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THE STOCHASTIC TRANSPORT EQUATION DRIVEN BY LÉVY WHITE NOISE *

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Abstract. In this paper we demonstrate how concepts of white noise analysis can be used to give an explicit solution to a stochastic transport equation driven by Lévy white noise.

Key words. Lévy processes, white noise analysis, stochastic partial differential equations.

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1. Introduction

The theory of stochastic partial differential equations (SPDE's) has become increasingly important in the study of a vast number of random phenomena in natural sciences and mathematical finance. SPDE's driven by multiparameter Gaussian processes have been investigated by many authors. See e.g. [41], [25] and [23] to mention a few. However, from a modeling point of view the Gaussian setting is too special, since phenomena in porous media or pollution growth indicate the influence of noises of a different nature. SPDE's perturbed by Lévy processes can be e.g. found in [4], [32].

The object of this paper is to provide an application of the white noise framework, developed in [28], [29], to analyze an important type of SPDE's driven by multiparameter Lévy processes. More precisely we are interested in solving the stochastic transport equation driven by Lévy white noise, given by

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{1}{2}\sigma^2 \Delta U + \mathbf{W}_\phi(x) \diamond \nabla U + K(t,x) \diamond U + g(t,x);\ t>0, x\in\mathbb{R}^d \\ U(0,x) &= f(x);\ x\in\mathbb{R}^d. \end{aligned} \tag{1.1}$$

Here $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and ∇ is the gradient with respect to $x = (x_1,...,x_d) \in \mathbb{R}^d$. Further, σ is a constant and $K(t,x)$, $g(t,x)$, $f(x) \in (\mathcal{S})_{-1}$ are given stochastic distribution processes. The stochastic distribution space $(\mathcal{S})_{-1}$ is a Lévy version of the Kondratiev space, equipped with the product $\diamond : (\mathcal{S})_{-1} \times (\mathcal{S})_{-1} \longrightarrow (\mathcal{S})_{-1}$, which is a Lévy white noise analogue to the Wick product in the Gaussian case (see [23]). The process $\mathbf{W}_\phi(x)$ is the $d-$dimensional ϕ-smoothed Lévy white noise (see Section 2 for definitions). System (1.1) can be e.g. used as a stochastic model of environmental pollution (see [24]). In this model $U(t,x)$ can be thought of as the concentration of a pollutant that disperses at time t and location x in a turbulent medium. $\frac{1}{2}\sigma^2$ is the dispersion coefficient, $\mathbf{W}_\phi(x)$ describes the velocity of the medium, $K(t,x)$ is the leakage rate of the substance and $g(t,x)$ is the rate of increase of the concentration due to the deposits of the wasting substance. The function $f(x)$ models the initial concentration.

The theory of SPDE's driven by Gaussian white noise was initiated by the works of Gross [17], Daleckii [9], Malliavin [31], Fujisaki, Kallianpur, Kunita [13], Pardoux [36], Zakai [42] and Walsh [41].

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Even simple types of SPDE's as the stochastic Poisson equation do not possess solutions, which are regular stochastic processes, unless the dimension d is chosen to be low. Walsh [41] introduced a weak solution concept for the study of SPDE's. The approach, given by Walsh, is based on solutions $u(x,\omega)$ in the distributional sense, that is

$$\text{the map } (x \longmapsto u(x,\omega)) \text{ is a Sobolev distribution for a.a. } \omega. \tag{1.2}$$

Although the construction of Walsh supplies a useful tool for the study of linear SPDE's, its applicability to nonlinear equations is limited. However, one can utilize ideas of Colombeau's nonlinear theory of distributions to cope with certain types of nonlinear SPDE's (see [7] and [1]).

Another approach comes from white noise analysis (see e.g. [23]), where generalized solutions $u(x,\omega)$ are in the sense that

$$(\omega \longmapsto u(x,\omega)) \text{ is a stochastic distribution for a.a. } x. \tag{1.3}$$

An advantage of this method is that one can establish a theory of nonlinear operations on distributions to handle a wide class of nonlinear SPDE's by using the Wick product.

In this paper we will give a solution to system (1.1) in the sense of (1.3). We will apply the framework in [28] to show that (1.1) admits a unique strong solution $U(x,\omega)$ with values in the stochastic distribution space $(\mathcal{S})_{-1}$, which is a Lévy version of the Kondratiev space. The proof of this result rests on the use of the Hermite transform and the Feynman-Kac formula.

In the Gaussian case similar equations to (1.1), involving multiplication by means of Stratonovich integration instead of the Wick product, were investigated e.g. in [6], [33] and [37]. See also the work of [38], based on the Hitsuda-Skorohod interpretation. A more general method was given in [10]. The stochastic transport equation (1.1) driven by Gaussian white noise was treated in [14]. The latter, that is the Gaussian white noise analogue to (1.1), arose from a combination of two cases, studied in [14]and [24]. Our solution is a generalization of [15] to the case of Lévy white noise.

In Section 2 we recall some results and definitions in [28], where a white noise approach for the study of SPDE's driven by multi-parameter pure jump Lévy prossses is presented. Then in Section 3 we illustrate how the framework in Section 2 can be applied to solve the stochastic transport equation driven by a pure jump Lévy noise (Theorem 3.1). Finally, in Section 4 we explain the extension of Theorem 3.1 to the case of a combination of Gaussian and pure jump Lévy noise.

2. A white noise approach for Lévy processes

White noise analysis has proved to be a vital and important area of mathematics. The pioneering works of T. Hida (see [18] for an account) have stimulated a breathtaking development of white noise analysis and its applications. Particularly, it has served as a very useful tool in applications to mathematical physics. See [19] and the references therein. As another example of a variety of applications white noise theory has been successfully used in the study of stochastic partial differential equations (SPDE's). See e.g. [23], [20] and other researchers. More recently, white noise methods in connection with Malliavin calculus have fostered interesting applications to mathematical finance (see e.g. [3], [8] or [35]). See also [2].

In this section we review some white noise concepts for the study of stochastic partial differential equation driven by multi-parameter Lévy processes, developed in

[28], [29]. In Section 3 we will demonstrate how this framework can be utilized to give a solution to the stochastic transport equation driven by Lévy noise. The notational style of this paper is leaned on [23], where Gaussian white noise is investigated. For excellent treatments of general white noise theory the reader may consult [19], [27] and [34].

In this section we first focus on square integrable (d-parameter) pure jump Lévy processes without drift. In Section 4 we outline a framework for the study of combinations of such processes and multi-parameter Brownian motion.

