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Schur Duality in the Toroidal Setting

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Abstract: The classical Frobenius–Schur duality gives a correspondence between finite dimensional representations of the symmetric and the linear groups. The goal of the present paper is to extend this construction to the quantum toroidal setup with only elementary (algebraic) methods. This work can be seen as a continuation of [J, D1 and C2] (see also [C-P and G-R-V]) where the cases of the quantum groups $\mathbf{U}_q(\mathfrak{sl}(\mathfrak{n}))$, $\mathbf{Y}(\mathfrak{sl}(n))$ (the Yangian) and $\mathbf{U}_q(\mathfrak{sl}(n))$ are given. In the toroidal setting the two algebras involved are deformations of Cherednik's double affine Hecke algebra introduced in [C1] and of the quantum toroidal group as given in [G-K-V]. Indeed, one should keep in mind the geometrical construction in [G-R-V] and [G-K-V] in terms of equivariant K-theory of some flag manifolds. A similar K-theoretic construction of Cherednik's algebra has motivated the present work. At last, we would like to lay emphasis on the fact that, contrary to [J, D1 and C2], the representations involved in our duality are infinite dimensional. Of course, in the classical case, i.e., $q = 1$, a similar duality holds between the toroidal Lie algebra and the toroidal version of the symmetric group.

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1. Definition of the Toroidal Hecke Algebra

For any positive integer k set $[k] = \{0, 1, 2, \ldots, k\}$ and $[k]^\times = \{1, 2, \ldots, k\}$.

1.1. Definition. *The toroidal Hecke algebra of type* $\mathfrak{gl}(l)$, $\ddot{\mathbf{H}}_{\mathscr{A}}$, *is the unital associative algebra over* $\mathscr{A} = \mathbb{C}[\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}, \mathbf{q}^{\pm 1}]$ *with generators*

$$\mathrm{T}_i^{\pm 1}, \mathrm{X}_j^{\pm 1}, \mathrm{Y}_j^{\pm 1}, \quad i \in [l-1]^\times,\ j \in [l]^\times\ ,$$

and the following relations:

$$\mathrm{T}_i \mathrm{T}_i^{-1} = \mathrm{T}_i^{-1} \mathrm{T}_i = 1, \qquad (\mathrm{T}_i + 1)(\mathrm{T}_i - \mathbf{q}^2) = 0\ ,$$

$$\mathrm{T}_i \mathrm{T}_{i+1} \mathrm{T}_i = \mathrm{T}_{i+1} \mathrm{T}_i \mathrm{T}_{i+1}\ ,$$

$$\mathrm{T}_i\mathrm{T}_j = \mathrm{T}_j\mathrm{T}_i \quad \textit{if } |i-j|>1\,,$$

$$\mathrm{X}_0\mathrm{Y}_1 = \mathbf{x}\mathrm{Y}_1\mathrm{X}_0, \qquad \mathrm{X}_i\mathrm{X}_j = \mathrm{X}_i\mathrm{X}_j, \qquad \mathrm{Y}_i\mathrm{Y}_j = \mathrm{Y}_j\mathrm{Y}_i\,,$$

$$\mathrm{X}_j\mathrm{T}_i = \mathrm{T}_i\mathrm{X}_j, \qquad \mathrm{Y}_j\mathrm{T}_i = \mathrm{T}_i\mathrm{Y}_j, \quad \textit{if } j \neq i, i+1$$

$$\mathrm{T}_i\mathrm{X}_i\mathrm{T}_i = \mathbf{q}^2\mathrm{X}_{i+1}, \qquad \mathrm{T}_i^{-1}\mathrm{Y}_i\mathrm{T}_i^{-1} = \mathbf{q}^{-2}\mathrm{Y}_{i+1}\,,$$

$$\mathrm{X}_2\mathrm{Y}_1^{-1}\mathrm{X}_2^{-1}\mathrm{Y}_1 = \mathbf{q}^{-2}\mathbf{y}\mathrm{T}_1^2\,,$$

where $\mathrm{X}_0 = \mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_l$.

1.2. Remarks. (*i*) When $\mathbf{x}$ is taken to be 1 the toroidal Hecke algebra is nothing but the double affine Hecke algebra introduced by Cherednik (see [C1]), which should be seen as a quotient of a central extension of the braid group over a torus. Given a permutation $w \in \mathfrak{S}_l$ let $\mathrm{T}_w \in \ddot{\mathbf{H}}_{\mathscr{A}}$ be the usual element defined in terms of a reduced expression of w and, given an l-tuple of integers $\mathbf{r} = (r_1, r_2, \ldots, r_l)$, denote by $\mathrm{X}^{\mathbf{r}}$ and $\mathrm{Y}^{\mathbf{r}}$ the corresponding monomials in the X_i's and Y_i's. Then the elements $\mathrm{X}^{\mathbf{s}}\mathrm{Y}^{\mathbf{r}}\mathrm{T}_w$ give a basis of $\ddot{\mathbf{H}}_{\mathscr{A}}$ as a $\mathscr{A}$-module (see [C1, Theorem 2.6.(a)] for the case $\mathbf{x} = 1$).

(*ii*) Note that the map $\mathrm{T}_i \mapsto \mathrm{T}_i^{-1}$, $\mathrm{X}_i \mapsto \mathrm{Y}_i$, $\mathrm{Y}_i \mapsto \mathrm{X}_i$, $\mathbf{x} \mapsto \mathbf{x}^{-1}$, $\mathbf{y} \mapsto \mathbf{y}^{-1}$, $\mathbf{q} \mapsto \mathbf{q}^{-1}$, extends to an automorphism, Ξ, of $\ddot{\mathbf{H}}_{\mathscr{A}}$ over $\mathbb{C}$.

1.3. Given $1 \leqq i \leqq j < l$ set $\mathrm{T}_{i,j} = \mathrm{T}_i\mathrm{T}_{i+1}\cdots\mathrm{T}_j$ and $\mathrm{T}_{j,i} = \mathrm{T}_j\mathrm{T}_{j-1}\cdots\mathrm{T}_i$. Then, put $\mathrm{Q} = \mathrm{X}_1\mathrm{T}_{1,l-1} \in \ddot{\mathbf{H}}_{\mathscr{A}}$. Clearly, $\mathrm{T}_i^{\pm 1}, \mathrm{Y}_j^{\pm 1}, \mathrm{Q}^{\pm 1}$ $(i \in [l-1]^\times, j \in [l]^\times)$ is a system of generators of $\ddot{\mathbf{H}}_{\mathscr{A}}$. Besides, for any $i \in [l-1]^\times$ a direct computation gives $\mathrm{Q}\mathrm{Y}_i\mathrm{Q}^{-1} = \mathbf{y}^{-1}\mathrm{Y}_{i+1}$, and $\mathrm{Q}\mathrm{Y}_l\mathrm{Q}^{-1} = \mathbf{x}\mathbf{y}^{l-1}\mathrm{Y}_1$. Indeed we have (see [C1]).

Proposition. *The toroidal Hecke algebra* $\ddot{\mathbf{H}}_{\mathscr{A}}$ *admits an equivalent presentation in terms of generators*

$$\mathrm{T}_i^{\pm 1}, \mathrm{Y}_j^{\pm 1}, \mathrm{Q}^{\pm 1}, \quad i \in [l-1]^\times,\ j \in [l]^\times\,,$$

with relations:

$$\mathrm{T}_i\mathrm{T}_i^{-1} = \mathrm{T}_i^{-1}\mathrm{T}_i = 1, \qquad (\mathrm{T}_i + 1)(\mathrm{T}_i - \mathbf{q}^2) = 0\,,$$

$$\mathrm{T}_i\mathrm{T}_{i+1}\mathrm{T}_i = \mathrm{T}_{i+1}\mathrm{T}_i\mathrm{T}_{i+1}\,,$$

$$\mathrm{T}_i\mathrm{T}_j = \mathrm{T}_j\mathrm{T}_i \quad \textit{if } |i-j|>1\,,$$

$$\mathrm{Y}_i\mathrm{Y}_j = \mathrm{Y}_j\mathrm{Y}_i, \qquad \mathrm{T}_i^{-1}\mathrm{Y}_i\mathrm{T}_i^{-1} = \mathbf{q}^{-2}\mathrm{Y}_{i+1}\,,$$

$$\mathrm{Y}_j\mathrm{T}_i = \mathrm{T}_i\mathrm{Y}_j, \quad \textit{if } j \neq i, i+1$$

$$\mathrm{Q}\mathrm{T}_{i-1}\mathrm{Q}^{-1} = \mathrm{T}_i \ (1 < i < l-1), \qquad \mathrm{Q}^2\mathrm{T}_{l-1}\mathrm{Q}^{-2} = \mathrm{T}_1\,,$$

$$\mathrm{Q}\mathrm{Y}_i\mathrm{Q}^{-1} = \mathbf{y}^{-1}\mathrm{Y}_{i+1} \ (1 \leqq i \leqq l-1), \qquad \mathrm{Q}\mathrm{Y}_l\mathrm{Q}^{-1} = \mathbf{x}\mathbf{y}^{l-1}\mathrm{Y}_1\,.$$

1.4. Let $\dot{\mathbf{H}}_{\mathscr{A}}^{(1)}, \dot{\mathbf{H}}_{\mathscr{A}}^{(2)} \subset \ddot{\mathbf{H}}_{\mathscr{A}}$ be the subalgebras generated respectively by $\mathrm{T}_i^{\pm 1}$, $\mathrm{Y}_j^{\pm 1}$ and $\mathrm{T}_i^{\pm 1}$, $\mathrm{X}_j^{\pm 1}$ $(i \in [l-1]^\times, j \in [l]^\times)$. Then, $\dot{\mathbf{H}}_{\mathscr{A}}^{(1)}$ and $\dot{\mathbf{H}}_{\mathscr{A}}^{(2)}$ are isomorphic to the affine Hecke algebra over $\mathscr{A}$ of type $\mathfrak{gl}(l)$, simply denoted $\dot{\mathbf{H}}_{\mathscr{A}}$ (see Remark 1.2(i)).

