Unitarity of the Knizhnik–Zamolodchikov–Bernard Connection and the Bethe Ansatz for the Elliptic Hitchin systems

Fernando Falceto[1], Krzysztof Gawędzki[2]

[1] Depto. Física Teórica, Univ. Zaragoza, E-50009 Zaragoza, Spain
[2] I.H.E.S., C.N.R.S., F-91440 Bures-sur-Yvette, France

Abstract: We work out finite-dimensional integral formulae for the scalar product of genus one states of the group G Chern–Simons theory with insertions of Wilson lines. Assuming convergence of the integrals, we show that unitarity of the elliptic Knizhnik–Zamolodchikov–Bernard connection with respect to the scalar product of CS states is closely related to the Bethe Ansatz for the commuting Hamiltonians building up the connection and quantizing the quadratic Hamiltonians of the elliptic Hitchin system.

1. Introduction

The present paper continues the program [17, 10, 19, 20] aimed at analysis of the scalar product of states in the Chern–Simons (CS) theory. It extends the considerations of ref. [9] where we treated the $SU(2)$ CS theory on the elliptic curve (times the time-line and with insertions of time-like Wilson lines) to the case of a general group G. As in the previous papers of the series, the point is to express the formal scalar product of the CS theory, given by a functional integral over gauge fields, as a multiple finite-dimensional integral. The latter, if convergent for every state, provides the space of CS states W with a Hilbert space structure and the holomorphic vector bundle $\mathscr{W}$, obtained by varying the modulus τ of the elliptic curve and the positions z_n of insertions, with a hermitian structure.

The integral expressions for the scalar product of the CS states are close cousins of the contour integral expressions for the conformal blocks of the corresponding WZW conformal theory. In the elliptic case, the contour integral representations were recently studied in ref. [12]. Our approach elucidates the origin of the complicated expressions which appear in such representations for a general group: they are induced by a simple trick, already used in [10], which handles a change of variables in the functional integral.

The WZW conformal blocks are holomorphic sections θ of the bundle $\mathscr{W}$ of the CS state spaces satisfying in the elliptic case the Knizhnik–Zamolodchikov–Bernard

(KZB) equations [23, 3]

$$\left(\partial_\tau + \frac{1}{\kappa}H_0\right)\theta = 0, \qquad \left(\partial_{z_n} + \frac{1}{\kappa}H_n\right)\theta = 0, \quad n = 1,\ldots,N.\tag{1}$$

Above, $H_n \equiv H_n(\tau,\underline{z})$ are operators acting on the CS states and κ is a coupling constant. The KZB equations may be interpreted as the horizontality equations for a KZB connection in bundle $\mathcal{W}$. The consistency of the equations (the flatness of the KZB connection) requires that the operators H_n, $n \geq 0$, commute for fixed $(\tau,\underline{z})$. In fact, H_n, $n \geq 1$, are quantum versions of the quadratic classical Hamiltonians h_n of the elliptic Hitchin system which Poisson-commute. In the case of an elliptic curve with no insertions the conformal blocks coincide with the (linear combinations of) characters of the integrable representations of the affine algebras. H_0 in Eq. (1) is then proportional to the Laplacian on the Cartan algebra and the KZB equations reduce to the well known heat equation for the elliptic theta functions. In the case of an elliptic curve with one insertion, $H_0(\tau)$ becomes a version of the Calogero–Sutherland Hamiltonian [6]. For many insertions, operators $H_n(\tau,\underline{z})$, $n \geq 1$, are elliptic versions of the Gaudin Hamiltonians [15].

A hermitian structure on a holomorphic vector bundle induces a unique unitary connection of the $1,0$ type. One expects [18] that the scalar product of the CS states induces this way the KZB connection. For the elliptic case with no insertions, that was shown implicitly in [21] where it was proven that the affine characters form an orthonormal basis of the space of CS states. For group $SU(2)$ in the elliptic case with one insertion the unitarity of the KZB connection w.r.t. the scalar product of the CS states was proven in [9]. In the present paper we study this problem for the general elliptic situation. We show that, assuming convergence of the integrals giving the scalar product, the unitarity of the elliptic KZB connection follows from a result announced in [12] which provides a basis for the Bethe–Ansatz diagonalization of the commuting operators $H_n(\tau,\underline{z})$, $n \geq 0$. The relation of the integral representations for the genus zero conformal blocks of the WZW theory to the Bethe Ansatz was observed in [1], see also [2, 26, 11]. The elliptic $SU(2)$ case counterparts of these relations go back to XIXth century works of Hermite on the Lamé operator, as noticed in [6]. Recently, the integral representations for the elliptic conformal blocks were used in [12] to obtain the Bethe–Ansatz treatment of the elliptic Calogero–Sutherland model, an open problem till then. Our work exhibits an intrinsic connection between the Bethe–Ansatz and the unitarity of the KZB connection.

The paper is organized as follows. In Sect. 2, we describe briefly the space of CS states in the holomorphic quantization and identify the states on the elliptic curve as vector-valued theta-functions. Section 3 is devoted to the scalar product of the elliptic CS states. As realized in [21] for the case with no insertions, the formal functional integral over the gauge fields giving the scalar product may be computed as an iterative Gaussian integral. The insertions for general group G are handled by combining the methods of refs. [21, 10]. In Sect. 4, we describe the KZB connection and recall its relations to the Hitchin integrable systems. Section 5 discusses the unitarity of the KZB connection and finally, in Sect. 6, we consider the relations to the Bethe Ansatz.

2. Elliptic CS States

Let us recall the description of the CS states on the elliptic curve $\mathcal{T}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with insertion points z_n, given in [8, 9], with due modifications required by the replacement of the group $SU(2)$ by G. For simplicity, we assume G to be simple, connected and simply connected and will denote by $\mathbf{g}$ its Lie algebra. Let $\mathscr{A}$ denote the (complex) vector space of the 0,1-components $A \equiv A_{\bar z} d\bar z$ of the $\mathbf{g}$-valued gauge fields. The group $\mathscr{G}$ of complex gauge transformations $g : \mathcal{T}_\tau \to G^{\mathbb{C}}$ acts on $\mathscr{A}$ by

$$A \mapsto {}^g A = gAg^{-1} + g\bar\partial g^{-1}, \tag{2}$$

with $\bar\partial = d\bar z \partial_{\bar z}$. The Chern–Simons states Ψ are holomorphic functionals on $\mathscr{A}$. For the case with insertions of time-like Wilson lines, they take values in the tensor product $\otimes_n V_{\lambda_n} \equiv \underline{V}$ of the irreducible representation spaces of G of highest weights λ_n, associated to the insertions. States Ψ verify the chiral Ward identity

$$\Psi({}^g A) = e^{kS(g^{-1}, A)} \otimes_n g(z_n)_{(n)} \Psi(A), \tag{3}$$

where $S(g, A)$ is the action of the Wess–Zumino–Witten model coupled to A, see [8], and the subsubscript (n) indicates that the group element acts in the factor V_{λ_n} of the tensor product space $\underline{V}$.

Restricting functionals Ψ to connections $A_u = \pi u d\bar z/\tau_2$ for u in the Cartan algebra $\mathbf{h}^{\mathbb{C}}$ and $\tau_2 \equiv \operatorname{Im}\tau$, we can assign to every state a holomorphic map $\gamma : \mathbf{h}^{\mathbb{C}} \to \underline{V}$ related to Ψ by the equation

$$\Psi(A_u) = e^{\pi k |u|^2/(2\tau_2)} \otimes_n (e^{\pi(z_n - \bar z_n)u/\tau_2})_{(n)} \gamma(u), \tag{4}$$

where $|u|^2 \equiv \langle u, u \rangle$ with the Killing form $\langle \cdot, \cdot \rangle$ normalized so that $|\alpha|^2 = 2$ for long roots α (we identify $\mathbf{g}$ with its dual). Holomorphic maps γ corresponding to states Ψ satisfy the following conditions:

$$\gamma(u + q^\vee) = \gamma(u) \quad \text{for } q^\vee \text{ in the coroot lattice } Q^\vee, \tag{5.a}$$

$$\gamma(u + \tau q^\vee) = e^{-\pi i k \langle q^\vee, \tau q^\vee + 2u \rangle} \otimes_n (e^{-2\pi i z_n q^\vee})_{(n)} \gamma(u), \tag{5.b}$$

$$0 = \oplus_n (h)_{(n)} \gamma(u) \quad \text{for } h \in \mathbf{h}, \tag{5.c}$$

$$\gamma(wuw^{-1}) = \otimes_n (w)_n \gamma(u) \quad \text{for } w \in G \text{ normalizing } \mathbf{h}, \tag{5.d}$$

and

$$\left(\sum_n e^{-2\pi i s z_n} (e_\alpha)_{(n)} \right)^p \gamma(u + tp^\vee) = \mathscr{O}(t^p) \tag{6}$$

for any root α, coweight $p^\vee$ satisfying $\langle p^\vee, \alpha \rangle = 1$, u s.t. $\langle u, \alpha \rangle = m + \tau s$ with m, s integers, $p = 1, 2, \ldots$ and $t \to 0$. Here and below e_α is the step generator of $\mathbf{g}^{\mathbb{C}}$ corresponding to root α.

We shall denote the space of maps $\gamma(u)$ satisfying the above conditions by $W_{\tau, z, \lambda}$. Conditions (5.a) and (5.b) mean that γ is a vector of theta-functions which due to condition (5.c) take values in the zero weight subspace $\underline{V}_0 \subset \underline{V}$ and which by

condition (5.d) are Weyl group covariant. Equations (6) specify their regularity on the hyperplanes $\langle u, \alpha \rangle \in \mathbb{Z} + \tau \mathbb{Z}$, where $\alpha \in \mathbf{h}$ are roots of the algebra $\mathbf{g}$. They have been obtained for $G = SU(2)$ in [8] and for general G in [7], see also [14]. They follow by demanding the regularity at $t = 0$ of the maps

$$t \mapsto \Psi(^{h_{ms}}A_{u',t}) = \Psi(^{g_{ms}}A_{u+tp^\vee}),$$

where $t \mapsto A_{u',t} = \frac{\pi}{\tau_2}(u' + tp^\vee + e_\alpha)d\bar{z}$, with $u' = u - (m + \tau s)p^\vee$, is a 1-parameter holomorphic family of gauge fields, $h_{ms}(z) = \exp[\frac{\pi}{\tau_2}((m + \bar{\tau}s)z - (m + \tau s)\bar{z})p^\vee]$ is a multivalued gauge transformation and $g_{ms}(z) = h_{ms}(z)\exp[-t^{-1}e_\alpha]h_{ms}^{-1}(z)$ is a uni-valued one. Since $\langle u', \alpha \rangle = 0$, $A_{u'+tp^\vee}$ may be gauge transformed to $A_{u',t}$ for $t \neq 0$ by a constant gauge transformation $\exp[t^{-1}e_\alpha]$. Gauge fields $^{h_{ms}}A_{u',0}$ lie on codimension one strata in the space of gauge fields which cannot be attained by gauge transform-ing A_u's. Conditions (6) assure the global regularity of Ψ but, due to the properties (5.a–d), they are not independent for different m and s. For example, for all simple groups different from $SU(2)$ or $SO(2r + 1)$ it is enough to take $m = s = 0$.

