

STABILITY OF A TRAFFIC FLOW MODEL WITH NONCONVEX RELAXATION*

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Abstract. This paper is concerned with the nonlinear stability of traveling wave solutions for a quasi-linear relaxation model with a nonconvex equilibrium flux. The study is motivated by and the results are applied to the well-known dynamic continuum traffic flow model, the Payne and Whitham (PW) model with a nonconcave fundamental diagram. The PW model is the first of its kind and it has been widely adopted by traffic engineers in the study of stability and instability phenomena of traffic flow. The traveling wave solutions are shown to be asymptotically stable under small disturbances and under the sub-characteristic condition using a weighted energy method. The analysis applies to both non-degenerate case and the degenerate case where the traveling wave has exponential decay rates at infinity and has an algebraic decay rate at infinity, respectively.

Key words. Stability, quasi-linear relaxation model, nonconvex equilibrium flux, weighted energy method, nonconcave fundamental diagram, traffic flow.

AMS subject classifications. 35B30, 35B40, 35L65, 76L05, 90B20.

1. Introduction

We study the following quasi-linear relaxation model

$$\begin{cases} u_t + v_x = 0, \\ v_t + g(u)_x = -\frac{1}{\epsilon}(v - f(u)) \end{cases} \quad (1.1)$$

subject to the initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \rightarrow (u_{\pm}, v_{\pm}) \text{ as } x \rightarrow \pm\infty, \quad v_{\pm} = f(u_{\pm}) \quad (1.2)$$

where

$$g'(u) > 0 \quad (1.3)$$

$\epsilon > 0$, and the equilibrium flux f is nonconvex. A strict sub-characteristic condition

$$-\sqrt{g'(u)} < f'(u) < \sqrt{g'(u)} \quad (1.4)$$

is imposed for all u under consideration. Sub-characteristic condition (1.4) is a necessary condition for linear stability (Whitham [24]) and for nonlinear stability with convex equilibrium fluxes (Liu [13]).

The purpose of this paper is to show the existence and stability of the traveling wave solutions of (1.1) satisfying (1.4) and (1.12) with any nonconvex equilibrium flux f . When f is a nonconvex function, the standard energy method used in [13, 16] does not work. To overcome this difficulty, a weighted energy method, in the spirit of Matsumura and Nishihara [19], was developed in Liu, Wang and Yang [18] for semi-linear relaxation system, i.e. (1.1) with $g(u) = au$. For the quasi-linear relaxation system with nonconvex relaxation to be studied in this paper, we are able to adapt the weighted energy method performed in [18] to the current setting. As will be clear

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in the course of our weighted energy analysis, the nonlinearity of g together with the nonconvexity of f does require tricky estimates.

Our motivation for this study comes from the study of traffic flow. The system of equations (1.1) arises from a nonequilibrium continuum model of traffic flows with a nonconcave fundamental diagram: Payne [22] and Whitham [24] (PW) model

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + vv_x + \frac{c_0^2}{\rho} \rho_x = \frac{1}{\tau} (v_e(\rho) - v) \end{cases} \quad (1.5)$$

where $\tau > 0$ is the relaxation time, c_0 is the traffic sound speed coefficient and $v_e(\rho)$ is the desired speed. The first equation in (1.5) is a conservation law, while the second one describes drivers' acceleration behavior. The acceleration consists of a relaxation to the static equilibrium speed-density relation and an anticipation which expresses the effect of drivers reacting to conditions downstream. The PW model (1.5) is the first of its kind, namely, dynamic continuum models. It has been adopted by many traffic engineers in their study of stability and instability phenomena of traffic flow Helbing [2], Jin and Zhang [4] and Kerner and Konhäuser [7]. It has also been studied by mathematicians Schochet [23], Lattanzio and Marcati [9] and Li [12].

In the relaxation limit, $\tau \rightarrow 0^+$, the leading order of the relaxation system (1.5) is the LWR (Lighthill, Whitham and Richards) model

$$\rho_t + (q(\rho))_x = 0 \quad (1.6)$$

where

$$q(\rho) = \rho v_e(\rho) \quad (1.7)$$

is the equilibrium flux which is the so-called fundamental diagram in traffic flow.

Fundamental diagram gives a correspondence of vehicle density to the flow rate in traffic. Nonconcave fundamental diagrams are observed in real traffic Helbing [2], Knospe, Santen, Schadschneider and Schreckenberg [8] and Figure 1.1. Furthermore, a nonconcave fundamental diagram is a necessary condition in obtaining complicated traffic flow patterns such as stop-and-go waves, the self-organized oscillatory behavior and chaotic behavior in traffic Jin and Zhang [4], Kerner and Konhäuser [7] and Li [12], in *the unstable regions*, i.e., outside the stable region (1.4), or in the so-called super-characteristic case Jin and Katsoulakis [5]. It would be interesting to investigate, under the sub-characteristic condition (1.4), the nonlinear stability of traveling wave solutions for PW model (1.5) with a nonconcave fundamental diagram (1.7). The nonlinear stability result is a direct consequence of the sub-characteristic condition (1.4) and the weighted energy method in the spirit of [10, 14, 18, 19].

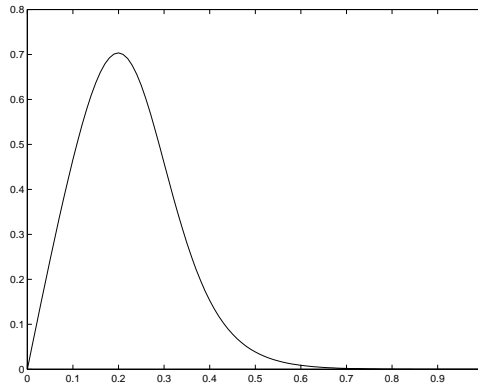
Using Chapman-Enskog expansion [13], the leading order of the relaxation system (1.1) is

$$v = f(u),$$

$$u_t + f(u)_x = 0. \quad (1.8)$$

The first order approximation to (1.1) is

$$v = f(u) - \varepsilon (g'(u) - f'(u)^2) u_x,$$

FIG. 1.1. A Fundamental Diagram $q(\rho)$ of Traffic Flow

$$u_t + f(u)_x = \varepsilon((g'(u) - f'(u)^2)u_x)_x. \quad (1.9)$$

Since (1.9) is dissipative provided that sub-characteristic condition (1.4) is satisfied, then similar to the diffusion, the relaxation term has smoothing and dissipative effects for the hyperbolic conservation laws. The stability of viscous shock waves with non-convex flux was investigated by several authors, cf. [6], [14], [19], [20]. Subsequently the stability of relaxation shock waves for a semi-linear relaxation system [3] with nonconvex flux was studied in [18, 17]. Other related results include: Li [10] established the nonlinear stability of a planar gaseous Chapman-Jouguet(CJ) detonation wave by the weighted energy method. Li [11] also obtained the well-posedness results for a relaxation model of traffic flow with nonconcave fundamental diagrams. Lattanzio and Marcati [9] showed the convergence to the equilibrium solution as the relaxation parameter τ tends to zero for (1.5) with a concave (quadratic) fundamental diagram. The asymptotic behavior of solutions of a p -system with convex relaxation was recently studied in [25]. Consult [15] for a bird's-eye view of results obtained for a class of relaxation systems, in particular of those introduced in [3].