1.5. Let $\mathbf{H}_{\mathscr{A}} \subset \ddot{\mathbf{H}}_{\mathscr{A}}$ be the subalgebra generated by $\mathrm{T}_i^{\pm 1}$ ($i \in [l-1]^{\times}$). Then, $\mathbf{H}_{\mathscr{A}}$ is the Hecke algebra over $\mathscr{A}$ of finite type $\mathfrak{gl}(l)$ (see Remark 1.2.(i)).

1.6. Given complex numbers $q,x,y \in \mathbb{C}^{\times}$, let $\mathbf{H}, \ddot{\mathbf{H}}, \ldots$ be the algebras obtained from $\mathbf{H}_{\mathscr{A}}, \ddot{\mathbf{H}}_{\mathscr{A}}, \ldots$ by specializing $\mathbf{q}, \mathbf{x}, \mathbf{y}$ to q, x, y.

2. Definition of the Toroidal Quantum Group

2.1. Definition. *The toroidal quantum group of type* $\mathfrak{sl}(n+1)$, $\ddot{\mathbf{U}}_{\mathscr{B}}$, *is the unital associative algebra over* $\mathscr{B} = \mathbb{C}[\mathbf{c}^{\pm 1}, \mathbf{p}^{\pm 1}](\mathbf{q})$ *with generators*

$$\mathbf{e}_{i,k},\ \mathbf{f}_{i,k},\ \mathbf{k}_{i,l},\ \mathbf{k}_i^{\pm 1},\ \mathbf{h}_i^{\pm 1}\ ,$$

where $i \in [n]$, $k, l \in \mathbb{Z}$ *and* $l \neq 0$. *The relations are expressed in terms of the formal series*

$$\mathbf{e}_i(z) = \sum_{k \in \mathbb{Z}} z^{-k} \cdot \mathbf{e}_{i,k}, \qquad \mathbf{f}_i(z) = \sum_{k \in \mathbb{Z}} z^{-k} \cdot \mathbf{f}_{i,k}, \qquad \mathbf{k}_i^{\pm}(z) = \mathbf{k}_i^{\pm 1} + \sum_{l>0} z^{\mp l} \cdot \mathbf{k}_{i,\pm l}\ ,$$

as follows

$$\mathbf{h}_i\mathbf{h}_i^{-1} = \mathbf{h}_i^{-1}\mathbf{h}_i = \mathbf{k}_i\mathbf{k}_i^{-1} = \mathbf{k}_i^{-1}\mathbf{k}_i = 1\ ,$$

$$[\mathbf{h}_i, \mathbf{h}_j] = [\mathbf{k}_i^{\pm}(z), \mathbf{k}_j^{\pm}(w)] = [\mathbf{h}_i, \mathbf{k}_j^{\pm}(z)] = 0\ ,$$

$$\mathbf{h}_i\mathbf{e}_j(z) = \mathbf{p}^{c_{ij}}\mathbf{e}_j(z)\mathbf{h}_i, \qquad \mathbf{h}_i\mathbf{f}_j(z) = \mathbf{p}^{-c_{ij}}\mathbf{f}_j(z)\mathbf{h}_i\ ,$$

$$\theta_{a_{ij}}(\mathbf{c}^{-2}\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{k}_i^{+}(z)\mathbf{k}_j^{-}(w) = \theta_{a_{ij}}(\mathbf{c}^{2}\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{k}_j^{-}(w)\mathbf{k}_i^{+}(z)\ ,$$

$$\mathbf{k}_i^{\pm}(z)\mathbf{e}_j(w) = \theta_{a_{ij}}(\mathbf{c}^{\pm 1}\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{e}_j(w)\mathbf{k}_i^{\pm}(\mathbf{p}^{-\delta_{ij}}z)\ ,$$

$$\mathbf{k}_i^{\pm}(z)\mathbf{f}_j(w) = \theta_{-a_{ij}}(\mathbf{c}^{\mp 1}\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{f}_j(w)\mathbf{k}_i^{\pm}(\mathbf{p}^{\delta_{ij}}z)\ ,$$

$$\mathbf{e}_i(\mathbf{p}^{-\delta_{ij}}z)\mathbf{f}_j(w) - \mathbf{f}_j(\mathbf{p}^{\delta_{ij}}w)\mathbf{e}_i(z) = \frac{\delta_{ij}}{\mathbf{q} - \mathbf{q}^{-1}}(\delta(\mathbf{c}^{-2}z/w)\mathbf{k}_i^{+}(\mathbf{c}w) - \delta(\mathbf{c}^{2}z/w)\mathbf{k}_i^{-}(\mathbf{c}z))\ ,$$

$$\mathbf{e}_i(z)\mathbf{e}_j(\mathbf{p}^{-\delta_{ij}}w) = \theta_{a_{ij}}(\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{e}_j(w)\mathbf{e}_i(\mathbf{p}^{-\delta_{ij}}z)\ ,$$

$$\mathbf{f}_i(z)\mathbf{f}_j(\mathbf{p}^{\delta_{ij}}w) = \theta_{-a_{ij}}(\mathbf{h}_{i-c_{ji}}^{m_{ji}}\, z/w)\mathbf{f}_j(w)\mathbf{f}_i(\mathbf{p}^{\delta_{ij}}z)\ ,$$

$$\{\mathbf{e}_i(z_1)\mathbf{e}_i(z_2)\mathbf{e}_j(w) - (\mathbf{q} + \mathbf{q}^{-1})\mathbf{e}_i(z_1)\mathbf{e}_j(w)\mathbf{e}_i(z_2) + \mathbf{e}_j(w)\mathbf{e}_i(z_1)\mathbf{e}_i(z_2)\}$$

$$+ \{z_1 \leftrightarrow z_2\} = 0, \quad \textit{if } a_{ij} = -1\ ,$$

$$\{\mathbf{f}_i(z_1)\mathbf{f}_i(z_2)\mathbf{f}_j(w) - (\mathbf{q} + \mathbf{q}^{-1})\mathbf{f}_i(z_1)\mathbf{f}_j(w)\mathbf{f}_i(z_2) + \mathbf{f}_j(w)\mathbf{f}_i(z_1)\mathbf{f}_i(z_2)\}$$

$$+ \{z_1 \leftrightarrow z_2\} = 0, \quad \textit{if } a_{ij} = -1\ ,$$

$$[\mathbf{e}_i(z), \mathbf{e}_j(w)] = [\mathbf{f}_i(z), \mathbf{f}_j(w)] = 0, \quad \textit{if} \quad \alpha_{ij} = 0\ ,$$

where $\mathbf{h}_{n+1}$ *stands for* $\mathbf{h}_0$, $\delta(z) = \sum_{n=-\infty}^{\infty} z^n$, $\theta_m(z) = \frac{\mathbf{q}^m z - 1}{z - \mathbf{q}^m}$ *and* a_{ij}, c_{ij}, m_{ij}, *are the entries of the following* $[n] \times [n]$*-matrices*:

$$\mathrm{C} = \begin{pmatrix} 1 & 0 & & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & & -1 & 1 \end{pmatrix}, \qquad \mathrm{A} = \mathrm{C}^t + \mathrm{C}, \qquad \mathrm{M} = \mathrm{C}^t - \mathrm{C}\,.$$

Remark. Given a simple complex Lie algebra $\mathfrak{g}$, let denote by $\ddot{\mathfrak{g}}$ the universal central extension of $\mathfrak{g}[x^{\pm 1}, y^{\pm 1}]$ (endowed with the structure of the current Lie algebra over the torus $\mathbb{C}^\times \times \mathbb{C}^\times$). A "one-loop" presentation of $\ddot{\mathfrak{g}}$ has been given in [M-R-Y]. The toroidal quantum group $\ddot{\mathbf{U}}_{\mathscr{B}}$ is a deformation of the enveloping algebra of $\ddot{\mathfrak{sl}}(n+1)$. When $\mathbf{p} = 1$ and $\mathbf{h}_0, \ldots, \mathbf{h}_n$, are specialized to a non-zero complex number d one recovers the toroidal quantum group introduced in [G-K-V].

2.2. Note that, given $a = (a_0, \ldots, a_n) \in (\mathbb{C}^\times)^{[n]}$, the map

$$\mathbf{e}_i(z) \mapsto \mathbf{e}_{i+1}(z/a_{i+1}), \qquad \mathbf{f}_i(z) \mapsto \mathbf{f}_{i+1}(z/a_{i+1}),$$

$$\mathbf{k}_i(z) \mapsto \mathbf{k}_{i+1}(z/a_{i+1}), \qquad \mathbf{h}_i \mapsto a_{i+1} a_i^{-1} \mathbf{h}_{i+1}\,,$$

($i \in [n]$, $n+1$ stands for 0), extends to an automorphism, Ψ_a, of the $\mathscr{B}$-algebra $\ddot{\mathbf{U}}_{\mathscr{B}}$.

