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A System of Difference Equations With Elliptic Coefficients and Bethe Vectors

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Abstract: An elliptic analogue of the q deformed Knizhnik–Zamolodchikov equations is introduced. A solution is given in the form of a Jackson-type integral of Bethe vectors of the XYZ-type spin chains.

Introduction

In this paper we introduce a holonomic system of difference equations associated to elliptic R matrices and give its solution in the form of a Jackson-type integral, following Reshetikhin's idea [R] for the trigonometric R matrices.

Reshetikhin constructed a solution to the q-deformed Knizhnik–Zamolodchikov equations [FR] by a Jackson-type integration of Bethe vectors of the XXZ-type spin chain models. Matsuo [Ma] also found the same kind of formulae from a different viewpoint.

On the other hand, the Bethe Ansatz method for the spin chain models associated to the elliptic R matrices has been studied since Baxter [B]. Hence a natural question is how to find an elliptic version of Reshetikhin's approach to the q-KZ equation. It turns out that the argument in [R] can be carried out for the elliptic R matrices as well, except for one point. In contrast to the trigonometric case, an elliptic spin chain model does not have a unique vacuum vector in its local state space but a series of "pseudo-vacua" which depend non-trivially on a spectral parameter. This dependence breaks down naive analogy.

We overcome this difficulty by introducing a "space of Bethe vectors" and a boundary operator which shifts a spectral parameter.

In the vertex picture the linear space of Bethe vectors depends on spectral parameters. Therefore we have to use the IRF picture in order to interpret the system as a holonomic matrix difference system in the sense of Aomoto [A]. In this context, our system is described in terms of representation of Felder's elliptic quantum groups [F].

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This paper is organized as follows. In the first section we recall several facts related to the elliptic R matrices and introduce a space of Bethe vectors and a boundary operator. We define a system of difference equations in the next section and show its holonomicity. The third section is devoted to construction of a solution of this system by a Jackson-type integral of Bethe vectors.

1. Space of Bethe Vectors

In this section we define a space of Bethe vectors of an elliptic spin chain and linear operators acting on this space. The state space $\mathscr{H}$ of a finite XYZ-type spin chain is defined to be a tensor product of the local state spaces:

$$\mathscr{H} := V^{l_1} \otimes V^{l_2} \otimes \cdots \otimes V^{l_N}\,, \tag{1.1}$$

where V^{l_j} $(j = 1,\dots,N)$ are the spin l_j representation spaces of the Sklyanin algebra. (See Appendix A for a review of the Sklyanin algebra and its representations.) We will introduce a direct sum of subspaces of $\mathscr{H}$ depending on spectral parameters which we will call a space of Bethe vectors. On this subspace act not only the R matrices but also a boundary operator which shifts the spectral parameters.

We assume that the parameter η which determines the structure constants of the Sklyanin algebra is a rational number, $\eta = r'/r$. This assumption is necessary to consider analytic solutions of the system later.

Space of Bethe vectors. Baxter [B] introduced vectors which intertwine vertex-type Boltzmann weight of the eight-vertex model and IRF-type Boltzmann weight. They are generalized to higher spin cases. (cf. [DJKMO], [T1][1])

Definition 1.1. *An* **(outgoing) intertwining vector**, $\phi^{(l)}_{\lambda,\lambda'}(u)$, *is a vector in the spin l representation space $V^l = \Theta_{00}^{4l+}$ of the Sklyanin algebra, defined by*:

$$\begin{aligned}
\phi^{(l)}_{\lambda,\lambda'}(u) &= \phi^{(l)}_{\lambda,\lambda'}(u; y)\\
&:= \prod_{j=1}^{l+m} \theta_{10}\left(y + \frac{\lambda-u}{2} + (2j-l-1)\eta\right)\theta_{10}\left(y - \frac{\lambda-u}{2} - (2j-l-1)\eta\right)\\
&\quad\times \prod_{j=1}^{l-m} \theta_{10}\left(y + \frac{\lambda'+u}{2} + (2j-l-1)\eta\right)\theta_{10}\left(y - \frac{\lambda'+u}{2} - (2j-l-1)\eta\right)\,,
\end{aligned} \tag{1.2}$$

where λ, λ' are parameters satisfying $\lambda - \lambda' = 4m\eta$, $m \in \{-l, -l+1, \dots, l\}$, and u is a complex parameter called a **spectral parameter**. *We call*

$$\omega^{(l)}_\lambda(u) = \phi^{(l)}_{\lambda,\lambda+4l\eta}(u) \tag{1.3}$$

a **local pseudo-vacuum**.

It is easy to see that generically $\{\phi_{\lambda+4m\eta,\lambda}(u)\}_{m=-l,-l+1,\dots,l}$ is a basis of V^l. Graphically an intertwining vector is denoted as in Fig. 1.

[1] The normalization in previous papers [T1] by the author has been improved. Several complicated factors are absent now. Vectors $\phi_{\lambda,\lambda'}(u)$ in those papers correspond to $\phi_{\lambda',\lambda}(u)$ in the present paper.

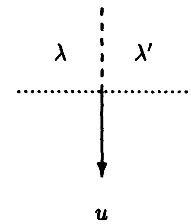

Fig. 1. An outgoing intertwining vector $\phi_{\lambda,\lambda'}(u)$

Definition 1.2. *A* **path vector** *is a vector in* $\mathscr{H}$ *defined as follows*:

$$|a_0, a_1, \ldots, a_N; z_1, \ldots, z_N; \mathring{\lambda}\rangle := \phi^{(l_1)}_{\lambda_0,\lambda_1}(z_1) \otimes \phi^{(l_2)}_{\lambda_1,\lambda_2}(z_2) \otimes \cdots \otimes \phi^{(l_N)}_{\lambda_{N-1},\lambda_N}(z_N)\,, \tag{1.4}$$

where $a_0, a_1, \ldots, a_N$ *are integers, satisfying the* **admissibility condition,**

$$a_{j-1} - a_j \in \{-2l_j, -2l_j + 2, \ldots, 2l_j - 2, 2l_j\}\,, \tag{1.5}$$

z_j*'s are complex parameters,* $\lambda_n = \mathring{\lambda} + 2a_n\eta$ *and* $\mathring{\lambda}$ *is a fixed parameter which we omit hereafter unless necessary. We call a path vector*

$$\begin{aligned}\Omega_a^{l_1,\ldots,l_N}(z_1, \ldots, z_N) &= |a_0, a_1, \ldots, a_N; z_1, \ldots, z_N\rangle \\ &= \omega^{l_1}_{\lambda_0}(z_1) \otimes \omega^{l_2}_{\lambda_1}(z_2) \otimes \cdots \otimes \omega^{l_N}_{\lambda_{N-1}}(z_N)\end{aligned} \tag{1.6}$$

with $a_0 = a$, $a_j = a + 2(l_1 + \cdots + l_j)$, $\lambda_j = \mathring{\lambda} + 2a_j\eta$, *a* **(global) pseudo-vacuum**.

We often denote $\Omega_a^{l_1,\ldots,l_N}(z_1, \ldots, z_N)$ by $\Omega_a(z_1, \ldots, z_N)$ for simplicity.
Since $\eta = r'/r$ is a rational number,

$$|a_0 + r, a_1 + r, \ldots, a_N + r; z_1, \ldots, z_N\rangle = |a_0, a_1, \ldots, a_N; z_1, \ldots, z_N\rangle\,, \tag{1.7}$$

because of the periodicity of theta functions.

A graphical notation for a path vector $|a_0, a_1, \ldots, a_N; z_1, \ldots, z_N\rangle$ is shown in Fig. 2. Though we should write λ_n in this figure instead of a_n for the consistency with Fig. 1, we use a_n for later convenience.

Definition 1.3. *The* **space of Bethe vectors** $\mathfrak{B}^{l_1,\ldots,l_N}_{z_1,\ldots,z_N}$ *with spectral parameters* $\vec{z} = (z_1, z_2, \ldots, z_N)$ *is a space of functions*

$$f : \mathbb{Z}/r\mathbb{Z} \ni \nu \mapsto f(\nu) \in \mathscr{H}$$

of the form

$$f(\nu) = \sum_{a=0}^{r-1} e^{2\pi i a\nu\eta} \sum_{\vec{a}} \varphi_{\vec{a}}\, |a_0, a_1, \ldots, a_N; z_1, \ldots, z_N\rangle\,,$$

where $\varphi_{\vec{a}}$ *are complex numbers and sequences of integers* $\vec{a} = (a_0, a_1, \ldots, a_N)$ *satisfy* $a_0 = a_N = a$ *and the admissibility condition* (1.5). (*cf.* (1.7).)