Under the scaling $(x, t) \rightarrow (\varepsilon x, \varepsilon t)$, equation (1.1) becomes

$$\begin{cases} u_t + v_x = 0, \\ v_t + g(u)_x = f(u) - v. \end{cases} \quad (1.10)$$

The behavior of the solution (u, v) of (1.1) and (1.2) at any fixed time t as $\varepsilon \rightarrow 0^+$ is equivalent to the long time behavior of (u, v) of (1.10) as $t \rightarrow \infty$.

This paper is organized as follows: in Section 2, we start with a discussion about how the system (1.1) arises from a nonequilibrium continuum model of traffic flows with a nonconcave fundamental diagram. We then show that there exist traveling wave solutions for (1.10), i.e.,

$$(u, v)(x, t) = (U, V)(x - st) \equiv (U, V)(z)$$

satisfying

$$(U, V)(z) \rightarrow (u_{\pm}, v_{\pm}) \text{ as } z \rightarrow \pm\infty, \quad v_{\pm} = f(u_{\pm}) \quad (1.11)$$

with shock speed s being sub-characteristic

$$-\sqrt{g'(u)} < s < \sqrt{g'(u)}. \quad (1.12)$$

The corresponding jump (u_-, u_+) is an admissible shock of (1.8). That is, the constants u_{\pm} and s satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \quad (1.13)$$

and the entropy condition

$$Q(u) \equiv f(u) - f(u_{\pm}) - s(u - u_{\pm}) \begin{cases} < 0 \text{ for } u_+ < u < u_-, \\ > 0 \text{ for } u_- < u < u_+. \end{cases} \quad (1.14)$$

Given such a traveling wave, the main result on its asymptotic stability is stated in Section 2. The stability results are established for both the non-degenerate case, $f'(u_+) < s < f'(u_-)$, and the degenerate case, say, $f'(u_+) = s < f'(u_-)$. The meanings of stability are different which are dictated by different decay rates of the traveling wave at far fields. In the non-degenerate case, the traveling wave has exponential decay rates as $x \rightarrow \pm\infty$. Thus the traveling wave is stable against small initial perturbations which are in H^2 . However, in the degenerate case, the traveling wave has an algebraic decay rate as $x \rightarrow +\infty$. Thus it is stable against small initial perturbations which are in $H^2 \cap L^2_{(x)_+}$. Once the stability result is established, further convergence rates to the underlying traveling wave can be similarly explored as in [17], though details are not provided in this paper. In Section 3, the problem is reformulated in terms of perturbations to the underlying traveling wave. The main result thus follows from the local existence and the a priori estimate. Section 4 is devoted to establishing the desired a priori estimates.

Notations: Hereafter, C denotes a generic positive constant. L^2 denotes the space of square integrable functions on R with the norm

$$\|f\| = \left(\int_R |f|^2 dx \right)^{1/2}.$$

Without any ambiguity, the integral region R will be omitted. $H^j (j > 0)$ denotes the usual j -th order Sobolev space with norm

$$\|f\|_{H^j} = \|f\|_j = \left(\sum_{k=0}^j \|\partial_x^k f\|^2 \right)^{1/2}.$$

For a weight function $w > 0$, L_w^2 denotes the space of measurable functions f satisfying $\sqrt{w}f \in L^2$ with the norm

$$\|f\|_w = \left(\int w(x) |f(x)|^2 dx \right)^{1/2}.$$

H_w^j denotes the weighted Sobolev space with norm

$$\|f\|_{H_w^j} = \|f\|_j = \left(\sum_{k=0}^j \|\partial_x^k f\|_w^2 \right)^{1/2}.$$

When $w(x) = (1 + x^2)^{\alpha/2}$, we denote $L_w^2 = L_{\alpha}^2$.

2. Preliminaries and Theorem

First, we show how a system of equations (1.1) arises from a nonequilibrium continuum model (1.5) of traffic flows.

In Lagrangian formulation, PW model (1.5) is equivalent to

$$\begin{cases} \gamma_t - v_x = 0, \\ v_t + p(\gamma)_x = \frac{w(\gamma) - v}{\tau} \end{cases} \quad (2.1)$$

where $\gamma = \frac{1}{\rho}$, $p(\gamma) = \frac{c_0^2}{\gamma}$ and $w(\gamma) = v_*(\frac{1}{\gamma})$. Moreover

$$w''(\gamma) = \frac{1}{\gamma^3} q''\left(\frac{1}{\gamma}\right) \quad (2.2)$$

where the fundamental diagram q is defined in (1.7). Thus w is nonconcave since q is nonconcave.

Now replacing γ by u and v by $-v$ in (2.1), we arrive at the quasi-linear relaxation model (1.1) with $g(u) = -p(u)$, $f(u) = -w(u)$ and $\epsilon = \tau$.

Moreover,

$$g'(u) = -p'(u) = \frac{c_0^2}{u^2} > 0.$$

Therefore condition (1.3) is satisfied. The equilibrium flux f in (1.1) is nonconvex since w is nonconcave.

We now look for traveling wave solutions with shock profiles for the relaxation system (1.10). Substituting

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st$$

into (1.10), we have

$$\begin{cases} -sU_z + V_z = 0, \\ -sV_z + g(U)_z = f(U) - V. \end{cases} \quad (2.3)$$

Hence

$$(g'(U) - s^2)U_z = f(U) - V. \quad (2.4)$$

Integration of the first equation of (2.3) over $(\pm\infty, z)$ using boundary condition (1.11) yields

$$-sU + V = -su_{\pm} + v_{\pm} = -su_{\pm} + f(u_{\pm}). \quad (2.5)$$

Combining (2.4) and (2.5), we obtain

$$U_z = \frac{Q(U)}{g'(U) - s^2} \quad (2.6)$$

where

$$Q(U) \equiv f(U) - f(u_{\pm}) - s(U - u_{\pm}) \quad (2.7)$$

and

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Therefore the Rankine-Hugoniot condition for the equilibrium equation (1.8) is satisfied by the end states of the traveling wave of the relaxation system (1.10).