2.3. Let $\dot{\mathbf{U}}_{\mathscr{B}}^{(2)} \subset \ddot{\mathbf{U}}_{\mathscr{B}}$ be the subalgebra generated by $\mathbf{e}_i = \mathbf{e}_{i,0}$, $\mathbf{f}_i = \mathbf{f}_{i,0}$ and $\mathbf{k}_i^{\pm 1}$ ($i \in [n]$). Denote by $\dot{\mathbf{U}}_{\mathscr{B}}$ Drinfeld–Jimbo's quantization of the enveloping algebra of $\widehat{\mathfrak{sl}}(n+1)$. Then $\mathbf{e}_i$, $\mathbf{f}_i$ and $\mathbf{k}_i^{\pm 1}$ verify the relations of the Chevalley-type generators of $\dot{\mathbf{U}}_{\mathscr{B}}$ and $\dot{\mathbf{U}}_{\mathscr{B}}^{(2)}$ is a quotient of $\dot{\mathbf{U}}_{\mathscr{B}}$. Similarly let $\dot{\mathbf{U}}_{\mathscr{B}}^{(1)} \subset \ddot{\mathbf{U}}_{\mathscr{B}}$ be the subalgebra generated by the Fourier coefficients of

$$\tilde{\mathbf{e}}_i(z) = \mathbf{e}_i(z\mathbf{h}_1 \cdots \mathbf{h}_i), \qquad \tilde{\mathbf{f}}_i(z) = \mathbf{f}_i(z\mathbf{h}_1 \cdots \mathbf{h}_i),$$

$$\tilde{\mathbf{k}}_i^{\pm}(z) = \mathbf{k}_i^{\pm}(z\mathbf{h}_1 \cdots \mathbf{h}_i) \quad (i \in [n]^\times)\,.$$

These elements verify precisely the relations in Drinfeld's "new presentation" of $\dot{\mathbf{U}}_{\mathscr{B}}$ (see [D1, B]). In other words, $\dot{\mathbf{U}}_{\mathscr{B}}^{(1)}$ is a quotient of $\dot{\mathbf{U}}_{\mathscr{B}}$. At last, put $\mathbf{T}_{\mathscr{B}} = \mathscr{B}[\mathbf{h}_0^{\pm 1}, \mathbf{h}_1^{\pm 1}, \ldots, \mathbf{h}_n^{\pm 1}] \subset \ddot{\mathbf{U}}_{\mathscr{B}}$. The algebra $\ddot{\mathbf{U}}_{\mathscr{B}}$ is generated by $\dot{\mathbf{U}}_{\mathscr{B}}^{(1)}$, $\dot{\mathbf{U}}_{\mathscr{B}}^{(2)}$ and $\mathbf{T}_{\mathscr{B}}$.

2.4. The subalgebra of $\ddot{\mathbf{U}}_{\mathscr{B}}$ generated by $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{k}_i^{\pm 1}$ ($i \in [n]^\times$) is a quotient of $\mathbf{U}_{\mathscr{B}}$, the quantum enveloping algebra of $\mathfrak{sl}(n+1)$.

2.5. Given complex numbers $c, p, q \in \mathbb{C}^\times$ let $\mathbf{U}, \ddot{\mathbf{U}}, \ldots$ be the algebras obtained from $\mathbf{U}_{\mathscr{B}}, \ddot{\mathbf{U}}_{\mathscr{B}}, \ldots$ by specializing $\mathbf{c}, \mathbf{p}, \mathbf{q}$ to c, p, q. A $\ddot{\mathbf{U}}$-module is said to have *trivial central charge* if its restrictions both to $\dot{\mathbf{U}}^{(1)}$ and $\dot{\mathbf{U}}^{(2)}$ have (i.e., $c = 1$ and $\mathbf{k}_0 \mathbf{k}_1 \cdots \mathbf{k}_n = 1$).

2.6. Fix $q \in \mathbb{C}^\times$, and suppose that $l \leqq n$. Following [C-P, 2.5], a $\mathbf{U}$-module is said to be of **level** l if its irreducible components are isomorphic to irreducible components of $\mathbf{V}^{\otimes l}$. Similarly a $\dot{\mathbf{U}}$-module or a $\ddot{\mathbf{U}}$-module is said to be of level l if it is of level l as a $\mathbf{U}$-module.

2.7. Set $P_l = \{\chi = (\chi_0, \ldots, \chi_n) \in \mathbb{Z}^{[n]} | \chi_0 + \chi_1 + \cdots + \chi_n = l\}$. A $\ddot{\mathbf{U}}$-module M with trivial central charge and level l is *integrable of index d* if it splits in a direct sum $M = \bigoplus_{\chi \in P_l} M_\chi$ such that

$$M_\chi = \{m \in M |\ \mathbf{h}_i m = dp^{\chi_i} m\} = \{m \in M |\ \mathbf{k}_i m = q^{\chi_i - \chi_{i+1}} m\} ,$$

where $n+1$ stands for 0 and if its restrictions both to $\dot{\mathbf{U}}^{(1)}$ and $\dot{\mathbf{U}}^{(2)}$ are integrable (see [L, 3.5.1]). Then, the central element $\mathbf{d} = \mathbf{h}_0 \cdots \mathbf{h}_n$ acts on M by scalar multiplication by $d^{n+1} p^l$.

3. Definition of the Duality Functor

Fix $c, x, y, p, q \in \mathbb{C}^\times$.

3.1. Let $\mathbf{V}$ be the fundamental representation of $\mathbf{U}$. It has a basis $v_1, \ldots, v_{n+1}$ on which the action of $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i$ $(i \in [n]^\times)$ is the following:

$$\mathbf{e}_i(v_r) = \delta_{r,i+1} v_{r-1}, \qquad \mathbf{f}_i(v_r) = \delta_{r,i} v_{r+1}, \qquad \mathbf{k}_i(v_r) = q^{\delta_{i,r} - \delta_{i+1,r}} v_r .$$

Then, $\mathbf{V}^{\otimes l}$ is a left $\mathbf{U}$-module for the induced action given by the following coproduct:

$$\Delta(\mathbf{e}_i) = \mathbf{e}_i \otimes \mathbf{k}_i + 1 \otimes \mathbf{e}_i, \qquad \Delta(\mathbf{f}_i) = \mathbf{f}_i \otimes 1 + \mathbf{k}_i^{-1} \otimes \mathbf{f}_i, \qquad \Delta(\mathbf{k}_i) = \mathbf{k}_i \otimes \mathbf{k}_i .$$

This action commutes with the left $\mathbf{H}$-action given by $\mathrm{T}_i = 1^{\otimes i-1} \otimes \mathrm{T} \otimes 1^{\otimes l-i-1}$, where $\mathrm{T} \in \operatorname{End} \mathbf{V}^{\otimes 2}$ verifies

$$\mathrm{T}(v_r \otimes v_s) = \begin{cases} q^2 v_r \otimes v_s & \text{if } r = s , \\ q v_s \otimes v_r & \text{if } r < s, \\ q v_s \otimes v_r + (q^2 - 1) v_r \otimes v_s & \text{if } r > s . \end{cases}$$

3.2. For any $i \in [n+1]^\times$ define $\mathbf{t}'_i$ to be the automorphism of the algebra $\dot{\mathbf{U}}$ given on the Kac–Moody generators by the formula

$$\mathbf{t}'_i(\mathbf{e}_i) = -\mathbf{f}_i \mathbf{k}_i, \qquad \mathbf{t}'_i(\mathbf{e}_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s} \mathbf{e}_i^{(-a_{ij}-s)} \mathbf{e}_j \mathbf{e}_i^{(s)} \quad \text{if } i \neq j ,$$

$$\mathbf{t}'_i(\mathbf{f}_i) = -\mathbf{k}_i^{-1} \mathbf{e}_i, \qquad \mathbf{t}'_i(\mathbf{f}_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{s} \mathbf{f}_i^{(s)} \mathbf{f}_j \mathbf{f}_i^{(-a_{ij}-s)} \quad \text{if } i \neq j ,$$

$$\mathbf{t}'_i(\mathbf{k}_j) = \mathbf{k}_{s_i(j)} ,$$

where $\mathbf{e}_i^{(j)}, \mathbf{f}_i^{(j)}$ are the usual quantum divided powers (see [L, 3.1.1]) and $s_i \in \mathfrak{S}_{n+1}$ is the transposition $(i\ i+1)$. Let M' be an integrable $\dot{\mathbf{U}}$-module. Set $\mathbf{t}''_i \in \operatorname{Aut}_{\mathbb{C}}(M')$ $(i \in [n+1]^\times)$ to be the braid operator defined by

$$\mathbf{t}''_i(m') = \sum_{r-s+t=-k} (-1)^{s+k} q^{s-rt} \mathbf{e}_i^{(r)} \mathbf{f}_i^{(s)} \mathbf{e}_i^{(t)} \cdot m' ,$$

where $m' \in M'$ and $k \in \mathbb{Z}$ are such that $\mathbf{k}_i \cdot m' = q^k m'$ (see [L, 5.2.1 and 5.2.3]). We have (see [L, Chaps. 5 and 37])

$$\forall m' \in M', \quad \forall u \in \dot{\mathbf{U}}, \quad \mathbf{t}_i''(um') = \mathbf{t}_i'(u)\mathbf{t}_i''(m') . \tag{3.2.1}$$