3. Scalar Product of CS States

The scalar product of Chern–Simons states is formally given by the functional integral

$$\|\Psi\|^2 = \int |\Psi(A)|^2 e^{-\frac{ik}{2\pi}\int \operatorname{tr}(A^*A)}\, DADA^*. \tag{7}$$

We shall perform the above functional integration reducing the scalar product expression to a finite-dimensional integral which, if finite, will provide $W_{\tau,z,\lambda}$ with a natural structure of a Hilbert space. The finiteness has been proven in the special cases (general G with no insertions [21] or $SU(2)$ with one insertion [9]) and has been conjectured to hold in general [18]. The strategy for the calculation of the integral (7) will be as in [21] and [9] so we shall be brief discussing in detail only the treatment of the insertions essentially borrowed from [10].

3.1. Change of variables. We shall reparametrize the gauge fields in the functional integral (7) as

$$A = {}^{g^{-1}}A_u \tag{8}$$

with g a complex gauge transformation and u in some fundamental domain of the ac-tion of translations $u \mapsto u + q^\vee$, $u \mapsto u + \tau q^\vee$ on $\mathbf{h}^{\mathbb{C}}$. ($\mathbf{h}^{\mathbb{C}}/(Q^\vee + \tau Q^\vee)$/Weyl group is the space of gauge orbits of an open dense set of semistable gauge fields; ignoring the Weyl group action produces an overall factor in the scalar product equal to the order of the Weyl group.) We shall use the Iwasawa decomposition of the $G^{\mathbb{C}}$-valued field g:

$$g = \exp\left[\sum_{\alpha > 0} v_\alpha e_\alpha\right] \exp[\phi/2]\, U \equiv n \exp[\phi/2]\, U \equiv bU, \tag{9}$$

where ϕ takes values in the Cartan algebra $\mathbf{h}$ and U in the compact group G. Upon the change of variables, the field U decouples from the functional integral (7) due to the gauge invariance leaving us with a WZW-type functional integral over the

$G^{\mathbb{C}}/G$-valued fields b [21]

$$\|\Psi\|^2 = \int \langle \Psi(A_u), \otimes_n (bb^*)^{-1}_{(n)}(z_n) \Psi(A_u) \rangle$$

$$\times e^{kS(bb^*, A_u + A_u^*) - \frac{ik}{2\pi} \int \mathrm{tr}(A_u^* A_u)} \, j(u,b) \delta(\phi(0)) \, Db \, d^2u, \qquad (10)$$

where $Db = \prod_z db(z)$ is the formal product of $G^{\mathbb{C}}$-invariant measures on $G^{\mathbb{C}}/G$, the delta function fixes the remaining freedom in the parametrization (8) and the Jacobian of the change of variables

$$j(u,b) = \mathrm{const.} \; \tau_2^{-2r} e^{2h^\vee S(bb^*, A_u + A_u^*)} \det(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}).$$

Above, r stands for the rank of G, $h^\vee$ for its dual Coxeter number and $\bar{\partial}_{A_u} \equiv \bar{\partial} + [A_u, \cdot]$. The last determinant was computed in [21]:

$$\det(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}) = \mathrm{const.} \; \tau_2^{2r} e^{\pi h^\vee |u - u^*|^2/\tau_2} |\Pi(u)|^4,$$

where Π is the Kac–Weyl denominator:

$$\Pi(u) = e^{\pi i d\tau/12} \prod_{\alpha > 0} (e^{\pi i \langle u, \alpha \rangle} - e^{-\pi i \langle u, \alpha \rangle}) \prod_{l > 0} (1 - q^l)^r \prod_\alpha (1 - q^l e^{2\pi i \langle u, \alpha \rangle})$$

with d denoting the dimension of the group and $q \equiv e^{2\pi i \tau}$. The WZW action in the parametrization (9) takes the form

$$S(bb^*, A_u + A_u^*) = -\frac{i}{4\pi} \int \langle \partial\phi, \bar{\partial}\phi \rangle - \frac{i}{2\pi} \int \langle e^{\phi_u} (n_u^{-1} \bar{\partial} n_u)^* e^{-\phi_u}, n_u^{-1} \bar{\partial} n_u \rangle,$$

where

$$n_u = e^{-\pi(z - \bar{z})u/\tau_2} n \, e^{\pi(z - \bar{z})u/\tau_2}, \qquad \phi_u = \phi - \pi(z - \bar{z})(u - \bar{u})/\tau_2.$$

Note that

$$n_u = \exp\left[\sum_{\alpha > 0} e^{-\pi\langle u, \alpha \rangle(z - \bar{z})/\tau_2} v_\alpha e_\alpha \right] \equiv \exp\left[\sum_{\alpha > 0} v'_\alpha e_\alpha \right].$$

In terms of the Iwasawa variables, the invariant measure on $G^{\mathbb{C}}/G$ is

$$db = \prod_{j=1}^r d\phi^j \prod_{\alpha > 0} d^2(e^{-\langle \phi, \alpha \rangle/2} v_\alpha),$$

where $\phi^j = \langle h_j, \phi \rangle$ are the coordinates of ϕ w.r.t. an orthonormal basis (h_j) of $\mathbf{h}$. Using the holomorphic functions $\gamma(u)$ to represent Ψ and the parametrization (9), we obtain

$$\left\langle \Psi(A_u), \prod_n (bb^*)^{-1}_{(n)}(z_n) \Psi(A_u) \right\rangle = e^{\frac{\pi k}{\tau_2} \mathrm{Re}(|u|^2)} \left\langle \gamma(u), \otimes_n ((n_u^*)^{-1} e^{-\phi_u} n_u^{-1})_{(n)}(z_n) \gamma(u) \right\rangle.$$

Finally,

$$-\frac{ik}{2\pi} \int \mathrm{tr}(A_u^* A_u) = -\pi k \langle \bar{u}, u \rangle/\tau_2.$$

With all these ingredients,

$$\|\Psi\|^2 = \text{const.} \int \langle \gamma(u), \otimes_n ((n_u^*)^{-1} e^{-\phi_u} n_u^{-1})(n)(z_n) \gamma(u) \rangle$$

$$\times \exp\left[-\frac{i(k+2h^\vee)}{4\pi} \int \langle \partial\phi, \bar\partial\phi \rangle\right]$$

$$\times \exp\left[-\frac{i(k+2h^\vee)}{2\pi} \int \langle e^{\phi_u}(n_u^{-1}\bar\partial n_u)^* e^{-\phi_u}, n_u^{-1}\bar\partial n_u \rangle\right.$$

$$\left. + \frac{\pi(k+2h^\vee)}{2\tau_2}|u-\bar u|^2\right]$$

$$\times |\Pi(u)|^4 \delta(\phi(0)) d^{2r}u \prod_{j=1}^r D\phi^j \prod_{\alpha>0,z} d^2(e^{-\langle \phi_u(z),\alpha\rangle/2} v_\alpha'(z)). \qquad (11)$$

In order to render the n_u-dependent terms in the action of the last functional integral quadratic, we shall introduce new variables $(\eta_\alpha')_{\alpha>0}$ defined by

$$n_u^{-1}\bar\partial n_u = \sum_{\alpha>0} \bar\partial\eta_\alpha' e_\alpha. \qquad (12)$$

Since $n_u(z+1) = n_u(z)$ and $n_u(z+\tau) = e^{-2\pi i u} n_u(z) e^{2\pi i u}$, it follows that

$$\eta_\alpha'(z+1) = \eta_\alpha'(z) \qquad \eta_\alpha'(z+\tau) = e^{-2\pi i\langle u,\alpha\rangle} \eta_\alpha'(z) \qquad (13)$$

and that the change of variables is well defined since $\bar\partial$ is invertible (for generic u) on functions satisfying the periodicity conditions (13). It is easy to see that the change of variables $(v_\alpha') \mapsto (\eta_\alpha')$ has a triangular nature: $\eta_\alpha' = v_\alpha' + F_\alpha((v_\beta')_{\beta<\alpha})$, where $\beta < \alpha$ if $\alpha - \beta$ is a positive root. Hence, the formal volume element does not change:

$$\prod_{\alpha>0,z} d^2(e^{-\langle \phi_u(z),\alpha\rangle/2} v_\alpha'(z)) = \prod_{\alpha>0,z} d^2(e^{-\langle \phi_u(z),\alpha\rangle/2} \eta_\alpha'(z)).$$

Although the action in the functional integral (11), which was polynomial in variables v_α' (and their derivatives) becomes quadratic in (derivatives of) η_α', there persists the n_u-dependence of the insertions in the functional integral (11). Thus, we have to invert relation (12) in order to express n_u as a function of $\eta' \equiv \sum_{\alpha>0} \eta_\alpha' e_\alpha$. This will be done generalizing a trick of [10].

Let $\mathbf{n} = \bigoplus_{\alpha>0} \mathbb{C}e_\alpha$ be the nilpotent subalgebra of $\mathbf{g}^\mathbb{C}$. The corresponding group $N \subset G^\mathbb{C}$ may be mapped into the enveloping algebra $\mathscr{U}(\mathbf{n})$ (no completion problem arises if we work in highest weight irreducible representations of $G^\mathbb{C}$). $\mathscr{U}(\mathbf{n})$ is graded by the positive cone Q_+ in the root lattice Q:

$$\mathscr{U}(\mathbf{n}) = \bigoplus_{q\in Q_+} \mathscr{U}(\mathbf{n})_q$$

according to the eigenvalues of the adjoint action of $\mathbf{h}$, with $e_{\alpha_1}\cdots e_{\alpha_m} \in \mathscr{U}(\mathbf{n})_{\alpha_1+\cdots+\alpha_m}$. The map $z \to n_u^{-1}(z) \equiv v(z)$ may be viewed as taking values in $\mathscr{U}(\mathbf{n})$ and it satisfies then the twisted periodicity conditions,

$$v_q(z+1) = v_q(z), \qquad v_q(z+\tau) = e^{-2\pi i\langle u,q\rangle} v_q(z), \qquad (14)$$

where v_q denotes the $\mathcal{U}(\mathbf{n})_q$-component of v. Relation (12) may be now rewritten as an equation

$$\partial_{\bar{z}} n_u^{-1} = -(\partial_{\bar{z}}\eta') n_u^{-1},$$

for $\mathcal{U}(\mathbf{n})$-valued functions and solved with the use of the Green functions of the twisted $\partial_{\bar{z}}$-operator. Since $(n_u^{-1})_0 = 1$, we obtain

$$n_u^{-1} = \sum_{K=0}^{\infty} (-1)^K \partial_{\bar{z}}^{-1}(\partial_{\bar{z}}\eta') \cdots \underbrace{\partial_{\bar{z}}^{-1}(\partial_{\bar{z}}\eta')}_{K \text{ times}} \cdot 1. \tag{15}$$

The Green function of $\partial_{\bar{z}}$ acting on functions obeying conditions (14) may be easily expressed by the Jacobi theta function $\vartheta_1(z) = \sum_l (-1)^l e^{\pi i(l+1/2)^2 \tau + 2\pi i z(l+1/2)}$ satisfying

$$\vartheta_1(z+1) = -\vartheta_1(z), \qquad \vartheta_1(z+\tau) = -e^{-\pi i(\tau+2z)}\vartheta_1(z). \tag{16}$$

