is also clear that for any $\chi > 0$ we obtain $c^* - \epsilon A + C_1\chi|\tau| < -\delta^* + C_1\chi|\tau| < -\frac{\delta^*}{2}$, provided that $|\tau|$ is taken sufficiently small (that is $|\tau| < \sigma^*$, for a suitable threshold $\sigma^* = \sigma^*(\chi)$). At this step we have so proved the following

LEMMA 2.1. *For given $h, N, \delta^* > 0$ there exists a constant $A > 0$ such that for every $\chi > 0$ there exists $\sigma^* = \sigma^*(\chi) > 0$ for which*

$$(z')^*\Sigma_2 z' \le -\frac{\delta^*}{2}|z'|^2, \qquad z' \in \mathrm{Ker} B_2, \quad \eta \in \mathcal{V}, \quad |\tau| < \sigma^*. \tag{2.18}$$

The next step will be to make a useful choice of the constants h, N (and M) in (2.9) in order that the corresponding $K(\tau, \eta)$ will satisfy an estimate such as (1.11). To this end, we replace in (2.8) the expressions of $k_{i,j}$ given by (2.9) and take the real part of z^*Pz, for an arbitrary vector $z \in \mathbb{C}^5$. First of all we observe that $k_{1,1} = k_{2,2} = k_{3,3}$ (cf. (2.9)) yields

$$\begin{aligned}
&\Re(i\Theta\eta\bar{k}_{2,2}\bar{z}_4 z_1) + \Re(i\Theta\eta k_{1,1} z_4\bar{z}_1) = 0,\\
&\Re(i\sqrt{\lambda}\eta\bar{k}_{3,3}\bar{z}_5 z_2) + \Re(i\sqrt{\lambda}\eta k_{2,2} z_5\bar{z}_2) = 0,\\
&\Re(i\mu/c_P\eta\bar{k}_{2,2}\bar{z}_4 z_3) + \Re(i\mu/c_P\eta k_{3,3} z_4\bar{z}_3) = 0.
\end{aligned}$$

From (2.9) we derive directly $\sqrt{\lambda}k_{3,5} = \frac{\mu}{c_P}k_{2,4}$ and $\frac{\mu}{c_P}k_{5,3} = \sqrt{\lambda}k_{4,2}$; hence we get also

$$\begin{aligned}
&\Re(i\sqrt{\lambda}\eta k_{3,5}\bar{z}_3 z_2) + \Re(i\mu/c_P\eta k_{2,4} z_3\bar{z}_2) = 0,\\
&\Re(i\sqrt{\lambda}\eta k_{4,2}\bar{z}_4 z_5) + \Re(i\mu/c_P\eta k_{5,3} z_4\bar{z}_5) = 0.
\end{aligned}$$

Let us now evaluate the remaining terms of $\Re(z^*Pz)$, arising from (2.8), and apply repeatedly the Cauchy-Schwarz and Young inequalities. We have

$$\Re(\tau k_{1,1}|z_1|^2) = (h\gamma + \chi|\tau|^2)|z_1|^2;$$

$$\Re((\tau k_{2,1} + i\Theta\eta k_{2,4})\bar{z}_2 z_1) \ge -\frac{1}{2}\gamma^2|\eta|(|z_1|^2 + |z_2|^2)$$

and analogously

$$\begin{aligned}
&\Re((\tau k_{3,1} + i\Theta\tfrac{c_P}{\sqrt{\lambda}}\eta\bar{k}_{2,5})\bar{z}_3 z_1) \ge -\tfrac{1}{2}\gamma^2|\eta|(|z_1|^2 + |z_3|^2),\\
&\Re((\tau k_{5,1} + i\Theta\tfrac{c_P}{\sqrt{\lambda}}\eta\bar{k}_{2,3})\bar{z}_5 z_1) \ge -\tfrac{1}{2}\gamma^2|\eta|(|z_1|^2 + |z_5|^2);
\end{aligned}$$

$$\Re(\tau k_{4,1}\bar{z}_4 z_1) \ge -\frac{M}{2}|\eta|(\chi_1|\tau|^2|z_1|^2 + \frac{1}{\chi_1}|z_4|^2),$$

where χ_1 is a given positive constant that will be suitably fixed later on. By similar computations we get also

$$\begin{aligned}
&\Re((\tau k_{2,2}+i\sqrt{\lambda}\eta k_{2,5})|z_2|^2)=(h\gamma+\chi|\tau|^2)|z_2|^2;\\
&\Re((\tau k_{4,2}\bar z_4 z_2)\geq -\tfrac{A}{2}(\chi_2|\tau|^2|z_2|^2+\tfrac{1}{\chi_2}|z_4|^2);\\
&\Re(\tau\tfrac{c_P}{\sqrt{\lambda}}\bar k_{4,3}\bar z_5 z_2)\geq -\tfrac12\tfrac{c_P N|\eta|}{\sqrt{\lambda}}(\chi_3|\tau|^2|z_2|^2+\tfrac{1}{\chi_3}|z_5|^2);\\
&\Re(\tau k_{2,3}\bar z_2 z_3)\geq -\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|(|z_2|^2+|z_3|^2);\\
&\Re((\tau k_{3,3}+i\tfrac{\mu}{\sqrt{\lambda}}\eta\bar k_{2,5})|z_3|^2)=(h\gamma+\chi|\tau|^2)|z_3|^2;\\
&\Re(\tau k_{4,3}\bar z_4 z_3)\geq -\tfrac12 N|\eta|(\chi_4|\tau|^2|z_3|^2+\tfrac{1}{\chi_4}|z_4|^2);\\
&\Re((\tau k_{5,3}+i\tfrac{\mu}{\sqrt{\lambda}}\eta\bar k_{2,3})\bar z_5 z_3)\geq -\tfrac12\tfrac{\sqrt{\lambda}c_P A}{\mu}(\chi_5|\tau|^2|z_3|^2+\tfrac{1}{\chi_5}|z_5|^2)-\tfrac12\tfrac{\mu|\rho|\gamma}{\Theta c_P}|\eta|(|z_3|^2+|z_5|^2);\\
&\Re((i\Theta\eta k_{2,1}+i\tfrac{\mu}{c_P}\eta k_{2,3}+\tau k_{2,4})\bar z_2 z_4)\geq -\tfrac12\Theta\eta^2(\gamma^2|z_2|^2+|z_4|^2)\\
&-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|\gamma}{\Theta c_P^2}|\eta|(|z_2|^2+|z_4|^2)-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|(|z_2|^2+|z_4|^2);\\
&\Re((i\Theta\eta k_{3,1}+\tau\tfrac{c_P}{\sqrt{\lambda}}\bar k_{2,5})\bar z_3 z_4)\geq -\tfrac12\Theta\eta^2(\gamma^2|z_3|^2+|z_4|^2)-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|(|z_3|^2+|z_4|^2);\\
&\Re((i\Theta\eta k_{4,1}+i\tfrac{\mu}{c_P}\eta k_{4,3}+\tau\bar k_{2,2})|z_4|^2=[(\Theta M-\tfrac{\mu}{c_P}N)\eta^2+h\gamma+\chi(\gamma^2-\rho^2)]|z_4|^2;
\end{aligned}$$