Similarly, denote by τ the automorphism of the affine Dynkin diagram $A_n^{(1)}$ (with vertices indexed by $1,\dots,n,n+1$) given by $\tau(i) = i+1$ (if $i \neq n+1$), $\tau(n+1) = 1$. Let τ' be the automorphism of the algebra $\dot{\mathbf{U}}$ given, in terms of its Kac–Moody generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i$, $(i \in [n+1]^\times)$, by the following rule:

$$\tau'(\mathbf{e}_i) = \mathbf{e}_{\tau(i)}, \qquad \tau'(\mathbf{f}_i) = \mathbf{f}_{\tau(i)}, \qquad \tau'(\mathbf{k}_i) = \mathbf{k}_{\tau(i)} .$$

Put $\mathbf{t}'_{\omega_1} = \tau' \circ \mathbf{t}'_n \circ \mathbf{t}'_{n-1} \circ \cdots \circ \mathbf{t}'_1 \in \mathrm{Aut}(\dot{\mathbf{U}})$. Take a right $\dot{\mathbf{H}}$-module M. In particular, M is a right $\mathbf{H}$-module and we can consider the dual left $\mathbf{U}$-module $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$. This module is endowed with a structure of left $\dot{\mathbf{U}}$-module such that for any $m \in M$ and $\mathbf{v} \in \mathbf{V}^{\otimes l}$ (see [C-P])

$$\mathbf{e}_{n+1}(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m\mathrm{Y}_j^{-1} \otimes \mathbf{f}_{\theta,j}(\mathbf{v}), \qquad \mathbf{f}_{n+1}(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m\mathrm{Y}_j \otimes \mathbf{e}_{\theta,j}(\mathbf{v}) ,$$
$$\mathbf{k}_{n+1}(m \otimes \mathbf{v}) = m \otimes (\mathbf{k}_\theta^{-1})^{\otimes l}(\mathbf{v}) ,$$

where $\mathbf{e}_\theta, \mathbf{f}_\theta, \mathbf{k}_\theta \in \mathrm{End}_{\mathbb{C}}(\mathbf{V})$ are defined by means of

$$\mathbf{e}_\theta \cdot v_r = \delta_{r,n+1} v_1, \qquad \mathbf{f}_\theta \cdot v_r = \delta_{r,1} v_{n+1}, \qquad \mathbf{k}_\theta \cdot v_r = q^{\delta_{1,r} - \delta_{n+1,r}} v_r ,$$

and $\mathbf{f}_{\theta,j} = 1^{\otimes j-1} \otimes \mathbf{f}_\theta \otimes (\mathbf{k}_\theta^{-1})^{\otimes l-j}$, $\mathbf{e}_{\theta,j} = \mathbf{k}_\theta^{\otimes j-1} \otimes \mathbf{e}_\theta \otimes 1^{\otimes l-j}$. Until the end of Sect. 3.2 take $M' = M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$. For any l-tuple $\mathbf{j} \in ([n+1]^\times)^l$ set $\mathbf{v_j} = v_{j_1} \otimes \cdots \otimes v_{j_l}$. Define $\tau'' \in \mathrm{Aut}(M')$ such that

$$\tau''(m \otimes \mathbf{v_j}) = m\mathrm{Y}_1^{\delta_{n+1,j_1}} \mathrm{Y}_2^{\delta_{n+1,j_2}} \cdots \mathrm{Y}_l^{\delta_{n+1,j_l}} \otimes v_{j_1+1} \otimes v_{j_2+1} \otimes \cdots \otimes v_{j_l+1} ,$$

where v_{n+2} stands for v_1 and $\mathbf{j} = (j_1, \dots, j_l)$ is an l-tuple of integers in $[n+1]^\times$. A direct computation gives

$$\forall m' \in M', \quad \forall u \in \dot{\mathbf{U}}, \quad \tau''(um') = \tau'(u)\tau''(m') . \tag{3.2.2}$$

Put $\mathbf{t}''_{\omega_1} = \tau'' \circ \mathbf{t}''_n \circ \mathbf{t}''_{n-1} \circ \cdots \circ \mathbf{t}''_1 \in \mathrm{Aut}(M')$. As a consequence of (3.2.1) and (3.2.2),

$$\forall m' \in M', \quad \forall u \in \dot{\mathbf{U}}, \qquad \mathbf{t}''_{\omega_1}(um') = \mathbf{t}'_{\omega_1}(u)\mathbf{t}''_{\omega_1}(m') . \tag{3.2.3}$$

Example. If $l = 1$ we find

$$\mathbf{t}_i''(m \otimes v_j) = (-1)^{\delta_{i+1,j}} q^{\delta_{ij}} m \otimes v_{s_i(j)}, \quad i \in [n]^\times, \ j \in [n+1]^\times , \tag{3.2.4}$$

$$\mathbf{t}''_{\omega_1}(m \otimes v_j) = -m(-q^n \mathrm{Y}_1)^{\delta_{1,j}} \otimes v_j . \tag{3.2.5}$$

Lemma. *Let* $\mathbf{j}$ *be non decreasing, and set* $\mathrm{Y}_{1,s} = \mathrm{Y}_1 \cdot \mathrm{Y}_2 \cdots \mathrm{Y}_s$, *with* $\mathbf{j}^{-1}(1) =]0,s]$. *Then*

$$\mathbf{t}''_{\omega_1}(m \otimes \mathbf{v_j}) = (-1)^{l+s} q^{ns} m\mathrm{Y}_{1,s} \otimes \mathbf{v_j} .$$

Proof. From [L, 5.3.4], for all integrable $\dot{\mathbf{U}}$-modules N_1, N_2 and for all $n_1 \in N_1$, $n_2 \in N_2$,

$$\mathbf{e}_i(n_1) \otimes \mathbf{f}_i(n_2) = 0 \Rightarrow \mathbf{t}_i''(n_1 \otimes n_2) = \mathbf{t}_i''(n_1) \otimes \mathbf{t}_i''(n_2) \tag{3.2.6}$$

(note that Lusztig uses the opposite coproduct in [L]). Put $\mathbf{j_1} = (j_1, \ldots, j_{l-1})$ and for any permutation $\sigma \in \mathfrak{S}_{n+1}$, set $\sigma(\mathbf{j}) = (\sigma(j_1), \ldots, \sigma(j_l))$. We first compute $\mathbf{t}_n'' \circ \mathbf{t}_{n-1}'' \circ \cdots \circ \mathbf{t}_1''$. Note that $\mathbf{t}_n'' \circ \mathbf{t}_{n-1}'' \circ \cdots \circ \mathbf{t}_1''(m \otimes \mathbf{v_j}) = m \otimes \mathbf{t}_n'' \circ \mathbf{t}_{n-1}'' \circ \cdots \circ \mathbf{t}_1''(\mathbf{v_j})$. We have $\mathbf{e}_1(\mathbf{v}_{\mathbf{j_1}}) \neq 0$ if and only if $\mathbf{j_1}^{-1}(2) \neq \emptyset$ and $\mathbf{f}_1(v_{j_l}) \neq 0$ if and only if $j_l = 1$. Since $\mathbf{j}$ is non-decreasing, using (3.2.6) and (3.2.4) we find

$$\mathbf{t}_1''(\mathbf{v_j}) = \mathbf{t}_1''(\mathbf{v}_{\mathbf{j_1}}) \otimes \mathbf{t}_1''(v_{j_l}) = \cdots = \bigotimes_{k=1}^{l} \mathbf{t}_1''(v_{j_k}) = (-1)^{a_2} q^{a_1} \mathbf{v}_{s_1(\mathbf{j})} ,$$

where a_i is the cardinality of $\mathbf{j}^{-1}(i)$. Suppose that

$$\mathbf{t}_k'' \circ \cdots \circ \mathbf{t}_1''(\mathbf{v_j}) = (-1)^{a_2 + \cdots + a_{k+1}} q^{k a_1} \mathbf{v}_{s_k \cdots s_1(\mathbf{j})} .$$

Now $\mathbf{e}_{k+1}(\mathbf{v}_{s_k \cdots s_1(\mathbf{j_1})}) \otimes \mathbf{f}_{k+1}(v_{s_k \cdots s_1(j_l)}) \neq 0$ if and only if $(s_k \cdots s_1(\mathbf{j_1}))^{-1}(k+2) \neq \emptyset$ and $j_l = 1$, that is if and only if $\mathbf{j_1}^{-1}(k+2) \neq \emptyset$ and $j_l = 1$. Since $\mathbf{j}$ is non-decreasing, using (3.2.6) and (3.2.4) we get the same formula as above for $\mathbf{t}_{k+1}'' \circ \cdots \circ \mathbf{t}_1''(\mathbf{v_j})$. Then, finally

$$\mathbf{t}_n'' \circ \cdots \circ \mathbf{t}_1''(m \otimes \mathbf{v_j}) = (-1)^{l-s} q^{ns} m \otimes v_{j_1 - 1} \otimes \cdots \otimes v_{j_l - 1} ,$$

where v_0 stands for v_{n+1}. Applying τ'' we find the result. □

3.3. Take a right $\dot{\mathbf{H}}$-module M. The $\dot{\mathbf{U}}$-module structure on $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is given in terms of Drinfeld generators by (see [G-R-V]).

