AN ENTROPY DISSIPATION–ENTROPY ESTIMATE FOR A THIN FILM TYPE EQUATION*

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Abstract. We prove a lower bound on the rate of relaxation to equilibrium in the H^1 norm for a thin film equation. We find a two stage relaxation, with power law decay in an initial interval, followed by exponential decay, at an essentially optimal rate, for large times. The waiting time until the exponential decay sets in is explicitly estimated.

Key words. thin films, Lyapunov functional, dissipation

MSC Classification Numbers. 35Q35, 26D10.

1. Introduction

The following fourth-order nonlinear parabolic equation

$$f_t = -(f^n f_{xxx})_x , \qquad -a \le x \le a , \tag{1.1}$$

with either periodic or "no flux" boundary conditions; i.e., $f_x(\pm a) = f_{xxx}(\pm a) = 0$, arises in modeling the evolution of thin films. The particular case $n = 1$, on which we shall focus, is used to model the flow in a Hele-Shaw cell; see [9].

This equation has been studied recently by many authors, e.g. [3],[1],[4],[2],[6]. In particular, in [2] Bernis and Friedman established the existence of a class of Hölder continuous weak solutions, and proved their existence. This was done for $n \ge 1$, and using "no flux" boundary conditions. They showed moreover that these weak solutions are actually classical solutions as long as they stay positive, and showed that if $f_0 \ge 0$, then this property is preserved for all time, and also that $\int_{-a}^{a} f(x,t) \mathrm{d}x$ is independent of t. The same equation has been studied under periodic boundary conditions in [5]. These authors prove that the weak solutions become classical, regular and positive in a finite time.

Several papers [5],[1],[8] have addressed the question of long–time behavior of solutions, and have shown that if f_0 is positive, then

$$\lim_{t \to \infty} f(x,t) = \frac{1}{2a} \int_{-a}^{a} f_0(x) \mathrm{d}x. \tag{1.2}$$

The steady state or "equilibrium" solutions to (1.1) are the constants. Since $\int_{-a}^{a} f(x,t) \mathrm{d}x$ is constant for any smooth solution $f(x,t)$ to (1.1), the equilibrium solution corresponding to the initial data f_0 is the constant on the right hand side in (1.2). Thus, (1.2) expresses the fact that solutions eventually relax to equilibrium. However, the proofs of (1.2) in [5],[1] rely on compactness arguments, and give no rate information. The analysis by Lopez, Soler and Toscani in [8] is based on an ingenious, but somewhat intricate, use of "entropy functionals" of the sort developed by Bernis

and Friedman in their fundamental work [2]. They provide an explicit exponential decay bound in the L^1 norm.

In this paper, we prove quantitative bounds of the relaxation rate to the equilibrium solution for $n=1$ with a finite domain $[-a,a]$. We do this in a stronger norm that in [8], namely the $H^1([a,a])$ Sobolev norm, and the rate we obtain is essentially "best possible", as it corresponds to the rate at which the linearized equation relaxes. As we show, the relaxation proceeds in two stages: There is an initial slow stage with power law decay. Then, once the solution has entered a neighborhood of the equilibrium that is sufficiently small for the errors made in linearization to be controlled, the relaxation proceeds at an exponential rate.

A straight forward analysis of the error terms in a linearization of (1.1) about the equilibrium shows that once the solution is close to equilibrium in the L^∞ norm, the distance to equilibrium begins to decay exponentially fast. The details are simple, but are provided below.

The interesting question concerns initial data that is far from equilibrium: *How long does it take solutions with initial data far from equilibrium to get sufficiently close to equilibrium that the linearization "takes over"* Indeed, one can ask if, in general, they *ever* get close enough. We answer this question and show that there is a power law bound on the distance from equilibrium valid for classical solutions even if they start far from equlibrium.

We do this by proving what has come to be known as an *entropy dissipation– entropy bound* for (1.1). The term "entropy" is frequently used for a Lyapunov functional whose rate of decrease can be bounded in terms of itself. That is, if $H(f)$ is some functional of f, and along the flow of some evolution we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f) \leq -\Phi(H(f)) \ , \tag{1.3}$$

with Φ some continuous strictly monotone increasing function on $\mathbb{R}_+$, then functional $H(f)$ is called an *entropy*, and the inequality (1.3) is called an *entropy dissipation– entropy inequality*. The point is that (1.3) can be used to quantitatively estimate the rate of decay of $H(f)$.