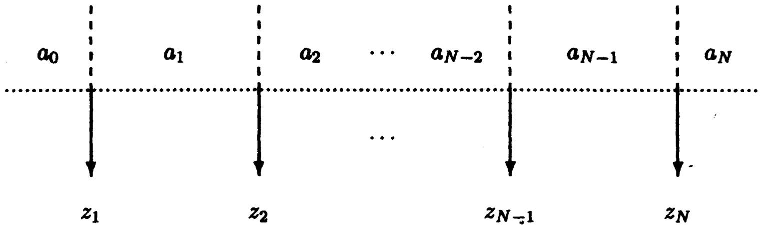

Fig. 2. Path vector $|a_0, a_1, \dots, a_N; z_1, \dots, z_N\rangle$

Generically,

$$\left\{ e^{2\pi i a v \eta} |a_0, \dots, a_N; z_1, \dots, z_N\rangle \;\middle|\; \begin{array}{l} a_0 = a_N = a \in \{0, \dots, r-1\}\,, \\ (a_0, \dots, a_N) \text{ satisfies } (1.5)\,. \end{array} \right\} \tag{1.8}$$

is a basis of $\mathfrak{B}^{l_1, \dots, l_N}_{z_1, \dots, z_N}$. Thus

$$\dim \mathfrak{B}^{l_1, \dots, l_N}_{z_1, \dots, z_N} = r \dim(\text{weight zero space of } W^{l_1} \otimes \cdots \otimes W^{l_N})\,, \tag{1.9}$$

where W^l are the spin l (i.e., $(2l+1)$-dimensional) irreducible representations of the Lie algebra $sl(2, \mathbf{C})$.

Remark 1.4. When η is not a rational number (or, exactly speaking, not a point of finite order on the elliptic curve $\mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$), the sum should be taken over all $a \in \mathbf{Z}$ and v is a continuous parameter. The sum might be considered as a formal series.

R matrices. **Elliptic** R **matrices** $R = R^{l,l'}(u)$ acting on the space $V^l \otimes V^{l'}$ are constructed by means of the fusion procedure (cf. [Ch, DJKMO, HZ, T2]). We recall the following most important properties and refer details to [T2].

(i) $R^{l,l'}(u)$ is a linear endomorphism of $V^l \otimes V^{l'}$ meromorphically depending on a complex parameter u (Fig. 3).

(ii) *Yang–Baxter equation.*

$$\begin{aligned} &R^{l,l'}_{12}(z_1 - z_2) R^{l,l''}_{13}(z_1 - z_3) R^{l',l''}_{23}(z_2 - z_3) \\ &= R^{l',l''}_{23}(z_2 - z_3) R^{l,l''}_{13}(z_1 - z_3) R^{l,l'}_{12}(z_1 - z_2) \end{aligned} \tag{1.10}$$

as an endomorphism of $V^l \otimes V^{l'} \otimes V^{l''}$ (Fig. 4).

(iii) *Unitarity.*

$$R^{l,l'}_{12}(u - v) R^{l',l}_{21}(v - u) = \mathrm{Id}_{V^l \otimes V^{l'}}\,, \tag{1.11}$$

as an endomorphism of $V^l \otimes V^{l'}$ (Fig. 5).

(iv) $R^{l,l'}(u)$ acts on the intertwining vectors as follows:

$$R^{l,l'}(u - v) \phi^{(l)}_{\lambda,\lambda'}(u) \otimes \phi^{(l')}_{\lambda',\mu}(v) = \sum_{\mu'} W\left(\begin{matrix} \lambda & \lambda' \\ \mu' & \mu \end{matrix} \middle| u - v \right) \phi^{(l)}_{\mu',\mu}(u) \otimes \phi^{(l')}_{\lambda,\mu'}(v)\,, \tag{1.12}$$

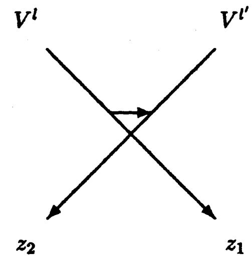

Fig. 3. R matrix $R^{l,l'}(z_1 - z_2)$

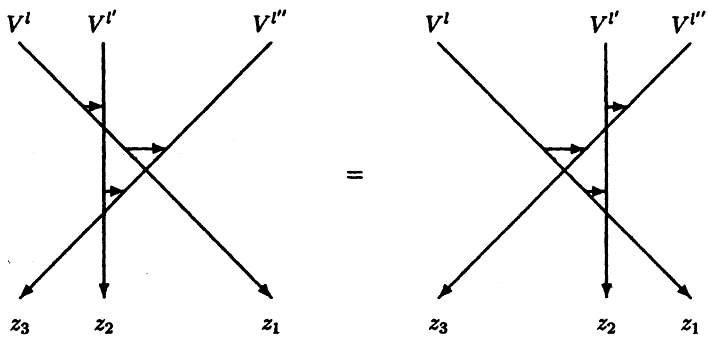

Fig. 4. Yang–Baxter equation

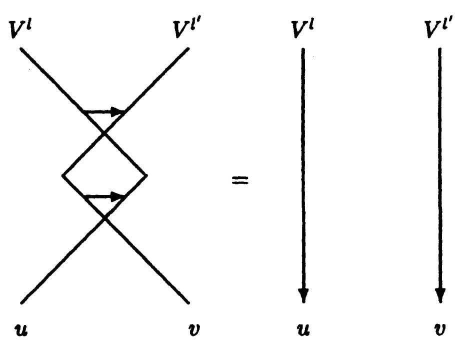

Fig. 5. Unitarity

where the sum on the right-hand side is taken over μ' satisfying $\mu' - \mu = 4m\eta$ ($m \in \{-l, -l+1, \dots, l\}$) and $\lambda - \mu' = 4m'\eta$ ($m \in \{-l', -l'+1, \dots, l'\}$). Scalar factors W are the Boltzmann weights of the IRF-type model. (Fig. 6. In the figure W is denoted by a crossing of dashed lines.)

Thanks to (iv), the R matrix defines a map between the spaces of Bethe vectors:

$$\check{R}_{j,j+1}(z_j - z_{j+1}) := P_{j,j+1} R_{j,j+1}(z_j - z_{j+1}) : \mathfrak{B}^{l_1,\dots,l_j,l_{j+1},\dots,l_N}_{z_1,\dots,z_j,z_{j+1},\dots,z_N} \to \mathfrak{B}^{l_1,\dots,l_{j+1},l_j,\dots,l_N}_{z_1,\dots,z_{j+1},z_j,\dots,z_N}, \tag{1.13}$$

where $P_{j,j+1}$ is a permutation operator of the j^{th} and the $(j+1)^{\text{st}}$ component of the tensor product $V^{l_1} \otimes \cdots \otimes V^{l_N}$. With respect to the basis (1.8), the R matrix is described by the IRF-type Boltzmann weight, and thus by a representation of Felder's elliptic quantum group [F]. For our purpose, the following special cases of (1.12) are important.

When both $\phi^{(l)}_{\lambda,\lambda'}(v)$ and $\phi^{(l')}_{\lambda',\mu}(u)$ are local pseudo-vacua (1.3), i.e., when $\lambda' = \lambda + 4l\eta$ and $\mu = \lambda' + 4l'\eta$, Eq. (1.12) is simply

$$R^{l,l'}(u-v)\omega^{(l)}_{\lambda}(u) \otimes \omega^{(l')}_{\lambda-4l\eta}(v) = \omega^{(l)}_{\lambda+4l'\eta}(u) \otimes \omega^{(l')}_{\lambda}(v). \tag{1.14}$$

When $l = 1/2$, the R matrix $R^{1/2,l}$ is expressed as the L operator (A.4) through the identification (A.9):

$$R^{1/2,l}(u) = \frac{\theta_{11}(2\eta)}{\theta_{11}(u+(2l+1)\eta)}\rho^l(L(u+\eta)) = \sum_{a=0}^{3} \frac{\theta_{11}(2\eta)W^L_a(u+\eta)}{\theta_{11}(u+(2l+1)\eta)}\sigma^a \otimes \rho^{(l)}(S^a). \tag{1.15}$$

In particular, $R^{1/2,1/2}(u)$ is proportional to Baxter's R matrix (A.5). Likewise the intertwining vectors $\phi^{(1/2)}_{\lambda,\lambda\pm 2\eta}(u)$ correspond to the following vectors in $\mathbb{C}^2$:

$$\phi^{(1/2)}_{\lambda\pm 2\eta,\lambda}(u-\eta) = C\begin{pmatrix} -\theta_{01}((\lambda \pm u)/2; \tau/2) \\ \theta_{00}((\lambda \pm u)/2; \tau/2) \end{pmatrix}, \tag{1.16}$$