$$\begin{aligned}
&\Re(i\Theta\eta k_{5,1}+\tau\tfrac{c_P}{\sqrt{\lambda}}\bar k_{2,3})\bar z_5 z_4)\geq -\tfrac12\Theta\gamma\eta^2(|z_4|^2+|z_5|^2)-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|(|z_4|^2+|z_5|^2);\\
&\Re(\tau k_{2,5}\bar z_2 z_5)\geq -\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|(|z_2|^2+|z_5|^2);\\
&\Re(\tau k_{3,5}\bar z_3 z_5)\geq -\tfrac12\tfrac{\mu|\rho|\gamma}{\sqrt{\lambda}c_P\Theta}|\tau|(|z_3|^2+|z_5|^2);\\
&\Re((ic_P\eta\bar k_{4,3}+\tau\bar k_{3,3})|z_5|^2)=(c_P N\eta^2+h\gamma+\chi(\gamma^2-\rho^2))|z_5|^2.
\end{aligned}$$

In the estimates listed above, $\chi_2,\chi_3,\chi_4,\chi_5$ are positive constants, to be precised later, with the same meaning as χ_1. This leads to

$$\Re(z^*Pz)\geq c_1|z_1|^2+c_2|z_2|^2+c_3|z_3|^2+c_4|z_4|^2+c_5|z_5|^2, \tag{2.19}$$

where $c_j=c_j(\tau,\eta)$ $(j=1,\dots,5)$ are determined as follows

$$c_1(\tau,\eta)=h\gamma+\chi|\tau|^2-\frac32\gamma^2|\eta|-\frac12\chi_1 M|\eta||\tau|^2; \tag{2.20}$$

$$\begin{aligned}
c_2(\tau,\eta)=&\,h\gamma+\chi|\tau|^2-\tfrac12\gamma^2|\eta|-\tfrac12 A\chi_2|\tau|^2-\tfrac12\tfrac{c_P N\chi_3|\eta|}{\sqrt{\lambda}}|\tau|^2-\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|\\
&-\tfrac12\Theta\eta^2\gamma^2-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|\gamma}{\Theta c_P^2}|\eta|-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|;
\end{aligned}\tag{2.21}$$

$$\begin{aligned}
c_3(\tau,\eta)=&\,h\gamma+\chi|\tau|^2-\tfrac12\gamma^2|\eta|-\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|-\tfrac12 N\chi_4|\eta||\tau|^2-\tfrac12\tfrac{\sqrt{\lambda}c_P A\chi_5}{\mu}|\tau|^2\\
&-\tfrac12\tfrac{\mu|\rho|\gamma}{\Theta c_P}|\eta|-\tfrac12\Theta\eta^2\gamma^2-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\tfrac{\mu|\rho|\gamma}{\sqrt{\lambda}c_P\Theta}|\tau|;
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
c_4(\tau,\eta)=&\,(\Theta M-\tfrac{\mu}{c_P}N)\eta^2+h\gamma+\chi(\gamma^2-\rho^2)-\tfrac12\tfrac{M}{\chi_1}|\eta|-\tfrac12\tfrac{A}{\chi_2}-\tfrac12\tfrac{N}{\chi_4}|\eta|\\
&-\Theta\eta^2-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|\gamma}{\Theta c_P^2}|\eta|-\tfrac32\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\Theta\eta^2\gamma;
\end{aligned}\tag{2.23}$$

$$\begin{aligned}
c_5(\tau,\eta)=&\,c_P N\eta^2+h\gamma+\chi(\gamma^2-\rho^2)-\tfrac12\gamma^2|\eta|-\tfrac12\tfrac{c_P N|\eta|}{\sqrt{\lambda}\chi_3}-\tfrac12\tfrac{\sqrt{\lambda}c_P A}{\mu\chi_5}-\tfrac12\tfrac{\mu|\rho|\gamma}{\Theta c_P}|\eta|\\
&-\tfrac12\Theta\eta^2\gamma-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|-\tfrac12\tfrac{\mu|\rho|\gamma}{\sqrt{\lambda}c_P\Theta}|\tau|.
\end{aligned}\tag{2.24}$$

We derive now suitable estimates of $c_j(\tau,\eta)$, $j=1,\dots 5$, for η belonging to a small neighborhood $\mathcal{V}$ of ±1 and $|\tau|$ sufficiently small such that $|\tau|^2+\eta^2=1$; actually, evaluating

$c_j(\tau,\pm1)$ for $j=1,\dots,5$ will be enough, in view of the continuity of the expressions (2.20)-(2.24). Remembering that $\gamma\geq 0$, we obtain for c_4, c_5

$$\begin{aligned}c_4(\tau,\pm1)&\geq(\Theta M-\tfrac{\mu}{c_P}N)-\Theta-\tfrac12\tfrac{M}{\chi_1}-\tfrac12\tfrac{A}{\chi_2}-\tfrac12\tfrac{N}{\chi_4}+\chi(\gamma^2-\rho^2)-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|\gamma}{\Theta c_P^2}\\&-\tfrac32\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\Theta\gamma;\end{aligned}\tag{2.25}$$

$$\begin{aligned}c_5(\tau,\pm1)&\geq c_PN-\tfrac12\tfrac{c_PN}{\sqrt{\lambda}\chi_3}-\tfrac12\tfrac{\sqrt{\lambda}c_PA}{\mu\chi_5}+\chi(\gamma^2-\rho^2)-\tfrac12\gamma^2-\tfrac12\tfrac{\mu|\rho|\gamma}{\Theta c_P}-\tfrac12\Theta\gamma\\&-\tfrac12\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|-\tfrac12\tfrac{\mu|\rho|\gamma}{\sqrt{\lambda}c_P\Theta}|\tau|.\end{aligned}\tag{2.26}$$