Theorem. *If* $\mathbf{j}$ *is non-decreasing and* $i \in [n]^\times$ *we have*

$$\mathbf{e}_i(z)(m \otimes \mathbf{v_j}) = q^{1-t+s} m \left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1} \right) \delta(q^{n+1-i} z \mathrm{Y}_{s+1}) \otimes \mathbf{v}_{\mathbf{j}^-} ,$$

$$\mathbf{f}_i(z)(m \otimes \mathbf{v_j}) = q^{1-s+r} m \left(1 + \sum_{k=r+1}^{s-1} \mathrm{T}_{k,s-1} \right) \delta(q^{n+1-i} z \mathrm{Y}_s) \otimes \mathbf{v}_{\mathbf{j}^+} ,$$

$$\mathbf{k}_i^\pm(z)(m \otimes \mathbf{v_j}) = m \prod_{j_k = i} \theta_1^\pm(q^{n+2-i} z \mathrm{Y}_k) \prod_{j_k = i+1} \theta_{-1}^\pm(q^{n-i} z \mathrm{Y}_k) \otimes \mathbf{v_j} ,$$

where $]r,s] = \mathbf{j}^{-1}(i)$, $]s,t] = \mathbf{j}^{-1}(i+1)$.

$$\mathbf{v}_{\mathbf{j}^-} = \mathbf{v}_{(j_1, \ldots, j_s, i, j_{s+2}, \ldots, j_l)} \quad \textit{if } s \neq t \quad \textit{and} \quad \mathbf{v}_{\mathbf{j}^-} = 0 \quad \textit{else} ,$$

$$\mathbf{v}_{\mathbf{j}^+} = \mathbf{v}_{(j_1, \ldots, j_{s-1}, i+1, j_{s+1}, \ldots, j_l)} \quad \textit{if } r \neq s \quad \textit{and} \quad \mathbf{v}_{\mathbf{j}^+} = 0 \quad \textit{else} ,$$

and $\theta_m^\pm(z)$ *stands for the Taylor expansion of* $\theta_m(z) = \frac{q^m z - 1}{z - q^m}$ *respectively at* ∞ *and* 0.

Proof. Let us prove the formula involving $\mathbf{e}_{i,h}$. We know that $\mathbf{e}_{1,h} = (-1)^{-h}\mathbf{t}'^{-h}_{\omega_1}(\mathbf{e}_1)$ (see [B, 4.6] and 2.3). So (3.2.3) gives

$$\mathbf{e}_{1,h} \circ \mathbf{t}''^{-h}_{\omega_1}(m \otimes \mathbf{v_j}) = (-1)^{-h}\mathbf{t}''^{-h}_{\omega_1} \circ \mathbf{e}_1(m \otimes \mathbf{v_j}) . \tag{3.3.1}$$

First of all note that formulas in Sect. 3.1 give (with $\mathbf{j}^{-1}(1) =]0, s]$ and $\mathbf{j}^{-1}(2) =]s, t]$)

$$\mathbf{e}_1(m \otimes \mathbf{v_j}) = q^{1-t+s} m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \otimes \mathbf{v}_{\mathbf{j}^-} .$$

We have

$$\mathbf{t}''^{-h}_{\omega_1} \circ \mathbf{e}_1(m \otimes \mathbf{v_j}) = q^{1-t+s}\mathbf{t}''^{-h}_{\omega_1}\left(m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \otimes \mathbf{v}_{\mathbf{j}^-}\right).$$

Thus, using Lemma 3.2, the right-hand side of (3.3.1) is

$$(-1)^{h(s+l)} q^{1-t+s-h(s+1)n}\, m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \mathrm{Y}^{-h}_{1,s+1} \otimes \mathbf{v}_{\mathbf{j}^-} .$$

Now Y_i commutes with $\mathrm{T}_{k,s+1}$ if $i \in [s]^\times$. Thus

$$\mathbf{t}''^{-h}_{\omega_1} \circ \mathbf{e}_1(m \otimes \mathbf{v_j}) = (-1)^{h(s+l)} q^{1-t+s-h(s+1)n}\, m\mathrm{Y}^{-h}_{1,s}\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \mathrm{Y}^{-h}_{s+1} \otimes \mathbf{v}_{\mathbf{j}^-} .$$

A similar computation for the left-hand side of (3.3.1) gives

$$\mathbf{e}_{1,h} \circ \mathbf{t}''^{-h}_{\omega_1}(m \otimes \mathbf{v_j}) = (-1)^{h(s+l)} q^{-hsn}\mathbf{e}_{1,h}(m\mathrm{Y}^{-h}_{1,s} \otimes \mathbf{v_j}) .$$

Thus we finally obtain

$$\mathbf{e}_{1,h}(m \otimes \mathbf{v_j}) = q^{1-t+s-hn} m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \mathrm{Y}^{-h}_{s+1} \otimes \mathbf{v}_{\mathbf{j}^-} .$$

For the other cases we proceed in a similar way. Namely, since the i^{th} fundamental weight ω_i verifies $\omega_i = \tau^i \cdot w^i$, where $w = s_n \cdot s_{n-1} \cdots s_1$, define $\mathbf{t}''_{\omega_i} = \tau''^i \circ \mathbf{t}''_{w^i}$. Then $\mathbf{e}_{i,h} = (-1)^{-ih}\mathbf{t}'^{-h}_{\omega_i}(\mathbf{e}_i)$ and

$$\mathbf{e}_{i,h} \circ \mathbf{t}''^{-h}_{\omega_i}(m \otimes \mathbf{v_j}) = (-1)^{-ih}\mathbf{t}''^{-h}_{\omega_i} \circ \mathbf{e}_i(m \otimes \mathbf{v_j}) .$$

Thus it suffices to compute $\mathbf{t}''_{\omega_i}(m \otimes \mathbf{v_j})$ using the following reduced decomposition of w^i:

$$w^i = u_{n+1-i} \cdot u_{n-i} \cdots u_1, \quad \text{with } u_j = s_j \cdot s_{j+1} \cdots s_{j+i-1} . \qquad \square$$

Corollary. *Fix $d_1, \ldots, d_n \in \mathbb{C}^\times$ and take a right $\dot{\mathbf{H}}$-module M. For any non-decreasing $\mathbf{j}$ and $i \in [n]^\times$ set (see notations in Theorem 3.3)*

$$\mathbf{h}_i(m \otimes \mathbf{v_j}) = d_i p^{s-r} m \otimes \mathbf{v_j}$$

$$\mathbf{e}_i(z)(m \otimes \mathbf{v_j}) = q^{1-t+s} m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right) \delta(q^{n+1-i} p^{-s-1} z\mathrm{Y}_{s+1}/d_1 \cdots d_i) \otimes \mathbf{v}_{\mathbf{j}^-} ,$$

$$\mathbf{f}_i(z)(m\otimes \mathbf{v_j}) = q^{1-s+r} m\left(1+\sum_{k=r+1}^{s-1} \mathrm{T}_{k,s-1}\right)\delta(q^{n+1-i}p^{-s+1}z\mathrm{Y}_s/d_1\cdots d_i)\otimes \mathbf{v}_{\mathbf{j}^+}\,,$$

$$\mathbf{k}_i^{\pm}(z)(m\otimes \mathbf{v_j}) = m\prod_{j_k=i}\theta_1^{\pm}(q^{n+2-i}p^{-s}z\mathrm{Y}_k/d_1\cdots d_i)$$

$$\cdot \prod_{j_k=i+1}\theta_{-1}^{\pm}(q^{n-i}p^{-s}z\mathrm{Y}_k/d_1\cdots d_i)\otimes \mathbf{v_j}\,,$$

where v_{n+1} stands for v_0. These operators on $M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l}$ verify the relations in 2.1 *(involving non-zero i and j).*

Proof. Follows from 2.3. □

Remark. In the context of the corollary the relations 2.1 give in particular

$$\mathbf{e}_1\mathbf{k}_{2,1}-q\mathbf{k}_{2,1}\mathbf{e}_1 = (q-q^{-1})c^{-1}\mathbf{h}_2\mathbf{e}_{1,1}\mathbf{k}_2\,.$$

Moreover, for any non-decreasing l-tuple $\mathbf{j}$ one gets

$$\mathbf{k}_i(m\otimes \mathbf{v_j}) = q^{-r-t}m\otimes \mathbf{v_j}\,,$$

$$\mathbf{k}_{i,1}(m\otimes \mathbf{v_j}) = q^{i-n-r-t}(1-q^{-2})p^{-s}d_1^{-1}\cdots d_i^{-1}$$

$$\cdot\, m\left(q^{-1}\sum_{\mathbf{j}_k=i}\mathrm{Y}_k^{-1}-q\sum_{\mathbf{j}_k=i+1}\mathrm{Y}_k^{-1}\right)\otimes \mathbf{v_j}\,,$$

where, as usual, $]r,s]=\mathbf{j}^{-1}(i)$, $]s,t]=\mathbf{j}^{-1}(i+1)$. Thus, the evaluation of the above formula on a non-zero vector of type $m\otimes v_1\otimes v_2\otimes\cdots\otimes v_l$ gives $c=1$. In other words, on the dual module $M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l}$ the central element c is trivial, even if M is not finite dimensional.