A Lévy process can be paraphrased as a stochastic process $\eta(t)$ on $\mathbb{R}_+$ with independent and stationary increments starting at zero, i.e. $\eta(0)=0$. Such a process can be thought of as a random walk in continuous time. The probability law of $\eta(t)$ is infinitely divisible and its characteristic function is given by the famous Lévy-Khintchine formula, that is

$$E\exp(i\lambda\eta(t))=\exp(-t\Psi(\lambda));\quad \lambda\in\mathbb{R} \tag{2.1}$$

with characteristic exponent

$$\Psi(\lambda)=ia\lambda+\frac{1}{2}\sigma\lambda+\int_{\mathbb{R}_0}(1-e^{i\lambda y}+i\lambda y\chi_{\{|y|<1\}})\nu(dy),$$

for constants $a\in\mathbb{R}$ and $\sigma\geq 0$. The measure ν on $\mathbb{R}_0:=\mathbb{R}-\{0\}$ integrates the function $1\wedge y^2$ and is referred to as *Lévy measure*. Moreover, $\eta(t)$ has a decomposition

$$\eta(t)=at+\sigma B(t)+\int_0^t\int_{\mathbb{R}_0}y\chi_{\{|y|<1\}}\widetilde{N}(ds,dx)+\int_0^t\int_{\mathbb{R}_0}y\chi_{\{|y|\geq 1\}}N(ds,dy), \tag{2.2}$$

where $B(t)$ is the Brownian motion and where

$$\widetilde{N}(ds,dy)=N(ds,dy)-\nu(dy)ds$$

is the compensated Poisson random measure of $\eta(t)$. In particular, relation (2.2) shows that a square integrable pure jump Lévy martingale takes the form

$$\eta(t)=\int_0^t\int_{\mathbb{R}_0}y\widetilde{N}(ds,dy). \tag{2.3}$$

For a comprehensive account of the theory of Lévy processes we recommend the books of [5] and [39].

We start with an explicit construction of a (space-time) Lévy process $\eta(x),x\in\mathbb{R}^d$ on the white noise space $\widetilde{\mathcal{S}}(X)$, which is a multi-parameter version of (2.3). We recall the definition of $\widetilde{\mathcal{S}}(X)$ (see for details [28]). Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space on $\mathbb{R}^d$ and let $\mathcal{S}'(\mathbb{R}^d)$ be its dual (the space of tempered distribution). Since $\mathcal{S}(\mathbb{R}^d)$ is a (countably Hilbertian) nuclear space its topology can be induced by a sequence of increasing pre-Hilbertian norms $\|\cdot\|_p$, $p\in\mathbb{N}$ (see [16]). Moreover, the space $\mathcal{S}(\mathbb{R}^d)$ is a nuclear algebra, that is $\mathcal{S}(\mathbb{R}^d)$ forms a topological algebra with respect to the pointwise multiplication of functions. In the following we set $X=\mathbb{R}^d\times\mathbb{R}_0$ and we define the space $\mathcal{S}(X)$ as a nuclear subalgebra of $\mathcal{S}(\mathbb{R}^{d+1})$ (with respect to the restrictions of the norms $\|\cdot\|_p$) by

$$\mathcal{S}(X):=\left\{\varphi\in\mathcal{S}(\mathbb{R}^{d+1}):\varphi(z_1,...,z_d,0)=(\frac{\partial}{\partial z_{d+1}}\varphi)(z_1,...,z_d,0)=0\right\} \tag{2.4}$$

For the Lebesgue measure $\lambda^{\times d}$ on $\mathbb{R}^d$ and ν for a Lévy measure on $\mathbb{R}_0$, we set $\pi = \lambda^{\times d} \times \nu$ and we define the closed ideal $\mathcal{N}_\pi$ of $\mathcal{S}(X)$ by

$$\mathcal{N}_\pi := \{\phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0\} \tag{2.5}$$

Then we define the space $\widetilde{\mathcal{S}}(X)$ to be the quotient ring

$$\widetilde{\mathcal{S}}(X) = \mathcal{S}(X)/\mathcal{N}_\pi. \tag{2.6}$$

The space $\widetilde{\mathcal{S}}(X)$ is again a (countably Hilbertian) nuclear algebra with the compatible system of norms

$$\left\|\widehat{\phi}\right\|_{p,\pi} := \inf_{\psi \in \mathcal{N}_\pi} \|\phi + \psi\|_p, \quad p \in \mathbb{N}. \tag{2.7}$$

See [16]. We indicate by $\widetilde{\mathcal{S}}'(X)$ the dual of $\widetilde{\mathcal{S}}(X)$. By the famous Bochner-Minlos theorem there exists a unique probability measure μ on the Borel sets of $\widetilde{\mathcal{S}}'(X)$, characterized by the property that

$$\int_{\widetilde{\mathcal{S}}'(X)} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp\left(\int_X (e^{i\phi} - 1) d\pi \right) \tag{2.8}$$

holds for all $\phi \in \widetilde{\mathcal{S}}(X)$, where $\langle \omega, \phi \rangle = \omega(\phi)$ is the action of $\omega \in \widetilde{\mathcal{S}}'(X)$ on $\phi \in \widetilde{\mathcal{S}}(X)$. We call the probability measure μ on $\Omega = \widetilde{\mathcal{S}}'(X)$ *(pure jump) Lévy white noise probability measure.*

The measure μ is *non-degenerate* and fulfills the *first condition of analyticity* (see [28]), entailing the existence of *generalized Charlier polynomials* $C_n(\omega)$ (see [26] for the definition). The function α defined by $\alpha(\phi) = \log(1+\varphi) \bmod \mathcal{N}_\pi$, if $\phi = \widehat{\varphi}$ for $\varphi(x) > -1$ is holomorphic at zero and invertible. Then the exponential $\widetilde{e}(\phi, \omega) := \frac{\exp\langle \omega, \alpha(\phi) \rangle}{E_\mu[e^{\langle \omega, \alpha(\phi) \rangle}]}$ is the generating function of the generalized Charlier polynomials $C_n(\omega)$, i.e.