Inequality (2.25) leads us to choose the constants $M,N>0$ so that

$$\widetilde{C}:=\Theta(M-1)-\frac{\mu}{c_P}N>0.\tag{2.27}$$

Once M,N have been fixed, for given $h>0$ and $\delta^*>0$ we may also find a positive constant A so that estimate (2.18) of Lemma 2.1 is fulfilled by Σ_2 for any $\chi>0$, provided $|\tau|$ is small enough. Next, we can choose χ_j, $j=1,\dots 5$, large enough so that

$$\begin{aligned}\nu_4&:=2\widetilde{C}-\tfrac{M}{\chi_1}-\tfrac{A}{\chi_2}-\tfrac{N}{\chi_4}>0,\\\nu_5&:=2c_PN-\tfrac{c_PN}{\sqrt{\lambda}\chi_3}-\tfrac{\sqrt{\lambda}c_PA}{\mu\chi_5}>0.\end{aligned}\tag{2.28}$$

In this way we get for $c_4(\tau,\pm1),c_5(\tau,\pm1)$

$$\begin{aligned}c_4(\tau,\pm1)&\geq\tfrac{\nu_4}{2}+\chi(\gamma^2-\rho^2)-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|\gamma}{\Theta c_P^2}-\tfrac32\tfrac{|\rho|\gamma}{\Theta}|\tau|-\tfrac12\Theta\gamma,\\c_5(\tau,\pm1)&\geq\tfrac{\nu_5}{2}+\chi(\gamma^2-\rho^2)-\tfrac12\gamma^2-\tfrac12\tfrac{\mu|\rho|\gamma}{\Theta c_P}-\tfrac12\Theta\gamma-\tfrac12\tfrac{|\rho|\gamma}{\Theta}-\tfrac12\tfrac{\sqrt{\lambda}|\rho|\gamma}{\Theta c_P}|\tau|-\tfrac12\tfrac{\mu|\rho|\gamma}{\sqrt{\lambda}c_P\Theta}|\tau|;\end{aligned}\tag{2.29}$$

since all the terms different from $\nu_4/2,\nu_5/2$, involved in the right-hand sides of (2.29), are $O(|\tau|)$ we may conclude that

$$c_4(\tau,\pm1)>\frac{\nu_4}{4},\qquad c_5(\tau,\pm1)>\frac{\nu_5}{4},\tag{2.30}$$

whatever is $\chi>0$, provided that $|\tau|$ is sufficiently small. We now turn to the coefficients c_j, $j=1,2,3$, for which the following estimates may be obtained

$$c_1(\tau,\pm1)\geq\gamma(h-\frac32\gamma)+(\chi-\frac12\chi_1M)|\tau|^2;\tag{2.31}$$

$$\begin{aligned}c_2(\tau,\pm1)&\geq(h-\tfrac12\gamma-\tfrac{\sqrt{\lambda}|\rho|}{\Theta c_P}|\tau|-\tfrac12\Theta\gamma-\tfrac12\tfrac{\sqrt{\lambda}\mu|\rho|}{\Theta c_P^2}-\tfrac12\tfrac{|\rho|}{\Theta}|\tau|)\gamma\\&+(\chi-\tfrac12A\chi_2-\tfrac12\tfrac{c_PN\chi_3}{\sqrt{\lambda}})|\tau|^2;\end{aligned}\tag{2.32}$$

$$\begin{aligned}c_3(\tau,\pm1)&\geq(h-\tfrac12\gamma-\tfrac12\tfrac{\sqrt{\lambda}|\rho|}{\Theta c_P}|\tau|-\tfrac12\tfrac{\mu|\rho|}{\Theta c_P}-\tfrac12\Theta\gamma-\tfrac12\tfrac{|\rho|}{\Theta}|\tau|-\tfrac12\tfrac{\mu|\rho|}{\sqrt{\lambda}c_P\Theta}|\tau|)\gamma\\&+(\chi-\tfrac12N\chi_4-\tfrac12\tfrac{\sqrt{\lambda}c_PA\chi_5}{\mu})|\tau|^2.\end{aligned}\tag{2.33}$$

Thus if we choose a positive χ such that

$$\chi>\frac12\max\left\{\chi_1M,A\chi_2+\frac{c_PN\chi_3}{\sqrt{\lambda}},N\chi_4+\frac{\sqrt{\lambda}c_PA\chi_5}{\mu}\right\},\tag{2.34}$$

provided that γ and $|\rho|$ (that is $|\tau|$) are sufficiently small, we get

$$c_j(\tau,\pm1) > \frac{h}{2}\gamma, \qquad j=1,2,3. \tag{2.35}$$

Therefore from (2.19) we obtain

$$\Re(z^*Pz) > \frac{h}{2}\gamma(|z_1|^2+|z_2|^2+|z_3|^2) + \frac{\nu_4}{4}|z_4|^2 + \frac{\nu_5}{4}|z_5|^2 > C^*\gamma|z|^2, \tag{2.36}$$

for every $z\in\mathbb{C}^5$ and (τ,η) in a small neighborhood of $(0,\pm1)$ on the unit hemi-sphere $|\tau|^2+\eta^2=1$, $\Re\tau=\gamma\geq 0$; here $C^*:=\min\{\frac{h}{2},\frac{\nu_4}{4},\frac{\nu_5}{4}\}>0$ is independent of γ,ρ,η. This last estimate just gives inequality (1.11). To be more clear, let us summarize the basic steps in the choice of the constants h,χ,A,M,N appearing in (2.9).

- Firstly, we choose the positive constants M,N in such a way that (2.27) is satisfied.
- Chosen also $\delta^*,h>0$, we find a constant $A>0$ for which $c^*-\epsilon A<-\delta^*$ holds; here the constant c^*, involved in (2.16), is a known positive function of the previously fixed constants N and h, while ϵ, involved in (2.15), depends only on μ, λ when η runs through a small neighborhood of ±1.
- After estimating $\Re(z^*Pz)$ by means of (2.19), we choose the positive numbers χ_j, $j=1,\dots 5$, involved in (2.20)-(2.24), in such a way that inequalities (2.28), and consequently (2.29), are true.
- After giving estimates (2.31)-(2.33), we take χ fulfilling (2.34).
- Lastly, by suitably restricting γ, $|\rho|$ (that is $|\tau|$) and also, if necessary, the neighborhood of ±1 which η belongs to, we obtain (2.18) of Lemma 2.1; moreover we find the inequalities (2.30) and (2.35), giving, together with (2.19), the estimate (1.11).