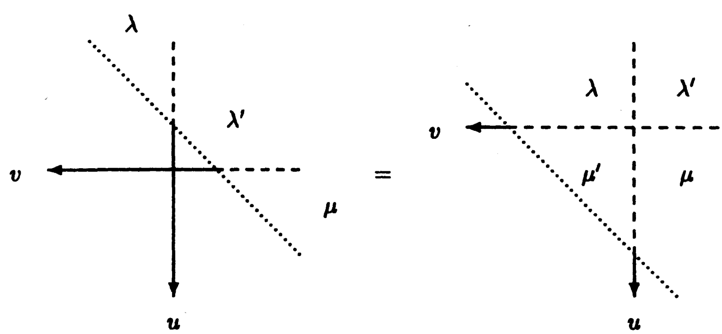

Fig. 6. R and IRF weight are intertwined

where C is a constant:

$$C = e^{-\pi i\tau/8} \frac{\theta_{00}(0;\tau)^2\theta_{01}(0;\tau)\theta_{10}(0;\tau)}{2\theta_{10}(0;\tau/2)\theta_{01}(0;2\tau)\theta_{10}((1+\tau)/4;\tau)\theta_{10}((1-\tau)/4;\tau)} .$$

Under this identification, the action of $R^{1/2,l}$ (1.12) can be stated in the following form which will be used later. Let us define a matrix of the gauge transformation by

$$M_\lambda(u) := \begin{pmatrix} -\theta_{01}((\lambda-u)/2;\tau/2) & -\theta_{01}((\lambda+u)/2;\tau/2) \\ \theta_{00}((\lambda-u)/2;\tau/2) & \theta_{00}((\lambda+u)/2;\tau/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta_{11}(\lambda;\tau)^{-1} \end{pmatrix}, \tag{1.17}$$

and a twisted L operator by

$$L_{\lambda,\lambda'}(u;v) = \begin{pmatrix} \alpha_{\lambda,\lambda'}(u;v) & \beta_{\lambda,\lambda'}(u;v) \\ \gamma_{\lambda,\lambda'}(u;v) & \delta_{\lambda,\lambda'}(u;v) \end{pmatrix} := M_\lambda(u)^{-1}L(u-v)M_{\lambda'}(u) . \tag{1.18}$$

Then the matrix elements of $L_{\lambda,\lambda'}(u;v)$ act on an intertwining vector as follows: Let $\lambda - \lambda' = 4m\eta$ ($m \in \{-l, -l+1, \dots, l\}$). Then,

$$\alpha_{\lambda,\lambda'}(u;v)\phi^{(l)}_{\lambda',\lambda}(v) = \frac{\theta_{11}(u-v+2m\eta)\theta_{11}(\lambda+2(l-m)\eta)}{\theta_{11}(\lambda)\theta_{11}(2\eta)}\phi^{(l)}_{\lambda'-2\eta,\lambda-2\eta}(v), \tag{1.19}$$

$$\beta_{\lambda,\lambda'}(u;v)\phi^{(l)}_{\lambda',\lambda}(v) = \frac{\theta_{11}(u-v+\lambda-2m\eta)\theta_{11}(2(l+m)\eta)}{\theta_{11}(\lambda)\theta_{11}(\lambda-4m\eta)\theta_{11}(2\eta)}\phi^{(l)}_{\lambda'+2\eta,\lambda-2\eta}(v) , \tag{1.20}$$

$$\gamma_{\lambda,\lambda'}(u;v)\phi^{(l)}_{\lambda',\lambda}(v) = \frac{\theta_{11}(u-v-\lambda+2m\eta)\theta_{11}(2(-l+m)\eta)}{\theta_{11}(2\eta)}\phi^{(l)}_{\lambda'-2\eta,\lambda+2\eta}(v) , \tag{1.21}$$

$$\delta_{\lambda,\lambda'}(u;v)\phi^{(l)}_{\lambda',\lambda}(v) = \frac{\theta_{11}(u-v-2m\eta)\theta_{11}(\lambda-2(l+m)\eta)}{\theta_{11}(\lambda-4m\eta)\theta_{11}(2\eta)}\phi^{(l)}_{\lambda'+2\eta,\lambda+2\eta}(v) . \tag{1.22}$$

This formulation was found in the context of the quantum inverse scattering method and the algebraic Bethe Ansatz [TF]. See also [T1]. Note that the column vectors of $M_\lambda(u)$ are essentially outgoing intertwining vectors. Therefore, defining the *incoming intertwining vectors* $\bar\phi_{\lambda,\lambda'}(u)$ as row vectors of $M_\lambda(u)^{-1}$:

$$M_\lambda(u)^{-1} = \begin{pmatrix} \bar\phi_{\lambda,\lambda-2\eta}(u-\eta) \\ \bar\phi_{\lambda,\lambda+2\eta}(u-\eta) \end{pmatrix}, \tag{1.23}$$

and denoting them as in Fig. 7, we can rewrite formulae (1.19)–(1.22) as in Fig. 8. (Exactly speaking, the normalization here is different from that in Fig. 6, which is not essential. In general, incoming intertwining vectors are defined as a dual basis of $\{\phi_{\lambda+4m\eta,\lambda}(u)\}_{m=-l,-l+1,\dots,l}$. See [H].)

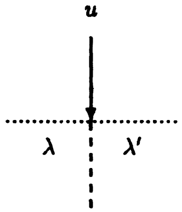

Fig. 7. An incoming intertwining vector $\bar{\phi}_{\lambda',\lambda}(u)$

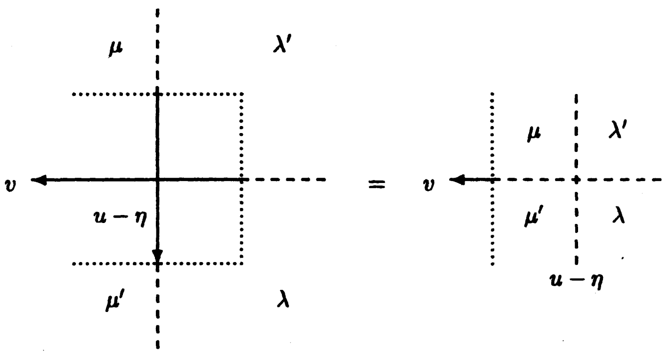

Fig. 8. An element of a twisted L operator acts on $\phi_{\lambda',\lambda}(v)$

Especially α, γ and δ act on a local pseudo vacuum as follows:

$$\alpha_{\lambda,\lambda'}(u;v)\omega^{(l)}_{\lambda'}(v) = \alpha^l(u-v)\omega^{(l)}_{\lambda'-2\eta}(v)\,, \tag{1.24}$$

$$\gamma_{\lambda,\lambda'}(u;v)\omega^{(l)}_{\lambda'}(v) = 0\,, \tag{1.25}$$

$$\delta_{\lambda,\lambda'}(u;v)\omega^{(l)}_{\lambda'}(v) = \delta^l(u-v)\omega^{(l)}_{\lambda'+2\eta}(v)\,, \tag{1.26}$$

where

$$\alpha^l(u) = \frac{\theta_{11}(u+2l\eta)}{\theta_{11}(2\eta)}, \qquad \delta^l(u) = \frac{\theta_{11}(u-2l\eta)}{\theta_{11}(2\eta)}\,. \tag{1.27}$$

Boundary operator. We introduce an operator Z which shifts a spectral parameter of a Bethe vector. Let κ and c be fixed complex parameters.