Since (2.6) is a scalar ordinary differential equation of U , the trajectories satisfying the boundary condition (1.11) necessarily connect adjacent equilibria u_- and u_+ . It is easy to check that there is a trajectory from u_- to u_+ if and only if condition

$$(u_+ - u_-) \frac{Q(U)}{g'(U) - s^2} > 0$$

holds for u lies strictly between u_+ and u_- . By virtue of (1.12) and entropy condition (1.14), this implies

$$Q(u)(u_+ - u_-) > 0$$

for u lies strictly between u_+ and u_- . Thus there is a trajectory from u_- to u_+ if and only if

$$u = \begin{cases} u_-, & x - st < 0 \\ u_+, & x - st > 0 \end{cases} \quad (2.8)$$

is an admissible shock for the equilibrium equation (1.8).

Without loss of generality, we study only the following case:

$$u_+ < u_- \text{ or } Q < 0 \text{ and } U_z < 0. \quad (2.9)$$

The shock wave satisfying (2.9) corresponds to a physical shock for the equilibrium traffic flow model (1.6). Then the ordinary differential equation (2.6) with boundary condition (1.11) has a unique smooth solution. Moreover, if

$$f'(u_+) < s < f'(u_-) \text{ or } Q'(u_{\pm}) \neq 0, \quad (2.10)$$

then

$$Q(U) \sim -|U - u_{\pm}| \text{ as } U \rightarrow u_{\pm}.$$

Hence

$$|(U - u_{\pm}, V - v_{\pm})(z)| \sim \exp(-c_{\pm}|z|) \text{ as } z \rightarrow \pm\infty \quad (2.11)$$

for some constants $c_{\pm} > 0$.

While

$$s = f'(u_+) \text{ or } Q'(u_+) = 0, \quad (2.12)$$

the so-called degenerate case, and as $U \rightarrow u_+$

$$Q(U) \sim -|U - u_+|^{1+k_+} \quad (2.13)$$

where $k_+ > 0$ is an integer such that

$$Q'(u_+) = \dots = Q^{(k_+)}(u_+) = 0 \text{ but } Q^{(k_++1)}(u_+) \neq 0, \quad (2.14)$$

then

$$|(U - u_+, V - v_+)(z)| \sim z^{-\frac{1}{k_+}} \text{ as } z \rightarrow +\infty. \quad (2.15)$$

Thus we have the following results on the existence of traveling wave solutions.

LEMMA 2.1. *Assume that (1.13), (2.9) hold. Then there exists a traveling wave solution $(U, V)(x - st)$ of (1.10) and boundary condition (1.11), which is unique up to a shift and the speed is sub-characteristic (1.12). Moreover, the convergence rates (2.11) and (2.15) hold when $k_+ = 0$ and $k_+ > 0$ in (2.13), respectively.*

For the initial disturbances, without loss of generality, we assume

$$\int_{-\infty}^{+\infty} (u_0 - U)(x) dx = 0. \quad (2.16)$$

For a pair of traveling wave solutions given by Lemma 2.1, we let

$$(\phi_0, \psi_0)(x) = \left(\int_{-\infty}^x (u_0 - U)(y) dy, (v_0 - V)(x) \right). \quad (2.17)$$

Our goal is to show that the solution $(u, v)(x, t)$ of (1.10), (1.2) exists globally and approaches the traveling wave solution $(U, V)(x - st)$ under the sub-characteristic condition (1.4) as $t \rightarrow \infty$, the main theorem is as follows.

THEOREM 2.2. *(Stability) Suppose that conditions (1.4), (1.13) and (1.14) hold. Let $(U, V)(x - st)$ be a traveling wave solution determined by (2.16) with speed satisfying (1.12). Then it holds:*

(i) *In the non-degenerate case, $f'(u_+) < s < f'(u_-)$. There exists a constant $\varepsilon_0 > 0$ independent of (u_{\pm}, v_{\pm}) such that if $\|u_0 - U\|_2 + \|v_0 - V\|_2 + \|\phi_0, \psi_0\| \leq \varepsilon_0$, the initial value problem (1.10), (1.2) has a unique global solution $(u, v)(x, t)$ satisfying*

$$(u - U, v - V) \in C^0(0, \infty; H^2) \cap L^2(0, \infty; H^2).$$

Furthermore, the solution satisfies

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (U, V)(x - st)| \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (2.18)$$

(ii) *In the degenerate case, $f'(u_+) = s < f'(u_-)$. There exists a positive constant ε_1 such that if $\|u_0 - U\|_2 + \|v_0 - V\|_2 + \|(\phi_0, \phi_{0,z}, \psi_0)\|_{\langle z \rangle_+} \leq \varepsilon_1$, then the Cauchy problem (1.10), (1.2) has a unique global solution $(u, v)(x, t)$ satisfying*

$$(u - U, v - V) \in C^0(0, \infty; H^2 \cap L^2_{\langle z \rangle_+}) \cap L^2(0, \infty; H^2 \cap L^2_{\langle z \rangle_+}),$$

and

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (U, V)(x - st)| \rightarrow 0 \text{ as } t \rightarrow +\infty,$$

where

$$\langle z \rangle_+ = \begin{cases} \sqrt{1 + z^2}, & z \geq 0, \\ 1, & z < 0. \end{cases}$$

Regarding this stability result, two remarks are in order.

REMARK 2.3. The above results will be proved by weighted L^2 estimates by taking an appropriate weight function $w = w(U)$ (3.8) which is dictated by the decay rates (2.11) or (2.15) of the traveling wave at far fields. In the non-degenerate case (i) $f'(u_+) < s < f'(u_-)$, the weight $w = w(U)$ is uniformly bounded

$$C^{-1} \leq w \leq C \quad (2.19)$$

where $C > 0$ is a constant. The traveling wave is stable against small initial perturbations which are in H^2 .

However, in the degenerate case (ii), say, $f'(u_+) = s < f'(u_-)$, the traveling wave has an algebraic decay rate

$$|(U - u_+, V - v_+)(z)| \sim z^{-1/k_+}, \quad \text{as } z \rightarrow +\infty$$

and $|Q(U)| \sim |U - u_+|^{1+k_+}$ as $U \rightarrow u_+$, as stated in Lemma 2.1. This when combined with our choice of the weight $w = w(U)$ defined in (3.8) leads to

$$w(U) \sim \langle z \rangle_+ \quad \text{as } z \rightarrow +\infty. \quad (2.20)$$

The traveling wave is stable against small initial perturbations which are in $H^2 \cap L^2_{(z)_+}$.

REMARK 2.4. With the above stability result at hand, convergence decay rates to underlying traveling waves can be further explored following the analysis in [17] for the study of a semi-linear relaxation system. The result may be stated as follows: if initial perturbations have additional decay rates in space, in the sense that

$$\left(\int_{-\infty}^x (u_0 - U)(y) dy, (u_0 - U)(x), (v_0 - V)(x) \right) \in L^2_\alpha$$

for any number $\alpha > 0$. Then such a spatial decay rate can be transformed into the corresponding decay rate in time, i.e.,

$$\sup_R |(u, v)(x, t) - (U, V)(x - st)| \leq C(1+t)^{-\alpha/2}$$

for some constant $C > 0$ depending on initial perturbations measured in the L^2_α norm. Technical details are chosen to be omitted.