The matrix valued function $K(\tau,\eta)$ that we have built near the central points $(0,\pm1)$ on the unit hemi-sphere $|\tau|^2+\eta^2=1$, $\Re\tau\geq 0$ takes the following form

$$K(\tau,\eta)=\begin{pmatrix} h+\chi\bar\tau & 0 & 0 & 0 & 0\\ \gamma\eta & h+\chi\bar\tau & -\frac{\sqrt{\lambda}\rho\gamma}{\Theta c_P} & -\frac{\rho\gamma}{\Theta} & -\frac{\sqrt{\lambda}\rho\gamma}{\Theta c_P}\\ \gamma\eta & 0 & h+\chi\bar\tau & -\frac{\rho\gamma}{\Theta} & -\frac{\mu\rho\gamma}{\sqrt{\lambda}\Theta c_P}\\ -iM\eta & -A & iN\eta & h+\chi\tau & 0\\ \gamma\eta & -\frac{c_P}{\sqrt{\lambda}}iN\eta & -\frac{\sqrt{\lambda}c_PA}{\mu} & -\frac{\rho\gamma}{\Theta} & h+\chi\tau \end{pmatrix},$$

where h,χ,A,M,N are positive constants to be fixed as was previously explained. Once a Kreiss symmetrizer $K(\tau,\eta)$ of (1.8) has been made, by usual computations (for which we address to [1], Chapter 4), we find that any function $u\in C^\infty(\mathbb{R}^2\times\mathbb{R})$, with compact support in $\overline{\mathbb{R}}^2_+\times\mathbb{R}$, satisfies for every $T\in\mathbb{R}$ and $\gamma>0$

$$\begin{aligned}&e^{-2\gamma T}\|u(T)\|_{L^2}+\gamma\int\int_{\mathbb{R}^2_+\times\mathbb{R}} e^{-2\gamma t}|u(x_1,x_2,t)|^2dx_1dx_2dt+\int\int_{\mathbb{R}\times\mathbb{R}} e^{-2\gamma t}|A^2u(x_1,0,t)|^2dx_1dt\\ &\leq C\left(\frac{1}{\gamma}\int\int_{\mathbb{R}^2_+\times\mathbb{R}} e^{-2\gamma t}|Lu(x_1,x_2,t)|^2dx_1dx_2dt+\int\int_{\mathbb{R}\times\mathbb{R}} e^{-2\gamma t}|Bu(x_1,0,t)|^2dx_1dt\right),\end{aligned} \tag{2.37}$$

where the constant C does not depend on γ,T and u. By a duality argument relying on the previous estimates for an "adjoint" ibvp, one shows the existence of a solution to the ibvp (1.8) in the weighted space $L^2_\gamma(\mathbb{R}^2_+\times\mathbb{R})$ for every $\gamma>0$. Let us recall that, for a given positive γ, $L^2_\gamma(\mathbb{R}^2_+\times\mathbb{R})$ is the space of all measurable functions $u(x_1,x_2,t)$ for which the norm $\|u\|^2_\gamma := \int\int_{\mathbb{R}^2_+\times\mathbb{R}} e^{-2\gamma t}|u(x_1,x_2,t)|^2 dx_1dx_2dt$ is finite. The uniqueness of the solution into the space $L^2(\mathbb{R}^2_+\times(0,T))$, for a finite $T>0$, then follows by arguing directly on estimates (2.37). Finally, the a priori estimates (1.9) are derived from (2.37) themselves, by a density argument. We refer to [1] (see Chapter 4, §4 there) for a detailed proof of the analogous to Theorem 1.1 in the non-characteristic case.

As a concluding remark, let us observe that Ohkubo also studied in [7] the Sobolev regularity of solutions to an L^2-well posed symmetric ibvp, with characteristic boundary, under the aforementioned assumption $a_2(\eta)\equiv 0$ (cf. Theorem 2 in [7]); in the case treated here, it is not clear whether (and under what additional assumptions) an analogous regularity result might be still valid.

3. The linear elasticity system in a bounded domain

Let Ω be a bounded open subset of $\mathbb{R}^2$ lying (locally) on one side of its smooth boundary $\partial\Omega=\Gamma$. We consider the initial boundary value problem (ibvp)

$$Lu(x,t)=f(x,t), \quad x\in\Omega, t>0 \tag{3.1}$$

$$Bu(x,t)=g(x,t), \quad x\in\Gamma, t>0, \tag{3.2}$$

$$u(x,0)=a(x), \quad x\in\Omega, \tag{3.3}$$

where L is the linear partial differential operator defined by (1.5)-(1.7), while $f(x,t), g(x,t)$ and $a(x)$ are given smooth functions. Let us denote by $\nu(x)=(\nu_1(x),\nu_2(x))^T$ the unit outward normal vector to any point x of the boundary Γ. In view of (1.6), (1.7) an explicit computation gives that $A(\xi)$ has eigenvalues $0, \pm c_P|\xi|, \pm\sqrt{\lambda}|\xi|$, for every ξ. Since $|\nu(x)|=1$ for all $x\in\Gamma$, it follows that the *boundary matrix* $A(\nu(x))$ has real eigenvalues $0,\pm c_P,\pm\sqrt{\lambda}$; hence the boundary Γ is *characteristic with constant rank*, namely

$$\mathrm{rank}A(\nu(x))=4, \quad x\in\Gamma. \tag{3.4}$$

Note that, since the eigenvalues of $A(\xi)$ depend on ξ only through its norm, actually the boundary Γ is uniformly characteristic in the sense of Majda-Osher [5]; indeed, by considering an extension of the unit outward normal $\nu(x)$ to a C^1 unit vector field defined in a neighborhood of Γ, the eigenvalues of $A(\nu(x))$ are constant near Γ. In the boundary condition (3.2), $B=B(\nu(x))$ is assumed to be a 2×5 real matrix, smoothly dependent on the unit outward normal $\nu(x)$, such that $\mathrm{rank}B(\nu(x))=2$ for all boundary points $x\in\Gamma$ (2 is the *number of incoming characteristics* of (3.1)). As in section 2, we assume the reflexivity property

$$\mathrm{Ker}A(\nu(x))\subset\mathrm{Ker}B(\nu(x)), \tag{3.5}$$

for every $x\in\Gamma$. Lastly, we require that the boundary matrix B satisfies the (UKL) condition. The following counterpart of Theorem 1.1 will be proved here.