Definition 1.5. *A* **boundary operator** $Z = Z^{l_1,\dots,l_N}_{z_1,\dots,z_N}(\kappa,c)$ *is a linear map from* $\mathfrak{B}^{l_1,\dots,l_N}_{z_1,\dots,z_N}$ *to* $\mathfrak{B}^{l_N,l_1,\dots,l_{N-1}}_{z_N+\kappa,z_1,\dots,z_{N-1}}$ *defined by*

$$\begin{aligned} &Z(e^{2\pi i a_N v\eta}|a_0,\dots,a_N;z_1,\dots,z_N\rangle) \\ &\quad = e^{2\pi i a_{N-1}v\eta}e^{c(a_N-a_{N-1}-2l_N)}|a_{N-1},a_0,\dots,a_{N-1};z_N+\kappa,z_1,\dots,z_{N-1}\rangle\,. \end{aligned} \tag{1.28}$$

(*Note that* $a_0 = a_N$ *by Definition* 1.3, *cf. Fig.* 9)

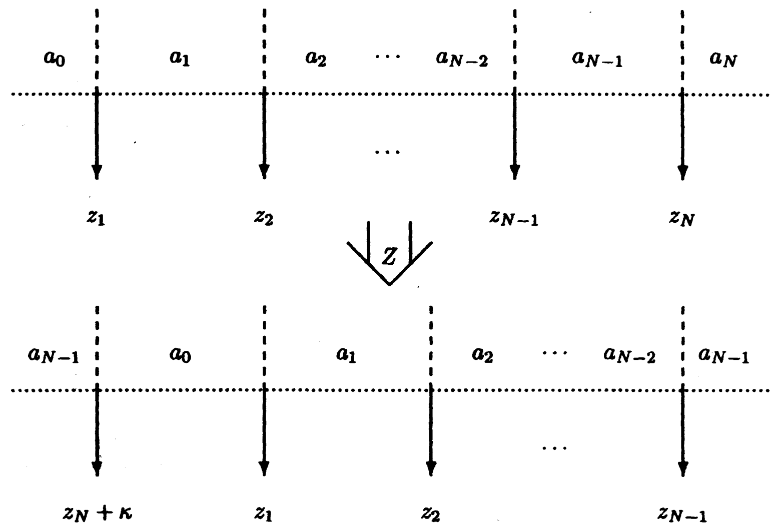

Fig. 9. A boundary operator

With respect to the basis (1.8), Z is a composite of a permutation and a diagonal matrix.

The following property is important.

Lemma 1.6. *The operator Z commutes with an R matrix as follows:*

$$\check{R}_{12}^{l_{N-1},l_N}(z_{N-1}-z_N)Z^{l_N,l_1,\dots,l_{N-1}}_{z_N+\kappa,z_1,\dots,z_{N-1}}(\kappa,c)Z^{l_1,\dots,l_N}_{z_1,\dots,z_N}(\kappa,c)$$
$$= Z^{l_{N-1},l_1,\dots,l_{N-2},l_N}_{z_{N-1}+\kappa,z_1,\dots,z_{N-2},z_N}(\kappa,c)Z^{l_1,\dots,l_{N-2},l_N,l_{N-1}}_{z_1,\dots,z_{N-2},z_N,z_{N-1}}(\kappa,c)\check{R}^{l_{N-1},l_N}_{N-1,N}(z_{N-1}-z_N) \quad (1.29)$$

as a map from $\mathfrak{B}^{l_1,\dots,l_N}_{z_1,\dots,z_N}$ to $\mathfrak{B}^{l_N,l_{N-1},l_1,\dots,l_{N-2}}_{z_N+\kappa,z_{N-1}+\kappa,z_1,\dots,z_{N-2}}$.

This is easily proved by comparing the action of the both sides on the basis (1.8). A graphical notation for this equality is shown in Fig. 10.

2. Difference Equation

In this section we introduce a system of difference equations which is an elliptic analogue of the equations introduced in [FR], and show holonomicity of this system in the sense of Aomoto [A].

Let us first define a linear operator $A_j(\vec{z})$, $\vec{z}=(z_1,\dots,z_N)$, from $\mathfrak{B}^{l_1,\dots,l_N}_{z_1,\dots,z_N}$ to $\mathfrak{B}^{l_1,\dots,l_j,\dots,l_N}_{z_1,\dots,z_j+\kappa,\dots,z_N}$ by

$$A_j(\vec{z}) := \check{R}^{l_j,l_{j-1}}_{j-1,j}(z_j+\kappa-z_{j-1})\cdots\check{R}^{l_j,l_1}_{12}(z_j+\kappa-z_1)$$
$$\cdot\ Z^{l_1,\dots,l_{j-1},l_{j+1},\dots,l_N,l_j}_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_N,z_j}(\kappa,c)\check{R}^{l_j,l_N}_{N-1,N}(z_j-z_N)\cdots\check{R}^{l_j,l_{j+1}}_{j,j+1}(z_j-z_{j+1}), \quad (2.1)$$

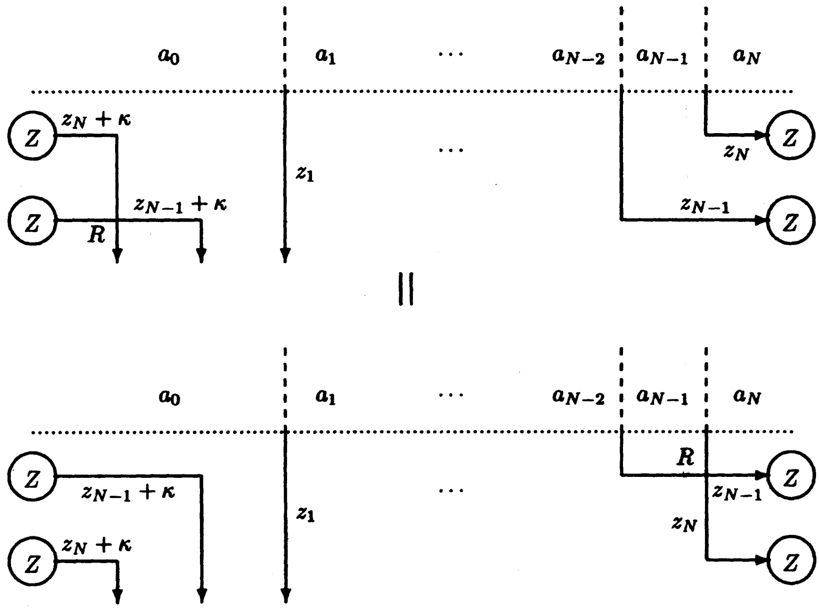

Fig. 10. $RZZ = ZZR$, Lemma 1.6

or, equivalently,

$$A_j(\vec{z}) := R^{l_j,l_{j-1}}_{j,j-1}(z_j+\kappa-z_{j-1})\cdots R^{l_j,l_1}_{j,1}(z_j+\kappa-z_1)$$

$$\cdot\ Z^{(j)}(\kappa,c)R^{l_j,l_N}_{j,N}(z_j-z_N)\cdots R^{l_j,l_{j+1}}_{j,j+1}(z_j-z_{j+1})\,, \tag{2.2}$$

where

$$Z^{(j)}(\kappa,c) = P_{j-1,j}P_{j-2,j-1}\cdots P_{12}$$

$$\cdot\ Z^{l_1,\dots,l_{j-1},l_{j+1},\dots,l_N,l_j}_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_N,z_j}(\kappa,c)P_{N-1,N}\cdots P_{j+1,j+2}P_{j,j+1}\,. \tag{2.3}$$

Then we can define our main object, a system of difference equations:

$$f(\vec{z}_j) = A_j(\vec{z})f(\vec{z})\,, \tag{2.4}$$

for $j=1,\dots,N$, where $\vec{z}_j = (z_1,\dots,z_j+\kappa,\dots,z_N)$ and $f(\vec{z}) \in \mathfrak{B}^{l_1,\dots,l_N}_{z_1,\dots,z_N}$. Expanding f as in (1.3),

$$f(\vec{z}) = \sum_{a=0}^{r-1} e^{2\pi i a v \eta}\sum_{\vec{a}} f_{\vec{a}}(\vec{z})\,|\vec{a};\vec{z}\rangle\,,$$

we can regard the system (2.4) as a system of equations for $f_{\vec{a}}(\vec{z})$. In this IRF picture, this system is holonomic in the sense of Aomoto [A] due to the following:

Proposition 2.1. *Operators $A_j(\vec{z})$ are compatible*:

$$A_j(\vec{z}_k)A_k(\vec{z}) = A_k(\vec{z}_j)A_j(\vec{z})\,.$$

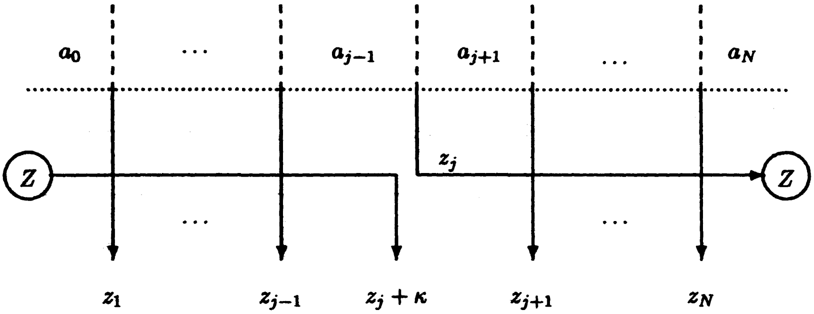

Fig. 11. Operator $A_j(\bar{z})$

This follows from the Yang–Baxter equation (1.10), the unitarity (1.11) and the commutativity of R and Z (Lemma 1.6) as in Theorem 5.4 of [FR]. Symbolically, we have only to change the order of crossings of lines in Fig. 12, using the procedures of Fig. 4, Fig. 5 and Fig. 10.