$$\widetilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \left\langle C_n(\omega), \phi^{\otimes n} \right\rangle, \tag{2.9}$$

for all ϕ in an open neighbourhood of zero in $\widetilde{\mathcal{S}}(X)$, where $\phi^{\otimes n} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$ (n-th completed symmetric tensor product of $\widetilde{\mathcal{S}}(X)$ with itself). We can identify the elements of $\widetilde{\mathcal{S}}(X)$ with functions $f \in \mathcal{S}(X^n)$ modulo $\mathcal{N}_{\pi^{\times n}}$ such that $f = f(x_1, ..., x_n)$ is symmetric with respect to the variables $x_1, ..., x_n \in X$. Based on relation (2.9) it can be deduced that the set $\{\langle C_n(\omega), \phi^{(n)} \rangle : \phi^{(n)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}, n \in \mathbb{N} \cup \{0\}\}$ is total in $L^2(\mu)$ and that the orthogonality relation

$$\int_{\widetilde{\mathcal{S}}'(X)} \left\langle C_n(\omega), \phi^{(n)} \right\rangle \left\langle C_m(\omega), \psi^{(m)} \right\rangle d\mu(\omega) = \delta_{n,m} n! (\phi^{(n)}, \psi^{(n)})_{L^2(X^n)} \tag{2.10}$$

holds, where $\phi^{(n)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} m}$. Using a density argument and the orthogonality relation (2.10) one sees the extensibility of the functional $\langle C_n(\omega), f_n \rangle$ from $f_n \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$ to symmetric $f_n \in L^2(X^n, \pi^{\times n})$. Note, that $C_1(\omega) = \omega - 1 \otimes \dot{\nu}$, where $1 \otimes \dot{\nu} \in \widetilde{\mathcal{S}}'(X)$ is given by $\left\langle 1 \otimes \dot{\nu}, \phi \right\rangle = \int_X \phi(x,y) \pi(dx, dy)$ (see [28]). Since the Lévy white noise measure μ has a Poissonian characteristic functional with intensity π, it is natural to define the stochastic integral of a $\phi \in L^2(\pi)$ with respect to the compensated Poisson random measure $\widetilde{N}$ associated with π as

$$\int_X \phi(x,y)\widetilde{N}(dx,dy) := \langle \omega - 1 \otimes \dot{\nu}, \phi \rangle . \tag{2.11}$$

The latter definition gives rise to an explicit construction of a *$d-$parameter (pure jump) Lévy process* or *space-time (pure jump) Lévy process*, denoted by $(\eta(x))_{x\in\mathbb{R}^d}$, which we choose to have right continuous paths with existing left limits in each component of $x=(x_1,...,x_d)$. That is we select a càdlàg version of the random field

$$\widetilde{\eta}(x) := \int_X \chi_{[0,x_1]\times\ldots\times[0,x_d]}(x)\cdot y\widetilde{N}(dx,dy) \text{ for } x=(x_1,...,x_d)\in\mathbb{R}^d, \tag{2.12}$$

where $[0,x_i]$ is understood to be $[x_i,0]$, if $x_i<0$ and where the Lévy measure $\nu(dy)$ is supposed to integrate y^2. Note that the dimension d of the parameter set shall admit the interpretation of either the time, space or space-time dimension of the system. We proceed to define the multidimensional version of a $d-$parameter Lévy process. For $m\in\mathbb{N}$ we consider the probability space

$$(\widetilde{\mathcal{S}}', \mathcal{B}, \mu_m) := (\times_{i=1}^m \widetilde{\mathcal{S}}'(X), \times_{i=1}^m \mathcal{B}(\widetilde{\mathcal{S}}'(X)), \times_{i=1}^m \mu). \tag{2.13}$$

The triplet $(\widetilde{\mathcal{S}}', \mathcal{B}, \mu_m)$ is called the *$d-$parameter multidimensional Lévy white noise probability space.* We define the *$m-$dimensional* ($d-$*parameter pure jump*) *Lévy process* $\eta(x)$ as a $m-$tupel of independent copies of the $1-$dimensional Lévy processes by the process

$$\eta(x,\omega) = (\eta(x,\omega_1),...,\eta(x,\omega_m)) \tag{2.14}$$

for $\omega=(\omega_1,..., \omega_m)\in\widetilde{\mathcal{S}}'$ on $(\widetilde{\mathcal{S}}', \mathcal{B}, \mu_m)$. Using (2.11) and (2.12) we introduce a stochastic process, called the *(d-parameter) smoothed white noise process* $W_\phi(x,\omega)$ by

$$W_\phi(x,\omega) := \int_{\mathbb{R}^d} \phi_x(u)d\eta(u), \tag{2.15}$$

where $\phi_x(u)=\phi(u-x)$ is the $x-$shift of $\phi\in\mathcal{S}(\mathbb{R}^d)$, $u,x\in\mathbb{R}^d$ and where the stochastic integral with respect to $\eta(u)$ is defined as

$$\int_{\mathbb{R}^d} \varphi(u)d\eta(u) = \langle \omega - 1\otimes\dot{\nu}, \varphi(u)\cdot y\rangle$$

for $\varphi\in L^2(\mathbb{R}^d)$. The *m-dimensional smoothed white noise process* $\mathbf{W}_\phi(x,\omega)$ is constructed as

$$\begin{aligned}\mathbf{W}_\phi(x,\omega) &= (W_\phi^{(1)}(x,\omega),...,W_\phi^{(m)}(x,\omega)) \\ &= (W_{\phi_1}(x,\omega_1),...,W_{\phi_m}(x,\omega_m)),\end{aligned} \tag{2.16}$$

for $\phi=(\phi_1,...,\phi_m)\in \times_{i=1}^m\mathcal{S}(\mathbb{R}^d)$, $\omega=(\omega_1,..., \omega_m)\in\widetilde{\mathcal{S}}'$. The process $\{\mathbf{W}_\phi(x,\omega)\}_{x\in\mathbb{R}^d}$ can be used as a mathematical model for (coloured) white noise, which one encounters in random phenomena. We need the definition of various spaces of stochastic test functions and stochastic distributions, based on chaos expansions. In the sequel we indicate by $\mathcal{J}$ the set of all multi-indices $\alpha=(\alpha_1,\alpha_2,...)$ with finitely many non-zero elements $\alpha_i \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Define $Index(\alpha)=\max\{i:\alpha_i\neq 0\}$ and $|\alpha| = \sum_i \alpha_i$ for $\alpha\in\mathcal{J}$. Denote by $\{\xi_k\}_{k\geq 1}$the Hermite functions (for its definition see e.g. [40]) and take a bijection $h:\mathbb{N}^d \longrightarrow \mathbb{N}$. Then choose an orthonormal basis $\{\zeta_k\}_{k\geq 1}$of $L^2(\mathbb{R}^d)$, by defining $\zeta_k(x_1,...,x_d)=\xi_{i_1}(x_1)\cdot ...\cdot \xi_{i_d}(x_d)$, if $k=h(i_1,...,i_d)$ for $i_j\in\mathbb{N}$. Further let$\{\pi_j\}_{j\geq 1}\subset \mathcal{S}(X)$, $d=0$ (or $\subset L^2(\nu)$) be an orthonormal basis of $L^2(\nu)$. Using the bijective map