Explicitly, it is equal to

$$P_x(z) = \frac{\vartheta_1'(0)\vartheta_1(x+z)}{\pi\vartheta_1(x)\vartheta_1(z)} \tag{17}$$

with $x = \langle u, q\rangle$. Hence Eq. (15) takes the form

$$n_u^{-1}(z) = \sum_{K=0}^{\infty} (-1)^K \sum_{(\alpha_1,\dots,\alpha_K)} \int P_{\langle u, \alpha_1+\cdots+\alpha_K\rangle}(z-y_1)(\partial_{\bar{z}}\eta'_{\alpha_1})(y_1) P_{\langle u, \alpha_2+\cdots+\alpha_K\rangle}(y_1-y_2)$$

$$\times (\partial_{\bar{z}}\eta'_{\alpha_2})(y_2) \cdots P_{\langle u, \alpha_K\rangle}(y_{K-1}-y_K)(\partial_{\bar{z}}\eta'_{\alpha_K})(y_K) e_{\alpha_1} \dots e_{\alpha_K} d^2 y_1 \cdots d^2 y_K. \tag{18}$$

In the $\gamma(u)$ matrix element, we may insert the partition of unity $\otimes_n (\sum_{a_n} |\mu_{a_n}, a_n\rangle \langle \mu_{a_n}, a_n|)$, where vectors $|\mu_{a_n}, a_n\rangle$ corresponding to weights μ_{a_n} form an orthonormal basis in the representation spaces V_{λ_n}. This gives

$$\langle \gamma(u), \otimes_n ((n_u^*)^{-1} e^{-\phi_u} n_u^{-1})_{(n)}(z_n)\gamma(u)\rangle$$

$$= \sum_{\underline{a}} \prod_n e^{-\langle \phi_u(z_n), \mu_{a_n}\rangle} |\otimes_n \langle \mu_{a_n}, a_n| \otimes_n n_u^{-1}(z_n)\gamma(u)\rangle|^2$$

into which we may insert the expressions (18) for $n_u^{-1}(z_n)$. As in [10], the final renormalization of the functional integral over the Cartan algebra-valued field ϕ will kill most of the terms obtained this way leaving only the ones with $|\mu_{a_n}, a_n\rangle$ equal to the highest weight vectors $|\lambda_n\rangle$ and the sequences of positive roots $(\alpha_1, \dots, \alpha_p)$ composed uniquely of simple roots. We may then write

$$\langle \gamma(u), \otimes_n ((n_u^*)^{-1} e^{-\phi_u} n_u^{-1})_{(n)}(z_n)\gamma(u)\rangle = \prod_n e^{-\langle \phi_u(z_n), \lambda_n\rangle} \left| \sum_{\underline{K}} \sum_{\underline{\alpha}} \int F_{\underline{K},\underline{\alpha}}(\tau, u, \underline{z}, \underline{y}) \right.$$

$$\times \left. \prod_n \prod_{i=1}^{K_n} (\partial_{\bar{z}}\eta'_{\alpha_{n,i}})(y_{n,i}) d^2 y_{n,i} \langle \lambda | \otimes_n (e_{\alpha_{n,1}} \cdots e_{\alpha_{n,K_n}})\gamma(u)\rangle \right|^2 + \cdots, \tag{19}$$

where $\langle\lambda| \equiv \otimes_n\langle\lambda_n|$, $\underline{\alpha} = (\alpha_{1,1},\ldots,\alpha_{1,K_1},\alpha_{2,1},\ldots,\alpha_{N,K_N}) \equiv (\alpha_1,\ldots,\alpha_K)$ is a sequence of $K = \sum_n K_n$ simple roots satisfying

$$\sum_{s=1}^{K}\alpha_s = \sum_{n=1}^{N}\lambda_n\,, \tag{20}$$

$\underline{y} = (y_{1,1},\ldots,y_{1,K_1},y_{2,1},\ldots,y_{N,K_N}) \equiv (y_1,\ldots,y_K)$ is a sequence of K points in $\mathcal{T}_\tau$ and the kernels $F_{\underline{p},\underline{\alpha}}$ are composed of the Green functions of twisted $\bar{\partial}$

$$F_{\underline{K},\underline{\alpha}}(\tau,u,\underline{z},\underline{y}) = \prod_n P_{\langle u,\alpha_{n,1}+\cdots+\alpha_{n,K_n}\rangle}(z_n - y_{n,1})P_{\langle u,\alpha_{n,2}+\cdots+\alpha_{n,K_n}\rangle}(y_{n,1} - y_{n,2})$$
$$\times \cdots \times P_{\langle u,\alpha_{n,K_n}\rangle}(y_{n,K_n-1} - y_{n,K_n})\,. \tag{21}$$

"$\cdots$" in Eq. (19) contains the terms that will drop under the renormalization of the ϕ-integral.

3.2. Functional integration.
The above use of Eq. (15) to express the η' dependence of the insertions reduces the η' integral to the form

$$\int \prod_{s=1}^{K}(\partial_{\bar{z}}\eta'_{\alpha_s})(y_s)\overline{(\partial_{\bar{z}}\eta'_{\beta_s})(v_s)} \exp\left[\frac{-i(k+2h^\vee)}{2\pi}\sum_{\alpha>0}\int e^{-\langle\phi_u,\alpha\rangle}\overline{\partial}\eta'_\alpha\partial\eta'_\alpha\right]$$
$$\times \prod_{\alpha>0,z}d^2(e^{-\langle\phi_u,\alpha\rangle/2}\eta'_\alpha)$$
$$= \left(\frac{\pi}{k+2h^\vee}\right)^K\sum_{\sigma\in S_K}\prod_{s=1}^{K}e^{\langle\phi_u(y_s),\alpha_s\rangle}\delta_{\alpha_s,\beta_{\sigma(s)}}\delta(y_s - v_{\sigma(s)})\prod_{\alpha>0}\det(\bar{\partial}_\alpha^\dagger\bar{\partial}_\alpha)^{-1} \tag{22}$$

with σ running over the permutations of K points and with $\bar{\partial}_\alpha \equiv e^{-\langle\phi_u,\alpha\rangle/2}\bar{\partial}e^{\langle\phi_u,\alpha\rangle/2}$. The determinants are well known

$$\prod_{\alpha>0}\det(\bar{\partial}_\alpha^\dagger\bar{\partial}_\alpha)^{-1} = \text{const.}\,e^{-\pi r\tau_2/6}|\Pi(u)|^{-2}\prod_{l=1}^{\infty}|1-q^l|^{2r}$$
$$\times \exp\left[\frac{ih^\vee}{4\pi}\int\langle\partial\phi,\bar{\partial}\phi\rangle - \frac{\pi h^\vee}{2\tau_2}|u-\bar{u}|^2\right]\,.$$

After the η'_α-integration and an easy combinatorial manipulation, see [10], trading the sum over root sequences $\underline{\alpha}$ into sums over permutations (two $\underline{\alpha}$'s satisfying Eq. (20) differ necessarily only by a permutation), the scalar product formula (11) becomes

$$\|\Psi\|^2 = \text{const.}\,e^{-\pi r\tau_2/6}\prod_{l=1}^{\infty}|1-q^l|^{2r}\int\prod_{n=1}^{N}e^{-\langle\phi_u(z_n),\lambda_n\rangle}\prod_{s=1}^{K}e^{\langle\phi_u(y_s),\alpha_s\rangle}$$
$$\times \exp\left[-\frac{i\kappa}{4\pi}\int\langle\partial\phi,\bar{\partial}\phi\rangle + \frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2\right]\delta(\phi(0))\prod_{j=1}^{r}D\phi^j\,|\Pi(u)|^2$$
$$\times \left|\sum_{\underline{K}}\sum_{\sigma\in S_K}F_{\underline{K},\sigma\underline{\alpha}}(\tau,u,\underline{z},\sigma\underline{y})\langle\lambda|\otimes_n(e_{(\sigma\alpha)_{n,1}}\cdots e_{(\sigma\alpha)_{n,K_n}})_{(n)}\gamma(u)\rangle\right|^2 d^{2r}u\prod_{s=1}^{K}d^2y_s\,, \tag{23}$$

where $\kappa \equiv k + h^\vee$ and $\underline{\alpha}$ is a fixed sequence of K simple roots satisfying (20).

The remaining ϕ-integral is of the Gaussian (Coulomb gas) type and may be easily performed:

$$\int e^{\int \Sigma_j \phi^j(y) f^j(y)} \exp\left[-\frac{i\kappa}{4\pi}\int \langle \partial\phi, \bar{\partial}\phi\rangle\right] \delta(\phi(0)) \prod_{j=1}^r D\phi^j$$

$$= \text{const. } \tau_2^{r/2} \det'(\bar{\partial}^\dagger\bar{\partial})^{-r/2} e^{-\frac{1}{2(k+hV)}\Sigma_j \int\int f^j(y)\mathscr{G}(y-y')f^j(y')}$$

$$= \text{const. } \tau_2^{-r/2} e^{\pi r\tau_2/6} \prod_{l=1}^\infty |1-q^l|^{-2r} e^{-\frac{1}{2(k+hV)}\Sigma_j \int\int f^j(y)\mathscr{G}(y-y')f^j(y')}, \quad (24)$$

where we should take $f = -\sum_n \lambda_n \delta_{z_n} + \sum_s \alpha_s \delta_{y_s}$, which corresponds in the Coulomb gas jargon to external charges at points z_n and screening charges at points y_s. The Green function is

$$\mathscr{G}(z) = \ln|\vartheta_1(z)|^2 + \frac{\pi}{2\tau_2}(z-\bar{z})^2 + \text{const.} \quad (25)$$

Renormalizing the divergences due to the singularities at coinciding points by point-splitting, as explained in [10 and 9], we finally obtain for the scalar product of CS states the following finite-dimensional integral expression:

$$\|\Psi\|^2 = \text{const. } \tau_2^{-r/2} \int e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left|e^{-\frac{1}{\kappa}S(\tau,\underline{z},\underline{y})}\langle G(\tau,u,\underline{z},\underline{y}), \theta(u)\rangle\right|^2 d^2u \prod_{s=1}^K d^2y_s, \quad (26)$$

where $\theta(u) \equiv \Pi(u)\gamma(u)$,

$$w \equiv u + \frac{1}{\kappa}\sum_{n=1}^N z_n\lambda_n - \frac{1}{\kappa}\sum_{s=1}^K y_s\alpha_s. \quad (27)$$

S is a multivalued holomorphic function of τ and non-coincident z_n's and y_s's and G a holomorphic multivalued map with values in the dual space $\underline{V}^*$ of $\underline{V}$ given, respectively, by

$$S(\tau,\underline{z},\underline{y}) = \sum_{n<n'} \langle\lambda_n,\lambda_{n'}\rangle \ln\tilde{\vartheta}_1(z_n-z_{n'}) - \sum_{n,s}\langle\lambda_n,\alpha_s\rangle \ln\tilde{\vartheta}_1(z_n-y_s)$$

$$+ \sum_{s<s'}\langle\alpha_s,\alpha_{s'}\rangle \ln\tilde{\vartheta}_1(y_s-y_{s'}) \quad (28)$$

with $\tilde{\vartheta}_1(z) \equiv \vartheta_1(z)/\vartheta_1'(0)$ and

$$G(\tau,u,\underline{z},\underline{y}) = \sum_{\underline{K}}\sum_{\sigma\in S_K} F_{\underline{K},\sigma\underline{\alpha}}(\tau,u,\underline{z},\sigma\underline{y})\langle\lambda|\otimes_n(e_{(\sigma\alpha)_{n,1}}\cdots e_{(\sigma\alpha)_{n,K_n}})(n)\cdot \quad (29)$$