3. Jackson-Type Integral Solution

In this section we give a solution of the system of difference equations (2.4) which is expressed as a Jackson-type integral of Bethe vectors. This result is an elliptic analogue of that of [R]. We assume that $M := l_1 + \cdots + l_N$ is an integer.

First let us define the *monodromy matrix* $T(u;z_1,\ldots,z_N)$ by

$$T^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) = \begin{pmatrix} A^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) & B^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) \\ C^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) & D^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) \end{pmatrix}$$
$$:= L_N^{l_N}(u-z_N)\cdots L_2^{l_2}(u-z_2)L_1^{l_1}(u-z_1) \tag{3.1}$$

as an endomorphism of $\mathbb{C}^2 \otimes \mathscr{H} = \mathbb{C}^2 \otimes V^{l_1} \otimes \cdots \otimes V^{l_N}$. Here $L_j^{l_j}(u)$ acts non-trivially only on $\mathbb{C}^2$ and V^{l_j}:

$$L_j^{l_j}(u) = \sum_{a=0}^{3} W_a^L(u)\sigma^a \otimes \rho_j^{l_j}(S^a),$$

$$\rho_j^{l_j} = 1 \otimes \cdots \otimes 1 \otimes \rho^{l_j} \otimes 1 \otimes \cdots \otimes 1 : U_{\tau,\eta}(sl(2)) \to \mathrm{End}(\mathscr{H}).$$

Bethe vectors of the XYZ type spin chains are constructed by means of the twisted monodromy matrix defined as follows:

$$T_{\lambda,\lambda'}^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) = \begin{pmatrix} A_{\lambda,\lambda'}^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) & B_{\lambda,\lambda'}^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) \\ C_{\lambda,\lambda'}^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) & D_{\lambda,\lambda'}^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1) \end{pmatrix}$$
$$:= M_\lambda(u)^{-1}T^{l_N,\ldots,l_1}(u;z_N,\ldots,z_1)M_{\lambda'}(u), \tag{3.2}$$

where $M_\lambda(u)$ is defined by (1.17). We often denote, for example, $B_{\lambda+2a\eta,\lambda+2a'\eta}^{l_N,\ldots,l_1}$ by $B_{a,a'}^{l_N,\ldots,l_1}$ or $B_{a,a'}$ for simplicity, and graphically as in Fig. 13 (cf. Fig. 8).

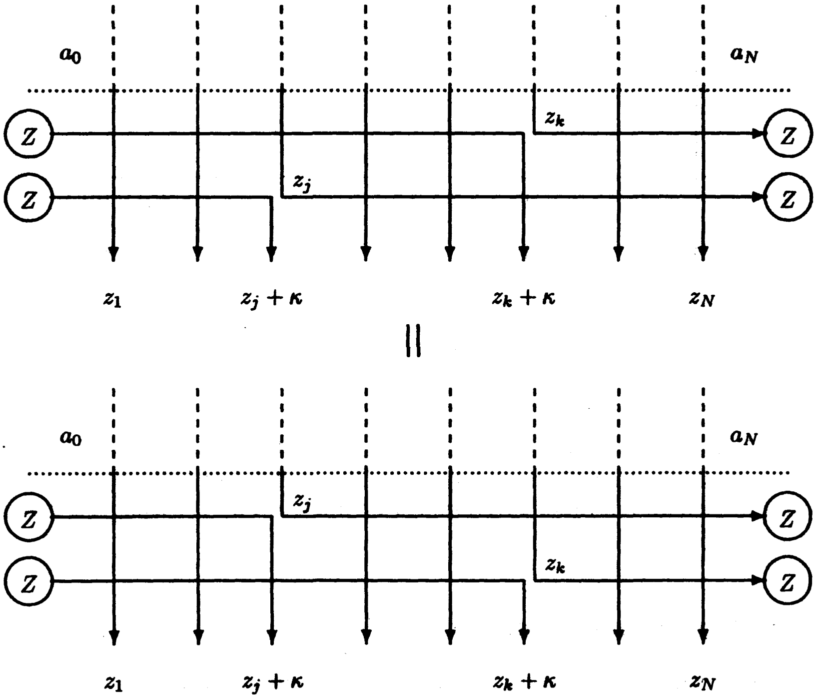

Fig. 12. Holonomicity of the system (2.4)

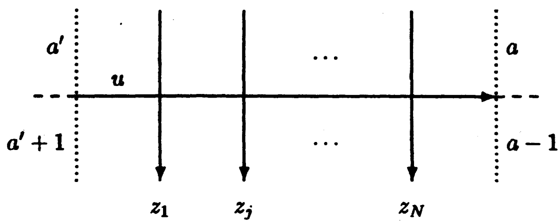

Fig. 13. Operator $B^{l_N,\dots,l_1}_{a,a'}(u; z_N, \dots, z_1)$

It follows from (1.24), (1.25), (1.26) that

$$A^{l_N,\dots,l_1}_{a+2M,a}(u; z_N, \dots, z_1)\Omega_a(\vec{z}) = \left(\prod_{j=1}^{N} \alpha^{l_j}(u - z_j)\right) \Omega_{a-1}(\vec{z}) , \tag{3.3}$$

$$C^{l_N,\dots,l_1}_{a+2M,a}(u; z_N, \dots, z_1)\Omega_a(\vec{z}) = 0 , \tag{3.4}$$

$$D^{l_N,\dots,l_1}_{a+2M,a}(u; z_N, \dots, z_1)\Omega_a(\vec{z}) = \left(\prod_{j=1}^{N} \delta^{l_j}(u - z_j)\right) \Omega_{a+1}(\vec{z}) . \tag{3.5}$$

By a standard argument in [TF], these operators satisfy the following commutation relations:

$$\begin{aligned}&B_{a,a'+1}(u;z_N,\dots,z_1)B_{a+1,a'}(v;z_N,\dots,z_1)\\&=B_{a,a'+1}(v;z_N,\dots,z_1)B_{a+1,a'}(u;z_N,\dots,z_1)\,,\end{aligned}\tag{3.6}$$

$$\begin{aligned}&A_{a,a'+1}(u;z_N,\dots,z_1)B_{a+1,a'}(v;z_N,\dots,z_1)\\&=\alpha(u-v)B_{a,a'-1}(v;z_N,\dots,z_1)A_{a+1,a'}(u;z_N,\dots,z_1)\\&\quad+\beta_{a'}(u-v)B_{a,a'-1}(u;z_N,\dots,z_1)A_{a+1,a'}(v;z_N,\dots,z_1)\,,\end{aligned}\tag{3.7}$$

$$\begin{aligned}&D_{a-1,a'}(u;z_N,\dots,z_1)B_{a,a'-1}(v;z_N,\dots,z_1)\\&=\alpha(v-u)B_{a+1,a'}(v;z_N,\dots,z_1)D_{a,a'-1}(u;z_N,\dots,z_1)\\&\quad-\beta_a(u-v)B_{a+1,a'}(u;z_N,\dots,z_1)D_{a,a'-1}(v;z_N,\dots,z_1)\,,\end{aligned}\tag{3.8}$$

where

$$\alpha(u)=\frac{\theta_{11}(u-2\eta)}{\theta_{11}(u)},\qquad \beta_a(u)=\frac{\theta_{11}(u-\mathring{\lambda}-2a\eta)\theta_{11}(2\eta)}{\theta_{11}(u)\theta_{11}(\mathring{\lambda}+2a\eta)}\,.\tag{3.9}$$

Let us recall the definition of N-cycle with step κ in [R].

Definition 3.1. *Let Ψ be a function on $\mathbf{C}^N$ and $\mathscr{C}$ be a subset of $\{(x_1+m_1\kappa,\dots,x_N+m_N\kappa)\,|\,(m_1,\dots,m_N)\in\mathbf{Z}^N\}$ for a certain $(x_1,\dots,x_N)$. We call $\mathscr{C}$ an N-***cycle with step*** κ **for** Ψ *if*

$$\sum_{\vec{t}\in\mathscr{C}}\Psi(\vec{t}+\vec{n}\kappa)=\sum_{\vec{t}\in\mathscr{C}}\Psi(\vec{t})\,,$$

for any $\vec{n}\in\mathbf{Z}^N$.