$$\kappa:\mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N};\ (i,j)\longmapsto j+(i+j-2)(i+j-1)/2. \tag{2.17}$$

we define the function δ_k by

$$\delta_k(x,y)=\zeta_i(x)\pi_j(y), \tag{2.18}$$

if $k=\kappa(i,j)$ for $i,j\in\mathbb{N}$. Let $Index(\alpha)=j$ and $|\alpha|=m$ for $\alpha\in\mathcal{J}$ and introduce the function $\delta^{\otimes\alpha}$ as

$$\begin{aligned} &\delta^{\otimes\alpha}((x_1,y_1),...,(x_m,y_m)) \\ &=\delta_1^{\otimes\alpha_1}\otimes...\otimes\delta_j^{\otimes\alpha_j}((x_1,y_1),...,(x_m,y_m)) \\ &=\delta_1(x_1,y_1)\cdot...\cdot\delta_1(x_{\alpha_1},y_{\alpha_1}) \\ &\quad\cdot...\cdot\delta_j(x_{\alpha_1+...+\alpha_{j-1}+1},y_{\alpha_1+...+\alpha_{j-1}+1})\cdot...\cdot\delta_j(x_m,y_m), \end{aligned} \tag{2.19}$$

where we set $\delta_i^{\otimes 0}=1$. Then the *symmetrized tensor product* of the δ_k's, denoted by $\delta^{\widehat{\otimes}\alpha}$, is defined as the symmetrization of $\delta^{\otimes\alpha}$ with respect to the variables $(x_1,y_1),...,(x_m,y_m)$. Then, if we set

$$K_\alpha(\omega)=\left\langle C_{|\alpha|}(\omega),\delta^{\widehat{\otimes}\alpha}\right\rangle, \tag{2.20}$$

we obtain an orthogonal $L^2(\mu)$ basis $\{K_\alpha\}_{\alpha\in\mathcal{J}}$ $(K_0:=1)$. Now, for $\gamma=(\gamma^{(1)},...,\gamma^{(m)})\in \mathcal{J}^m := \mathcal{J}\times...\times\mathcal{J}$ and $\omega=(\omega_1,...,\omega_m)\in\widetilde{\mathcal{S}}'$ put

$$\mathbf{K}_\gamma(\omega)=\prod_{i=1}^m K_{\gamma^{(i)}}(\omega_i) \tag{2.21}$$

Then the family $\{\mathbf{K}_\gamma(\omega)\}_{\gamma\in\mathcal{J}^m}$ constitutes an orthogonal basis for $L^2(\mu_m)$. So every $F\in L^2(\mu_m)$ has the unique representation

$$F=\sum_{\gamma\in\mathcal{J}^m} c_\gamma \mathbf{K}_\gamma \tag{2.22}$$

for $c_\gamma\in\mathbb{R}$ with norm expression

$$\|F\|^2_{L^2(\mu_m)}=\sum_{\gamma\in\mathcal{J}^m}\gamma! c_\gamma^2, \tag{2.23}$$

where $\gamma! := \gamma^{(1)}!\gamma^{(2)}!...$ for $\gamma=(\gamma^{(1)},\gamma^{(2)},...)\in\mathcal{J}^m$ and where $\gamma^{(j)}!=\gamma_1^{(j)}!\ \gamma_2^{(j)}!...$, $j=1,...,m$. Using a second quantization argument we introduce the following stochastic test function and distribution spaces: for $0\leq\rho\leq 1$ the *Kondratiev test function space* $(\mathcal{S})_\rho$ consists of all $f=\sum_{\gamma\in\mathcal{J}^m} c_\gamma\mathbf{K}_\gamma \in L^2(\mu_m)$ such that

$$\|f\|^2_{\rho,k} := \sum_{\gamma\in\mathcal{J}^m}(\gamma!)^{1+\rho}c_\gamma^2(2\mathbb{N})^{k\gamma}<\infty, \tag{2.24}$$

for all $k\in\mathbb{N}_0$, where $(2\mathbb{N})^{k\gamma}=(2\mathbb{N})^{k\gamma^{(1)}}\cdot\ldots\cdot(2\mathbb{N})^{k\gamma^{(m)}}$ and $(2\mathbb{N})^{k\beta}=(2\cdot 1)^{k\beta_1}(2\cdot 2)^{k\beta_2}\ldots(2\cdot m)^{k\beta_l}$, if $Index(\beta)=l$.

Similarly, the *Kondratiev distribution space* $(\mathcal{S})_{-\rho}$ can be described as the set of all formal series $F=\sum_{\gamma\in\mathcal{J}^m} b_\gamma \mathbf{K}_\gamma$ such that

$$\|F\|^2_{-\rho,-k}:=\sum_{\gamma\in\mathcal{J}^m}(\gamma!)^{1-\rho}c_\gamma^2(2\mathbb{N})^{-k\gamma}<\infty \tag{2.25}$$

for a $k\in\mathbb{N}_0$. $(\mathcal{S})_\rho$ is endowed with the projective and $(\mathcal{S})_{-\rho}$ with the inductive topology, based on the above seminorms $\{\|\cdot\|_{\rho,k}\}$. The space $(\mathcal{S})_{-\rho}$ can be regarded as the dual of $(\mathcal{S})_\rho$ via the action

$$\langle F,f\rangle=\sum_{\gamma\in\mathcal{J}^m} b_\gamma c_\gamma \gamma! \tag{2.26}$$

for $F=\sum_{\gamma\in\mathcal{J}^m} b_\gamma \mathbf{K}_\gamma\in(\mathcal{S})_{-\rho}$ and $f=\sum_{\gamma\in\mathcal{J}^m} b_\gamma \mathbf{K}_\gamma\in(\mathcal{S})_\rho$. For general $0\leq\rho\leq 1$ we have by definition the following chain of inclusions

$$(\mathcal{S})_1\hookrightarrow(\mathcal{S})_\rho\hookrightarrow(\mathcal{S})_0\hookrightarrow L^2(\mu_m)\hookrightarrow(\mathcal{S})_{-0}\hookrightarrow(\mathcal{S})_{-\rho}\hookrightarrow(\mathcal{S})_{-1}$$