Using the transformation properties (16), it is straightforward to verify that for $H \equiv e^{-\frac{1}{\kappa}S}\langle G,\theta\rangle$, $\underline{\delta}^n = (0,\ldots\underset{n}{1}\ldots,0)$ and $\underline{\delta}^s$ defined similarly,

$$H(\tau,u+q^\vee,\underline{z},\underline{y}) = H(\tau,u,\underline{z},\underline{y}),$$

$$H(\tau,u+\tau q^\vee,\underline{z},\underline{y}) = e^{-\pi i\kappa\langle q^\vee,\tau q^\vee+2w\rangle}H(\tau,u,\underline{z},\underline{y}),$$

$$H(\tau,u,\underline{z}+\underline{\delta}^n,\underline{y}) = (-1)^{\langle\lambda_n,\lambda_n\rangle/\kappa}H(\tau,u,\underline{z},\underline{y}),$$

$$H(\tau, u, \underline{z} + \tau \underline{\delta}^n, \underline{y}) = (-1)^{\langle \lambda_n, \lambda_n \rangle / \kappa} e^{-\pi i \tau \langle \lambda_n, \lambda_n \rangle / \kappa + 2\pi i \langle w, \lambda_n \rangle} H(\tau, u, \underline{z}, \underline{y}),$$

$$H(\tau, u, \underline{z}, \underline{y} + \underline{\delta}^s) = (-1)^{\langle \alpha_s, \alpha_s \rangle / \kappa} H(\tau, u, \underline{z}, \underline{y}),$$

$$H(\tau, u, \underline{z}, \underline{y} + \tau \underline{\delta}^s) = (-1)^{\langle \alpha_s, \alpha_s \rangle / \kappa} e^{-\pi i \tau \langle \alpha_s, \alpha_s \rangle / \kappa - 2\pi i \langle w, \alpha_s \rangle} H(\tau, u, \underline{z}, \underline{y}). \qquad (30)$$

It is then easy to see that the under-integral expression in (26) is univalued under the $u \mapsto u + (\tau) q^\vee$, $z_n \mapsto z_n + (\tau) 1$, $y_s \mapsto y_s + (\tau) 1$ transformations.

It is useful to compare the above expression for the scalar product of the CS states with the genus zero ones obtained in [10]. The genus zero states are determined by their values $\Psi(0) \equiv \gamma$ at $A = 0$ belonging to the G-invariant subspace $\underline{V}^G$ of the tensor product $\underline{V}$ of the representation spaces. Adapting the notations to the ones of the present paper, Eqs. (19.20) of [10] read

$$\| \Psi \|^2 = \text{const.} \int \left| e^{-\frac{1}{\kappa} S^0(\underline{z}, \underline{y})} \langle G^0(\underline{z}, \underline{y}), \gamma \rangle \right|^2 \prod_{s=1}^{K} d^2 y_s, \qquad (31)$$

where

$$S^0(\underline{z}, \underline{y}) = \sum_{n < n'} \langle \lambda_n, \lambda_{n'} \rangle \ln(z_n - z_{n'}) - \sum_{n,s} \langle \lambda_n, \alpha_s \rangle \ln(z_n - y_s)$$

$$+ \sum_{s < s'} \langle \alpha_s, \alpha_{s'} \rangle \ln(y_s - y_{s'}) \qquad (32)$$

and

$$G^0(\underline{z}, \underline{y}) = \sum_{\underline{k}} \sum_{\sigma \in S_K} F_{\underline{K}}(\underline{z}, \sigma \underline{y}) \langle \lambda | \otimes_n (e_{(\sigma\alpha)_{n,1}} \cdots e_{(\sigma\alpha)_{n,K_n}})_{(n)} \qquad (33)$$

with

$$F_{\underline{K}}(\underline{z}, \underline{y}) = \frac{1}{\pi^K} \prod_n \frac{1}{z_n - y_{n,1}} \frac{1}{y_{n,1} - y_{n,2}} \cdots \frac{1}{y_{n,K_n-1} - y_{n,K_n}}. \qquad (34)$$

The similarity to the genus one case is obvious.

4. Knizhnik–Zamolodchikov(–Bernard) Connection and the Hitchin Systems

The trivial bundle with the fiber $\underline{V}$ over the space $X_N^0 \equiv \mathbb{C}^N \backslash \Delta$, where Δ contains vectors $\underline{z}$ with coinciding components carries a 1-parameter family of flat holomorphic connections defined by

$$\nabla_{\bar{z}_n} = \partial_{\bar{z}_n}, \qquad \nabla_{z_n} = \partial_{z_n} + \frac{1}{\kappa} H_n^0(\underline{z}),$$

where $H_n^0(\underline{z})$ are the Gaudin Hamiltonians [15]:

$$H_n^0(\underline{z}) = \sum_{a=1}^{d} \sum_{n' \neq n} \frac{t_{(n)}^a t_{(n')}^a}{z_{n'} - z_n} \qquad (35)$$

with (t^a) forming an orthonormal basis of Lie algebra $\mathbf{g}$. Connection ∇ appeared (implicitly) for the first time in ref. [23]: it was shown there that the genus zero conformal blocks γ of the WZW theory satisfy the (Knizhnik–Zamolodchikov) equations $\nabla\gamma = 0$. In fact, the WZW conformal blocks are horizontal sections of a (generally proper) subbundle $\mathcal{W}^0 \subset X_N^0 \times \underline{V}^G$. The fibers W_z^0 of $\mathcal{W}^0$ may be identified by the assignment $\Psi \to \Psi(0)$ with the genus zero CS state spaces. The subbundle $\mathcal{W}^0$ may be described by giving explicit algebraic conditions, depending holomorphically on $\underline{z}$, on the invariant tensors in $\underline{V}^G$ [29, 16]. The KZ connection ∇ preserves the subbundles $X_N^0 \times \underline{V}^G$ and $\mathcal{W}^0$ of the trivial bundle $X_N^0 \times \underline{V}$.

The extension of the Knizhnik–Zamolodchikov connection to the genus one case was first obtained in ref. [3] and elaborated further in [8, 6, 14, 12]. We shall use the description of the genus one CS states by the theta functions $\theta(u) = \Pi(u)\gamma(u)$. The spaces $W_{\tau,\underline{z}}$ of states form a holomorphic bundle over the space of pairs $(\tau, \underline{z})$ with no coincidences in z_n's viewed as points in $\mathcal{T}_\tau$. The holomorphic sections of $\mathcal{W}$ correspond to holomorphic families $(\tau, \underline{z}) \to \theta(\tau, \underline{z}, \cdot)$. The KZB connection is given by the formulae

$$\nabla_{\bar{\tau}} = \partial_{\bar{\tau}}, \qquad \nabla_{\bar{z}_n} = \partial_{\bar{z}_n}, \qquad \nabla_{\tau} = \partial_{\tau} + \frac{1}{\kappa}H_0(\tau, \underline{z}), \qquad \nabla_{z_n} = \partial_{z_n} + \frac{1}{\kappa}H_n(\tau, \underline{z}), \quad (36)$$

compare to Eqs. (1). Explicitly [6, 14, 12],

$$H_0(\tau, \underline{z}) = \frac{i}{4\pi}\Delta_u + \sum_{n,n'=1}^{N}\left(\frac{i}{4}\sum_\alpha \partial_x P_{\langle u,\alpha\rangle}(z_n - z_{n'})(e_\alpha)_{(n)}(e_{-\alpha})_{(n')}\right.$$

$$\left. + \frac{i}{8\pi}\sum_{j=1}^{r}(\rho(z_n - z_{n'})^2 + \rho'(z_n - z_{n'}))h_{(n)}^j h_{(n')}^j\right), \qquad (37)$$

$$H_n(\tau, \underline{z}) = -\sum_{j=1}^{r}h_{(n)}^j \partial_{u^j} - \sum_{n' \neq n}\left(\sum_\alpha \pi P_{\langle u,\alpha\rangle}(z_n - z_{n'})(e_\alpha)_{(n)}(e_{-\alpha})_{(n')}\right.$$

$$\left. + \sum_{j=1}^{r}\rho(z_n - z_{n'})h_{(n)}^j h_{(n')}^j\right), \qquad (38)$$

where $\Delta_u \equiv \sum_j(\partial_{u^j})^2$ and $\rho \equiv \vartheta_1'/\vartheta_1$. The expressions for H_n, $n \geq 1$, reduce to the ones for the Gaudin Hamiltonians (35) in the limit $\tau \to i\infty$. The operators $H_n, n \geq 0$, acting, say, on meromorphic functions of $u \in \mathbf{h}^{\mathbb{C}}$ with values in $\underline{V}_0$ commute, see [14]. Their commutation forms part of the conditions assuring the flatness of the KZB connection (36). Although the coefficient functions in H_n have poles on the hyperplanes $\langle u, \alpha\rangle \in \mathbb{Z} + \tau\mathbb{Z}$, the connection maps holomorphic families of genus one CS states into families with the same property, due to the increased regularity (6) of the CS states on the singular hyperplanes, see [8, 14].

The commuting operators H_n^0 for genus zero or H_n at genus one are quantizations of classical Poisson-commuting Hamiltonians of the, respectively, genus zero and genus one Hitchin integrable systems [22, 4, 24, 5]. Let us briefly recall this relation. Given a Riemann surface Σ and a group G, let $\mathcal{A}$ denote the corresponding space of $0,1$-gauge fields and $\mathcal{G}$ the group of complex gauge transformations, as in the beginning of Sect. 2. Both are (infinite dimensional) complex manifolds and we shall work in the holomorphic category. The cotangent bundle $T^*\mathcal{A}$ is composed of pairs (A, Φ), where Φ is a $1,0$ form on Σ with values in Lie algebra $\mathbf{g}^{\mathbb{C}}$. The

action (2) of $\mathscr{G}$ on $\mathscr{A}$ lifts to the symplectic action $(A, \Phi) \mapsto ({}^{g}\!A, {}^{g}\Phi \equiv g\Phi g^{-1})$ on $T^*\mathscr{A}$ with the moment map

$$M(A, \Phi) = \bar{\partial}A + A \wedge \Phi + \Phi \wedge A$$

(the 2-forms on Σ with values in $\mathbf{g}^{\mathbb{C}}$ form the space dual to the Lie algebra of $\mathscr{G}$). If $\underline{z}$ is a sequence of N insertion points in Σ and $\underline{\lambda}$ a corresponding sequence of Cartan algebra elements, then we may symplectically reduce $T^*\mathscr{A}$ w.r.t. the $\mathscr{G}$-coadjoint orbit of $\sum_n \lambda_n \delta_{z_n}$ defining the reduced phase space

$$\mathscr{P}_{\underline{z}, \underline{\lambda}} = M^{-1}\left(\left\{\sum_n \lambda_n \delta_{z_n}\right\}\right) \Big/ \mathscr{G}_{\underline{z}, \underline{\lambda}},$$

where $\mathscr{G}_{\underline{z}, \underline{\lambda}}$ is the subgroup of $\mathscr{G}$ fixing $\sum_n \lambda_n \delta_{z_n}$. The symplectic form on $T^*\mathscr{A}$ descends to (the non-singular part of) $\mathscr{P}_{\underline{z}, \underline{\lambda}}$ turning it into a finite dimensional (complex) symplectic manifold (we shall ignore here the singularities of $\mathscr{P}_{\underline{z}, \underline{\lambda}}$). For a $G^{\mathbb{C}}$ invariant homogeneous polynomial p on $\mathbf{g}^{\mathbb{C}}$ of degree d_p, the assignment

$$(A, \Phi) \mapsto p(\Phi)$$

defines a map on $T^*\mathscr{A}$ with values in the space of d_p-differentials (sections of the d_p^{th} symmetric power of the canonical bundle on Σ). All those vector-valued maps Poisson-commute since they depend only on Φ. They descend to $\mathscr{P}_{\underline{z}, \underline{\lambda}}$ giving a system of Poisson-commuting maps h_p with values in the (finite-dimensional) spaces of meromorphic d_p-differentials with poles of order $\leq d_p$ at the insertion points z_n. Their components form a maximal set of classical Hamiltonians in involution turning $\mathscr{P}_{\underline{z}, \underline{\lambda}}$ into an integrable system introduced and analyzed in [22] for the case without insertions, see [24, 5] for the generalization including the insertions.