Functions $F^l(t)$ and $\Phi(t)$ are defined as solutions of the ordinary difference equations:

$$F^l(t+\kappa)=\frac{\delta^l(t)}{\alpha^l(t+\kappa)}F^l(t)\,,\tag{3.10}$$

$$\Phi(t+\kappa)=\frac{\alpha(-t)}{\alpha(t+\kappa)}\Phi(t)\,.\tag{3.11}$$

We shall give an explicit solution to these equations later in terms of infinite products.

The main theorem of this paper is:

Theorem 3.2. *Let*

$$\varphi(\vec{z}|\vec{t})=e^{c\sum_{i=1}^M t_i}\cdot\prod_{1\leq i<j\leq M}\Phi(t_i-t_j)\cdot\prod_{j=1}^M\prod_{n=1}^N F^{l_n}(t_j-z_n)\,,\tag{3.12}$$

where $\vec{t} = (t_1, \dots, t_M)$. *Then*

$$f(\vec{z}) = \sum_{\vec{t} \in \mathscr{C}} \varphi(\vec{z}|\vec{t})\Psi(\vec{t}) \tag{3.13}$$

is a solution of (2.4), *where*

$$\Psi(\vec{t}) = \sum_{a=0}^{r-1} e^{2\pi i a v \eta} B_{a+1,a-1}(t_1; z_N, \dots, z_1) B_{a+2,a-2}(t_2; z_N, \dots, z_1)$$
$$\cdots B_{a+M,a-M}(t_M; z_N, \dots, z_1)\Omega_{a-M}(\vec{z}) \tag{3.14}$$

and $\mathscr{C}$ *is an* N*-cycle with step* κ *for* $\varphi(\vec{z}|\vec{t})\Psi(\vec{t})$.

Graphically, a summand in (3.14) is denoted as in Fig. 14.

Remark 3.3. The vector $\Psi(\vec{t})$ gives an eigenvector of the transfer matrix of the XYZ type spin chains when $\{v, t_1, \dots, t_M\}$ satisfies so-called Bethe equations. (See [B, TF, T1]).

The proof of the theorem is essentially the same as that of Theorem 1.4 of [R]. We first reduce Eq. (2.4) including the operator A_j ((2.1) or (2.2)) to the system

$$\check{R}_{12}^{l_1,l_j}(z_1 - z_j - \kappa) \cdots \check{R}_{j-1,j}^{l_{j-1},l_j}(z_{j-1} - z_j - \kappa) f(\vec{z}_j)$$
$$= Z^{l_1,\dots,l_{j-1},l_{j+1},\dots,l_N,l_j}_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_N,z_j}(\kappa, c)\check{R}^{l_j,l_N}_{N-1,N}(z_j - z_N) \cdots \check{R}^{l_j,l_{j+1}}_{j,j+1}(z_j - z_{j+1}) f(\vec{z}), \tag{3.15}$$

using the unitarity (1.11).

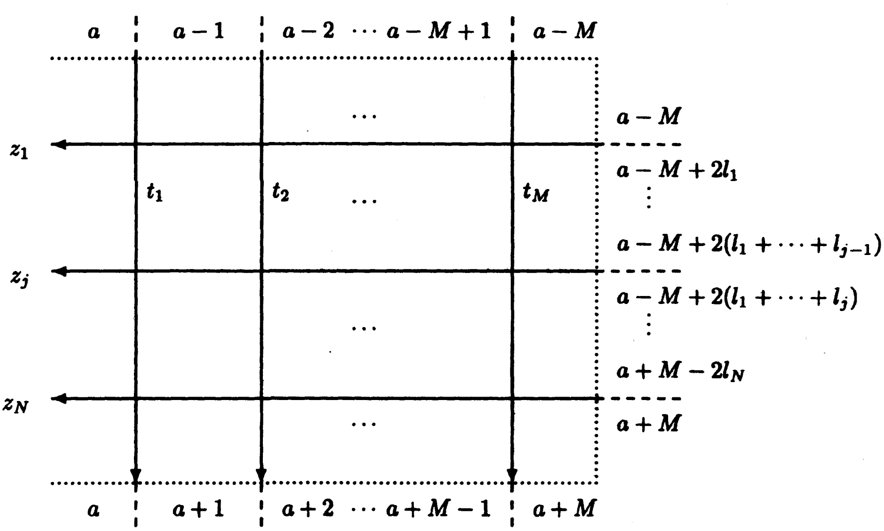

Fig. 14. $B_{a+1,a-1}(t_1) \cdots B_{a+M,a-M}(t_M)\Omega_{a-M}(\vec{z})$

Assuming the form of the solution (3.13), we can show that (3.15) is equivalent to

$$\sum_{a,\vec{t}} \varphi(\vec{z}_j|\vec{t}) e^{2\pi i a v \eta} B^{l_N,\dots,l_1,l_j}_{a+1,a-1}(t_1;z_N,\dots,z_1,z_j+\kappa)$$

$$\cdots B^{l_N,\dots,l_1,l_j}_{a+M,a-M}(t_M;z_N,\dots,z_1,z_j+\kappa)\Omega^{l_j,l_1,\dots,l_N}_{a-M}(z_j+\kappa,z_1,\dots,z_N)$$

$$= \sum_{a,\vec{t}} \varphi(\vec{z}|\vec{t}) Z(e^{2\pi i a v \eta} B^{l_j,l_N,\dots,l_1}_{a+1,a-1}(t_1;z_j,z_N,\dots,z_1)$$

$$\cdots B^{l_j,l_N,\dots,l_1}_{a+M,a-M}(t_M;z_j,z_N,\dots,z_1)\Omega^{l_1,\dots,l_N,l_j}_{a-M}(z_1,\dots,z_N,z_j))\,. \qquad (3.16)$$

Here, for example, $(z_N,\dots,z_1,z_j)$ means $(z_N,\dots,z_{j+1},z_{j-1},\dots,z_1,z_j)$. Equivalence of the left-hand side of (3.15) and (3.16) is proved as illustrated in Fig. 15. We move the line with the spectral parameter $z_j+\kappa$, repeatedly using the procedures Fig. 4, and then Fig. 6. Because of the admissibility condition (1.5) and Eq. (1.14), the IRF-type weights appearing in Fig. 15 (crossings of dashed lines) are equal to the unity. The right-hand side of (3.16) is derived from the right-hand side of (3.15) in the same manner.

In the next step we need a counterpart of Lemma 2.3 of [R].

Lemma 3.4 (*Two-side formula*). *Fix* n $(1 \leqq n \leqq N-1)$ *and* m $(1 \leqq m \leqq M)$ *and let* a *be an integer,* $a_0 = a$ *and* $a_j = a + 2(l_1 + \dots + l_j)$ *for all* $j \in \{1,\dots,N\}$. *Then*

$$B^{l_N,\dots,l_1}_{a_N-m+1,a_0+m-1}(t_m;z_N,\dots,z_1)\cdots B^{l_N,\dots,l_1}_{a_N,a_0}(t_1;z_N,\dots,z_1)\Omega^{l_1,\dots,l_N}_a(\vec{z})$$

$$= \sum_{\{1,\dots,m\}=\mathrm{I}\sqcup\mathrm{II}} \prod_{i\in\mathrm{I},i'\in\mathrm{II}} \alpha(t_i,t_{i'}) \prod_{i\in\mathrm{I}} \prod_{k=n+1}^{N} \alpha^{l_k}(t_i-z_k) \prod_{i'\in\mathrm{II}} \prod_{k=1}^{n} \delta^{l_k}(t_{i'}-z_k)$$

$$\times B^{l_n,\dots,l_1}_{a_n-\#(\mathrm{I})+\#(\mathrm{II})+1,a_0+m-1}(t_{i_1};z_n,\dots,z_1)$$

$$\cdots B^{l_n,\dots,l_1}_{a_n+\#(\mathrm{II}),a_0+\#(\mathrm{II})}(t_{i_{\#(\mathrm{I})}};z_n,\dots,z_1)\Omega^{l_1,\dots,l_n}_{a_0+\#(\mathrm{II})}(z_1,\dots,z_n)$$

$$\otimes B^{l_N,\dots,l_{n+1}}_{a_N-m+1,a_n-\#(\mathrm{I})+\#(\mathrm{II})-1}(t_{i'_1};z_N,\dots,z_{n+1})$$

$$\cdots B^{l_N,\dots,l_{n+1}}_{a_N-\#(\mathrm{I}),a_n-\#(\mathrm{I})}(t_{i'_{\#(\mathrm{II})}};z_N,\dots,z_{n+1})\Omega^{l_{n+1},\dots,l_N}_{a_n-\#(\mathrm{I})}(z_{n+1},\dots,z_N)\,, \qquad (3.17)$$

where $\{1,\dots,m\} = \mathrm{I}\sqcup\mathrm{II}$ *is a partition,* $\#$ *denotes the number of elements,* $i_\bullet$ *designates an element of* I *and* $i'_\bullet$ *an element of* II.