THEOREM 3.1. *Let us consider the ibvp* (3.1)-(3.3)*; let the boundary matrix $B=B(\nu)$ satisfy the (UKL) condition. Then for every data $f\in L^2(\Omega\times(0,T))$, $g\in L^2(\Gamma\times(0,T))$ and $a\in L^2(\Omega)$, with arbitrary $T>0$, there exists one, and only one, solution $u\in L^2(\Omega\times(0,T))$ of* (3.1)-(3.3) *such that:*

a. $u\in C([0,T];L^2(\Omega))$;

b. $A(\nu)u$ *admits a trace* $\gamma_0 A(\nu)u$ *on the boundary* Γ *of* Ω *which is of class* $L^2(\Gamma\times(0,T))$.

Finally, for any positive real number γ *sufficiently large, the following a priori estimate holds true*

$$e^{-2\gamma T}\|u(T)\|^2_{L^2}+\|u\|^2_{\gamma,T}\leq C\left(\|a\|^2_{L^2}+\int_0^T e^{-2\gamma t}\left(\frac{1}{\gamma}\|f(t)\|^2_{L^2}+\|g(t)\|^2_{L^2}\right)dt\right), \quad (3.6)$$

where the constant $C>0$ *does not depend on* f,g,a *and* γ,T. *In* (3.6) $\|.\|_{L^2}$ *denotes the norm in either* $L^2(\Omega)$ *or* $L^2(\Gamma)$ *and*

$$\|u\|^2_{\gamma,T}:=\int_0^T\int_\Gamma e^{-2\gamma t}|\gamma_0 A(\nu)u(x,t)|^2dxdt+\gamma\int_0^T\int_\Omega e^{-2\gamma t}|u(x,t)|^2dxdt. \quad (3.7)$$

We follow here the usual approach consisting of reducing the original ibvp into a finite number of ibvps on the half-plane, by use of a smooth partition of unity and local changes of coordinates.

Before starting to reduce our problem, it is convenient to focus on some relevant invariance properties displayed by the linear elasticity system (3.1). Actually, this system is invariant under translations of $\mathbb{R}^2$. It is even worthwhile to remark that (3.1) is invariant under rotations of $\mathbb{R}^2$, provided that a rotation is coupled with an appropriate linear change of unknowns. In order to see that, for every unit vector $\zeta=(\zeta_1,\zeta_2)^T$ let $U=U(\zeta)$ be a real orthogonal matrix such that

$$U\left(\frac{\xi}{|\xi|}\right)^*A(\xi)U\left(\frac{\xi}{|\xi|}\right)=|\xi|\mathrm{diag}(0,+\sqrt{\lambda},-\sqrt{\lambda},+c_P,-c_P), \quad \xi\neq 0; \quad (3.8)$$

in the right-hand side of (3.8) $\mathrm{diag}(0,+\sqrt{\lambda},-\sqrt{\lambda},+c_P,-c_P)$ stands, as usual, for the diagonal matrix with diagonal entries equal to $0,\pm\sqrt{\lambda},\pm c_P$ respectively. Observing that $A_2=A((0,1)^T)$, from (3.8) we straightforwardly derive

$$|\xi|T\left(\frac{\xi}{|\xi|}\right)A_2T\left(\frac{\xi}{|\xi|}\right)^*=A(\xi), \quad \xi\neq 0, \quad (3.9)$$

where for every unit vector ζ we have set

$$T(\zeta):=U(\zeta)U((0,1)^T)^*. \quad (3.10)$$

By an explicit computation, we find for $|\zeta|=1$

$$T(\zeta)=\begin{pmatrix} \zeta_2^2-\frac{\mu}{c_P^2}\zeta_1^2 & \frac{\Theta}{\sqrt{\lambda}}\zeta_1\zeta_2 & \frac{\Theta}{c_P}\zeta_1^2 & 0 & 0\\ -\frac{\Theta}{\sqrt{\lambda}}\zeta_1\zeta_2 & \zeta_2^2-\zeta_1^2 & \frac{2\sqrt{\lambda}}{c_P}\zeta_1\zeta_2 & 0 & 0\\ \frac{\Theta}{c_P}\zeta_1^2 & -\frac{2\sqrt{\lambda}}{c_P}\zeta_1\zeta_2 & \zeta_2^2+\frac{\mu}{c_P^2}\zeta_1^2 & 0 & 0\\ 0 & 0 & 0 & \zeta_2 & \zeta_1\\ 0 & 0 & 0 & -\zeta_1 & \zeta_2 \end{pmatrix}. \quad (3.11)$$

Moreover, for every nonzero $\xi=(\eta,\xi_2)^T$ we compute that

$$|\xi|T\left(\frac{\xi}{|\xi|}\right)A_1T\left(\frac{\xi}{|\xi|}\right)^*=A((\xi_2,-\eta)^T)=\xi_2A_1-\eta A_2. \quad (3.12)$$

Let us also remark that for every unit vector $\zeta=(\zeta_1,\zeta_2)^T$ we get $T(\zeta_1,\zeta_2)^*=T(-\zeta_1,\zeta_2)$. Now we come back to the ibvp (3.1)-(3.3) and take an arbitrary rotation of $\mathbb{R}^2$; in fact, due to the translation invariance of the problem, we may always assume that the origin of $\mathbb{R}^2$ is the center of the rotation. Thus the equations of the rotation become