The spaces $(\mathcal{S})_0$ and $(\mathcal{S})_{-0}$ coincide with Lévy versions of the *Hida spaces* $(\mathcal{S})$ and $(\mathcal{S})^*$, respectively (see [19] and [23]). Next, if we choose in (2.18) a $L^2(\nu)$ basis $(\pi_j)_{j\geq 1}$ with $\pi_1(y)=y$, we can represent $\eta(t)$ as

$$\eta(x)=\sum_{k\geq 1} m_0\int_0^{x_d}\ldots\int_0^{x_1}\zeta_k(x_1,\ldots x_d)dx_1\ldots dx_d\cdot K_{\epsilon^{\kappa(k,1)}}, \tag{2.27}$$

where $m_0=\|y\|^2_{L^2(\nu)}$, the map κ is as in (2.17) and where $\epsilon^l\in\mathcal{J}$ is defined by

$$\epsilon^l(j)=\begin{cases}1 \text{ for } j=l\\ 0 \text{ else}\end{cases},l\geq 1. \tag{2.28}$$

Then relation (2.27) gives us the motivation for the definition of a *d-parameter (pure jump)* or *space-time Lévy white noise* $\overset{\bullet}{\eta}(x)=\frac{\partial^d}{\partial x_1\ldots\partial x_d}\eta(x)$ as a space-time derivative of $\eta(x)$ in $(\mathcal{S})^*$, introduced as the formal expansion

$$\overset{\bullet}{\eta}(x,\omega)=m_0\sum_{k\geq 1}\zeta_k(x)K_{\epsilon^{\kappa(k,1)}}\in(\mathcal{S})^* \text{ for all } x. \tag{2.29}$$

The $m-$*dimensional* ($d-$*parameter pure jump*) *Lévy white noise* $\overset{\bullet}{\eta}(x)$ can be defined by

$$\overset{\bullet}{\eta}(x,\omega)=(\overset{\bullet}{\eta}(x,\omega_1),\ldots,\overset{\bullet}{\eta}(x,\omega_m)),\quad \omega=(\omega_1,\ldots,\omega_m)\in\widetilde{\mathcal{S}}' \ . \tag{2.30}$$

A more general definition, which covers $\overset{\bullet}{\eta}(x)$, is the *(d−parameter) white noise* $\overset{\bullet}{\widetilde{N}}(x,y)$ of the Poisson random measure $\widetilde{N}(dx,dy)$ (see [35]). The Lévy noise $\overset{\bullet}{\eta}(x)$ can be viewed as a marginal case of the smoothed white noise $W_\phi(x,\omega)$ in the following sense: if $\phi_n dy$ tends to the Dirac measure δ_0 in the weak star topology in the space of measures on $\mathbb{R}^d$ for $n\longrightarrow\infty$, then $W_\phi(x,\omega)\longrightarrow\overset{\bullet}{\eta}(x)$ in $(\mathcal{S})^*$. This can be seen just as in the Gaussian case (see [23]). The properties of $W_\phi(x,\omega)$ serve as a useful tool in the study of stability questions of SPDE's (compare [23]). Further, we endow the Kondratiev spaces with the structure of a topological algebra with respect to the *Lévy Wick product,* $\diamond: (\mathcal{S})_{-\rho} \times (\mathcal{S})_{-\rho} \longrightarrow (\mathcal{S})_{-\rho}$, defined by

$$(K_\gamma \diamond K_\beta)(\omega) = (K_{\gamma+\beta})(\omega), \ \gamma,\beta \in \mathcal{J}^m \tag{2.31}$$

The product is linearly extended to the whole space. Note that e.g.

$$\langle C_n(\omega), f_n \rangle \diamond \langle C_m(\omega), g_m \rangle = \left\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \right\rangle \tag{2.32}$$

for symmetric functions $f_n \in L^2(\pi^{\times n})$ and $g_m \in L^2(\pi^{\times m})$ (see [28]). The Wick product reveals an interesting relation to Itô-Skorohod integration, that is

$$\int_0^T Y(x)\delta\eta(x) = \int_0^T Y(x) \diamond \overset{\bullet}{\eta}(x)dx. \tag{2.33}$$

The left hand side is a Skorohod integral of $Y(x) = Y(x,\omega)$ with respect to the Lévy process, satisfying the condition $E\int_0^T Y^2(x,\omega)dx < \infty$. The right hand side of (2.33) is in terms of a Bochner-integral on $(\mathcal{S})_{-1}$ (see [8] for definitions). The *Lévy Hermite transform* (see [28]) gives a description of the Kondratiev space $(\mathcal{S})_{-1}$ by means of power series in infinitely many complex variables. The definition of this transform utilizes the expansion along the basis $\{\mathbf{K}_\gamma\}_{\gamma\in\mathcal{J}^m}$, just as in the Gaussian case, which was initiated in [30]. Let $F = \sum_{\gamma\in\mathcal{J}^m} a_\gamma \mathbf{K}_\gamma \in (\mathcal{S})_{-1}$ with $a_\gamma \in \mathbb{R}$. The *Lévy Hermite transform of* F, denoted by $\mathcal{H}F$, is defined by

$$\mathcal{H}F(z) = \sum_{\gamma\in\mathcal{J}^m} a_\gamma z^\gamma \in \mathbb{C}, \tag{2.34}$$

where $z^\gamma = z_1^{\gamma^{(1)}} ... z_m^{\gamma^{(m)}}$, $z_j = (z_{j,1}, z_{j,2}, ...) \in \mathbb{C}^{\mathbb{N}}$ and $z_j^\alpha = z_{j,1}^{\alpha_1} z_{j,2}^{\alpha_2} ... z_{j,n}^{\alpha_n} ...$ for $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{J}$, $j = 1,...,m$ $(z_{i,j}^0 := 1))$ under the assumption that the series converges. The Hermite transform converges e.g. for some $0 < q, R < \infty$ on the infinite-dimensional neighborhood $\mathbb{K}_q(R)$ in $(\mathbb{C}^{\mathbb{N}})^m$, given by

$$\mathbb{K}_q(R) = \{\xi = (\xi_1, ..., \xi_m) \in (\mathbb{C}^{\mathbb{N}})^m : \sum_{\gamma\neq 0} |\xi^\gamma|^2 (2\mathbb{N})^{q\gamma} < R^2\}. \tag{2.35}$$

By the characterization theorem (see Theorem 2.3.8 in [28]) any element in $(\mathcal{S})_{-1}$ is uniquely determined through its $\mathcal{H}$-transform. For example the Hermite transform of the Lévy white noise $\overset{\bullet}{\eta}(x)$ is

$$\mathcal{H}(\overset{\bullet}{\eta})(x,z) = m_0 \sum_{k\geq 1} \zeta_k(x) \cdot z_{\kappa(k,1)}, \ z \in \mathbb{C}^{\mathbb{N}}. \tag{2.36}$$

The Hermite transform $\mathcal{H}$ is an algebra-homomorphism between $(\mathcal{S})_{-1}$ and the algebra of power series in infinitely many complex variables. In particular we have the relation