3.4. In the two next sections set $c=1, p=y$, take a right $\ddot{\mathbf{H}}$-module M and fix $d_1,\ldots,d_n\in\mathbb{C}^\times$. In particular M is a right $\dot{\mathbf{H}}^{(1)}$-module and the operators on $M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l}$ given in Corollary 3.3 verify the relations 2.1 involving non-zero i and j. Let $\psi: M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l}\to M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l}$ be the linear map defined for any l-tuple $\mathbf{j}=(j_1,j_2,\ldots,j_l)$ and any $m\in M$ by

$$\psi(m\otimes \mathbf{v_j}) = m\mathrm{X}_1^{-\delta_{n+1,j_1}}\mathrm{X}_2^{-\delta_{n+1,j_2}}\cdots\mathrm{X}_l^{-\delta_{n+1,j_l}}\otimes v_{1+j_1}\otimes v_{1+j_2}\otimes\cdots\otimes v_{1+j_l}\,,$$

where v_{n+2} stands for v_1. An easy computation gives

$$\psi^{-1}\circ\mathbf{h}_i\circ\psi = d_id_{i-1}^{-1}\mathbf{h}_{i-1}\quad (i=2,3,\ldots,n),\quad \psi^{-2}\circ\mathbf{h}_1\circ\psi^2 = d_1d_n^{-1}\mathbf{h}_n\,. \tag{3.4.1}$$

Proposition. *Let $p=y$. Given $i=2,3,\ldots,n$, we have the following identities in* $\mathrm{End}(M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l})$

$$\psi^{-1}\circ\mathbf{e}_i(z)\circ\psi = \mathbf{e}_{i-1}(z/qd_i),\qquad \psi^{-2}\circ\mathbf{e}_1(xz)\circ\psi^2 = \mathbf{e}_n(q^{n-1}d_2\cdots d_nz)\,,$$

$$\psi^{-1}\circ\mathbf{f}_i(z)\circ\psi = \mathbf{f}_{i-1}(z/qd_i),\qquad \psi^{-2}\circ\mathbf{f}_1(xz)\circ\psi^2 = \mathbf{f}_n(q^{n-1}d_2\cdots d_nz)\,,$$

$$\psi^{-1}\circ\mathbf{k}_i^{\pm}(z)\circ\psi = \mathbf{k}_{i-1}^{\pm}(z/qd_i),\qquad \psi^{-2}\circ\mathbf{k}_1^{\pm}(xz)\circ\psi^2 = \mathbf{k}_n^{\pm}(q^{n-1}d_2\cdots d_nz)\,.$$

In order to prove this proposition we need the following.

Lemma. *For any* $1 \leqq i \leqq j < l$ *put* $Q_{i,j} = X_i \cdot T_{i,j} \in \ddot{\mathbf{H}}_{\mathscr{A}}$. *Then, if* $i \leqq r \leqq j$ *and* $i < t < j$,

$$Q_{i,j} Y_r Q_{i,j}^{-1} = \mathbf{y}^{-1} Y_{r+1}, \quad Q_{i,j} T_{t-1} Q_{i,j}^{-1} = T_t .$$

Proof. We prove the first equality, the second being similar. First, suppose that $i = 1$ and use a decreasing induction on j. If $j = l-1$ we get the relation $QY_rQ^{-1} = \mathbf{y}^{-1}Y_{r+1}$. Take r such that $1 \leqq r \leqq j-1$, then $r \leqq j$ and by induction we get

$$X_1 T_{1,j-1} T_j Y_r T_j^{-1} T_{1,j-1}^{-1} X_1^{-1} = \mathbf{y}^{-1} Y_{r+1} .$$

Since $r \neq j, j+1$, we have $T_j Y_r = Y_r T_j$ and we are done.

Fix now j and make induction on i, the case $i = 1$ being proved before. Consider $i+1 \leqq r \leqq j$. Then $i \leqq r$ and by induction we have

$$X_i T_i T_{i+1,j} Y_r T_{i+1,j}^{-1} T_i^{-1} X_i^{-1} = \mathbf{y}^{-1} Y_{r+1} .$$

Now $X_i T_i = q^2 T_i^{-1} X_{i+1}$, then we find

$$T_i^{-1}(X_{i+1} T_{i+1,j} Y_r T_{i+1,j}^{-1} X_{i+1}) T_i = \mathbf{y}^{-1} Y_{r+1} ,$$

i.e.,

$$Q_{i+1,j} Y_r Q_{i+1,j}^{-1} = \mathbf{y}^{-1} T_i Y_{r+1} T_i^{-1} .$$

Since $1 + r \neq i, i+1$ we have $T_i Y_{r+1} = Y_{r+1} T_i$ and we are through. □

Proof of the proposition. We prove the relation involving $\mathbf{e}_i$. Suppose first that $i \neq 0, 1$. Take a non-decreasing l-tuple $\mathbf{j} = (j_1, j_2, \ldots, j_l)$ and put

$$]r,s] = \mathbf{j}^{-1}(i-1), \qquad]s,t] = \mathbf{j}^{-1}(i), \qquad]p,l] = \mathbf{j}^{-1}(n+1) .$$

Consider the following l-tuples

$$\mathbf{j}_1 = (1+j_1, \ldots, 1+j_p, 1, \ldots, 1), \qquad \mathbf{j}_2 = (1, \ldots, 1, 1+j_1, \ldots, 1+j_p) ,$$

$$\mathbf{j}_3 = (1, \ldots, 1, 1+j_1, \ldots, 1+j_{l-p+s}, j_{l-p+s+1}, 1+j_{l-p+s+2}, \ldots, 1+j_p) .$$

Set $R_p = q^{p(p-l)} T_{p,1} T_{p+1,2} \cdots T_{l-1,l-p}$. Then

$$\begin{aligned}
\mathbf{e}_i(z) \circ \psi(m \otimes \mathbf{v}_\mathbf{j}) &= \mathbf{e}_i(z)(m X_{p+1}^{-1} X_{p+2}^{-1} \cdots X_l^{-1} \otimes \mathbf{v}_{\mathbf{j}_1}) \\
&= \mathbf{e}_i(z)(m X_{p+1}^{-1} X_{p+2}^{-1} \cdots X_l^{-1} R_p \otimes \mathbf{v}_{\mathbf{j}_2}) \\
&= q^{1-t+s} m X_{p+1}^{-1} \cdots X_l^{-1} R_p \left(1 + \sum_{k=l-p+s+1}^{l-p+t-1} T_{k,l-p+s+1}\right) \\
&\quad \cdot \delta(q^{n+1-i} y^{p-l-s-1} z Y_{l-p+s+1}/d_1 \cdots d_i) \otimes \mathbf{v}_{\mathbf{j}_3} .
\end{aligned}$$

In another hand

$$\psi \circ \mathbf{e}_{i-1}(z/qd_i)(m \otimes \mathbf{v_j})$$

$$= \psi\left(q^{1-t+s} m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right)\delta(q^{n+1-i}y^{-s-1}z\mathrm{Y}_{s+1}/d_1 \cdots d_i) \otimes \mathbf{v_{j-}}\right)$$

$$= q^{1-t+s} m\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right)\delta(q^{n+1-i}y^{-s-1}z\mathrm{Y}_{s+1}/d_1 \cdots d_i)\mathrm{X}_{p+1}^{-1} \cdots \mathrm{X}_l^{-1}\mathrm{R}_p \otimes \mathbf{v_{j_3}}\,.$$

Now, $\mathrm{R}_p^{-1}\mathrm{X}_l\mathrm{X}_{l-1} \cdots \mathrm{X}_{p+1} = q^{p(p-l)}\mathrm{P}_p$, where $\mathrm{P}_p = \mathrm{Q}_{l-p,l-1} \cdots \mathrm{Q}_{2,p+1}\mathrm{Q}_{1,p}$. Thus the relation follows from

$$\mathrm{P}_p\left(1 + \sum_{k=s+1}^{t-1} \mathrm{T}_{k,s+1}\right)\delta(y^{-s-1}\mathrm{Y}_{s+1})\mathrm{P}_p^{-1}$$

$$= \left(1 + \sum_{k=l-p+s+1}^{l-p+t-1} \mathrm{T}_{k,l-p+s+1}\right)\delta(y^{p-l-s-1}\mathrm{Y}_{l-p+s+1})\,,$$

which is a consequence of the preceding lemma (note that $1 \leqq s+1 \leqq p$ and $0 < k < p-1$ for any $k \in]s,t[$).

Suppose now that $i = 1$. Set $]r,s] = \mathbf{j}^{-1}(n)$, $]s,l] = \mathbf{j}^{-1}(n+1)$, and consider the following l-tuples:

$$\mathbf{j}_1 = (1,\dots,1,2,\dots,2,j_1+2,\dots,j_r+2), \qquad \mathbf{j}_1^{-1}(1) =]0,s-r],$$
$$\mathbf{j}_1^{-1}(2) =]s-r,l-r]\,,$$
$$\mathbf{j}_2 = (1,\dots,1,2,\dots,2,j_1+2,\dots,j_r+2), \qquad \mathbf{j}_2^{-1}(1) =]0,s-r+1],$$
$$\mathbf{j}_2^{-1}(2) =]s-r+1,l-r]\,,$$
$$\mathbf{j}_3 = (j_1,\dots,j_r,n,\dots,n,n+1,\dots,n+1), \qquad \mathbf{j}_3^{-1}(n) =]r,s+1],$$
$$\mathbf{j}_3^{-1}(n+1) =]s+1,l]\,,$$
$$\mathbf{j}_4 = (j_1+2,\dots,j_r+2,1,\dots,1,2,\dots,2), \qquad \mathbf{j}_4^{-1}(1) =]r,s+1],$$
$$\mathbf{j}_4^{-1}(2) =]s+1,l]\,.$$

Then

$$\mathbf{e}_1(xz) \circ \psi^2(m \otimes \mathbf{v_j}) = \mathbf{e}_1(xz)(m\mathrm{X}_{r+1}^{-1} \cdots \mathrm{X}_l^{-1}\mathrm{R}_r \otimes \mathbf{v_{j_1}})$$

$$= q^{1-l+s} m\mathrm{X}_{r+1}^{-1} \cdots \mathrm{X}_l^{-1}\mathrm{R}_r\left(1 + \sum_{k=s-r+1}^{l-r-1} \mathrm{T}_{k,s-r+1}\right)\delta(q^n y^{r-s-1}xz\mathrm{Y}_{s-r+1}/d_1) \otimes \mathbf{v_{j_2}}\,.$$