The entropy functional we employ has been discovered by Laugesen [7]: For $p \geq 0$, define $H_p(f)$ by

$$H_p(f) := \int_{-a}^{a} \frac{(f_x)^2}{f^p} \mathrm{d}x \ .$$

Notice that with $g = (1-p/2)f^{1-p/2}$, $H_p(f) = \int_{-a}^{a} g_x^2 \mathrm{d}x$, so the precise definition of $H_p(f)$ is that it is infinite unless g has a square integrable distributional derivative, and otherwise, it is the square integral of the distributional derivative of g. However, as in [7], we shall be working with positive classical solutions where the formula can be taken literally.

Laugesen showed that for $0 \leq p \leq 1/2$, H_p is a Lyapunov functional; i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) \leq 0 \tag{1.4}$$

for any positive classical solution of (1.1) with $n=1$. (This was already known for $p=0$. The cases with $0 < p \leq 1/2$ are more subtle.) In fact, Laugesen did more: He showed that for all n with $1/2 < n < 3$, there is a range of values of p, $0 < p < p(n)$,

for which $H_p(f)$ is non increasing for positive classical solutions of (1.1). We have focused on the Hele–Shaw case $n=1$ because of our own interests, and to keep the formulae that follow both simple and explicit. However, we note that it is possible to prove strict entropy dissipation–entropy inequalities such as the one we prove in Theorem 1.1 below also for other values of n in this range.

The inequality (1.4), while pertinent to (1.2), gives no rate information on the rate of approach, and generally it is not easy to pass from (1.4) to (1.3), since the bound on the dissipation, which is generally a complicated functional, must be made in terms of the original entropy functional. For example, a simple calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}H_0(f) = -2\int_{-a}^{a} f(f_{xxx})^2 \mathrm{d}x \ . \tag{1.5}$$

There is no useful lower bound on the integral on the right hand side in terms of $H_0(f) = \|f_x\|_2^2$, without assuming that f is close to its constant equilibrium value.

Building on Laugesen's work, we show that for certain values of p, his Lyapunov functionals are better behaved so that its rate of decrease can be bounded below *in terms of themselves*, as in (1.3).

THEOREM 1.1. *Consider any positive classical solution of (1.1) with $n=1$. Suppose the initial data f_0 is such that $H_p(f_0)$ is finite for some p with $0 < p < (9+4\sqrt{15})/53$. Then there is a strictly positive constant $B_p(f_0)$ depending on f_0 only through $\int_{-a}^{a} f_0(x)\mathrm{d}x$ and $H_p(f_0)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}H_p(f(\cdot,t)) \leq -B_p(f_0)H_p(f(\cdot,t))^3 \tag{1.6}$$

and hence

$$H_p(f(\cdot,t)) \leq \left[2B_p(f_0)t + (H_p(f_0))^{-2}\right]^{-1/2} \ . \tag{1.7}$$

We mention that an explicit computation of $B_p(f_0)$ is provided in the proof that follows. It should also be noted that since $(9+4\sqrt{15})/53 \approx 0.4621119507$, (1.6) only shows that $H_p(f)$ is decreasing for a more restricted range than that obtained by Laugesen. However, within this range, we obtain a bound on the decrease of $H_p(f)$ that can be expressed in terms of $H_p(f)$ itself, and this is crucial for obtaining bounds of the rate of approach to the equilibrium solution.

We now apply Theorem 1.1 to quantify the rate of convergence in (1.2). Given positive, continuous initial data f_0, let M denote the mean height; i.e.,

$$M = (2a)^{-1}\int_{-a}^{a} f_0 \mathrm{d}x \ , \tag{1.8}$$

so that M is the constant value of the equilibrium solution corresponding to f_0. For the classical solution f with positive initial data f_0, clearly $(2a)^{-1}\int_{-a}^{a} f(x,t)\mathrm{d}x = M$ for all t.

It is not hard to see that when $H_p(f)$ is small, then so is $\|f - M\|_\infty$. This fact will be used below, and so we give a formal statement in the following lemma, which provides a sort of Poincare–Sobolev inequality for the functional H_p.