Due to (3.6), *the right-hand side of* (3.17) *is well-defined.*

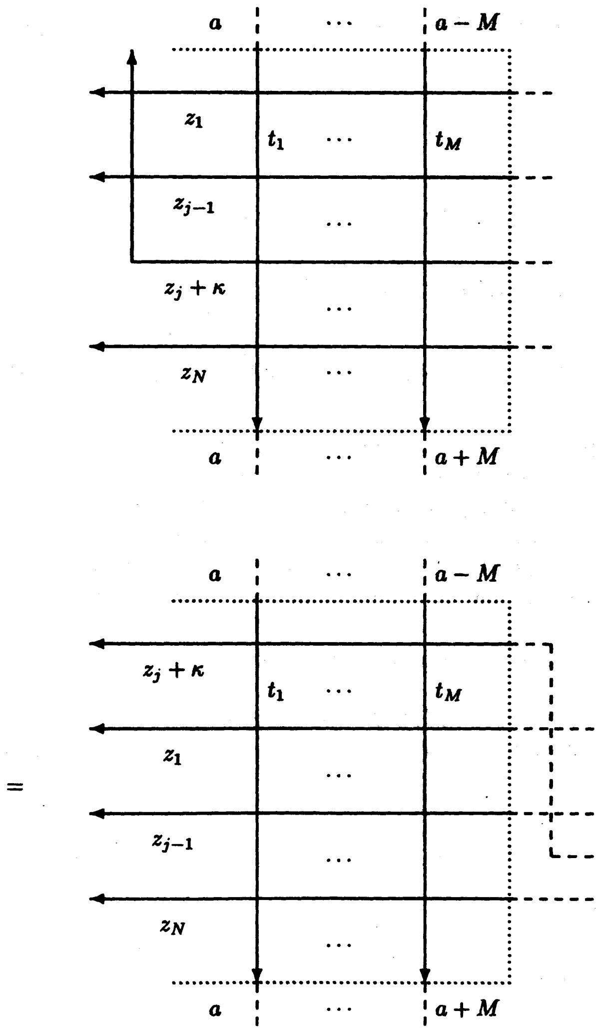

Fig. 15. Equivalence of (3.15) and (3.16)

One can prove this lemma by induction on m. Using (3.7), (3.8) and a formula

$$B_{a,a'}^{l_N,\dots,l_1}(u;z_N,\dots,z_1) = A_{a,b}^{l_N,\dots,l_{n+1}}(u;z_N,\dots,z_{n+1})B_{b,a'}^{l_n,\dots,l_1}(u;z_n,\dots,z_1)$$
$$+B_{a,b}^{l_N,\dots,l_{n+1}}(u;z_N,\dots,z_{n+1})D_{b,a'}^{l_n,\dots,l_1}(u;z_n,\dots,z_1)$$

for any integer b, which is easily derived from (3.2), one can apply the formula (5.6) of [TF] to this case.

Applying (3.17) for $(m,n) = (M,1)$ and $(M,N-1)$ to Eqs. (3.16), using (1.28) and comparing the coefficients of

$$e^{2\pi i(a+2\sharp(\mathrm{II})-2l_j)\nu} B^{l_j}_{a+1,a+2\sharp(\mathrm{II})-2l_j-1}(t_{i'_1}+\kappa; z_j+\kappa)$$

$$\cdots B^{l_j}_{a+\sharp(\mathrm{II}),a+\sharp(\mathrm{II})-2l_j}(t_{i'_{\sharp(\mathrm{II})}}+\kappa; z_j+\kappa)\Omega^{l_j}_{a+\sharp(\mathrm{II})}(z_j+\kappa)$$

$$\otimes B^{l_N,\ldots,l_{j+1},l_{j-1},\ldots,l_1}_{a+2\sharp(\mathrm{II})-2l_j+1,a-1}(t_{i_1}; z_N\ldots,z_{j+1},z_{j-1},\ldots,z_1)$$

$$\cdots B^{l_N,\ldots,l_{j+1},l_{j-1},\ldots,l_1}_{a+M-2l_j+\sharp(\mathrm{II}),a-\sharp(\mathrm{I})}(t_{i_{\sharp(\mathrm{I})}}; z_N\ldots,z_{j+1},z_{j-1},\ldots,z_1)$$

$$\times\, \Omega^{l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_N}_{a+M-2l_j+\sharp(\mathrm{II})}(z_1\ldots,z_{j-1},z_{j+1},\ldots,z_N)$$

of the resulting equation for a partition $\{1,\ldots,M\}=\mathrm{I}\sqcup\mathrm{II}$, we can show that (3.13) gives a solution to (2.4) if $\varphi(\vec{z}\,|\,\vec{t})$ satisfies

$$\varphi(\vec{z}_j\,|\,t_i+\kappa\delta_{i,\mathrm{II}})\prod_{i\in\mathrm{I},i'\in\mathrm{II}}\alpha(t'_i+\kappa,t_i)\prod_{i'\in\mathrm{II}}\prod_{\substack{k=1\\k\neq j}}^{N}\alpha^{l_k}(t_{i'}+\kappa-x_k)\prod_{i\in\mathrm{I}}\delta^{l_j}(t_i-x_j-\kappa)$$

$$=\varphi(\vec{z}\,|\,\vec{t})e^{-2c\sharp(\mathrm{II})}\prod_{i\in\mathrm{I},i'\in\mathrm{II}}\alpha(t_i,t'_i)\prod_{i\in\mathrm{I}}\alpha^{l_j}(t_i-x_j)\prod_{i'\in\mathrm{II}}\prod_{\substack{k=1\\k\neq j}}^{N}\delta^{l_k}(t_{i'}-x_k)\,. \tag{3.18}$$

Here in the left-hand side $\delta_{i,\mathrm{II}}=0$ if $i\in\mathrm{I}$ and $=1$ if $i\in\mathrm{II}$. The function defined by (3.12) is a solution of (3.18). Thus we have proved the theorem.

Ordinary difference equations and cycles. We now return to Eqs. (3.10) and (3.11) and give the solution to them in the form of infinite products.

First let us consider Eq. (3.10). Using the infinite product formula of the theta functions (see, e.g., [Mu]), we have

$$\frac{\delta^l(t)}{\alpha^l(t+\kappa)}=e^{-4\pi il\eta}\frac{(e^{-2\pi it+4\pi il\eta};p)_\infty}{(q^{-1}e^{-2\pi it-4\pi il\eta};p)_\infty}\frac{(pe^{2\pi it-4\pi il\eta};p)_\infty}{(pqe^{2\pi it+4\pi il\eta};p)_\infty}\,, \tag{3.19}$$

where $p=\exp(2\pi i\tau)$, $q=\exp(2\pi i\kappa)$ and $(x;p)_\infty=\prod_{m=0}^{\infty}(1-p^m x)$. It is easy to see from (3.19) that

$$F^l(t)=e^{-4\pi il\eta t/\kappa}\frac{(qe^{-2\pi it+4\pi il\eta};p,q)_\infty}{(e^{-2\pi it-4\pi il\eta};p,q)_\infty}\frac{(pqe^{2\pi it+4\pi il\eta};p,q)_\infty}{(pe^{2\pi it-4\pi il\eta};p,q)_\infty} \tag{3.20}$$

satisfies (3.10), at least formally, where $(x;p,q)_\infty=\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}(1-p^mq^nx)$.