In another hand,

$$\psi^2 \circ \mathbf{e}_n(q^{n-1}d_2\cdots d_n z)(m\otimes \mathbf{v_j})$$

$$= \psi^2\left(q^{1-l+s}m\left(1+\sum_{k=s+1}^{l-1}\mathrm{T}_{k,s+1}\right)\delta(q^n y^{-s-1}z\mathrm{Y}_{s+1}/d_1)\otimes \mathbf{v}_{\mathbf{j}_3}\right)$$

$$= q^{1-l+s}m\left(1+\sum_{k=s+1}^{l-1}\mathrm{T}_{k,s+1}\right)\delta(q^n y^{-s-1}z\mathrm{Y}_{s+1}/d_1)\mathrm{X}_{r+1}^{-1}\cdots \mathrm{X}_l^{-1}\otimes \mathbf{v}_{\mathbf{j}_4}$$

$$= q^{1-l+s}m\left(1+\sum_{k=s+1}^{l-1}\mathrm{T}_{k,s+1}\right)\delta(q^n y^{-s-1}z\mathrm{Y}_{s+1}/d_1)\mathrm{X}_{r+1}^{-1}\cdots \mathrm{X}_l^{-1}\mathrm{R}_r\otimes \mathbf{v}_{\mathbf{j}_2}\,.$$

Thus it suffices to prove that

$$\mathrm{P}_r\left(1+\sum_{k=s+1}^{l-1}\mathrm{T}_{k,s+1}\right)\delta(y^{-s-1}\mathrm{Y}_{s+1})\mathrm{P}_r^{-1} = \left(1+\sum_{k=s-r+1}^{l-r-1}\mathrm{T}_{k,s-r+1}\right)\delta(xy^{r-s-1}\mathrm{Y}_{s-r+1})\,. \tag{3.4.2}$$

Formula (3.4.2) follows from

Lemma. *For any* $r<l$ *set* $\mathrm{P}_r = \mathrm{Q}_{l-r,l-1}\cdots \mathrm{Q}_{2,r+1}\mathrm{Q}_{1,r}\in \ddot{\mathbf{H}}_{\mathscr{A}}$. *Then, if* $r<s+1$ *and* $r<k<l$,

$$\mathrm{P}_r\mathrm{Y}_{s+1}\mathrm{P}_r^{-1} = \mathbf{x}\mathbf{y}^r\mathrm{Y}_{s-r+1}, \qquad \mathrm{P}_r\mathrm{T}_k\mathrm{P}_r^{-1} = \mathrm{T}_{k-r}\,.$$

Proof. By definition of P_r one gets

$$\mathrm{X}_0\mathrm{P}_r^{-1} = q^{2r(l-r)}\mathrm{X}_1\mathrm{X}_2\cdots \mathrm{X}_r\mathrm{T}_{r,1}\mathrm{T}_{r+1,2}\cdots \mathrm{T}_{l-1,l-r}\,.$$

Then a direct computation gives $\mathrm{P}_r = q^{2r(r-l)}\mathrm{Q}_{1,l-r}^{-1}\mathrm{Q}_{2,l-r+1}^{-1}\cdots \mathrm{Q}_{r,l-1}^{-1}\mathrm{X}_0$, and the result follows from the preceding lemma. □

3.5. Fix $d_0\in\mathbb{C}^\times$ and let $\mathbf{e}_0(z),\mathbf{f}_0(z),\mathbf{k}_0^{\pm}(z),\mathbf{h}_0\in \mathrm{End}(M\otimes_{\mathbf{H}}\mathbf{V}^{\otimes l})[[z^{\pm 1}]]$ be such that

$$\mathbf{e}_0(z) = \psi^{-1}\circ \mathbf{e}_1(qd_1z)\circ\psi, \qquad \mathbf{f}_0(z) = \psi^{-1}\circ \mathbf{f}_1(qd_1z)\circ\psi\,,$$
$$\mathbf{k}_0^{\pm}(z) = \psi^{-1}\circ \mathbf{k}_1^{\pm}(qd_1z)\circ\psi, \qquad \mathbf{h}_0 = d_0d_1^{-1}\psi^{-1}\circ\mathbf{h}_1\circ\psi\,.$$

The operators are defined in such a way that if $i\in[n]$ and $x = d_0\cdots d_nq^{n+1}$,

$$\begin{aligned}\psi^{-1}\circ\mathbf{e}_i(z)\circ\psi &= \mathbf{e}_{i-1}(z/qd_i), & \psi^{-1}\circ\mathbf{f}_i(z)\circ\psi &= \mathbf{f}_{i-1}(z/qd_i),\\ \psi^{-1}\circ\mathbf{k}_i^{\pm}(z)\circ\psi &= \mathbf{k}_{i-1}^{\pm}(z/qd_i), & \psi^{-1}\circ\mathbf{h}_i\circ\psi &= d_id_{i-1}^{-1}\mathbf{h}_{i-1}\end{aligned} \tag{3.5.1}$$

where $\mathbf{e}_{-1}(z)$, $\mathbf{f}_{-1}(z)$, $\mathbf{k}_{-1}^{\pm}(z)$ and $\mathbf{h}_{-1}$ stand for $\mathbf{e}_n(z)$, $\mathbf{f}_n(z)$, $\mathbf{k}_n^{\pm}(z)$ and $\mathbf{h}_n$.

Example. In particular we get

$$\mathbf{k}_0(m\otimes\mathbf{v_j}) = q^{l-s-r}m\otimes\mathbf{v_j}, \qquad \mathbf{h}_0(m\otimes\mathbf{v_j}) = d_0p^{l-s}m\otimes\mathbf{v_j}\,,$$

and, if $\mathbf{j}$ is non-decreasing,

$$\mathbf{e}_0(m \otimes \mathbf{v}_\mathbf{j}) = q^{l-r+1} m \left(\sum_{k=1}^{r} \mathrm{T}_{s,k}^{-1} \right) \mathrm{X}_{s+1} \otimes \mathbf{v}_{\mathbf{j}^-} ,$$

$$\mathbf{f}_0(m \otimes \mathbf{v}_\mathbf{j}) = q^{s} m \left(\sum_{k=s+1}^{l} \mathrm{T}_{r+1,k-1}^{-1} \right) \mathrm{X}_{r+1}^{-1} \otimes \mathbf{v}_{\mathbf{j}^+} ,$$

where $]0,r] = \mathbf{j}^{-1}(1)$, $]s,l] = \mathbf{j}^{-1}(n+1)$ and

$$\mathbf{v}_{\mathbf{j}^-} = \mathbf{v}_{(j_1,\dots,j_{r-1},j_{r+1},\dots,j_l,n+1)} \quad \text{if } r \neq 0 \text{ and } \mathbf{v}_{\mathbf{j}^-} = 0 \text{ else} ,$$

$$\mathbf{v}_{\mathbf{j}^+} = \mathbf{v}_{(j_1,\dots,j_r,\,1,j_{r+1},\dots,j_{l-1})} \quad \text{if } s \neq l \text{ and } \mathbf{v}_{\mathbf{j}^+} = 0 \text{ else} .$$

Thus $\mathbf{k}_0\mathbf{k}_1 \cdots \mathbf{k}_n(m \otimes \mathbf{v}) = m \otimes \mathbf{v}$.

The main result of this section is the following theorem

Theorem. *Put $c = 1$, $x = d_0 \cdots d_n q^{n+1}$, $y = p$ and $n > 1$. Then for any right $\ddot{\mathbf{H}}$-module M, the preceding formulas give a left $\ddot{\mathbf{U}}$-module structure on $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$. The module $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is of level l and has trivial central charge. Moreover, if $d_0 = \cdots = d_n = d$, $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is integrable with index d.*

Proof. By construction, the operators $\mathbf{e}_i(z), \mathbf{f}_i(z), \mathbf{k}_i^{\pm}(z), \mathbf{h}_i \in \mathrm{End}(M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l})[[z^{\pm 1}]]$ with non-zero i, defined in Corollary 3.3, verify relations 2.1. In order to verify all the relations 2.1, it's sufficient to prove the relations involving $\Psi_a^k(\mathbf{e}_i(z))$, $\Psi_a^k(\mathbf{f}_i(z))$, $\Psi_a^k(\mathbf{k}_i^{\pm}(z))$ and $\Psi_a^k(\mathbf{h}_i)$ for $k = 1, 2, \dots, n$ and $a_i = q^{-1} d_i^{-1}$ (see 2.2). But from (3.4.1), (3.5.1) these elements are equal respectively to $\psi^k \circ \mathbf{e}_i(z) \circ \psi^{-k}$, $\psi^k \circ \mathbf{f}_i(z) \circ \psi^{-k}$, $\psi^k \circ \mathbf{k}_i^{\pm}(z) \circ \psi^{-k}$ and $\psi^k \circ \mathbf{h}_i \circ \psi^{-k}$ ($i \in [n+1]^{\times}$). Thus we are done. For the trivial central charge see Remark 3.3 and Example 3.5. The integrability of $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ follows from the integrability of the $\mathbf{U}$-module $\mathbf{V}$. □

4. Definition of an Inverse Functor

Fix $c, x, y, d, p, q \in \mathbb{C}^{\times}$ and $l, n, \in \mathbb{N}$.