3. Reformulation of the Problem

The proof of Theorem 2.2 is based on weighted L^2 energy estimates. The key condition for the stability is the sub-characteristic condition (1.4). Depending on the decay rates (2.11) or (2.15) of the traveling wave U at far fields, the meanings of stability are different in the sense that U is stable against small initial perturbations which are in H^2 or in $H^2 \cap L^2_{(x)_+}$, respectively. Proper choice of weight function w is a crucial step in our proof. Such a weight function w is chosen so that we prove stability for both non-degenerate and degenerate cases as stated in Theorem 2.2.

We firstly rewrite the problem (1.10), (1.2) using the moving coordinate $z = x - st$. Under the assumption of (2.16), we will look for a solution of the following form:

$$(u, v)(x, t) = (U, V)(z) + (\phi_z, \psi)(z, t), \quad (3.1)$$

where (ϕ, ψ) is in some space of integrable functions which will be defined later.

We substitute (3.1) into (1.10), by virtue of (2.3), and integrate the first equation once with respect to z , to have that the perturbation (ϕ, ψ) satisfies

$$\begin{cases} \phi_t - s\phi_z + \psi = 0, \\ \psi_t - s\psi_z + (g(U + \phi_z) - g(U))_z = f(U + \phi_z) - f(U) - \psi. \end{cases} \quad (3.2)$$

The first equation of (3.2) gives

$$\psi = -(\phi_t - s\phi_z). \quad (3.3)$$

Substituting (3.3) into the second equation of (3.2), we get a closed equation for ϕ

$$L(\phi) \equiv (\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - (g'(U)\phi_z)_z + \phi_t + \lambda\phi_z = -F(U, \phi_z) \quad (3.4)$$

where

$$\lambda = Q'(U) = f'(U) - s$$

and

$$F(U, \phi_z) = F_1 + F_2$$

with

$$F_1 := f(U + \phi_z) - f(U) - f'(U)\phi_z \quad (3.5)$$

$$F_2 := -(g(U + \phi_z) - g(U) - g'(U)\phi_z)_z = (G(U, \phi_z)\phi_z^2)_z \quad (3.6)$$

and

$$G(U, \phi_z) := - \int_0^1 \int_0^1 g''(U + \theta\eta\phi_z)\theta d\theta d\eta$$

is the error term due to nonlinearity of function g .

The corresponding initial data for (3.4) becomes

$$\phi(z, 0) = \phi_0(z), \quad \phi_t(z, 0) = s\phi'_0(z) - \psi_0 = \phi_1(z). \quad (3.7)$$

The asymptotic stability of the profile (U, V) means that the perturbation (ϕ_z, ψ) decays to zero as $t \rightarrow +\infty$. The left hand side of (3.4) contains a first-order term with speed λ which plays the essential role of governing the large-time behavior of the solution.

First we choose the weight function $w(U)$ introduced in [19] for the scalar viscous conservation laws with nonconvex flux

$$w(U) = \frac{(U - u_+)(U - u_-)}{Q(U)}. \quad (3.8)$$

Then $w \in C^2[u_+, u_-]$. By (2.9), $w \geq 0$. Note that w is uniformly bounded in the non-degenerate case $f'(u_-) > s > f'(u_+)$, see (2.19). and it becomes unbounded as $z \rightarrow +\infty$ in the degenerate case $f'(u_-) > s = f'(u_+)$, see (2.20)

Now, we introduce the solution space of the problem (3.4), (3.7) as follows:

$$X(0, T) =$$

$$\{\phi(z, t) : \phi \in C^0([0, T]; H^3 \cap L_w^2) \cap C^1(0, T; H^2 \cap L_w^2), \phi_z, \phi_t \in L^2(0, T; H^2 \cap L_w^2)\}$$

with $0 < T \leq +\infty$.

By virtue of (3.3), we have

$$\psi \in C^0([0, T]; H^2 \cap L_w^2) \cap L^2(0, T; H^2 \cap L_w^2).$$

By the Sobolev embedding theorem, if we let

$$N(t) = \sup_{0 \leq \tau \leq t} \{ \|\phi(\tau)\|_3 + \|\phi_t(\tau)\|_2 + \|\phi(\tau)\|_{H_w^1} + \|\phi_t(\tau)\|_{L_w^2} \}, \quad (3.9)$$

then

$$\sup_{z \in R} \{ |\phi|, |\phi_z|, |\phi_{zz}|, |\phi_t|, |\phi_{tz}| \} \leq CN(t). \quad (3.10)$$

Thus Theorem 2.2 is a consequence of the following theorem.

THEOREM 3.1. *Under the conditions of Theorem 2.2, there exists a positive constant δ_1 such that if $N(0) \leq \delta_1$, then the problem (3.4), (3.7) has a unique global solution $\phi \in X(0, +\infty)$ satisfying*

$$\|\phi(t)\|_3^2 + \|\phi_t\|_2^2 + \|\phi\|_{H_w^1}^2 + \|\phi_t\|_{L_w^2}^2 + \int_0^t \|(\phi_t, \phi_z)(\tau)\|_2^2 d\tau \leq CN^2(0) \quad (3.11)$$

for $t \in [0, +\infty)$. Furthermore,

$$\sup_{z \in R} |(\phi_z, \phi_t)(z, t)| \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (3.12)$$

For the solution ϕ in the above theorem, we define (ϕ, ψ) by (3.3). Then it becomes a global solution of the problem (3.2) with $(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z)$, and consequently we have the desired solution of the problem (1.10), (1.2) through the relation (3.1). On the other hand the solution of (1.10) is unique in the space $C^0(0, T; H^2 \cap L_w^2)$, therefore Theorem 2.2 follows from Theorem 3.1. Global existence for ϕ will be derived from the following local existence theorem for ϕ combined with an a priori estimate. The estimate (3.11) gives

$$\|\phi_t, \phi_z\|_1^2 \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (3.13)$$

whence we have

$$\begin{aligned} \phi_t^2 + \phi_z^2 &= \int_{-\infty}^z (2\phi_t \phi_{tz} + 2\phi_z \phi_{zz})(y, t) dy \\ &\leq \left(\int_{-\infty}^{+\infty} (\phi_t^2 + \phi_z^2) dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} (\phi_{tz}^2 + \phi_{zz}^2) dy \right)^{1/2} \rightarrow 0 \text{ as } t \rightarrow \infty. \end{aligned}$$

PROPOSITION 3.2. *(Local existence) For any $\delta_0 > 0$, there exists a positive constant T_0 depending on δ_0 , such that if $\phi_0 \in H^3 \cap H_w^1$ and $\phi_1 \in H^2 \cap L_w^2$, with $N(0) < \delta_0/2$, then the problem (3.4), (3.7) has a unique solution $\phi \in X(0, T_0)$ satisfying*

$$N(t) < 2N(0) \quad (3.14)$$

for any $0 \leq t \leq T_0$.