Let us specify first the above construction to the genus zero case. In that case, (almost) each gauge field A is in the gauge orbit of $A = 0$, i.e. it is of the form

$$A = h^{-1}\bar{\partial}h$$

for $h \in \mathscr{G}$ with h determined modulo $h \mapsto h_0 h$ with constant h_0. Equation $M(A, \Phi) = \sum_n \lambda_n \delta_{z_n}$ becomes now

$$\bar{\partial}({}^{h}\Phi) = \sum_n h(z_n)\lambda_n h^{-1}(z_n)\delta_{z_n},$$

which has a unique solution

$$({}^{h}\Phi)(z) = \frac{1}{2\pi i}\sum_n \frac{h(z_n)\lambda_n h^{-1}(z_n)}{z - z_n}dz,$$

provided that the sum of the residues vanishes:

$$\sum_n h(z_n)\lambda_n h^{-1}(z_n) = 0.$$

The group $\mathscr{G}_{\underline{z}, \underline{\lambda}}$ is composed of arbitrary gauge transformations g s.t. $g(z_n) \in G_{\lambda_n}^{\mathbb{C}}$, acting on h by $h \mapsto hg^{-1}$. Above, $G_{\lambda_n}^{\mathbb{C}}$ denotes the subgroups of the $G^{\mathbb{C}}$ stabilizing

λ_n under the adjoint action. For the symplectically reduced phase space $P^0_{\underline{z},\underline{\lambda}}$ (the superscript 0 referring to the genus zero case) we obtain

$$ P^0_{\underline{z},\underline{\lambda}} \cong \left\{ \underline{\mu} \,\middle|\, \mu_n \in \mathcal{O}_{\lambda_n},\; \sum_n \mu_n = 0 \right\} \middle/ G^{\mathbb{C}} , $$

where $\mu_n \equiv h(z_n)\lambda_n h^{-1}(z_n)$ run through the (co)adjoint orbits $\mathcal{O}_{\lambda_n}$. The latter are naturally complex symplectic manifolds and it is not difficult to check that, as a symplectic manifold, $P^0_{\underline{z},\underline{\lambda}}$ is the reduction of $\times_n \mathcal{O}_{\lambda_n}$ by the diagonal action of $G^{\mathbb{C}}$. Since for an invariant polynomial p, $p(\Phi) = p(^h\Phi)$, the corresponding Poisson-commuting Hamiltonians on $P^0_{\underline{z},\underline{\lambda}}$ are

$$ h^0_p(\underline{\mu})(z) = p\left(\frac{1}{2\pi i} \sum_n \frac{\mu_n}{z - z_n} \right) (dz)^{d_p} . $$

In particular, for the quadratic polynomial given by the Killing form $p_2 = \langle \cdot, \cdot \rangle$, we obtain the quadratic meromorphic differential

$$ h^0_{p_2}(\underline{\mu})(z) = -\frac{1}{4\pi^2} \sum_{n,n'} \frac{\langle \mu_n, \mu_{n'} \rangle}{(z - z_n)(z - z_{n'})}(dz)^2 $$

$$ = \sum_n \left(-\frac{\langle \mu_n, \mu_n \rangle}{4\pi^2(z - z_n)^2} + \frac{1}{z - z_n}h^0_n \right)(dz)^2 , $$

where the residues at $z = z_n$ are

$$ h^0_n = \frac{1}{2\pi^2} \sum_{n' \neq n} \frac{\langle \mu_n, \mu_{n'} \rangle}{z_{n'} - z_n} . $$

These are, up to normalization, the classical versions of the Gaudin Hamiltonians of Eq. (35). The latter may be obtained from h^0_n's by replacing the coordinates $\mu^a_n \equiv \langle \mu_n, t^a \rangle$ on the coadjoint orbit $\mathcal{O}_{\lambda_n}$ by the generators $t^a_{(n)}$ of $\mathbf{g}$ acting in the irreducible representation V_{λ_n} obtained by geometric quantization of $\mathcal{O}_{\lambda_n}$.

Similarly at genus one, for $\Sigma = \mathcal{T}_\tau$, (almost) each gauge field is in the gauge orbit of the gauge fields A_u, i.e. it is of the form

$$ A = {}^{h^{-1}}A_u = (h_u h)^{-1}\bar{\partial}(h_u h) $$

with $u \in \mathbf{h}^{\mathbb{C}}$ and $h_u \equiv e^{\pi \bar{z}(u - \bar{u})/\tau_2}$. Hence the gauge fields A may be parametrized by pairs (u, h) with the identifications

$$ (u, h) = (wuw^{-1}, wh) = (u + q^{\vee}, h^{-1}_{q^{\vee}}h) = (u + \tau q^{\vee}, h^{-1}_{\tau q^{\vee}}h) \tag{39} $$

for $q^{\vee}$ in the coroot lattice $Q^{\vee}$ and w in the normalizer $N(\mathbf{h}^{\mathbb{C}}) \subset G^{\mathbb{C}}$ of $\mathbf{h}^{\mathbb{C}}$. Equation $M(A, \Phi) = \sum_n \lambda_n \delta_{z_n}$ becomes now

$$ \bar{\partial}^{(h_u h)}\Phi = \sum_n (h_u h)(z_n)\lambda_n(h_u h)^{-1}(z_n)\delta_{z_n} . \tag{40} $$

Upon decomposing

$$ (h_u h)(z_n)\lambda_n(h_u h)^{-1}(z_n) = \sum_\alpha \mu^{-\alpha}_n e_\alpha + \mu^0_n \equiv \mu_n $$

with μ_n^0 in the Cartan subalgebra $\mathbf{h}^{\mathbb{C}}$, Eq. (40) may be solved, provided that $\sum_n \mu_n^0 = 0$, with use of the Green functions of the twisted and untwisted $\bar{\partial}$-operator on $\mathscr{T}_\tau$:

$$^{(h_u h)}\Phi(z) = \Phi^0 dz + \frac{1}{2\pi i} \sum_n \left(\pi \sum_\alpha P_{\langle u, \alpha \rangle}(z - z_n)\mu_n^{-\alpha} e_\alpha + \rho(z - z_n)\mu_n^0 \right) dz ,$$

where Φ^0 is an arbitrary constant in $\mathbf{h}^{\mathbb{C}}$. The symplectically reduced phase space $P_{\underline{z}, \underline{\lambda}}$ becomes

$$P_{\underline{z}, \underline{\lambda}} = \left\{ (u, \Phi^0, \underline{\mu}) \in T^* \mathbf{h}^{\mathbb{C}} \times (\times_n \mathcal{O}_{\lambda_n}) \Big| \sum_n \mu_n^0 = 0 \right\} \Big/ N(\mathbf{h}^{\mathbb{C}}) \rtimes (Q^\vee + \tau Q^\vee)$$

with the identifications

$$(u, \Phi^0, \underline{\mu}) = (wuw^{-1}, w\Phi^0 w^{-1}, w\underline{\mu}w^{-1}) = (u + (\tau)q^\vee, \Phi^0, (h_{(\tau)q^\vee}^{-1}(z_n)\mu_n h_{(\tau)q^\vee}(z_n))) .$$

As a symplectic manifold, $P_{\underline{z}, \underline{\lambda}}$ is a symplectic reduction of $T^* \mathbf{h}^{\mathbb{C}} \times (\times_n \mathcal{O}_{\lambda_n})$ by the group $N(\mathbf{h}^{\mathbb{C}}) \rtimes (Q^\vee + \tau Q^\vee)$. The Hitchin Hamiltonians become

$$h_p(u, \Phi^0, \underline{\mu})(z)$$

$$= p \left(\Phi^0 + \frac{1}{2\pi i} \sum_n \left(\pi \sum_\alpha P_{\langle u, \alpha \rangle}(z - z_n)\mu_n^{-\alpha} e_\alpha + \rho(z - z_n)\mu_n^0 \right) \right) (dz)^{d_p}$$

which for $p = p_2$ and upon writing $h_{p_2}(u, \Phi^0, \underline{\mu})(z) = \tilde{h}_{p_2}(u, \Phi^0, \underline{\mu})(z)(dz)^2$, reduces to

$$\tilde{h}_{p_2}(u, \Phi^0, \underline{\mu})(z) = \langle \Phi^0, \Phi^0 \rangle + \frac{1}{\pi i} \sum_n \rho(z - z_n)\langle \mu_n^0, \Phi^0 \rangle$$

$$- \frac{1}{4\pi^2} \sum_{n, n'} \left(\sum_\alpha \pi^2 P_{\langle u, \alpha \rangle}(z - z_n) P_{-\langle u, \alpha \rangle}(z - z_{n'}) \mu_n^{-\alpha} \mu_{n'}^\alpha + \rho(z - z_n)\rho(z - z_{n'})\langle \mu_n^0, \mu_{n'}^0 \rangle \right)$$

$$= \frac{1}{4\pi^2} \sum_n \rho'(z - z_n)\langle \mu_n, \mu_n \rangle + \sum_n \rho(z - z_n)h_n(u, \Phi^0, \underline{\mu}) + h_0(u, \phi^0, \underline{\mu}) ,$$

with the residues at z_n

$$h_n(u, \Phi^0, \underline{\mu}) = \frac{1}{\pi i}\langle \mu_n^0, \Phi^0 \rangle - \frac{1}{2\pi^2} \sum_{n' \neq n} \left(\sum_\alpha \pi P_{\langle u, \alpha \rangle}(z_n - z_{n'})\mu_n^\alpha \mu_{n'}^{-\alpha} \right.$$

$$\left. + \rho(z_n - z_{n'})\langle \mu_n^0, \mu_{n'}^0 \rangle \right)$$

and the holomorphic (actually z independent) piece

$$h_0(u, \phi^0, \underline{\mu}) = \langle \Phi^0, \Phi^0 \rangle - \frac{1}{8\pi^2} \sum_{n, n'} \left(2\pi \sum_\alpha \partial_x P_{\langle u, \alpha \rangle}(z_n - z_{n'})\mu_n^\alpha \mu_{n'}^{-\alpha} \right.$$

$$\left. + (\rho(z_n - z_{n'})^2 + \rho'(z_n - z_{n'}))\langle \mu_n^0, \mu_{n'}^0 \rangle \right) .$$

Hamiltonians h_n, $n \geq 0$ are the classical versions of the elliptic Gaudin Hamiltonians H_n of Eqs. (37) and (38), see [24, 25].

The $SU(n)$ elliptic Hitchin system corresponding to one insertion has unexpectedly appeared recently in the description of the low energy sector of supersymmetric Yang–Mills theories [4].