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}(F)(z) \cdot \mathcal{H}(G)(z). \tag{2.37}$$

Relation (2.37) naturally elicits an extension to Wick versions of complex analytical functions: if g has a Taylor expansion around $\xi_0 = \mathcal{H}(F)(0)$ with real coefficients, then the characterization theorem (see [28]) implies that there exists a unique $Y\in(\mathcal{S})_{-1}$ such that

$$\mathcal{H}(Y)(z)=(g\circ\mathcal{H}(F))(z) \tag{2.38}$$

holds. We define the Wick version $g^{\diamond}(F)$ of g applied to F by setting $g^{\diamond}(F)=Y$. As an example the Wick version of the exponential function exp can written as

$$\exp^{\diamond}(F)=\sum_{n\geq 0}\frac{1}{n!}F^{\diamond n}$$

with the n'th Wick power of F, given by $F^{\diamond n}=F\diamond F\diamond\ldots\diamond F$ (n times).

3. The stochastic transport equation perturbed by Lévy white noise

In this section we want to examine strong solutions of the generalized stochastic transport equation in the Kondratiev distribution space $(\mathcal{S})_{-1}$, that is we consider the SPDE

$$\begin{aligned}\frac{\partial U}{\partial t}&=\frac{1}{2}\sigma^2\Delta U+\mathbf{W}_{\phi}(x)\diamond\nabla U+K(t,x)\diamond U+g(t,x);\ t>0,\ x\in\mathbb{R}^d\\ U(0,x)&=f(x);\ x\in\mathbb{R}^d,\end{aligned} \tag{3.1}$$

where $\mathbf{W}_{\phi}(x)=(W_{\phi}^{(1)}(x),\ldots,\,W_{\phi}^{(d)}(x))$ is the $d-$dimensional $\phi-$smoothed Lévy white noise in (2.16), σ a constant and where $K(t,x)$, $g(t,x)$ and $f(x)$ are $(\mathcal{S})_{-1}-$valued stochastic processes.

For notational convenience we indicate by $\widetilde{X}$ the Hermite transform $\mathcal{H}(X)$ of a distribution $X\in(\mathcal{S})_{-1}$. In the following let us require that the processes $K:\mathbb{R}_+\times\mathbb{R}^d\longrightarrow(\mathcal{S})_{-1}$, $g:\mathbb{R}_+\times\mathbb{R}^d\longrightarrow(\mathcal{S})_{-1}$ and $f:\mathbb{R}^d\longrightarrow(\mathcal{S})_{-1}$ in (3.1) fulfill the conditions

(*i*) $\left|\widetilde{K}(t,x,z)\right|+|\widetilde{g}(t,x,z)|+\left|\widetilde{f}(x,z)\right|$ is uniformly bounded on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{K}_q(R)$ for some q and $R<\infty$.
(*ii*) For all $z\in\mathbb{K}_q(R)$ there exists a $\gamma=\gamma(z)\in(0,1)$ such that
$\widetilde{K}(\cdot,\cdot,z)\in C^{1,\gamma}(\mathbb{R}_+\times\mathbb{R}^d),\widetilde{g}(\cdot,\cdot,z)\in C^{1,\gamma}(\mathbb{R}_+\times\mathbb{R}^d)$ and $\widetilde{f}(\cdot,\cdot,z)\in C^{2+\gamma}(\mathbb{R}^d)$, (3.2)

where $C^{l,\gamma}(U)$ resp. $C^{k+\gamma}(U)$ (U an open set of $\mathbb{R}^n$)denotes the space of all $C(U)-$functions that are Hölder continuous of order γ and continuously differentiable with respect to the first variable resp. the space of all functions in $C^k(U)$, whose partial derivatives up to order k are Hölder continuous of order γ.

THEOREM 3.1. *Let K, g and f be processes as in (3.2). Then there exists a unique solution U of the system (3.1). Moreover, the solution is explicitly given by*

$$\begin{aligned}U(t,x)=\hat{E}^x\Bigg[\bigg(&f(\sigma B_t)\diamond\exp^{\diamond}[\int_0^t K(t-v,\sigma B_v)dv]\\&+\int_0^t g(t-s,\sigma B_v)dv\diamond\exp^{\diamond}[\int_0^s K(s-v,\sigma B_v)dv]ds\bigg)\diamond F_t^{\diamond}\Bigg],\end{aligned} \tag{3.3}$$

with

$$F_t^{\diamond}=\exp^{\diamond}[\sum_{k=1}^d\sigma^{-1}\int_0^t W_{\phi}^{(k)}(\sigma B_s)dB_s^{(k)}-\frac{1}{2}\sum_{k=1}^d\sigma^{-2}\int_0^t\left(W_{\phi}^{(k)}(\sigma B_s)\right)^{\diamond 2}ds]. \tag{3.4}$$

where $(B_t)_{t\geq 0}=(B_t^{(1)},...,B_t^{(d)})_{t\geq 0}$ *is a* $d-$*dimensional Brownian motion, starting at* x, *with probability law* $\hat{P}^x$ *on a filtered measurable space* $(\Omega^*,\mathcal{F}^*),\{\mathcal{F}_t^*\}_{t\geq 0}$ *and where the notation* $\hat{E}^x[F]$ *stands for a Bochner integral of an integrable* $F:\Omega^*\longrightarrow(\mathcal{S})_{-1}$ *with respect to the measure* $\hat{P}^x$.

Proof. In proving the result we proceed as in [14], [15], [21] and [22], where the Gaussian case is treated.

To determine a solution candidate of (3.1) we (formally) apply the Hermite transform to both sides of the equation and get the deterministic partial differential equation

$$\begin{aligned}\frac{\partial u}{\partial t}&=\frac{1}{2}\sigma^2\Delta u+\widetilde{\mathbf{W}}_\phi(x)\cdot\nabla u+\widetilde{K}(t,x)\cdot u+\widetilde{g}(t,x);\ t>0,\ x\in\mathbb{R}^d\\ u(0,x)&=\widetilde{f}(x);\ x\in\mathbb{R}^d,\end{aligned}\tag{3.5}$$

where $u=u(t,x,z)=\widetilde{U}(t,x,z)$ for $z\in(\mathbb{C}_c^{\mathbb{N}})^d$. The space $(\mathbb{C}_c^{\mathbb{N}})^d$ denotes the set of complex-valued sequences $\beta=(\beta_1,\beta_2,...)$ with finitely many non-zero entries. We want to solve equation (3.5), by using the Feynman-Kac formula. For this purpose consider first the real part of system (3.5), i.e. assume that $z=\alpha=(\alpha_1,...,\alpha_m)\in(\mathbb{R}_c^{\mathbb{N}})^d$. Then define the second-order differential operator

$$A^\alpha=\sum_{k=1}^d\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_k^2}+\sum_{k=1}^d\widetilde{W}_\phi^{(k)}(x,\alpha)\frac{\partial}{\partial x_k}.\tag{3.6}$$