PROPOSITION 3.3. (A priori estimate) Let $\phi \in X(0, T)$ be a solution for a positive constant T , then there exists a positive constant δ_2 independent of T such that if

$$N(t) < \delta_2, \quad t \in [0, T],$$

then ϕ satisfies (3.11) for any $0 \leq t \leq T$.

Proposition 3.2 can be proved in the standard way, so we omit the proof, cf [21]. To prove Proposition 3.3 is our main task in the following section.

Theorem 3.1 can be proved by the continuation arguments based on Proposition 3.2 and Proposition 3.3, cf. [18].

4. Energy Estimates

In this section, we will complete the proof of our stability theorem. The stability result is a direct consequence of the sub-characteristic condition (1.4) and the weighted energy method in the spirit of [10, 14, 18, 19]. We establish the weighted L^2 estimate while taking the nonlinearity of function g into account.

LEMMA 4.1. Under the conditions of Theorem 2.2, there is a positive constant C such that if sub-characteristic condition (1.4) is satisfied for $u \in [u_+, u_-]$ and $|u_+ - u_-|$ is sufficiently small, then

$$\begin{aligned} & \|\phi(t)\|_{H_w^1}^2 + \|\phi_t(t)\|_w^2 + \int_0^t \|(\phi_t, \phi_z)(\tau)\|_w^2 d\tau + \int_0^t \int_R |U_z| \phi^2 dz d\tau \\ & \leq C \left\{ \|\phi_0\|_{H_w^1}^2 + \|\phi_1\|_w^2 + \int_0^t \int_R w |F| (|\phi| + |(\phi_t, \phi_z)|) dz d\tau \right\} \end{aligned} \quad (4.1)$$

holds for $t \in [0, T]$.

Proof. Let $w := w(U) > 0$ be a weight function to be determined.

Multiplying (3.4) by $2w(U)\phi$, we obtain

$$2w(U)\phi \cdot L(\phi) = -2Fw(U)\phi. \quad (4.2)$$

The left hand side of (4.2) can be reduced to

$$\begin{aligned} & 2[(\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - (g'(U)\phi_z)_z]w\phi + 2(\phi_t + \lambda\phi_z)w\phi \\ & = [2w\phi(\phi_t - s\phi_z)]_t - 2w\phi_t(\phi_t - s\phi_z) - 2s[w\phi(\phi_t - s\phi_z)]_z \\ & \quad + 2sw_z\phi(\phi_t - s\phi_z) + 2sw\phi_z(\phi_t - s\phi_z) - 2(wg'(U)\phi\phi_z)_z + 2wg'(U)\phi_z^2 \\ & \quad + (g'(U)w_z\phi^2)_z - (g'(U)w_z)_z\phi^2 + (w\phi^2)_t + (\lambda w\phi^2)_z - \phi^2(\lambda w)_z \\ & = [w\phi^2 + 2w\phi(\phi_t - s\phi_z)]_t - 2w(\phi_t - s\phi_z)^2 + 2wg'(U)\phi_z^2 - (g'(U)w_z)_z\phi^2 \\ & \quad - (\lambda w)_z\phi^2 + sw_z(\phi^2)_t - s^2\{w_z(\phi^2)\}_z + s^2w_{zz}\phi^2 \\ & \quad + \{-2sw\phi(\phi_t - s\phi_z) - 2g'(U)w\phi\phi_z + g'(U)w_z\phi^2 + \lambda w\phi^2\}_z \\ & = [w\phi^2 + 2w\phi(\phi_t - s\phi_z) + sw_z\phi^2]_t - 2w(\phi_t - s\phi_z)^2 + 2g'(U)w\phi_z^2 + A\phi^2 + \{\dots\}_z \end{aligned} \quad (4.3)$$

where

$$A = s^2w_{zz} - (g'(U)w_z)_z - (\lambda w)_z \quad (4.4)$$

and $\{\dots\}_z$ denotes the terms which will disappear after integration with respect to $z \in R$.

Secondly, we calculate

$$2(\phi_t - s\phi_z)w \cdot L(\phi) = -2F(\phi_t - s\phi_z)w. \quad (4.5)$$

The left hand side of (4.5) is

$$\begin{aligned} & 2[(\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - (g'(U)\phi_z)_z]w(\phi_t - s\phi_z) \\ & \quad + 2w(\phi_t - s\phi_z)(\phi_t - s\phi_z + f'(U)\phi_z) \\ = & [w(\phi_t - s\phi_z)^2]_t - s[w(\phi_t - s\phi_z)^2]_z + sw_z(\phi_t - s\phi_z)^2 \\ & \quad - 2[wg'(U)\phi_z(\phi_t - s\phi_z)]_z + 2g'(U)w_z\phi_z(\phi_t - s\phi_z) + 2g'(U)w\phi_z(\phi_t - s\phi_z)_z \\ & \quad + 2w(\phi_t - s\phi_z)^2 + 2wf'(U)\phi_z(\phi_t - s\phi_z) \\ = & [w(\phi_t - s\phi_z)^2]_t + (2w + sw_z)(\phi_t - s\phi_z)^2 + 2g'(U)w_z\phi_z(\phi_t - s\phi_z) \\ & \quad + 2wf'(U)\phi_z(\phi_t - s\phi_z) + [g'(U)w\phi_z^2]_t - [swg'(U)\phi_z^2]_z + s(wg'(U))_z\phi_z^2 \\ & \quad - [sw(\phi_t - s\phi_z)^2 + 2g'(U)w\phi_z(\phi_t - s\phi_z)]_z \\ = & [wg'(U)\phi_z^2 + w(\phi_t - s\phi_z)^2]_t + (2w + sw_z)(\phi_t - s\phi_z)^2 \\ & \quad + s(wg'(U))_z\phi_z^2 + 2f'(U)w\phi_z(\phi_t - s\phi_z) + 2g'(U)w_z\phi_z(\phi_t - s\phi_z) \\ & \quad - [sw(\phi_t - s\phi_z)^2 + 2g'(U)w\phi_z(\phi_t - s\phi_z) + swg'(U)\phi_z^2]_z. \end{aligned} \quad (4.6)$$