5. Unitarity of the KZB Connection

One of the essential features of the structures discussed above should be the compatibility of the KZ and KZB connections with the scalar product of CS states. The integrals in Eq. (31) have been conjectured in [17] (and proved in many cases) to converge precisely for invariant tensors $\gamma \in W_z^0$ and to equip bundle $\mathcal{W}^0$ with the hermitian structure preserved by the KZ connection. The latter condition means that for all local holomorphic sections $z \mapsto \gamma(z)$ of bundles $\mathcal{W}^0$, corresponding to holomorphic families $z \mapsto \Psi(z)$ of CS states,

$$\partial_{z_n} \|\Psi\|^2 = (\Psi, \nabla_{z_n} \Psi) . \tag{41}$$

In order to see why one should expect such a relation, it will be convenient to express the scalar product integral in the language of differential forms, following refs. [27, 28]. Let $\omega(y) \equiv \frac{1}{\pi} y^{-1} dy$. Introduce the V^*-valued K forms

$$\Omega^0_{K,\underline{\alpha}}(\underline{z}, \underline{y}) = \omega(z_1 - y_{1,1}) \wedge \omega(y_{1,1} - y_{1,2}) \wedge \cdots \wedge \omega(y_{1,K_1-1} - y_{1,K_1})$$

$$\wedge \cdots \wedge \omega(z_N - y_{N,1}) \wedge \omega(y_{N,1} - y_{N,2}) \wedge \cdots \wedge \omega(y_{N,K_N-1} - y_{N,K_N})$$

$$\times \langle \lambda | \otimes_n (e_{\alpha_{n,1}} \cdots e_{\alpha_{n,K_n}})(n)$$

and

$$\Omega^0(\underline{z}, \underline{y}) = \sum_{\underline{K}} \sum_{\sigma \in S_K} (-1)^{|\sigma|} \Omega_{\underline{K}, \sigma\underline{\alpha}}(\underline{z}, \sigma \underline{y}) .$$

We may rewrite the scalar product formula (31) as

$$\|\Psi\|^2 = \text{const.} \int_{Y_{\underline{z}}} \left| e^{-\frac{1}{\kappa} S^0} \langle \Omega^0, \gamma \rangle \right|^2 ,$$

where $Y_{\underline{z}}$ stands for the space of $\underline{y}$'s with y_s's not coinciding among themselves and with z_n's. We use the conventions that $|\Omega^0|^2 \equiv (-1)^{K(K-1)/2} (\frac{i}{2})^K \Omega^0 \wedge \overline{\Omega^0}$ and that the integral of the forms of degree lower than the dimension of the cycle is zero. Assuming a sufficiently strong convergence of the integrals, we may enter with the holomorphic exterior derivative under the integral so that

$$\partial \|\Psi\|^2 = \text{const.} \int_{Y_{\underline{z}}} \partial \left| e^{-\frac{1}{\kappa} S^0} \langle \Omega^0, \gamma \rangle \right|^2$$

$$= \text{const.} \int_{Y_{\underline{z}}} \partial (e^{-\frac{1}{\kappa} S^0} \langle \Omega^0, \gamma \rangle) \wedge \overline{e^{-\frac{1}{\kappa} S^0} \langle \Omega^0, \gamma \rangle} ,$$

where the ∂-operator on the l.h.s. acts on the z-variables and the one under the integral may be taken as acting on both z- and y-variables. The $\underline{V}^*$-valued holomorphic multivalued forms Ω^0 have two basic properties:

$$\partial\Omega^0 = 0 \quad \text{and} \quad (\partial S^0) \wedge \Omega^0 + \sum_{n=1}^{N} dz_n \wedge H_n^0 \Omega^0 = 0 \tag{42}$$

with the contragradient action of the Gaudin Hamiltonians on the $\underline{V}^*$-valued form Ω^0. The first relation is trivial. The second, more involved one, has been proven in [28]. Using Eqs. (42), we obtain

$$\partial\|\Psi\|^2 = \text{const.} \int_{Y_{\underline{z}}} \left|e^{-\frac{1}{\kappa}S^0}\right|^2 \sum_n dz_n \wedge \left\langle \Omega^0, \left(\partial_{z_n} + \frac{1}{\kappa}H_n\right)\gamma\right\rangle \wedge \overline{e^{-\frac{1}{\kappa}S^0}\langle\Omega^0,\gamma\rangle} ,$$

which implies the relation (41).

The above analysis has its counterpart for the elliptic case. We again conjecture that the integrals in Eq. (26) converge for θ corresponding to CS states (in fact exactly when the regularity conditions (6) are fulfilled) and that the resulting hermitian structure on the bundle $\mathscr{W}$ renders the KZB connection unitary, see [9] for the proof of this conjecture for the case of $G = SU(2)$ and one insertion. In order to substantiate the conjecture for the case of general G and arbitrary insertions, let us rewrite the genus one scalar product integral (26) in the language of differential forms. Define

$$\omega_q(z) \equiv P_{\langle u,q\rangle}(z)dz - \frac{i}{2\pi}\partial_x P_{\langle u,q\rangle}(z)d\tau ,$$

see Eq. (17). It is easy to see that $\partial\omega_q = 0$, if ∂ differentiates only the variables (τ,z). Set

$$\Omega_{\underline{K},\underline{\alpha}}(\tau,u,\underline{z},\underline{y}) = \omega_{\beta_{1,1}}(z_1 - y_{1,1}) \wedge \omega_{\beta_{1,2}}(y_{1,1} - y_{1,2})$$

$$\wedge \cdots \wedge \omega_{\beta_{1,K_1}}(y_{1,K_1-1} - y_{1,K_1}) \wedge \cdots \wedge \omega_{\beta_{N,1}}(z_N - y_{N,1})$$

$$\wedge \omega_{\beta_{N,2}}(y_{N,1} - y_{N,2}) \wedge \cdots \wedge \omega_{\beta_{N,K_N}}(y_{N,K_N-1} - y_{N,K_N})\langle\lambda| \otimes_n (e_{\alpha_{n,1}} \cdots e_{\alpha_{n,K_n}})(n) \tag{43}$$

with $\beta_{n,i} = \sum_{i'=i}^{K_n} \alpha_{n,i'}$ and

$$\Omega(\tau,u,\underline{z},\underline{y}) = \sum_{\underline{K}} \sum_{\sigma\in S_K} (-1)^{|\sigma|}\Omega_{\underline{K},\sigma\underline{\alpha}}(\tau,u,\underline{z},\sigma\underline{y}) .$$

We still have to dress Ω with the du^j differentials. The convenient way to do this is to define the form

$$\Omega = e^{-\frac{1}{\kappa}S}\left(dw^1 \wedge \cdots \wedge dw^r \wedge \langle\Omega,\theta\rangle + \frac{1}{2\pi i\kappa}\sum_{j=1}^{r} dw^1 \wedge \cdots \underset{\widehat{j}}{d\tau} \cdots \wedge dw^r \partial_{u^j}\langle\Omega,\theta\rangle\right) , \tag{44}$$

where $w^j \equiv \langle h^j, w \rangle$ with w given by Eq. (27) and, as before, $\theta \equiv \Pi\gamma$. It is easy to see that Eq. (26) may be rewritten as

$$\|\Psi\|^2 = \text{const.} \, \tau_2^{-r/2} \int\limits_{UY_{\tau,\underline{z}}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left| e^{-\frac{1}{\kappa}S} \langle \Omega, \theta \rangle \right|^2 d^{2r}u$$

$$= \text{const.} \, \tau_2^{-r/2} \int\limits_{UY_{\tau,\underline{z}}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} |\Omega|^2 \ . \tag{45}$$

This holds since the terms with $d\tau$ differentials do not contribute to the integral over the cycle $UY_{\tau,\underline{z}}$ composed of points $(u, \underline{y})$ with identifications $u = u + (\tau)q^\vee$, $y_s = y_s + (\tau)1$ which form the $(r + K)$-dimensional torus $\mathcal{T}_\tau^{r+K}$ (with coincidences of y_s's among themselves and with z_n's removed). The gain from adding the terms with $d\tau$'s is that the form Ω transforms under the maps $u \mapsto u + (\tau)q^\vee$, $y_s \mapsto y_s + (\tau)1$ (and $z_n \mapsto z_n + (\tau)1$) exactly as the functions H, see Eq. (30), and, as the result, the form under the last integral in Eq. (45) is well defined on $UY_{\tau,\underline{z}}$.

We would like to compute the holomorphic differential of the norms squared $\|\Psi\|^2$ of a holomorphic family of states. Due to the fact that the integration cycles $UY_{\tau,\underline{z}}$ depend nontrivially on the differentiation variables, this requires a more subtle consideration than in the genus zero case where we could ignore this problem (assuming enough convergence at the coinciding points). Ignoring again the coinciding points, the geometric setup is as follows. We are given a bundle $\varpi : \mathcal{N} \to \mathcal{M}$ of multidimensional tori over the space $\mathcal{M}$ of pairs $(\tau, \underline{z})$. Let us forget for a moment the complex structures on these spaces considering them as real manifolds. On $\mathcal{N}$ we are given a differential form η of degree equal to the dimension of the fiber (we shall not need an obvious generalization of the story to forms of arbitrary degree) and we are studying a function f on the base obtained by fiber-wise integration of η

$$f(n) = \int\limits_{\varpi^{-1}(\{n\})} \eta \ .$$

In local trivialization of the bundle, only the components of η with differentials along the fiber contribute to f. If there are no convergence problems, then obviously

$$df(n) = \int\limits_{\varpi^{-1}(\{n\})} d^\perp \eta = \int\limits_{\varpi^{-1}(\{n\})} d\eta , \tag{46}$$

where the differential $d^\perp$ in the transverse directions is defined using a trivialization and the second equality follows since $d^\parallel \eta \equiv (d - d^\perp)\eta$ is a closed form when restricted to the fibers so that its fiber-wise integral vanishes (again assuming no convergence problems).

Let us first see how this argument may be used in the simplest case with no insertions, where we have

$$\|\Psi\|^2 = \text{const.} \, \tau_2^{-r/2} \int\limits_{\mathcal{T}_\tau^r} e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} |\theta(u)|^2 d^{2r}u = \text{const.} \, \tau_2^{-r/2} \int\limits_{\mathcal{T}_\tau^r} e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} |\Omega|^2 ,$$

where $\theta(u + q^\vee) = \theta(u)$ and $\theta(u + \tau q^\vee) = e^{-\pi i \kappa \langle q^\vee, \tau q^\vee + 2u \rangle} \theta(u)$, i.e. θ is an r-dimensional theta-function of degree 2κ, and

$$\Omega = \theta \, du^1 \wedge \cdots \wedge du^r + \frac{1}{2\pi i \kappa} \sum_{j=1}^r \partial_{u^j} \theta \, du^1 \wedge \cdots \underset{\widehat{j}}{d\tau} \cdots \wedge du^r$$

is a holomorphic r-form which under $u \mapsto u + (\tau) q^\vee$ transforms the same way as the theta-function θ. Note the relations:

$$du^j \wedge \Omega = \frac{i}{2\pi\kappa} d\tau \wedge \partial_{u^j} \Omega, \qquad \sum_j du^j \wedge \partial_{u^j} \Omega = \frac{i}{2\pi\kappa} d\tau \wedge \Delta_u \Omega \ . \tag{47}$$

The $(2r)$-form $\eta = e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} |\Omega|^2$ is well defined on the bundle $\mathcal{N}$ of tori obtained by varying $\mathcal{T}_\tau^r$ with τ. Applying the above geometrical considerations to the case at hand, we obtain

$$d\|\Psi\|^2 = \text{const.} \, \tau_2^{-r/2} \int_{\mathcal{T}_\tau^r} \left(d + \frac{ri}{4\tau_2}(d\tau - d\bar{\tau})\wedge \right) \left(e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} |\Omega|^2 \right) \ .$$