LEMMA 1.2. *For any p with $0 < p < 2$, and any postitive function f for which $H_p(f)$ is finite,*

$$\|f - M\|_\infty^2 \leq 2a\left(M^{1-p/2} + (1-p/2)(2a)^{1/2}(H_p(f))^{1/2}\right)^{2p/(2-p)} H_p(f) \ . \tag{1.9}$$

Proof. Notice that with g defined by $g = (1-p/2)^{-1} f^{1-p/2}$, $H_p(f) = \int_{-a}^{a} (g_x)^2 \mathrm{d}x$. Then since $g(b) = (1-p/2)^{-1} M^{1-p/2}$ for some b with $-a < b < a$,

$$\|g - (1-p/2)^{-1} M^{1-p/2}\|_{\infty}^2 \leq \left(\int_{-a}^{a} |g_x| \mathrm{d}x \right)^2 \leq 2a H_p(f) . \qquad (1.10)$$

Introduce $h = f - M$. Then

$$|g - (1-p/2)^{-1} M^{1-p/2}| = (1-p/2)^{-1} |(M+h)^{1-p/2} - M^{1-p/2}| \qquad (1.11)$$
$$\geq |h|/K \qquad\qquad\qquad$$
$$(1.12)$$

where K is the maximum of $M^{p/2}$ and $\|M+h\|_{\infty}^{p/2}$. Since $(M+h)^{p/2} = f^{p/2} = ((1-p/2)g)^{p/(2-p)}$, we have from (1.10) that $K \leq \left(M^{1-2/p} + (1-p/2)(2a)^{1/2}(H_p(f))^{1/2} \right)^{2p/(2-p)}$. Combining this with (1.11), which says that

$$\|f - M\|_{\infty} \leq K \|g - (1-p/2)^{-1} M^{1-p/2}\|_{\infty} ,$$

and then with (1.10), we obtain the result.   □

Since by (1.7), $H(f)$ decays to zero like $t^{-1/2}$, $h = f - M$ decay to zero, uniformly in x, like $t^{-1/4}$, assuming only that f_0 is positive and that $H_p(f)$ is finite for some p in the range indicated in Theorem 1.1 After this polynomial decay has gone on long enough, we reach a sufficiently small neighborhood of the equilibrium that it is possible to control the errors in linearization, and from this point onward, the decay is exponentially fast.

The following theorem, which makes this precise, is relatively easy to prove. However it is meaningful only on account of Theorem 1.1 and Lemma 1.2 that guarantee its applicability to solutions of our equation with initial data in a fairly general class.

THEOREM 1.3. *For any positive classical solution of (1.1) with $n = 1$, let M be the corresponding equilibrium value, and suppose the initial data f_0 is such that $H_p(f_0) < \infty$ for some p with $0 < p < (9 + 4\sqrt{15})/53$. Then for any $\epsilon > 0$, there is a finite time T_ϵ, explicitly computable in terms of M and $H_p(f_0)$ so that for all $t > T_\epsilon$, $\|f - M\|_{\infty} \leq \epsilon$. Morover, for all $t > T_\epsilon$, we have*

$$H_0(f) \leq H_0(f_0) e^{-(t-T_\epsilon)2(M-\epsilon)(\pi/a)^4}$$

in case we are using periodic boundary conditions, and

$$H_0(f) \leq H_0(f_0) e^{-(t-T_\epsilon)2(M-\epsilon)(\pi/2a)^4}$$

in case we are using "no flux" boundary conditions.

Recall that $H_0(f) = \int f_x^2 \mathrm{d}x$, so Theorem 1.3 proves that this Sobolev norm decays to zero exponentially fast. Of course $\|f - M\|_{\infty}^2 \leq (2a) H_0(f)$, and so Theorem 1.3 also ensures an exponential rate of convergence in (1.2) in the uniform norm. As will be clear from the proof, which is based on linearization, the rates are essentially best possible, as one cannot hope for faster convergence than one would get from the linearized equation. While explicit exponential convergence in (1.2) was obtained in the L^1 norm earlier, the rates were considerably slower. To our knowledge, Theorem 1.3 provides the first proof that $\int f_x^2 \mathrm{d}x$ decreases to zero at any rate for any class of initial data that is not already close to equilibrium.

2. Proofs

Proof of Theorem 1.1

Proof. The proof is given in several steps.