Hence, the combination (4.2) $\times \mu$ + (4.5) with a positive constant μ yields

$$\begin{aligned} & \{E_1(\phi, (\phi_t - s\phi_z)) + E_3(\phi_z)\}_t + E_2(\phi_z, (\phi_t - s\phi_z)) + E_4(\phi) + \{\dots\}_z \\ & = -2Fw\{\mu\phi + (\phi_t - s\phi_z)\} \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} E_1(\phi, (\phi_t - s\phi_z)) &= w(\phi_t - s\phi_z)^2 + 2\mu w\phi(\phi_t - s\phi_z) + \mu(w + sw_z)\phi^2, \\ E_3(\phi_z) &= wg'(U)\phi_z^2, \\ E_2(\phi_z, (\phi_t - s\phi_z)) &= (2w + sw_z - 2\mu w)(\phi_t - s\phi_z)^2 \\ & \quad + 2(f'(U)w + g'(U)w_z)\phi_z(\phi_t - s\phi_z) \\ & \quad + (2\mu wg'(U) + s(wg'(U))_z)\phi_z^2, \\ E_4(\phi) &= \mu A\phi^2 \end{aligned} \quad (4.8)$$

where A is defined in (4.4). Due to $(g'(U) - s^2)U_z = Q(U)$, we have

$$\begin{aligned} A &= -\{(g'(U) - s^2)w'(U)U_z + \lambda w\}_z \\ &= -\{w'(U)Q(U) + Q'(U)w\}_z \\ &= -\{wQ\}''(U)U_z. \end{aligned} \quad (4.9)$$

The monotonicity of the shock profile U (2.9) requires that

$$(wQ)''(U) \geq \nu > 0. \quad (4.10)$$

On the other hand, we need to choose a constant $\mu > 0$ and w such that the discriminants of E_j ($j=1,2$) are negative, that is, the inequalities

$$\sup_j D_j < 0, \quad j=1,2 \quad (4.11)$$

hold uniformly on $[u_+, u_-]$, where D_j is the discriminants of the quadratics $E_j (j = 1, 2)$, respectively.

$$D_1 = 4\mu w[(\mu - 1)w - sw_z],$$

$$D_2 = 4\{(f'(U)w + g'(U)w_z)^2 - (2\mu wg'(U) + s(g'(U)w)_z)(2w + sw_z - 2\mu w)\},$$

and $2\mu wg'(U) + s(wg'(U))_z > 0$.

For this choice of μ and w , there exist positive constants c and C such that

$$\begin{cases} cw\{\phi^2 + (\phi_t - s\phi_z)^2\} \leq E_1 \leq Cw\{\phi^2 + (\phi_t - s\phi_z)^2\}, \\ cw\{\phi_z^2 + (\phi_t - s\phi_z)^2\} \leq E_2. \end{cases} \quad (4.12)$$

Furthermore, (1.3), (4.9) and (4.10) yield

$$\begin{cases} E_3 = wg'(U)\phi_z^2 \geq 0, \\ E_4 \geq \mu\nu|U_z|\phi^2 \geq 0. \end{cases} \quad (4.13)$$

Thus the equality (4.7) together with the estimates (4.12), (4.13) give the desired estimate (4.1) after integration with respect to t and z .

It remains to check conditions (4.10) (4.11). By definition of w (3.8), $(wQ)''(U) = \nu = 2$, i.e., (4.10) holds.

Furthermore, choosing $\mu = \frac{1}{2}$ and noting (1.12), the two inequalities in (4.11) are equivalent to

$$1 + 2s\frac{w_z}{w} > 0, \quad (4.14)$$

$$(f'(U) + g'(U)\frac{w_z}{w})^2 < (1 + s\frac{w_z}{w})(g'(U) + s\frac{(wg'(U))_z}{w}). \quad (4.15)$$

By (2.6) (2.7) (2.19) (2.20) (3.8), we have

$$\frac{w_z}{w} = \frac{w'}{w}U_z = \frac{w'}{w} \frac{Q}{g'(U) - s^2} = O(|u_+ - u_-|)$$

and

$$\frac{(wg'(U))_z}{w} = \frac{(wg'(U))_U}{w}U_z = O(|u_+ - u_-|)$$

which are small provided $|u_+ - u_-|$ is suitably small. Using this fact and sub-characteristic condition (1.4), we derive inequalities (4.14) and (4.15). Thus condition (4.11) is satisfied. This completes the proof of Lemma 4.1. \square

Next we estimate the higher order derivatives of ϕ . Multiplying the derivative of (3.4) with respect to z by ϕ_z and $(\phi_t - s\phi_z)_z$ respectively, we have

$$2\partial_z L(\phi) \cdot \phi_z = -2F_z \phi_z,$$

$$2\partial_z L(\phi) \cdot (\phi_t - s\phi_z)_z = -2F_z (\phi_t - s\phi_z)_z.$$

Letting $\phi_z = \Phi$, then

$$\begin{aligned}\partial_z L(\phi) &= (\phi_{zt} - s\phi_{zz})_t - s(\phi_{zt} - s\phi_{zz})_z - (g'(U)\phi_z)_{zz} + \phi_{zt} + \lambda\phi_{zz} + \lambda_z\phi_z \\ &= L(\phi_z) + \lambda_z\phi_z - (g'(U)_z\phi_z)_z = L(\Phi) + \lambda_z\Phi - (g'(U)_z\Phi)_z.\end{aligned}\quad (4.16)$$

By a similar argument used in obtaining (4.3) and (4.6) with $w=1$, we have

$$\begin{aligned}[\Phi^2 + 2\Phi(\Phi_t - s\Phi_z)]_t + 2g'(U)\Phi_z^2 - 2(\Phi_t - s\Phi_z)^2 - \lambda_z\Phi^2 + 2\lambda_z\Phi^2 \\ + 2g'(U)_z\Phi\Phi_z + \{\dots\}_z = -2F_z\Phi\end{aligned}\quad (4.17)$$

and

$$\begin{aligned}[(\Phi_t - s\Phi_z)^2 + g'(U)\Phi_z^2]_t + 2(\Phi_t - s\Phi_z)^2 + 2f'(U)\Phi_z(\Phi_t - s\Phi_z) \\ + sg'(U)_z\Phi_z^2 + 2\lambda_z\Phi(\Phi_t - s\Phi_z) - 2(g'(U)\Phi)_z(\Phi_t - s\Phi_z) + \{\dots\}_z \\ = -2F_z(\Phi_t - s\Phi_z).\end{aligned}\quad (4.18)$$

The combination (4.17) $\times \frac{1}{2}$ + (4.18) yields

$$\begin{aligned}\{E_1(\Phi, (\Phi_t - s\Phi_z)) + E_2(\Phi_z)\}_t + E_3(\Phi_z, (\Phi_t - s\Phi_z)) + G + \{\dots\}_z \\ = -F_z\{\Phi + 2(\Phi_t - s\Phi_z)\}\end{aligned}\quad (4.19)$$

where

$$\begin{aligned}G &= \frac{\lambda_z}{2}\Phi^2 + 2\lambda_z\Phi(\Phi_t - s\Phi_z) + g'(U)_z\Phi\Phi_z - 2(g'(U)\Phi)_z(\Phi_t - s\Phi_z), \\ E_1(\Phi, (\Phi_t - s\Phi_z)) &= (\Phi_t - s\Phi_z)^2 + \Phi(\Phi_t - s\Phi_z) + \frac{1}{2}\Phi^2, \\ E_2(\Phi_z) &= g'(U)\Phi_z^2, \\ E_3(\Phi_z, (\Phi_t - s\Phi_z)) &= (\Phi_t - s\Phi_z)^2 + 2f'\Phi_z(\Phi_t - s\Phi_z) + (g' + sg'_z)\Phi_z^2.\end{aligned}\quad (4.20)$$