Clearly, only the holomorphic exterior derivative ∂ will contribute under the integral to $\partial\|\Psi\|^2$. Explicit differentiation gives:

$$\partial\|\Psi\|^2 = \text{const.} \, \tau_2^{-r/2} \int_{\mathcal{T}_\tau^r} e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} \left(\left(\frac{\pi i \kappa}{4\tau_2^2}|u-\bar{u}|^2 d\tau \wedge + \frac{\pi\kappa}{\tau_2}\sum_j (u^j - \bar{u}^j) du^j \wedge \right. \right.$$

$$\left. \left. + \frac{ri}{4\tau_2} d\tau \wedge + d\tau \wedge \partial_\tau + \sum_j du^j \wedge \partial_{u^j} \right) \Omega \right) \wedge \overline{\Omega}$$

$$= \text{const.} \, \tau_2^{-r/2} \int_{\mathcal{T}_\tau^r} e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} \left(\left(\frac{\pi i \kappa}{4\tau_2^2}|u-\bar{u}|^2 d\tau \wedge + \frac{i}{2\tau_2} d\tau \wedge \sum_j (u^j - \bar{u}^j) \partial_{u^j} \right. \right.$$

$$\left. \left. + \frac{ri}{4\tau_2} d\tau \wedge + d\tau \wedge \partial_\tau + \frac{i}{2\pi\kappa} d\tau \wedge \Delta_u \right) \Omega \right) \wedge \overline{\Omega} \ , \tag{48}$$

where to obtain the last equality we have used the relations (47). The latter expression may be simplified if we notice that

$$\frac{i}{4\pi\kappa} d\tau \wedge \sum_j \partial_{u^j} \left(e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} \left(\partial_{u^j} + \frac{\pi\kappa}{\tau_2}(u^j - \bar{u}^j) \right) \Omega \wedge \overline{\Omega} \right)$$

$$= e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2} \left(\left(\frac{\pi i \kappa}{4\tau_2^2}|u-\bar{u}|^2 d\tau \wedge + \frac{i}{2\tau_2} d\tau \wedge \sum_j (u^j - \bar{u}^j)\partial_{u^j} \right. \right.$$

$$\left. \left. + \frac{i}{4\pi i\kappa} d\tau \wedge \Delta_u + \frac{ri}{4\tau_2} d\tau\wedge \right) \Omega \right) \wedge \overline{\Omega} \tag{49}$$

with $e^{\frac{\pi\kappa}{2\tau_2}|u-\bar{u}|^2}((\partial_{u^j} + \frac{\pi\kappa}{\tau_2}(u^j - \bar{u}^j))\Omega \wedge \overline{\Omega}$ being a $(2r)$-form well defined on $\mathcal{N}$. It follows that the fiber-wise integral of both sides of Eq. (49) vanishes and,

consequently, that

$$\partial_\tau \|\Psi\|^2 = \text{const.}\, \tau_2^{-r/2} \int_{\mathcal{T}_\tau^r} e^{\frac{\pi \kappa}{2\tau_2}|u-\bar{u}|^2} \left(\left(\partial_\tau + \frac{i}{4\pi\kappa}\Delta_u \right) \Omega \right) \wedge \overline{\Omega}$$

$$= \text{const.}\, \tau_2^{-r/2} \int_{\mathcal{T}_\tau^r} e^{\frac{\pi \kappa}{2\tau_2}|u-\bar{u}|^2} \overline{\theta(u)} \left(\partial_\tau + \frac{i}{4\pi\kappa}\Delta_u \right) \theta(u) d^{2r}u = (\Psi, \nabla_\tau \Psi),$$

which proves the unitarity of the KZB connection for the case with no insertions in a way that maybe was not the simplest (two straightforward integrations by parts would do) but which has the virtue of generalizing to the case with arbitrary insertions (modulo convergence problems).

To treat the general case, we shall apply the formula (46) to the differential form $\eta = e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left| e^{-\frac{1}{\kappa}S}\Omega \right|^2$ with Ω given by Eq. (44) and satisfying the generalization of relations (47) with du^j replaced by dw^j. As before, we obtain, denoting by ∂' the holomorphic exterior derivative in all but u directions,

$$\partial \|\Psi\|^2 = \text{const.}\, \tau_2^{-r/2} \int_{UY_{\tau,z}} \left(\partial + \frac{ri}{4\tau_2}d\tau \wedge \right) (e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} |\Omega|^2)$$

$$= \text{const.}\, \tau_2^{-r/2} \int_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left(\left(\frac{\pi i\kappa}{4\tau_2^2}|w-\bar{w}|^2 d\tau \wedge \right.\right.$$

$$+ \frac{\pi\kappa}{\tau_2}\sum_j (w^j - \bar{w}^j)dw^j \wedge + \frac{ri}{4\tau_2}d\tau \wedge + \partial' + \sum_j du^j \wedge \partial_{u^j} \bigg) \Omega \bigg) \wedge \overline{\Omega}$$

$$= \text{const.}\, \tau_2^{-r/2} \int_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left(\left(\frac{\pi i\kappa}{4\tau_2^2}|w-\bar{w}|^2 d\tau \wedge \right.\right.$$

$$+ \frac{i}{2\tau_2}d\tau \wedge \sum_j (w^j - \bar{w}^j)\partial_{u^j} + \frac{ri}{4\tau_2}d\tau \wedge + \partial' + \frac{i}{2\pi\kappa}d\tau \wedge \Delta_u$$

$$- \frac{1}{\kappa}\sum_n \lambda_n^j dz_n \wedge \partial_{u_j} + \frac{1}{\kappa}\sum_s \alpha_s^j dy_s \wedge \partial_{u_j} \bigg) \Omega \bigg) \wedge \overline{\Omega}.$$

A large part of the right-hand side vanishes by integration by parts with use of the relation generalizing (49):

$$\frac{i}{4\pi\kappa}d\tau \wedge \sum_j \partial_{u^j} \left(e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left((\partial_{u^j} + \frac{\pi\kappa}{\tau_2}(w^j - \bar{w}^j)) \Omega \wedge \overline{\Omega} \right) \right)$$

$$= e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left(\left(\frac{\pi i\kappa}{4\tau_2^2}|w-\bar{w}|^2 d\tau \wedge + \frac{i}{2\tau_2}d\tau \wedge \sum_j (w^j - \bar{w}^j)\partial_{u^j} \right.\right.$$

$$+ \frac{i}{4\pi i\kappa}d\tau \wedge \Delta_u + \frac{ri}{4\tau_2}d\tau \wedge \bigg) \Omega \bigg) \wedge \overline{\Omega}, \tag{50}$$

and we obtain

$$\partial \|\Psi\|^2 = \text{const.}\,\tau_2^{-r/2} \int\limits_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left(\left(\partial' + \frac{i}{4\pi\kappa}d\tau \wedge \Delta_u - \frac{1}{\kappa}\sum_n \lambda_n^j dz_n \wedge \partial_{u_j} \right.\right.$$
$$\left.\left. + \frac{1}{\kappa}\sum_s \alpha_s^j dy_s \wedge \partial_{u_j} \right) \Omega \right) \wedge \overline{\Omega}$$
$$= \text{const.}\,\tau_2^{-r/2} \int\limits_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left(\left(\partial' + \frac{i}{4\pi\kappa}d\tau \wedge \Delta_u - \frac{1}{\kappa}\sum_n \lambda_n^j dz_n \wedge \partial_{u_j} \right.\right.$$
$$\left.\left. + \frac{1}{\kappa}\sum_s \alpha_s^j dy_s \wedge \partial_{u_j} \right) e^{-\frac{1}{\kappa}S}\langle \Omega, \theta \rangle \right) \wedge \overline{e^{-\frac{1}{\kappa}S}\langle \Omega, \theta \rangle}\; d^{2r}u\,. \tag{51}$$

The ∂_{u^j}-terms may be transformed with use of relation

$$\sum_n \sum_{i=1}^{K_n} \beta_{n,i}^j d(y_{n,i-1} - y_{n,i}) \wedge \Omega = \frac{i}{2\pi}d\tau \wedge \partial_{u^j}\Omega\,, \tag{52}$$

in the notations of Eq. (43) and with $y_{n,0} \equiv z_n$, which follows easily from the definitions of the forms Ω and ω_q. Equation (52) may be rewritten as

$$\left(\sum_n \lambda_n^j dz_n - \sum_s \alpha_s^j dy_s \right) \wedge \Omega = \frac{i}{2\pi}d\tau \wedge \partial_{u^j}\Omega - \sum_n dz_n \wedge h_{(n)}^j \Omega\,, \tag{53}$$

with the contragradient action of $h_{(n)}^j$ on the $\underline{V}^*$-valued form Ω. The last relation, upon substitution to Eq. (51), yields

$$\partial\|\Psi\|^2 = \text{const.}\,\tau_2^{-r/2} \int\limits_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left| e^{-\frac{1}{\kappa}S} \right|^2 \left(\partial'\langle\Omega,\theta\rangle - \frac{1}{\kappa}\partial S \wedge \langle\Omega,\theta\rangle \right.$$
$$+ \frac{i}{4\pi\kappa}d\tau \wedge \Delta_u\langle\Omega,\theta\rangle - \frac{i}{2\pi\kappa}d\tau \wedge \sum_j \partial_{u^j}(\langle\partial_{u^j}\Omega,\theta\rangle)$$
$$\left. + \frac{1}{\kappa}\sum_n dz_n \wedge \partial_{u^j}(\langle h_{(n)}^j\Omega,\theta\rangle) \right) \wedge \overline{\Omega}\; d^{2r}u$$
$$= \text{const.}\,\tau_2^{-r/2} \int\limits_{UY_{\tau,z}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left| e^{-\frac{1}{\kappa}S} \right|^2 \left(\left\langle \partial'\Omega - \frac{1}{\kappa}\partial S \wedge \Omega \right.\right.$$
$$\left. - \frac{i}{4\pi\kappa}d\tau \wedge \Delta_u\Omega + \frac{1}{\kappa}\sum_n dz_n \wedge h_{(n)}^j\partial_{u^j}\Omega, \theta \right\rangle$$
$$\left. + (-1)^K \left\langle \Omega\,, \partial'\theta + \frac{i}{4\pi\kappa}d\tau \wedge \Delta_u\theta - \frac{1}{\kappa}\sum_n dz_n \wedge h_{(n)}^j\partial_{u_j}\theta \right\rangle \right) \wedge \overline{\Omega}\; d^{2r}u\,.$$

The crucial result is the following equalities:

$$\partial'\Omega = 0 \quad \text{and} \quad (\partial S) \wedge \Omega + d\tau \wedge H_0\Omega + \sum_n dz_n \wedge H_n\Omega = 0 \tag{54}$$

with the contragradient action of H_n's. The first of these equalities is a straightforward consequence of the closedness of the forms ω_q from which Ω is built. The second, more technical one, has been announced in [12] (as Prop. 9). Using these

relations, we finally obtain

$$\partial \| \Psi \|^2 = \text{const.} \, \tau_2^{-r/2} \int_{UY_{\tau,\underline{z}}} e^{\frac{\pi\kappa}{2\tau_2}|w-\bar{w}|^2} \left| e^{-\frac{1}{\kappa}S} \right|^2 (-1)^K \left\langle \Omega, \partial'\theta + \frac{1}{\kappa} d\tau \wedge H_0\theta \right.$$

$$\left. + \frac{1}{\kappa} \sum_n dz_n \wedge H_n\theta \right\rangle \wedge \bar{\Omega} \, d^{2r}u = (\Psi, \nabla\Psi) \, ,$$

which proves the unitarity of the KZ connection w.r.t. the scalar product (26) modulo the control of convergence of the integrals.