Similarly, Eq. (3.11) has a formal solution

$$\Phi(t)=e^{4\pi i\eta t/\kappa}(e^{-2\pi it};p)_\infty(pe^{2\pi it};p)_\infty\frac{(qe^{-2\pi it-4\pi i\eta};p,q)_\infty}{(e^{-2\pi it+4\pi i\eta};p,q)_\infty}\frac{(pqe^{2\pi it-4\pi i\eta};p,q)_\infty}{(pe^{2\pi it+4\pi i\eta};p,q)_\infty}\,. \tag{3.21}$$

In fact, the infinite products in (3.20) and (3.21) gives meromorphic functions on the whole complex plane, provided that $\operatorname{Im}\kappa>0$. The function $F^l(t)$ has zeros at

$$\{t = n + m_1\tau + m_2\kappa + 2l\eta \mid n \in \mathbf{Z}, m_1 \in \mathbf{Z}_{\geqq 0}, m_2 \in \mathbf{Z}_{>0}\}$$
$$\cup \{t = n - m_1\tau - m_2\kappa - 2l\eta \mid n \in \mathbf{Z}, m_1 \in \mathbf{Z}_{>0}, m_2 \in \mathbf{Z}_{>0}\}, \qquad (3.22)$$

and poles at

$$\{t = n + m_1\tau + m_2\kappa - 2l\eta \mid n \in \mathbf{Z}, m_1 \in \mathbf{Z}_{\geqq 0}, m_2 \in \mathbf{Z}_{\geqq 0}\}$$
$$\cup \{t = n - m_1\tau - m_2\kappa + 2l\eta \mid n \in \mathbf{Z}, m_1 \in \mathbf{Z}_{>0}, m_2 \in \mathbf{Z}_{\geqq 0}\}. \qquad (3.23)$$

Taking these properties of $F^l(t)$ into account, we can choose an N-cycle in (3.13) so that the sum over $t \in \mathscr{C}$ reduces to a finite sum, if κ satisfies a certain rationality condition. Suppose that there exist integers $(n, m_0, m_1) \in \mathbf{Z}_{>0} \times \mathbf{Z} \times \mathbf{Z}_{>0}$ satisfying

$$n\kappa = m_0 + m_1\tau + 4l_j\eta \qquad (3.24)$$

for a certain $j \in \{1, \ldots, N\}$. Then all but finite points in the set

$$\mathscr{C}' := \{t = m_0^0 + m_1^0\tau + m\kappa + 2l_j\eta \mid m \in \mathbf{Z}\} \qquad (3.25)$$

fall into the set of zeros of $F^l(t)$ (3.22) for any integers m_0^0 and $m_1^0 \geqq 0$. We can choose a cycle $\mathscr{C}$ so that $t_n - x_j \in \mathscr{C}'$ for all $n = 1, \ldots, M$ and, thanks to the zeros of function F^{l_j}, the sum over $t \in \mathscr{C}$ in (3.13) is essentially a finite sum. Hence Eq. (3.13) gives an analytic solution of the difference equation (2.4).

4. Comments

In recent years several difference equations with elliptic coefficients related to the q-Knizhnik–Zamolodchikov equations have been proposed:

- Etingof's equation: Etingof [E] showed that a (modified) trace of certain intertwining operators of representations of quantum affine universal enveloping algebras, $U_q(\hat{\mathfrak{g}})$ satisfies a difference equation with elliptic coefficients.
- Jimbo–Miwa–Nakayashiki's equation: Jimbo, Miwa and Nakayashiki [JMN] found a difference equation which should be satisfied by correlation functions of eight vertex models.

Unfortunately we do not know the relation of our difference equation (2.4) with any one of above equations. It might be possible that one system turns into another by specialization of parameters. We can also expect that a quasi-classical limit of our system (2.4) is related to the Knizhnik–Zamolodchikov–Bernard equation. In fact, Felder introduced a q-KZB equation in [F] whose semiclassical limit is the KZB equation, and our equation should be related to it by a vertex-IRF correspondence.[2]

It is a challenging problem to give a representation theoretical interpretation to the solution (3.13) like that of the integral solution of the (q-)KZ equations [Ma, FFR].

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[2] The author thanks Professor G. Felder for pointing out this relation.

Appendix A. Review of the Sklyanin Algebra

In this appendix we recall several facts on the Sklyanin algebra and its representations from [S1] and [S2]. We use notations in [Mu] for theta functions:

$$\theta_{ab}(z;\tau) = \sum_{n\in\mathbf{Z}} \exp\left(\pi i\left(\frac{a}{2}+n\right)^2\tau + 2\pi i\left(\frac{a}{2}+n\right)\left(\frac{b}{2}+z\right)\right), \tag{A.1}$$

where τ is a complex number such that $\mathrm{Im}\tau > 0$. The Pauli matrices are defined as usual:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The **Sklyanin algebra**, $U_{\tau,\eta}(sl(2))$ is generated by four generators S^0, S^1, S^2, S^3, satisfying the following relations:

$$R_{12}(u-v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u-v). \tag{A.3}$$

Here u, v are complex parameters, the L **operator**, $L(u)$, is defined by

$$L(u) = \sum_{a=0}^{3} W_a^L(u)\sigma^a \otimes S^a, \tag{A.4}$$

where

$$W_0^L(u) = \frac{\theta_{11}(u;\tau)}{2\theta_{11}(2\eta;\tau)\theta_{11}(\eta;\tau)}, \qquad W_1^L(u) = \frac{\theta_{10}(u;\tau)}{2\theta_{11}(2\eta;\tau)\theta_{10}(\eta;\tau)},$$
$$W_2^L(u) = \frac{\theta_{00}(u;\tau)}{2\theta_{11}(2\eta;\tau)\theta_{00}(\eta;\tau)}, \qquad W_3^L(u) = \frac{\theta_{01}(u;\tau)}{2\theta_{11}(2\eta;\tau)\theta_{01}(\eta;\tau)},$$

$R(u) = R(u;\tau)$ is **Baxter's R matrix** defined by

$$R(u) = \sum_{a=0}^{3} W_a^R(u)\sigma^a \otimes \sigma^a, \qquad W_a^R(u) := \theta_{11}(2\eta;\tau)W_a^L(u+\eta), \tag{A.5}$$

and indices $\{0,1,2\}$ denote the spaces on which operators act non-trivially: for example,

$$R_{12}(u) = \sum_{a=0}^{3} W_a^R(u) 1 \otimes \sigma^a \otimes \sigma^a, \qquad L_{13}(u) = \sum_{a=0}^{3} W_a^L(u)\sigma^a \otimes 1 \otimes S^a.$$

The above relation (A.3) contains u and v as parameters, but the commutation relations among S^a $(a = 0,\dots,3)$ do not depend on them:

$$[S^\alpha, S^0]_- = -iJ_{\alpha,\beta}[S^\beta, S^\gamma]_+, \qquad [S^\alpha, S^\beta]_- = i[S^0, S^\gamma]_+, \tag{A.6}$$

where (α,β,γ) stands for any cyclic permutation of $(1,2,3)$, $[A,B]_\pm = AB \pm BA$, and $J_{\alpha,\beta} = (W_\alpha^2 - W_\beta^2)/(W_\gamma^2 - W_0^2)$ depend on τ and η but not on u.

The **spin l representation** of the Sklyanin algebra, $\rho^l : U_{\tau,\eta}(sl(2)) \to \mathrm{End}_{\mathbf{C}}(\Theta_{00}^{4l})$ is defined as follows: The representation space V^l is

$$V^l = \Theta_{00}^{4l+} := \{f(y) \mid f(y+1) = f(-y) = f(y), f(y+\tau) = \exp^{-4l\pi i(2y+\tau)} f(y)\}. \tag{A.7}$$

It is easy to see that $\dim V^l = 2l+1$. The generators of the algebra act on this space as difference operators:

$$(\rho^l(S^a)f)(y) = \frac{s_a(y-l\eta)f(y+\eta) - s_a(-y-l\eta)f(y-\eta)}{\theta_{11}(2y;\tau)}, \tag{A.8}$$

where

$$s_0(y) = \theta_{11}(\eta;\tau)\theta_{11}(2y;\tau), \qquad s_1(y) = \theta_{10}(\eta;\tau)\theta_{10}(2y;\tau),$$
$$s_2(y) = i\theta_{00}(\eta;\tau)\theta_{00}(2y;\tau), \qquad s_3(y) = \theta_{01}(\eta;\tau)\theta_{01}(2y;\tau).$$

These representations reduce to the usual spin l representations of $U(sl(2))$ for $J_{\alpha\beta} \to 0$ ($\eta \to 0$). In particular, in the case $l = 1/2$, S^a are expressed by the Pauli matrices σ^a: Let us identify Θ_{00}^{2+} and $\mathbb{C}^2$ by

$$\theta_{00}(2y;2\tau) - \theta_{10}(2y;2\tau) \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\theta_{00}(2y;2\tau) + \theta_{10}(2y;2\tau) \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.9}$$

Under this identification S^a have matrix forms

$$\rho^{1/2}(S^a) = \theta_{11}(2\eta;\tau)\sigma^a. \tag{A.10}$$

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