Using (3.6) we can reformulate (3.5) as

$$\begin{aligned}-\frac{\partial u}{\partial t}+A^\alpha u+\widetilde{K}u&=-\widetilde{g};\ t>0,x\in\mathbb{R}^d\\ u(0,x)&=\widetilde{f}(x);\ x\in\mathbb{R}^d.\end{aligned}\tag{3.7}$$

The stochastic differential equation, which is used to give a stochastic representation of the solution of the Cauchy problem (3.7), takes the form

$$dX_t^\alpha=\widetilde{\mathbf{W}}_\phi(X_t^\alpha,\alpha)dt+\sigma dB_t;\ t\geq 0,X_0^\alpha=0.\tag{3.8}$$

on a filtered probability space $(\Omega^*,\mathcal{F}^*,P^*),\{\mathcal{F}_t^*\}_{t\geq 0}$. Since the k'th component of $\widetilde{\mathbf{W}}_\phi(x,\alpha)$, namely $\widetilde{W}_\phi^{(k)}(x,\alpha)$, can be represented as

$$\widetilde{W}_\phi^{(k)}(x,\alpha)=\sum_{j\geq 1}(\phi_k(\cdot-x),\zeta_j)_{L^2(\mathbb{R}^d)}\alpha_{k,\kappa(j,1)}\tag{3.9}$$

with κ in (2.17) and ζ_j in (2.18), one verifies that $\widetilde{\mathbf{W}}_\phi(x,\alpha)$ is Lipschitz continuous and satisfies a linear growth condition. Hence (3.8) has a unique strong solution $X_t^\alpha=X_t^{\alpha,x}$. Then the Feynman-Kac theorem (see [12]) in connection with the Girsanov transformation provides the existence of a unique solution u to (3.7) with

$$\begin{aligned}u(t,x,\alpha) &= E^x_{P^*}\Big[\widetilde{f}(X^\alpha_t,\alpha)\cdot\exp[\int_0^t \widetilde{K}(t-v,X^\alpha_v,\alpha)dv] \\ &\quad + \int_0^t g(t-s,X^\alpha_s,\alpha)\cdot\exp[\int_0^s \widetilde{K}(s-v,X^\alpha_v,\alpha)dv]ds\Big] \\ &= \hat{E}^x\Big[\Big(\widetilde{f}(\sigma B_t,\alpha)\cdot\exp[\int_0^t \widetilde{K}(t-v,\sigma B_v,\alpha)dv] \\ &\quad + \int_0^t \widetilde{g}(t-s,\sigma B_s,\alpha)\cdot\exp[\int_0^s \widetilde{K}(s-v,\sigma B_v,\alpha)dv]ds\Big)\cdot F_t\Big], \qquad (3.10)\end{aligned}$$

where

$$F_t = \exp[\sum_{k=1}^d \sigma^{-1}\int_0^t \widetilde{W}^{(k)}_\phi(\sigma B_s,\alpha)dB^{(k)}_s - \frac{1}{2}\sum_{k=1}^d \sigma^{-2}\int_0^t \Big(\widetilde{W}^{(k)}_\phi(\sigma B_s,\alpha)\Big)^2 ds]. \qquad (3.11)$$

We observe in (3.10) that $u(t,x,\alpha)$ is real analytic in $\alpha \in (\mathbb{R}^{\mathbb{N}}_c)^d$. Thus $u(t,x,\alpha)$ admits a complex analytical extension to a function $u(t,x,z)$, $z \in (\mathbb{C}^{\mathbb{N}}_c)^d$. Next we verify the existence of a unique $U:\mathbb{R}_+\times\mathbb{R}^d \longrightarrow (\mathcal{S})_{-1}$ with $\mathcal{H}U = u$, by invoking (2.38) for $g = id$. By the characterization theorem for $(\mathcal{S})_{-1}$ in [28] boundedness of $u(t,x,z)$ in $z \in \mathbb{K}_q(R)$ for some q,R provides a sufficient condition for the validity of (2.38). However, taking (3.2) (i) into consideration, one infers the latter condition from the representation of u in the first equation of (3.10). By comparing the Hermite transforms we find that the process $U(t,x)$ coincides with the one in (3.3).

In the final step of the proof we check that U actually solves system (3.1). Theorem 2.78 in [11] yields the following estimate for $u(t,x)$ on every open and bounded set $G = (0,T)\times D \subset \mathbb{R}_+\times\mathbb{R}^d$:

$$\|u\|_{C^{1,2+\gamma}(G)} \le const.\cdot\Big(\|Lu\|_{C^{1,\gamma}(G)} + \Big\|\widetilde{f}\Big\|_{C^{2+\gamma}(\partial D)}\Big),$$

where L is the differential operator, given by

$$Lu(t,x) = \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2\Delta u - \widetilde{W}_\phi\cdot\nabla u - \widetilde{K}(t,x)u.$$

Since $Lu(t,x,z) = \widetilde{g}(t,x,z)$, the result follows from (3.2) and the next Lemma 3.2. □

Lemma 3.2. *Let G be a bounded open subset of $\mathbb{R}_+\times\mathbb{R}^d$. Assume a process $U:G\longrightarrow(\mathcal{S})_{-1}$ with $\mathcal{H}U = u$ such that u and its partial derivatives $\frac{\partial u}{\partial t}, (\frac{\partial u}{\partial x_j})_{j=1,\dots,d}, (\frac{\partial^2 u}{\partial x_j^2})_{j=1,\dots,d}$ are bounded on $G\times\mathbb{K}_q(R)$, continuous with respect to $(t,x)\in G$ for all $z\in\mathbb{K}_q(R)$ and analytic in $z\in\mathbb{K}_q(R)$ for all $(t,x)\in G$, $q<\infty$, $R>0$. Then*

$$\mathcal{H}(\frac{\partial U}{\partial t}) = \frac{\partial u}{\partial t},\ \mathcal{H}(\Delta U) = \Delta u \text{ and } \mathcal{H}(\nabla U) = \nabla U$$

on $\mathbb{K}_q(R)$.

Proof. The proof of the Lemma is based on the same arguments as in Lemma 2.8.4 of [23]. We give the proof for the derivative $\frac{\partial u}{\partial t}$. The case of higher order derivatives can be obtained by rerunning the arguments.