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \tilde\theta_2 & -\tilde\theta_1 \\ \tilde\theta_1 & \tilde\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.13}$$

where the real numbers $\tilde\theta_1,\tilde\theta_2$ satisfy $\tilde\theta_1^2+\tilde\theta_2^2=1$. Let us set $\tilde\theta=(\tilde\theta_1,\tilde\theta_2)^T$, $\tilde\theta^\perp:=(\tilde\theta_2,-\tilde\theta_1)^T$; if $u=u(x,t)$ solves (3.1)-(3.3), then $U(X,t):=u(\widetilde R^*X,t)$ must solve the differential system in the rotated space variables $X=(X_1,X_2)^T$

$$\frac{\partial U}{\partial t}+A(\tilde\theta^\perp)\frac{\partial U}{\partial X_1}+A(\tilde\theta)\frac{\partial U}{\partial X_2}=\widetilde f(X,t), \quad X\in\widetilde\Omega, t>0, \tag{3.14}$$

together with the new boundary and initial conditions

$$\begin{aligned} &B(\nu(\widetilde R^*X))U(X,t)=\widetilde g(X,t), \quad x\in\widetilde\Gamma, t>0, \\ &U(X,0)=\widetilde a(X), \quad X\in\widetilde\Omega; \end{aligned} \tag{3.15}$$

here we have set for brevity

$$\widetilde R:=\begin{pmatrix} \tilde\theta_2 & -\tilde\theta_1 \\ \tilde\theta_1 & \tilde\theta_2 \end{pmatrix},$$

$\widetilde f(X,t):=f(\widetilde R^*X,t)$, $\widetilde g(X,t):=g(\widetilde R^*X,t)$, $\widetilde a(X):=a(\widetilde R^*X)$, while $\widetilde\Omega:=\widetilde R(\Omega)$ and $\widetilde\Gamma=\widetilde R(\Gamma)$ are the rotated open bounded domain and its boundary respectively. Since from (3.12) and (3.9) we derive respectively

$$\begin{aligned} &T(\tilde\theta)^*A(\tilde\theta^\perp)T(\tilde\theta)=A_1 \\ &T(\tilde\theta)^*A(\tilde\theta)T(\tilde\theta)=A_2, \end{aligned}$$

multiplying (3.14) on the left by $T(\tilde\theta)^*$ yields that the new function $V(X,t)=T(\tilde\theta)^*U(X,t)$ must solve the original system (3.1) in the new space variables X, that is

$$\frac{\partial V}{\partial t}+A_1\frac{\partial V}{\partial X_1}+A_2\frac{\partial V}{\partial X_2}=T(\tilde\theta)^*\widetilde f(X,t), \quad X\in\widetilde\Omega, t>0. \tag{3.16}$$

The corresponding boundary and initial conditions for V become respectively

$$B(\nu(\widetilde R^*X))T(\tilde\theta)V(X,t)=\widetilde g(X,t), \quad X\in\widetilde\Gamma, t>0 \tag{3.17}$$

and

$$V(X,0)=T(\tilde\theta)^*\widetilde a(X), \quad X\in\widetilde\Omega. \tag{3.18}$$

The resulting ibvp (3.16)-(3.18) keeps the main features of the original one (3.1)-(3.3). Firstly, let us remark that the unit outward normal vector $\tilde\nu(X)$ to the rotated boundary $\widetilde\Gamma$, at the point $X=\widetilde Rx$, can be expressed in terms of the outward normal ν to the original boundary Γ by formula

$$\tilde\nu(X)=\widetilde R\nu(\widetilde R^*X). \tag{3.19}$$

Then the matrix involved in the boundary condition (3.17) can be rewritten as

$$B(\nu(\widetilde{R}^*X))T(\tilde{\theta}) = \widetilde{B}(\tilde{\nu}(X)), \tag{3.20}$$

where $\widetilde{B}(.) := B(\widetilde{R}^*.)T(\tilde{\theta})$ is smoothly dependent on $\tilde{\nu}$; moreover, it is clear that the rank of $\widetilde{B}(\tilde{\nu}(X))$ is maximal, for any $X \in \widetilde{\Gamma}$, since $T(\tilde{\theta})$ is an orthogonal matrix. Lastly, using (3.19) we compute

$$A(\tilde{\nu}(X)) = T(\tilde{\theta})^* A(\nu(\widetilde{R}^*X))T(\tilde{\theta}); \tag{3.21}$$

hence (3.4) and (3.5) easily imply

$$\begin{aligned} &\mathrm{rank} A(\tilde{\nu}(X)) = 4, \\ &\mathrm{Ker} A(\tilde{\nu}(X)) = \mathrm{Ker} A(\nu(\widetilde{R}^*X))T(\tilde{\theta}) \subset \mathrm{Ker}\widetilde{B}(\tilde{\nu}(X)), \end{aligned} \tag{3.22}$$

for every $X \in \widetilde{\Gamma}$.

Let us fix now an arbitrary point $\bar{x}$ of Γ and choose a small neighborhood $\mathcal{W}$ of $\bar{x}$; due to the assumptions made about the boundary Γ itself, $\Gamma \cap \mathcal{W}$ can be represented by an equation $x_2 = \gamma(x_1)$ for some smooth function γ, and we may assume that $x_2 > \gamma(x_1)$ holds true in $\Omega \cap \mathcal{W}$. We may always assume that $\gamma(0) = \gamma'(0) = 0$; indeed, denoting by $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2)^T$ the unit outward normal to Γ at $\bar{x}$, this can be obtained by translating the origin of $\mathbb{R}^2$ in $\bar{x}$, then performing a rotation around $\bar{x}$ that changes the orthonormal basis $\{(-\bar{\nu}_2, \bar{\nu}_1)^T, (-\bar{\nu}_1, -\bar{\nu}_2)^T\}$ into the canonical basis $\{(1,0)^T, (0,1)^T\}$ and passing to the corresponding problem (3.16)-(3.18) (where now $\tilde{\theta}$ is $-(\bar{\nu}_1, \bar{\nu}_2)^T$); however, later on we will always identify the starting ibvp (3.1)-(3.3) with the one obtained under the aforesaid rotation. Let $\mathcal{V}$ be a small neighborhood of the origin in $\mathbb{R}^2$; we define a transformation $x = \psi(y)$ on $\mathcal{V}$ by setting

$$\begin{aligned} &\psi_1(y_1, y_2) = y_1 - \nu_1(y_1, \gamma(y_1))y_2, \\ &\psi_2(y_1, y_2) = \gamma(y_1) - \nu_2(y_1, \gamma(y_1))y_2. \end{aligned} \tag{3.23}$$

For y_1 sufficiently small, the outward normal vector $\nu((y_1, \gamma(y_1)))$ to the point $(y_1, \gamma(y_1))$ is explicitly given by

$$\nu((y_1, \gamma(y_1))) = \nu(y_1) = \frac{1}{(1 + \gamma'(y_1)^2)^{\frac{1}{2}}} (\gamma'(y_1), -1)^T. \tag{3.24}$$

Let us observe also that $\psi(0) = 0$. Since the Jacobian of ψ, evaluated at 0, is equal to 1 provided that $\mathcal{V}$ is taken sufficiently small, ψ defines a diffeomorphism from $\mathcal{V}$ onto an open set $\psi(\mathcal{V}) =: \mathcal{U} \subseteq \mathcal{W}$; let us denote by ϕ the inverse transformation of ψ. We check that $\phi(\Gamma \cap \mathcal{U}) = \{y_2 = 0\} \cap \mathcal{V}$ and $\phi(\Omega \cap \mathcal{U}) = \{y_2 > 0\} \cap \mathcal{V}$.