After integration with respect to t and z , (4.19) together with (4.20) gives the following estimate

$$\begin{aligned}\|\Phi(t)\|_1^2 + \|\Phi_t(t)\|^2 + \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau \\ \leq C \left\{ \|\Phi_0\|_1^2 + \|\Phi_1\|^2 + \int_0^t \int |G| dz d\tau + \int_0^t \int_R |F_z| (|\Phi| + |(\Phi_t, \Phi_z)|) dz d\tau \right\},\end{aligned}\quad (4.21)$$

here $\Phi_0 = \phi'_0$ and $\Phi_1 = \phi'_1$.

Using the estimate (4.1) and the smallness of $|u_+ - u_-|$, we obtain

$$\begin{aligned}\int_0^t \int_R |G| dz d\tau \leq \frac{1}{2} \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau + C \int_0^t \int_R \Phi^2 dz d\tau \\ \leq \frac{1}{2} \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau +\end{aligned}$$

$$+C \left\{ \|\phi_0\|_{H_w^1}^2 + \|\phi_1\|_w^2 + \int_0^t \int w|F|(|\phi| + |(\phi_t, \phi_z)|) dz d\tau \right\} \quad (4.22)$$

where we have used Lemma 4.1 and the Young inequality.

Substituting (4.22) into (4.21) and replacing Φ by $\partial_z \phi$, we have the following lemma.

LEMMA 4.2. *Under the conditions of Theorem 2.2, there is a positive constant C such that if sub-characteristic condition (1.4) is satisfied for all $u \in [u_+, u_-]$ and $|u_+ - u_-|$ is sufficiently small, then*

$$\begin{aligned} & \|\partial_z \phi\|_1^2 + \|\partial_z \phi_t\|^2 + \frac{1}{2} \int_0^t \|(\partial_z \phi_t, \partial_z \phi_z)(\tau)\|^2 d\tau \\ & \leq C \left\{ \|\phi_0\|_2^2 + \|\phi_1\|_1^2 + \int_0^t \int |F_z|(|\partial_z \phi| + |(\partial_z \phi_t, \partial_z \phi_z)|) dz d\tau \right. \\ & \quad \left. + \|\phi_0\|_{H_w^1}^2 + \|\phi_1\|_w^2 + \int_0^t \int w|F|(|\phi| + |(\phi_t, \phi_z)|) dz d\tau \right\} \end{aligned} \quad (4.23)$$

holds for $t \in [0, T]$.

Next we calculate the equality

$$\partial_z^2 \phi \cdot \partial_z^2 L(\phi) + 2\partial_z^2(\phi_t - s\phi_z) \cdot \partial_z^2 L(\phi) = -F_{zz}(\partial_z^2 \phi + 2\partial_z^2(\phi_t - s\phi_z))$$

in the same way as in the proof of Lemma 4.2. Set $\Psi := \partial_z^2 \phi$, we have

$$\partial_z^2 L(\phi) = L(\Psi) - (g'(U)\Phi)_{zzz} + (g'(U)\Psi_z)_z + (\lambda\Phi)_{zz} - \lambda\Psi_z.$$

A straightforward calculation gives

$$\begin{aligned} & [(\Psi_t - s\Psi_z)^2 + g'(U)\Psi_z^2 + \Psi(\Psi_t - s\Psi_z) + \frac{1}{2}\Psi^2]_t + (\Psi_t - s\Psi_z)^2 \\ & + 2f'(U)\Psi_z(\Psi_t - s\Psi_z) + (g'(U) + sg'(U)_z)\Psi_z^2 + 4\lambda_z\Psi(\Psi_t - s\Psi_z) \\ & + \frac{3}{2}\lambda_z\Psi^2 + \lambda_{zz}\Psi\phi_z + 2\lambda_{zz}\phi_z(\Psi_t - s\Psi_z) + \{\dots\}_z \\ & = -F_{zz}[\Psi + 2(\Psi_t - s\Psi_z)] + J. \end{aligned} \quad (4.24)$$

By the smallness of $|u_+ - u_-|$

$$\begin{aligned} J & = [(g'(U)\Phi)_{zzz} - (g'(U)\Psi_z)_z] \cdot [\Psi + 2(\Psi_t - s\Psi_z)] \\ & \leq \frac{1}{3}|(\Phi_t, \Phi_z)|^2 + C|(\Phi, \Psi)|^2. \end{aligned}$$

Thus, noting $\Psi = \phi_{zz}$, we have from (4.24) that

$$\begin{aligned} & \|\partial_z^2 \phi(t)\|_1^2 + \|\partial_z^2 \phi_t\|^2 + \frac{1}{3} \int_0^t \|(\partial_z^2 \phi_t, \partial_z^2 \phi_z)(\tau)\|^2 d\tau - C \int_0^t (\|\partial_z^2 \phi\|^2 + \|\phi_z\|^2) d\tau \\ & \leq C \left\{ \|\phi_0\|_3^2 + \|\phi_1\|_2^2 + \left| \int_0^t \int F_{zz}(\partial_z^2 \phi + 2\partial_z^2(\phi_t - s\phi_z)) dz d\tau \right| \right\} \end{aligned} \quad (4.25)$$

where we have used the fact that $\partial_z^j \{\lambda, g'(U)\}$ is a smooth bounded function and the Young inequality for mixed terms such as $4\lambda_z \Psi(\Psi_t - s\Psi_z)$ and $2\lambda_{zz} \phi_z(\Psi_t - s\Psi_z)$. Combining successively the estimate (4.1), (4.23) and (4.25), we have

$$\begin{aligned} & \|\phi(t)\|_3^2 + \|\phi_t(t)\|_2^2 + \|\phi(t)\|_{H_w^1}^2 + \|\phi_t(t)\|_w^2 + \int_0^t \int |\lambda_z| \phi^2 dz d\tau + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau \\ & \leq C \left\{ \|\phi_0\|_3^2 + \|\phi_1\|_2^2 + \|\phi_0\|_{H_w^1}^2 + \|\phi_1\|_w^2 \right. \\ & \quad + \int_0^t \int (w|F|(|\phi| + |(\phi_t, \phi_z)|) + |F_z|(|\partial_z \phi| + |(\partial_z \phi_t, \partial_z \phi_z)|)) dz d\tau \\ & \quad \left. + \left| \int_0^t \int F_{zz}(\partial_z^2 \phi + 2\partial_z^2(\phi_t - s\phi_z)) dz d\tau \right| \right\}. \end{aligned} \quad (4.26)$$