(Step One): We first show that for $0 \le p < (9 + 4\sqrt{15})/53$

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) \le -C_p I_3 \tag{2.1}$$

where C_p is a strictly positive constant given explicitly in (2.14) below. Toward this end, we compute the rate of change of $H_p((f\cdot,t))$. (All integrals below are over the range $[-a,a]$. To facilitate reading the formulae, we drop the limits from our notation here.)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) &= \frac{\mathrm{d}}{\mathrm{d}t} \int \frac{f_x^2}{f^p} \mathrm{d}x \\
&= 2 \int \frac{f_x}{f^p} f_{tx} \mathrm{d}x - p \int \frac{f_x^2}{f^{p+1}} f_t \mathrm{d}x \\
&= -2 \int \frac{f_x}{f^p} (f f_{xxx})_{xx} \mathrm{d}x + p \int \frac{f_x^2}{f^{p+1}} (f f_{xxx})_x \mathrm{d}x \ .
\end{aligned} \tag{2.2}$$

Integrating by parts twice in the first integral and once in the second, so that f_{xxx} is the highest spatial derivative present, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2 \int \frac{f_{xxx}^2}{f^{p-1}} dx + 4p \int \frac{f_x f_{xx} f_{xxx}}{f^p} - p(p+1) \int \frac{f_x^3 f_{xxx}}{f^{p+1}} dx. \tag{2.3}$$

To show that the quantity in (2.3) is negative, we shall write it as a sum of negative multiples of integrals of perfect squares. This is also the basic strategy of Laugesen, though we make some different choices below, so as to arrive at (2.1).

For any numbers α, β and γ, to be chosen below, define the quantity

$$A = \int \left(\alpha f_{xxx} + \beta \frac{f_{xx} f_x}{f} + \gamma \frac{f_x^3}{f^2} \right)^2 f^{1-p} \mathrm{d}x \ . \tag{2.4}$$

This can be written as

$$A = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\beta J_{12} + 2\alpha\gamma J_{13} + 2\beta\gamma J_{23} \tag{2.5}$$

where

$$I_1 = \int \frac{f_{xxx}^2}{f^{p-1}} \mathrm{d}x, \qquad I_2 = \int \frac{f_x^2 f_{xx}^2}{f^{p+1}} \mathrm{d}x, \qquad I_3 = \int \frac{f_x^6}{f^{p+3}} \mathrm{d}x; \tag{2.6}$$

$$J_{12} = \int \frac{f_x f_{xx} f_{xxx}}{f^p} dx, \qquad J_{13} = \int \frac{f_x^3 f_{xxx}}{f^{p+1}} dx, \qquad J_{23} = \int \frac{f_x^4 f_{xx}}{f^{p+2}} dx. \tag{2.7}$$

Integration by parts yields the following relations:

$$I_2 = \left(\frac{1+p}{3} \right) J_{23} - \frac{1}{3} J_{13} \qquad \text{and} \qquad J_{23} = \left(\frac{2+p}{5} \right) I_3 \ . \tag{2.8}$$

There is no integration by parts identity relating J_{12} to other integrals in the lists (2.6) and (2.7) – integrating by parts in J_{12}, no matter how it is done, would introduce other integrals into the game.

Using the notation in (2.6) and (2.7) we can rewrite (2.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2I_1 + 4pJ_{12} - p(p+1)J_{13}. \tag{2.9}$$

Our plan is to use (2.5) to eliminate the J_{12} term in (2.9), since we have no integration by parts identity for J_{12}. To achieve this, chose $\alpha = 1$ and $\beta = -p$. Then by (2.5),

$$A = I_1 + p^2 I_2 + \gamma^2 I_3 - 2pJ_{12} + 2\gamma J_{13} - 2p\gamma J_{23} \ ,$$

or, what is the same,

$$I_1 - 2pJ_{12} = A - p^2 I_2 - \gamma^2 I_3 - 2\gamma J_{13} + 2p\gamma J_{23} \ . \tag{2.10}$$

Combining (2.9) and (2.10),

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2A + 2p^2 I_2 + 2\gamma^2 I_3 + (4\gamma - p(p+1))J_{13} - 4p\gamma J_{23} \ .$$

Next, use the first identity in (2.8) to eliminate I_2. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2A + \left(4\gamma - p(p+1) - \frac{2p^2}{3}\right) J_{13} + \left(2p^2 \frac{1+p}{3} - 4p\gamma\right) J_{23} + 2\gamma^2 I_3 \ .$$

Finally, using the second identity in (2.8) to eliminate J_{23}, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2A + \left(4\gamma - p(p+1) - \frac{2p^2}{3}\right) J_{13} \tag{2.11}$$

$$+ \left(\left(2p^2 \frac{1+p}{3} - 4p\gamma\right)\left(\frac{2+p}{5}\right) + 2\gamma^2\right) I_3 \ . \tag{2.12}$$

Since J_{13} can have either sign, we choose γ so that the multiple of J_{13} vanishes. That is, we chose

$$\gamma = \frac{3p(p+1) + 2p^2}{12} \ .$$

With this choice of γ, (2.11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p(f) = -2A - \frac{p^2}{360}\left(3 + 18p - 53p^2\right) I_3 \ . \tag{2.13}$$

(Notice that with the choices of β and γ made above that (2.13) reduces to (1.5) for $p = 0$). One now easily calculates that the roots of $3 + 18p - 53p^2 = 0$ are $p = (9 \pm 4\sqrt{15})/53$. Thus, if we define C_p by

$$C_p = \frac{p^2}{360}\left(3 + 18p - 53p^2\right) \ , \tag{2.14}$$

we have that $C_p > 0$ for $0 < p < (9 + 4\sqrt{15})/53$, and that (2.1) holds with this value of C_p.