We are now able to transform the ibvp (3.1)-(3.3) into a family of problems in the half-plane. Due to the compactness, we can cover $\overline{\Omega}$ by a finite family of open sets $\{\mathcal{U}_j\}_{j=0}^l$ such that $\mathcal{U}_j \cap \Gamma \neq \emptyset$, $j = 1, \ldots, l$ and $\overline{\mathcal{U}_0} \subset \Omega$. Next, we choose a partition of unity $\{\chi_j\}_{j=0}^l$, subordinated to the covering $\{\mathcal{U}_j\}_{j=0}^l$, such that $\sum_{j=0}^{l} \chi_j = 1$ over $\overline{\Omega}$ and $\chi_j \geq 0$, $j = 0, \ldots, l$. For every $j = 1, \ldots, l$, let $\psi^j = (\psi_1^j, \psi_2^j)$ be the diffeomorphism from $\{y_2 \geq 0\} \cap \mathcal{V}$ onto $\overline{\Omega} \cap \mathcal{U}_j$ defined by equations (3.23) (where the equation $x_2 = \gamma(x_1)$ has been replaced with the equation $x_2 = \gamma_j(x_1)$ of the piece of boundary $\Gamma \cap \mathcal{U}_j$) and $\phi^j = (\phi_1^j, \phi_2^j)$ the inverse transformation. For the sake of convenience, we denote also

by ϕ^0 a translation of $\mathbb{R}^2$ mapping $\mathcal{U}_0$ into $\phi^0(\mathcal{U}_0)\subseteq\mathcal{V}_0:=\{y:y_1>\delta,|y|<\frac{1}{\delta}\}$, for some small $\delta>0$, with inverse map ψ^0.
Hereafter, we will set $\tilde{v}(y)=v(\psi^j(y))$ for any fixed $j=0,\dots,l$ and all functions $v=v(x)$ defined on $\overline{\Omega}\cap\mathcal{U}_j$. Let $u=u(x,t)$ be a solution to (3.1)-(3.3); for every $j=0,\dots,l$, we set

$$u^j=u^j(y,t)=\widetilde{\chi_j u}(y,t). \tag{3.25}$$

We have that $\mathrm{supp}u^j\subset\{y_2\geq 0\}\cap\mathcal{V}$, $j=1,\dots,l$, and $\mathrm{supp}u^0\subset\mathcal{V}_0$.
For each index $j=1,\dots,l$ we find that u^j must solve an ibvp in the half-plane such as

$$\begin{aligned}&L^j u^j(y,t)=f^j(y,t), \quad y_1\in\mathbb{R},\, y_2,t>0,\\ &B^j u^j(y_1,0,t)=g^j(y_1,t), \quad y_1\in\mathbb{R},\, t>0,\\ &u^j(y,0)=a^j(y), \quad y_1\in\mathbb{R},\, y_2>0.\end{aligned} \tag{3.26}$$

Here L^j is the first order linear partial differential operator

$$L^j:=\frac{\partial}{\partial t}+A_1^j\frac{\partial}{\partial y_1}+A_2^j\frac{\partial}{\partial y_2}, \tag{3.27}$$

with variable coefficients

$$A_i^j=A_i^j(y):=A(\widetilde{\nabla_x\phi_i^j}(y)) \quad i=1,2, \tag{3.28}$$

where we have set for brevity $\nabla_x:=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)^T$. Furthermore

$$B^j=B^j(y_1):=B(\nu(\psi^j(y_1,0))) \tag{3.29}$$

and

$$\begin{aligned}&f^j(y,t):=\widetilde{\chi_j f}(y,t)+A(\widetilde{\nabla_x\chi_j})\widetilde{u}(y,t),\\ &g^j(y_1,t):=\widetilde{\chi_j g}(y_1,0,t),\\ &a^j(y)=\widetilde{\chi_j a}(y).\end{aligned} \tag{3.30}$$

For every $j=1,\dots,l$, (3.26) is a characteristic ibvp in the half-plane, since of course $\mathrm{rank}A_2^j(y)=4$. From equations (3.23) (with γ replaced by γ_j) we compute that, for $y_2=0$, the Jacobian $J_{\psi^j}(y_1,0)$ of the transformation ψ^j is equal to $(1+\gamma_j'(y_1)^2)^{\frac{1}{2}}$; using also (3.24) and the inverse mapping differentiation formulae we get

$$\nu(\psi^j(y_1,0))=-\widetilde{\nabla_x\phi_2}(y_1,0)=-J_\psi(y_1,0)\widetilde{\nabla_x^\perp\phi_1}(y_1,0), \tag{3.31}$$

where $\nabla_x^\perp:=\left(-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right)^T$. In view of (3.31), (3.5) yields the reflexivity property for (3.26)

$$\mathrm{Ker}A_2^j(y_1,0)=\mathrm{Ker}A(\nu(\psi^j(y_1,0)))\subset\mathrm{Ker}B^j(y_1). \tag{3.32}$$

Correspondingly to χ_0, we also find that u_0 must solve the pure Cauchy problem in $\mathbb{R}^2$

$$\begin{aligned}&Lu^0(y,t)=f^0(y,t), \quad y\in\mathbb{R}^2,t>0,\\ &u^0(y,0)=a^0(y), \quad y\in\mathbb{R}^2,\end{aligned} \tag{3.33}$$

where $L=\frac{\partial}{\partial t}+A_1\frac{\partial}{\partial y_1}+A_2\frac{\partial}{\partial y_2}$ is just the differential operator involved in the original system (3.1), while the data f^0 and a^0 are defined respectively by

$$f^0(y,t):=\widetilde{\chi_0 f}(y,t)+A(\widetilde{\nabla_x\chi_0})\widetilde{u}(y,t),\quad a^0(y)=\widetilde{\chi_0 a}(y). \tag{3.34}$$