6. Bethe Ansatz

The basic algebraic relations (42) responsible for the unitarity of the KZ connection turn out to be a disguised (and compact) form of the Bethe Ansatz solution of the eigenvalue problem for the Gaudin Hamiltonians (35). This was remarked in [1] in the context of contour integral representations for the solutions of the KZ equations and developed further in [2, 26] and in [11] where a relation between the Bethe Ansatz and the Wakimoto realization of the highest weight modules of Kac–Moody algebras was explained.

The elementary fact is that if, for fixed $\underline{z}$, a configuration $\underline{y}$ of K non-coincident points in the plane satisfies the equations

$$\partial_{y_s} S^0(\underline{z}, \underline{y}) = 0, \quad s = 1, \ldots, K \, , \tag{55}$$

then the second relation of (42) reduces to

$$\sum_n dz_n \wedge (\partial_{z_n} S^0(\underline{z}, \underline{y}) + H_n^0(\underline{z})) \Omega^0(\underline{z}, \underline{y}) = 0 \, .$$

The last equation gives the Bethe Ansatz solution for the common eigenvectors of the operators $H_n^0(\underline{z})$ acting by the contragradient representation in $\underline{V}^*$:

$$(\partial_{z_n} S^0(\underline{z}, \underline{y}) + H_n^0(\underline{z})) G^0(\underline{z}, \underline{y}) = 0 \tag{56}$$

in the notations of Eqs. (33, 34). The conditions (55) required for Eq. (56) to hold have the explicit form

$$\sum_n \langle \lambda_n, \alpha_s \rangle \frac{1}{z_n - y_s} = \sum_{s' \neq s} \langle \alpha_{s'}, \alpha_s \rangle \frac{1}{y_{s'} - y_s} \, . \tag{57}$$

The above genus zero story has its elliptic counterpart, as remarked in [6] for the case of $G = SU(2)$, see also [9], and in [12] for general simple groups. For fixed $\tau, \underline{z}$ and for $\underline{y}$ satisfying the equations

$$\partial_{y_s} S(\tau, \underline{z}, \underline{y}) = 0, \quad s = 1, \ldots, K \, , \tag{58}$$

or explicitly, with $\rho \equiv \vartheta_1'/\vartheta_1$,

$$\sum_n \langle \lambda_n, \alpha_s \rangle \rho(z_n - y_s) = \sum_{s' \neq s} \langle \alpha_{s'}, \alpha_s \rangle \rho(y_{s'} - y_s) \,, \tag{59}$$

the second relation of (54) reduces to

$$\left(d\tau \wedge (\partial_\tau S(\tau, \underline{z}, \underline{y}) + H_0(\tau, \underline{z})) + \sum_n dz_n \wedge (\partial_{z_n} S(\tau, \underline{z}, \underline{y}) + H_n(\tau, \underline{z})) \right) \Omega(\tau, u, \underline{z}, \underline{y}) = 0 \,. \tag{60}$$

Operators $H_n(\tau, \underline{z})$, $n \geq 0$, act in the space of meromorphic functions f of variable $u \in \mathbf{h}^{\mathbb{C}}$ taking values in $\underline{V}_0^*$, obeying the periodicity conditions

$$f(u + q^\vee) = f(u) \,, \tag{61}$$

with possible poles on the hyperplanes $\langle u, \alpha \rangle \in \mathbb{Z} + \tau\mathbb{Z}$. Equation (60) gives the Bethe Ansatz solutions for the common eigenvectors of H_n's in that space

$$(\partial_\tau S(\tau, \underline{z}, \underline{y}) + H_0(\tau, \underline{z})) G(\tau, u, \underline{z}, \underline{y}) = 0 \,,$$

$$(\partial_{z_n} S(\tau, \underline{z}, \underline{y}) + H_n(\tau, \underline{z})) G(\tau, u, \underline{z}, \underline{y}) = 0$$

in the notations of Eqs. (21, 29).

A slightly modified version of the above argument allows to extend this construction and also to find Bethe Ansatz eigenvectors of H_n's acting in the space of meromorphic $\underline{V}_0^*$-valued functions satisfying the twisted boundary conditions

$$f(u + q^\vee) = \mathrm{e}^{\langle \xi, q^\vee \rangle} f(u) \tag{62}$$

with $\xi \in \mathbf{h}^{\mathbb{C}}$. Denoting

$$\widetilde{\Omega} \equiv \mathrm{e}^{\langle u, \xi \rangle} \Omega, \qquad \widetilde{S} \equiv S + \frac{1}{4\pi i} |\xi|^2 \tau - \sum_n \langle \xi, \lambda_n \rangle z_n + \sum_s \langle \xi, \alpha_s \rangle y_s \,,$$

and combining the second equality of (54) with the relation (53), we obtain

$$(\partial\widetilde{S}) \wedge \widetilde{\Omega} + d\tau \wedge H_0\widetilde{\Omega} + \sum_n dz_n \wedge H_n\widetilde{\Omega} = 0 \,, \tag{63}$$

which results in the eigenvalue equations for the twisted-periodic Bethe Ansatz eigenvectors:

$$(\partial_\tau \widetilde{S}(\tau, \underline{z}, \underline{y}) + H_0(\tau, \underline{z})) \mathrm{e}^{\langle u, \xi \rangle} G(\tau, u, \underline{z}, \underline{y}) = 0 \,,$$

$$(\partial_{z_n} \widetilde{S}(\tau, \underline{z}, \underline{y}) + H_n(\tau, \underline{z})) \mathrm{e}^{\langle u, \xi \rangle} G(\tau, u, \underline{z}, \underline{y}) = 0$$

holding provided that $\partial_{y_s}\widetilde{S} = 0$, i.e. that

$$\sum_n \langle \lambda_n, \alpha_s \rangle \rho(z_n - y_s) + \langle \xi, \alpha_s \rangle = \sum_{s' \neq s} \langle \alpha_{s'}, \alpha_s \rangle \rho(y_{s'} - y_s) \,. \tag{64}$$

In particular, for ξ in the weight lattice P, we obtain more periodic Bethe Ansatz eigenvectors.

As explained in refs. [7 or 12], in the special case of $G = SU(n)$ and one insertion of the nm-fold symmetric power of the fundamental representation, operator H_0 acting on the V_0^*-valued functions reduces to the elliptic Calogero–Sutherland–Moser multi-body operator and the above techniques were used in [12, 13] to obtain its Bethe Ansatz diagonalization. The above case is a quantization of the very same elliptic Hitchin system which appeared in the effective low energy description of supersymmetric Yang–Mills theories [4]. For the $SU(2)$ case, one recovers this way [6] the classical Hermite results about the Lamé operator [30].

References

1. Babujian, H.M.: Off-Shell Bethe Ansatz Equation and N-point Correlators in the $SU(2)$ WZNW Theory. Preprint Bonn-HE-93-22
2. Babujian, H.M., Flume, R.: Off-Shell Bethe Ansatz Equation for Gaudin Magnets and Solutions of Knizhnik–Zamolodchikov Equations. Preprint Bonn-HE-93-30, hep-th/9310110
3. Bernard, D.: On the Wess–Zumino–Witten Models on the Torus. Nucl. Phys. **B 303**, 77–93 (1988)
4. Donagi, R., Witten, E.: Supersymmetric Yang-Mills Theory and Integrable Systems. hep-th/9510101
5. Enriquez, B., Rubtsov, V.: Hitchin Systems, Higher Gaudin Operators and r-Matrices. alg-geom/9503010
6. Etingof, P., Kirillov, A., Jr.: Representations of Affine Lie Algebras, Parabolic Differential Equations and Lamé Functions. Duke Math. J. **74**, 585–614 (1994)
7. Etingof, P., Frenkel, I., Kirillov, A., Jr.: Spherical Functions on Affine Lie Groups. hep-th/9407047
8. Falceto, F., Gawędzki, K.: Chern–Simons States at Genus One. Commun. Math. Phys. **159**, 549–579 (1994)
9. Falceto, F., Gawędzki, K.: Elliptic Wess–Zumino–Witten Model from Elliptic Chern–Simons Theory. Lett. Math. Phys., in press
10. Falceto, F., Gawędzki, K., Kupiainen, A.: Scalar Product of Current Blocks in WZW Theory. Phys. Lett. **B 260**, 101–108 (1991)
11. Feigin, B., Frenkel, E., Reshethikin, N.: Gaudin Model, Bethe Ansatz and Correlation Functions at the Critical Level. hep-th/9402022
12. Felder, G., Varchenko, A.: Integral Representations of Solutions of the Elliptic Knizhnik–Zamolodchikov–Bernard equations. hep-th
13. Felder, G., Varchenko, A.: Three Formulas for Eigenfunctions of Integrable Schroedinger Operators. hep-th/9511120
14. Felder, G., Wieczerkowski, C.: Conformal Blocks on Elliptic Curves and the Knizhnik–Zamolodchikov–Bernard Equations. hep-th/9411004
15. Gaudin, M.: Les fonctions d'onde de Bethe. Paris: Masson, 1983
16. Gawędzki, K.: Wess–Zumino–Witten Conformal Field Theories. In: Constructive quantum field theory II, eds. Wightman, A.A., Velo, G., New York: Plenum, 1990, pp. 89–120
17. Gawędzki, K.: Quadrature of Conformal Field Theories. Nucl. Phys. **B 328**, 733–752 (1989)
18. Gawędzki, K.: Constructive Conformal Field Theory. In: Functional integration, geometry and strings, eds. Haba, Z., Sobczyk, J. Basel, Boston, Berlin: Birkhäuser, 1989, pp. 277–302
19. Gawędzki, K.: $SU(2)$ WZW Theory at higher Genera. Commun. Math. Phys., in press
20. Gawędzki, K.: Coulomb Gas Representation of the $SU(2)$ WZW Correlators at Higher Genera. hep-th/9404012, Lett. Math. Phys., in press
21. Gawędzki, K., Kupiainen, A.: Coset Construction from Functional Integral. Nucl. Phys. **B 320**, 625–668 (1989)
22. Hitchin, N.: Stable Bundles and Integrable Systems. Duke Math. J. **54**, 91–114 (1987)
23. Knizhnik, V., Zamolodchikov, A.B.: Current Algebra and Wess–Zumino Model in Two Dimensions. Nucl. Phys. **B 247**, 83–103 (1984)
24. Nekrasov, N.: Holomorphic Bundles and Many-Body Systems. hep-th/9503157

25. Olshanetsky, M.: Generalized Hitchin Systems and Knizhnik–Zamolodchikov–Bernard Equation on Elliptic Curves. hep-th/9510143
26. Reshethikin, N., Varchenko, A.: Quasiclassical Asymptotics of Solutions of the KZ Equation. hep-th/9402126
27. Schechtman, V., Varchenko, A.: Integral Representations of N-point Conformal Correlators in the WZW Model. Preprint MPI/98-51 (1989)
28. Schechtman, V., Varchenko, A.: Arrangements of Hyperplanes and Lie Algebra Homology. Inv. Math. **106**, 139–194 (1991)
29. Tsuchiya, A., Kanie, Y.: Vertex Operators in the Conformal Field Theory on $P1$ and Monodromy Representations of the Braid Group. Adv. Stud. Pure Math. **16**, 297–372 (1988)
30. Whittaker, E.T., Watson, G.N.: Course of modern analysis. Cambridge: Cambridge Univ. Press, 1958