(Step Two): We show that there is a constant $K_p(f_0)$ depending on f_0 only through M and $H_p(f_0)$ so that

$$I_3 \geq K_p(f_0)\left(H_p(f)\right)^3 . \tag{2.15}$$

Toward this end, first notice that

$$I_3 = \int_{-a}^a \frac{f_x^6}{f^{3+p}}\mathrm{d}x = \int_{-a}^a \left(\frac{f_x^2}{f^p}\right)^3 \frac{1}{f^2}\frac{1}{f^{1-2p}}\mathrm{d}x \geq \frac{1}{\|f\|_\infty^{1-2p}}\int_{-a}^a \left(\frac{f_x^2}{f^p}\right)^3 \frac{1}{f^2} .$$

Letting $u = (f_x)^2/f^p$, and letting $v = f$, we have

$$I_3 \geq \frac{1}{\|f\|_\infty^{1-2p}}\int_{-a}^a u^3 v^{-2}\mathrm{d}x . \tag{2.16}$$

The function $(r,s) \mapsto r^3 s^{-2}$ is jointly convex, so that by Jensen's inequality,

$$\begin{aligned}
\frac{1}{2a}\int_{-a}^a u^3 v^{-2}\mathrm{d}x &\geq \left(\frac{1}{2a}\int_{-a}^a u\mathrm{d}x\right)^3\left(\frac{1}{2a}\int_{-a}^a v\mathrm{d}x\right)^{-2} \\
&= \frac{1}{2a\left(\int_{-a}^a f_0(x)\mathrm{d}x\right)^2}\left(H_p(f)\right)^3 \\
&= \frac{1}{2a}\frac{\left(H_p(f)\right)^3}{M^2} .
\end{aligned} \tag{2.17}$$

Combining this with (2.16), and using Lemma 1.2, together with the fact that $H_p(f) \leq H_p(f_0)$, to bound $\|f\|_\infty$ in terms of M and $H_p(f_0)$, we obtain (2.15). Combining this with (2.1), we obtain (1.6).

(Step Three): We now solve the differential inequality (1.6) to obtain (1.7). To do this, make a comparison with the solution of the differential equation

$$\dot{y}(t) = -C(y(t))^3 ,$$

whose solution is

$$y(t) = \left(2Ct + (y(0))^{-2}\right)^{-1/2} .$$

This results in (1.7). $\qquad\square$

Proof of Theorem 1.3

Proof. This is relatively simple. Our starting point is (1.5). Since if $\|f - M\|_\infty \leq \epsilon$ then $f \geq M - \epsilon$, we have from (1.5) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H_0(f) &= -2\int_{-a}^a f(f_{xxx})^2\mathrm{d}x \\
&\leq -2(M-\epsilon)\int_{-a}^a (f_{xxx})^2\mathrm{d}x .
\end{aligned} \tag{2.18}$$

Under periodic boundary conditions, f_x is orthogonal to the constant functions; i.e., the null space of the operator $-\mathrm{d}^2/\mathrm{d}x^2$ with periodic boundary conditions on $[-a,a]$. The least of the positive eigenvalues for this operator is $(\pi/a)^2$, so that under periodic boundary conditions, we obtain from (2.18) that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_0(f) \leq -2(M-\epsilon) \left(\frac{\pi}{a}\right)^4 H_0(f) \ .$$

Under the "no flux" boundary conditions, $f_x(\pm a) = f_{xxx}(\pm a) = 0$, f_x belongs to the domain of $-\mathrm{d}^2/\mathrm{d}x^2$ with Dirichlet boundary conditions on $[-a,a]$. Its smallest eigenvalue (in absolute value) is $(\pi/(2a))^2$. In this case we obtain from (2.18) that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_0(f) \leq -2(M-\epsilon) \left(\frac{\pi}{2a}\right)^4 H_0(f) \ .$$

$\square$

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