Summing up the preceding computations, by localization and flattening of the boundary Γ of the original ibvp (3.1)-(3.3), we are reduced to solve l characteristic ibvps in $\mathbb{R}^2_+$

$$\begin{array}{l} L^j v(y,t)=f^j(y,t),\quad y_1\in\mathbb{R},\, y_2,t>0,\\ B^j v(y_1,0,t)=g^j(y_1,t),\quad y_1\in\mathbb{R},\, t>0,\\ v(y,0)=a^j(y),\quad y_1\in\mathbb{R},\, y_2>0, \end{array} \tag{3.35}$$

with operators L^j,B^j and data f^j,g^j,a^j defined by (3.27)-(3.29) and (3.30) respectively, and the Cauchy problem in $\mathbb{R}^2$

$$\begin{array}{l} Lv(y,t)=f^0(y,t)\quad y\in\mathbb{R}^2,t>0,\\ v(y,0)=a^0(y),\quad y\in\mathbb{R}^2, \end{array} \tag{3.36}$$

with f^0,a^0 defined by (3.34).

Actually the above Cauchy problem is strongly L^2-well posed (in fact it is well posed in every Sobolev space H^s, $s\in\mathbb{R}$), due to the strict hyperbolicity of L.

Now we focus on the construction of a symbolic Kreiss symmetrizer for the variable coefficients ibvps (3.35), in order to get an L^2-well posedness result for each of them; this will be done starting from the dissipative symmetrizer that we already found in the constant coefficient case. For simplicity, hereafter we will drop the index j from the operators L^j,B^j and the data f^j,g^j,a^j in (3.35). Since the data f,g,a vanish outside some small neighborhood $\{y_2\geq 0\}\cap\mathcal{V}$, actually we are looking for a smooth function $\widetilde{K}:\{y_2\geq 0\}\cap\mathcal{V}\times\{(\tau,\eta)\in\mathbb{C}\times\mathbb{R}:\Re\tau\geq 0,|\tau|+|\eta|\neq 0\}\to\mathbf{M}_{5\times 5}(\mathbb{C})$, fulfilling the next properties:

i. for every $y\in\{y_2\geq 0\}\cap\mathcal{V}$, $\Re\tau\geq 0$ and $\eta\in\mathbb{R}$ with $|\tau|+|\eta|\neq 0$, the matrix $\Sigma(y,\tau,\eta):=K(y,\tau,\eta)A_2(y)$ is Hermitian;

ii. $\Sigma(y,\tau,\eta)$ must be non positive on $\mathrm{Ker}B(y_1)$ and its restriction to $\mathrm{Ker}B(y_1)$ vanishes only on $\mathrm{Ker}A_2(y_1,0)$, uniformly in (y_1,τ,η);

iii. For $P(y,\tau,\eta):=K(y,\tau,\eta)(\tau I_5+i\eta A_1(y))$ there exists a positive constant c_0 such that:

$$\Re P(y,\tau,\eta)\geq c_0(\Re\tau)I_5,\ \forall(y,\tau,\eta):y\in\{y_2\geq 0\}\cap\mathcal{V},\Re\tau\geq 0,|\tau|+|\eta|\neq 0,$$

where $\Re P:=\frac{1}{2}(P+P^*)$.

Since the matrix valued functions $A_1(y),A_2(y)$ converge to the matrices $A_1=A_1(0),A_2=A_2(0)$ respectively, as long as $y\to 0$, and putting $\xi=\widetilde{\nabla_x\phi_2}$ into (3.9) gives

$$A_2(y)=|\widetilde{\nabla_x\phi_2}(y)|\widetilde{T}(y)A_2\widetilde{T}(y)^*, \tag{3.37}$$

where we have set $\widetilde{T}(y):=T\left(\frac{\widetilde{\nabla_x\phi_2}(y)}{|\widetilde{\nabla_x\phi_2}(y)|}\right)$, we are led to define

$$\widetilde{K}(y,\tau,\eta):=\frac{1}{|\widetilde{\nabla_x\phi_2}(y)|}\widetilde{T}(y)K(\tau,\eta)\widetilde{T}(y)^*. \tag{3.38}$$

Here $K = K(\tau,\eta)$ is the dissipative Kreiss symmetrizer that we have constructed for the constant coefficient case, taking $B(0)$ as boundary matrix; notice that (3.32) yields $\mathrm{Ker}A_2 = \mathbb{R} \times \{\mathbf{0}_4\} \subset \mathrm{Ker}B(0)$, thus $B(0) = (\mathbf{0}_2, B_2)$ with $\mathrm{rank}B_2 = 2$. It is clear that $\widetilde{K}(y,\tau,\eta)A_2(y) = \widetilde{T}(y)K(\tau,\eta)A_2\widetilde{T}(y)^*$ is Hermitian since the same is true for $K(\tau,\eta)A_2$. Furthermore the function $y \mapsto \widetilde{K}(y,\tau,\eta)$ is smooth and converges to $\widetilde{K}(0,\tau,\eta) = K(\tau,\eta)$ uniformly with respect to τ,η, on the unit hemi-sphere $|\tau|^2 + \eta^2 = 1$, $\Re\tau \geq 0$, and then on the whole of $\{(\tau,\eta) \in \mathbb{C} \times \mathbb{R}; \Re\tau \geq 0, |\tau| + |\eta| \neq 0\}$ (recall that $K(\tau,\eta)$ may be constructed as a homogeneous function of degree 0 in (τ,η)). Since properties ii, iii are fulfilled by $\widetilde{K}(y,\tau,\eta)$ at $y = 0$, by shrinking the neighborhood $\mathcal{V}$ (if it is necessary), the same properties hold true in $\{y_2 \geq 0\} \cap \mathcal{V}$ due to the continuity of $\widetilde{K}$ with respect to y.

Adapting to the present framework the same arguments used in the constant coefficients case, by use of the above Kreiss symmetrizer and the pseudo-differential calculus we get the strong L^2-well posedness of (3.35). Hence the L^2-well posedness of the original ibvp (3.1)-(3.3) easily follows and the proof of Theorem 3.1 is complete.

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