The mean value theorem implies that

$$\frac{1}{\Delta t}(u(t+\Delta t,x,z)-u(t,x,z))=\frac{\partial u}{\partial t}(t+\xi\Delta t,x,z)$$

for some $\xi\in[0,1]$, for all $z\in\mathbb{K}_q(R)$. Because of the assumptions on u we conclude that

$$\frac{1}{\Delta t}(u(t+\Delta t,x,z)-u(t,x,z))\longrightarrow\frac{\partial u}{\partial t}(t,x,z) \text{ as } \Delta t\to 0$$

pointwise boundedly for $z\in\mathbb{K}_q(R)$. Since $(\mathcal{S})_1$ is a nuclear space, one can show as in Theorem 2.8.1 of [23] that the latter statement is equivalent to convergence in $(\mathcal{S})_{-1}$. That is

$$\frac{1}{\Delta t}(U(t+\Delta t,x)-U(t,x))\longrightarrow\frac{\partial U}{\partial t}(t,x)$$

for all (t,x). Since the Hermite transform is a continuous linear functional on $(\mathcal{S})_{-1}$, the result follows. □

REMARK 3.3. *System (3.1) reduces to the heat equation with stochastic potential K, when $\phi=0$ and $g=0$. In this case the solution in Theorem 3.1 simplifies to*

$$U(t,x)=\hat{E}^x[(f(\sigma B_t)\diamond\exp^{\diamond}[\int_0^t K(t-s,\sigma B_s)ds]]. \tag{3.12}$$

The stochastic heat equation was studied in [33], where K was chosen to be the Gaussian white noise. In [22] this equation was treated in the case of Gaussian Kondratiev spaces.

COROLLARY 3.4. *Consider the stochastic heat equation in Remark 3.3 for the potential term $K(t,x,\omega)=\int_X\varphi(s,y)\widetilde{N}(ds,dy),\varphi(s,y)>0$ with deterministic initial condition f as in (3.2). Then there exists a unique $L^2(\mu)$-solution of the heat equation, which takes the explicit form*

$$\begin{aligned}
&U(t,x)\\
&=\hat{E}^x[(f(\sigma B_t)\exp^{\diamond}[t\int_X\varphi(s,y)\widetilde{N}(ds,dy)]]\\
&=\hat{E}^x[(f(\sigma B_t)]\cdot\\
&\quad\cdot\exp[t\int_X\log(1+\varphi(s,y))\widetilde{N}(ds,dy)+t\int_X\log(1+\varphi(s,y))-\varphi(s,y)\pi(ds,dy)].
\end{aligned}$$

Proof. Since relation (2.8) implies that

$$\widetilde{e}(\varphi,\omega)=\exp^{\diamond}[\int_X\varphi(s,y)\widetilde{N}(ds,dy)],$$

we conclude by means of (3.13) the statement of the corollary. □

4. The stochastic transport equation driven by a mixture of Gaussian and pure jump Lévy noise

Let us briefly describe how the framework in Section 2 can be extended to the case of combinations of m Gaussian white noise and k pure jump Lévy white noise sources. For notational simplicity we restrict ourselves to the case $m=k=1$. We proceed as in the multidimensional pure jump Lévy noise case. Denote by μ_G the Gaussian white noise measure on the measurable space

$$(\Omega_G,\mathcal{F}_G)=(\mathcal{S}'(\mathbb{R}^d),\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))),$$

that is μ_G is defined as the unique probability measure on Ω_G such that

$$\int_{\Omega_G} e^{i\langle\omega,\phi\rangle}\mu_G(d\omega)=\exp(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d)})$$

for all $\phi\in\mathcal{S}(\mathbb{R}^d)$, where $\langle\omega,\phi\rangle=\omega(\phi)$ (see [23]). Recall that the stochastic polynomials $\{H_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$, given by

$$H_\alpha(\omega)=\prod_{j\geq 1} h_{\alpha_j}(\langle\omega,\zeta_j\rangle)$$

form an orthogonal $L^2(\mu_G)-$basis, where h_n is the n'th Hermite polynomial and ζ_j is a tensor product of j Hermite functions (see [23] for details). Let us indicate by μ_L the pure jump Lévy white noise measure on

$$(\Omega_L,\mathcal{F}_L)=(\widetilde{\mathcal{S}}'(X),\mathcal{B}(\widetilde{\mathcal{S}}'(X))).$$

Define the *Lévy white noise measure* μ to be the product measure $\mu_G\times\mu_L$ on the measurable space

$$(\Omega,\mathcal{F})=(\Omega_G\times\Omega_L,\mathcal{F}_G\otimes\mathcal{F}_L).$$

Further set

$$L_\gamma(\omega_1,\omega_2)=H_\alpha(\omega_1)K_\beta(\omega_2)$$

for $\gamma=(\alpha,\beta)\in\mathcal{J}^2$. Then we see that any $f\in L^2(\mu)$ can be uniquely written as

$$f(\omega_1,\omega_2)=\sum_{\gamma\in\mathcal{J}^2} c_\gamma L_\gamma(\omega_1,\omega_2),$$

Moreover, we have the isometry

$$\|f\|^2_{L^2(\mu)}=\sum_{\gamma\in\mathcal{J}^2} c_\gamma^2\gamma!$$

with $\gamma!=\alpha!\beta!$, if $\gamma=(\alpha,\beta)\in\mathcal{J}^2$. The definitions and results of Section 2, regarding the Kondratiev spaces, the Wick product and the Hermite transform carry over to the combined case of Gaussian and pure jump Lévy noise. If we denote by $\overset{\bullet}{B}$ the Gaussian noise (see [23]) and by $\overset{\bullet}{\eta}_L$ the pure jump Lévy white noise, then the *white noise* of the Lévy process $\eta=\sigma B+\eta_L$ can be introduced as

$$\overset{\bullet}{\eta}(\omega_1,\omega_2)=\sigma\overset{\bullet}{B}(\omega_1)+\overset{\bullet}{\eta}_L(\omega_2).$$

In the same way as in the last section we define the *(d-parameter) smoothed white noise process* by

$$\mathbf{W}_\phi(x,\omega_1,\omega_2) = (\langle \omega_1, \phi_1(u-x)\rangle, \left\langle \omega_2 - 1 \otimes \overset{\bullet}{\nu}, \phi_2(u-x)y \right\rangle)$$

for $\phi = (\phi_1,\phi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. With the above definitions one checks that Theorem 3.1 also holds for the stochastic transport equation driven by Gaussian and pure jump white Lévy white noise.

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