4.1. Remarks. (i) Suppose that q is not a root of unity. Then $\mathbf{H}$-modules and integrable $\mathbf{U}$-modules are direct sums of finite dimensional modules (see [L, 6.3.6] for the $\mathbf{U}$-case). Thus, if $l \leqq n$, the Schur duality in the finite case (see [J]) gives indeed an equivalence between the category of $\mathbf{H}$-modules and the category of integrable $\mathbf{U}$-modules of level l.

(ii) Similarly, if q is not a root of unity and $l \leqq n$, the affine Schur duality gives indeed an equivalence between the category of $\dot{\mathbf{H}}$-modules and the category of integrable $\dot{\mathbf{U}}$-module with trivial central charge and level l (see [C-P]).

4.2. Theorem. *Suppose that $c = 1$, $x = d^{n+1} q^{n+1}$ and $y = p$. Suppose moreover that $l + 1 < n$ and that q is not a root of unity. Let M' be an integrable $\ddot{\mathbf{U}}$-module with index d and level l. Then there exists a $\ddot{\mathbf{H}}$-module, M, such that $M' \simeq M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ as $\ddot{\mathbf{U}}$-modules.*

Proof. Given $i=1,2$, the restriction of M' to $\dot{\mathbf{U}}^{(i)}$ is integrable with trivial central charge and level l. Since $\dot{\mathbf{H}}^{(i)}$ is isomorphic to $\dot{\mathbf{H}}$, by affine Schur duality (see Remark 4.1 (ii)) one gets an $\dot{\mathbf{H}}^{(i)}$-module, $M^{(i)}$, such that $M' \simeq M^{(i)} \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ as $\dot{\mathbf{U}}^{(i)}$-modules. Moreover, the $M^{(i)}$ are isomorphic as $\mathbf{H}$-modules. So just denote them by M. By construction the action of $\mathbf{e}_0, \mathbf{f}_0, \mathbf{k}_0$ is as in Remark 3.5 and $\tilde{\mathbf{e}}_i(z), \tilde{\mathbf{f}}_i(z), \tilde{\mathbf{k}}_i(z)$ (see 2.3) act as in Theorem 3.3. Note that the action of $\mathrm{X}_i, \mathrm{Y}_j \in \ddot{\mathbf{H}}$ on M is given by the $\dot{\mathbf{H}}^{(2)}$-module and the $\dot{\mathbf{H}}^{(1)}$-module structure of M. In order to prove that these actions extend to a $\ddot{\mathbf{H}}$-module structure it's sufficient to verify that for any $m \in M$,

$$m\mathrm{Q}\mathrm{Y}_1\mathrm{Q}^{-1} = y^{-1}m\mathrm{Y}_2 \quad \text{and} \quad m\mathrm{Q}\mathrm{Y}_l\mathrm{Q}^{-1} = xy^{l-1}m\mathrm{Y}_1 \,, \tag{4.2.1}$$

where $\mathrm{Q} = \mathrm{X}_1\mathrm{T}_{1,l-1}$ (see 1.3). Recall that $M \mapsto M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is an equivalence from the category of $\mathbf{H}$-modules to the category of integrable $\mathbf{U}$-modules of level l (since q is not a root of unity, see 4.1). Thus, if $\mathbf{v} \in \mathbf{V}^{\otimes l}$ is a generator of $\mathbf{V}^{\otimes l}$, i.e. $\mathbf{V}^{\otimes l} = \mathbf{U} \cdot \mathbf{v}$, the map $M \ni m \mapsto m \otimes \mathbf{v} \in M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is injective for any $\mathbf{H}$-module M.

(i) Set $\mathbf{v} = v_1 \otimes v_2 \otimes v_4 \otimes \cdots \otimes v_{l+1}$ and $\mathbf{w} = v_2 \otimes v_4 \otimes \cdots \otimes v_{l+1} \otimes v_{n+1}$. Then,

$$\mathbf{e}_0(m \otimes \mathbf{v}) = q^{1-l} m\mathrm{Q} \otimes \mathbf{w} \,.$$

Since $\mathbf{w}$ is a generator of $\mathbf{V}$ $(l<n)$ the relation $m\mathrm{Q}\mathrm{Y}_1\mathrm{Q}^{-1} = y^{-1}m\mathrm{Y}_2$ will follow from $m\mathrm{Q}\mathrm{Y}_1\mathrm{Q}^{-1} \otimes \mathbf{w} = y^{-1}m\mathrm{Y}_2 \otimes \mathbf{w}$. In another hand the equality $\mathbf{e}_0\mathbf{k}_2^{\pm}(z) = \mathbf{k}_2^{\pm}(z)\mathbf{e}_0$ implies in particular that $\mathbf{e}_0\tilde{\mathbf{k}}_{2,-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1} = \tilde{\mathbf{k}}_{2,-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}\mathbf{e}_0$ (see 2.3). Now

$$\begin{aligned}\mathbf{e}_0\tilde{\mathbf{k}}_{2,-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}(m\otimes\mathbf{v}) &= q^n(q^{-2}-1)\mathbf{e}_0\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}(m\mathrm{Y}_2\otimes\mathbf{v}) \\ &= q^n(q^{-2}-1)y^{-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}\mathbf{e}_0(m\mathrm{Y}_2\otimes\mathbf{v}) \\ &= q^{n+1-l}(q^{-2}-1)y^{-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}(m\mathrm{Y}_2\mathrm{Q}\otimes\mathbf{w}) \,,\end{aligned}$$

and, since $\mathbf{h}_i$ and $\tilde{\mathbf{k}}_j$ commute,

$$\tilde{\mathbf{k}}_{2,-1}\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}\mathbf{e}_0(m\otimes\mathbf{v}) = q^{n+1-l}(q^{-2}-1)\mathbf{h}_1^{-1}\mathbf{h}_2^{-1}(m\mathrm{Q}\mathrm{Y}_1\otimes\mathbf{w}) \,.$$

Thus we are done.

(ii) Set $\mathbf{v} = v_1 \otimes v_3 \otimes v_4 \otimes \cdots \otimes v_{l+1}$ and $\mathbf{w} = v_3 \otimes v_4 \otimes \cdots \otimes v_{l+1} \otimes v_{n+1}$. Relations 2.1 give

$$\mathbf{h}_1(\mathbf{e}_0\mathbf{k}_{1,-1} - q^{-1}\mathbf{k}_{1,-1}\mathbf{e}_0)\mathbf{k}_1 = (q^{-1}-q)\mathbf{e}_{0,-1} = \mathbf{h}_0^{-1}(\mathbf{e}_0\mathbf{k}_{n,-1} - q^{-1}\mathbf{k}_{n,-1}\mathbf{e}_0)\mathbf{k}_n \,,$$

which evaluated on $m \otimes \mathbf{v}$ writes (see 2.3)

$$\mathbf{h}_1\mathbf{e}_0\tilde{\mathbf{k}}_{1,-1}\mathbf{h}_1^{-1}\mathbf{k}_1(m\otimes\mathbf{v}) = -q^{-1}\mathbf{h}_0^{-1}\tilde{\mathbf{k}}_{n,-1}\mathbf{h}_1^{-1}\cdots\mathbf{h}_n^{-1}\mathbf{e}_0\mathbf{k}_n(m\otimes\mathbf{v}) \,,$$

i.e.,

$$y^{-1}\mathbf{e}_0\tilde{\mathbf{k}}_{1,-1}\mathbf{k}_1(m\otimes\mathbf{v}) = -q^{-1}\mathbf{d}^{-1}\tilde{\mathbf{k}}_{n,-1}\mathbf{e}_0\mathbf{k}_n(m\otimes\mathbf{v}) \,.$$

Since $\mathbf{d} = \mathbf{h}_0 \cdots \mathbf{h}_n$ acts as $y^l d^{n+1}$ and $n > l+1$ we find

$$d^{n+1}q^{n+1}my^{-1}\mathrm{Y}_1\mathrm{Q}\otimes\mathbf{w} = y^{-l}m\mathrm{Q}\mathrm{Y}_l\otimes\mathbf{w} \,.$$

Thus, since $\mathbf{w}$ is a generator of $\mathbf{V}$, we get the second relation in (4.2.1).

We have proved that X_i and Y_j give a $\ddot{\mathbf{H}}$-module structure on M. In order to prove that M' is isomorphic to the dual module $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ we must now verify that $\mathbf{h}_i$ acts as in Corollary 3.3. It is a direct consequence of the integrability of $M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ since the equality $\mathbf{k}_i(m \otimes \mathbf{v_j}) = q^{\mathbf{j}^{-1}(i) - \mathbf{j}^{-1}(i+1)} m \otimes \mathbf{v_j}$ imply that $\mathbf{h}_i(m \otimes \mathbf{v_j}) = dp^{\mathbf{j}^{-1}(i)} m \otimes \mathbf{v_j}$. □

5. Conclusion

Using Theorems 3.5 and 4.2 we get the following duality theorem.

Theorem. *Suppose that $c = 1$, $x = d^{n+1} q^{n+1}$ and $y = p$. Suppose moreover that q is not a root of unity and that $l + 1 < n$. Then the functor $M \mapsto M \otimes_{\mathbf{H}} \mathbf{V}^{\otimes l}$ is an equivalence between the category of right $\ddot{\mathbf{H}}$-modules and the category of left integrable $\ddot{\mathbf{U}}$-modules with index d and level l.*

Remark. Suppose that q is not a root of unity. Since $\ddot{\mathbf{H}}$ does not admit finite dimensional representations if x is not a root of unity, the toroidal quantum group $\ddot{\mathbf{U}}$ does not admit either any finite dimensional representations if $d^{n+1} q^{n+1}$ is not a root of unity.

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