Noting (3.5) (3.6), we have

$$\begin{aligned} |F_1| &= O(1)(\phi_z^2), \quad |F_{1z}, F_2| = O(1)(\phi_z^2 + \phi_{zz}^2), \\ |F_{1zz}, F_{2z}| &= O(1)(\phi_z^2 + \phi_{zz}^2 + |\phi_z \phi_{zzz}|), \\ |F_{2zz}| &= O(1)(|\phi_{zz} \phi_{zzz}| + |\phi_z \phi_{zzzz}|). \end{aligned}$$

Due to the nonlinearity of function g , there are terms containing fourth order derivatives of ϕ in F_{2zz} . In order to close the energy estimate, the last term on the right hand side of (4.26)

$$\left| \int_0^t \int F_{zz}(\partial_z^2 \phi + 2\partial_z^2(\phi_t - s\phi_z)) dz d\tau \right|$$

needs to be handled via proper integration by parts. As an example, we estimate the following sub-term

$$\begin{aligned} & \int_0^t \int 2G(U, \phi_z) \phi_z \phi_{zzzz} \phi_{zzt} dz d\tau \\ &= 2 \int_0^t \int ((G(U, \phi_z) \phi_z \phi_{zzz} \phi_{zzt})_z - G(U, \phi_z) \phi_z \phi_{zzz} \phi_{zzzt} \\ & \quad - (G(U, \phi_z) \phi_z)_z \phi_{zzz} \phi_{zzt}) dz d\tau \\ &= \int_0^t \int (G(U, \phi_z) \phi_z)_t \phi_{zzz}^2 - 2(G(U, \phi_z) \phi_z)_z \phi_{zzz} \phi_{zzt} dz d\tau \\ & \quad - \int G(U, \phi_z) \phi_z \phi_{zzz}^2(z, t) dz + \int G(U, \phi_z) \phi_z \phi_{zzz}^2(z, 0) dz \end{aligned}$$

where $G(U, \phi_z)$ is defined in (3.6). Other sub-terms can be estimated similarly. We get rid of the fourth order derivatives and the energy estimates will be closed at the derivatives up to third order. Thus by virtue of (3.10), the integrals on the right hand side of (4.26) is majored by

$$CN(t) \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau,$$

then we have

$$N^2(t) + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau + \int_0^t \int |\lambda_z| \phi^2 dz d\tau \leq N^2(0) + CN(t) \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau.$$

Therefore, by assuming $N(T) \leq \frac{1}{2C}$, we obtain the desired estimate

$$N^2(t) + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau \leq CN^2(0), \quad \text{for } t \in [0, T].$$

Thus the proof of Proposition 3.3 is completed.

Q.E.D.

REFERENCES

- [1] G. Q. Chen, C. D. Levermore and T. P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., 47, 787-830, 1993.
- [2] Dirk Helbing, *Traffic and related self-driven many-particle systems*, Rev. Modern Phys., 73, 1067-1141, 2001.
- [3] Shi Jin and Zhouping Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math., 48, 555-563, 1995.
- [4] W. L. Jin and H. M. Zhang, *The formation and structure of vehicle clusters in the Payne-Whitham traffic flow model*, Transportation Research, B. 37, 207-223, 2003.
- [5] Shi Jin and Markos A. Katsoulakis, *Hyperbolic systems with supercharacteristic relaxations and roll waves*, SIAM J. Appl. Math., 61, 273-292, 2000.
- [6] C. Jones, R. Gardner and T. Kapitula, *Stability of traveling waves for non-convex scalar viscous conservation laws*, Comm. Pure Appl. Math., 46, 505-526, 1993.
- [7] B. S. Kerner and P. Konhäuser, *Structure and parameters of clusters in traffic flow*, Phys. Rev. E 50, 54-83, 1994.
- [8] W. Knospe, Ludger Santen, Andreas Schadschneider and Michael Schreckenberg, *Single-vehicle data of highway traffic: Microscopic description of traffic phases*, Phys. Rev. E 65, 056133-1-16, 2002.
- [9] C. Lattanzio and P. Marcati, *The zero relaxation limit for the hydrodynamic Whitham traffic flow model*, J. Diff. Eqns., 141, 150-178, 1997.
- [10] T. Li, *Rigorous asymptotic stability of a CJ detonation wave in the limit of small resolved heat release*, Combustion Theory and Modelling, 1, 259-270, 1997.
- [11] T. Li, *Global solutions of nonconcave hyperbolic conservation laws with relaxation arising from traffic flow*, J. Diff. Eqns., 190, 131-149, 2003.
- [12] T. Li, *Nonlinear dynamics of traffic jams*, Physica D, to appear.
- [13] T. P. Liu, *Hyperbolic conservation laws with relaxation*, Commun. Math. Phys., 108, 153-175, 1987.
- [14] H. L. Liu, *Asymptotic stability of shock profiles for non-convex convection-diffusion equation*, Appl. Math. Lett., 10, 129-134, 1997.
- [15] H. L. Liu, *Relaxation Dynamics, Scaling Limits and Convergence of Relaxation Schemes*, in press, Springer Verlag, 2004.
- [16] H. L. Liu and J. Wang, *Asymptotic stability of traveling wave solutions of a hyperbolic system with relaxation terms*, Beijing Math., 2, 119-130, 1996.
- [17] H. L. Liu, C. W. Woo and T. Yang, *Decay rate for traveling waves of a relaxation model*, J. Diff. Eqns., 134, 343-367, 1997.
- [18] H. L. Liu, J. Wang and T. Yang, *Stability for a relaxation model with a nonconvex flux*, SIAM J. Math. Anal., 29, 18-29, 1998.
- [19] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves of scalar viscous conservation laws with non-convex nonlinearity*, Commun. Math. Phys., 165, 83-96, 1994.
- [20] M. Mei, *Stability of shock profiles for non-convex scalar conservation laws*, Math. Models Method Appl. Sci., 5, 279-296, 1995.
- [21] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Publ. Math. D'osay 78, 02 Dept. de Math., Paris-sud, 1978.
- [22] H. J. Payne, *Models of freeway traffic and control*, Simulation Councils Proc. Ser.: Mathematical Models of Public Systems, 1, 51-61, 1971.

- [23] S. Schochet, *The instant-response limit in Whitham's nonlinear traffic-flow model: uniform well-posedness and global existence*, *Asymptotic Analysis*, 1, 263-282, 1988.
- [24] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
- [25] C. J. Zhu, *Asymptotic behavior of solutions for p-system with relaxation*, *J. Differential Equations*, 180, 2, 273-306, 2002.