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Strings from $N=2$ Gauged Wess–Zumino–Witten Models

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Abstract: We present an algebraic approach to string theory. An embedding of $sl(2|1)$ in a super Lie algebra together with a grading on the Lie algebra determines a nilpotent subalgebra of the super Lie algebra. Chirally gauging this subalgebra in the corresponding Wess–Zumino–Witten model, breaks the affine symmetry of the Wess–Zumino–Witten model to some extension of the $N=2$ superconformal algebra. The extension is completely determined by the $sl(2|1)$ embedding. The realization of the superconformal algebra is determined by the grading. For a particular choice of grading, one obtains in this way, after twisting, the BRST structure of a string theory. We classify all embeddings of $sl(2|1)$ into Lie super algebras and give a detailed account of the branching of the adjoint representation. This provides an exhaustive classification and characterization of both all extended $N=2$ superconformal algebras and all string theories which can be obtained in this way.

Contents

1. Introduction

By now, there is a plethora of different string theories. One way to categorize them is according to the gauge algebra on the worldsheet. Taking the Virasoro algebra as gauge algebra, one obtains the bosonic string, the $N = 1$ super Virasoro algebra gives rise to the superstring, the W_n algebra yields W_n-strings, ... In fact, it might be that to each extension of the Virasoro algebra, one can associate a string theory[1]. Given a gauge algebra, there is still a very large freedom which consists of the particular choice of the realization or string vacuum. Though all of these string theories are perfectly consistent in perturbation theory, only a very restricted set gives rise to phenomenologically acceptable theories. However, as long as we do not understand the non-perturbative behaviour of string theory, one should study all classical solutions, hoping that this provides hints to the real structure of string theory. Some glimpse of a more systematic structure was seen in [1], where it was shown that the bosonic string is a special choice of vacuum of an $N = 1$ superstring; the $N = 1$ superstring is then a special choice of vacuum of the $N = 2$ superstring. Similar patterns, involving other types of string theories were obtained later on. Though quite a fascinating observation, its relevance remains to be understood (see e.g. the remarks in [2]). A seemingly unrelated approach was initialized in [3]. There it was shown that the BRST structure of the bosonic string is encoded in a twisted $N = 2$ superconformal algebra. This seems to be a universal feature of string theories: for any string theory, the BRST structure is given by a twisted extension of the $N = 2$ superconformal algebra. This has been worked out in several concrete cases: the BRST structure of the superstring is given by the $N = 3$ superconformal algebra [4], the W_n strings by the corresponding twisted $N = 2$ W_n algebra [4] and strings with N supersymmetries have the Knizhnik–Bershadsky $SO(N + 2)$ superconformal symmetry [5]. Even topological strings exhibit such a structure [6]. The main idea here is that one adds both the BRST current and the anti-ghosts to the gauge algebra. The BRST-charge itself is then one of the supercharges,

$$\mathcal{Q}_{\mathrm{BRST}} \equiv G_0^+ = \frac{1}{2\pi i} \oint dz\, (cT + \cdots)\,, \tag{1.1}$$

while the Virasoro anti-ghost $b(z)$ is the conjugate supercurrent $G^-(z)$. This automatically ensures that $T(z) = \{\mathcal{Q}_{\mathrm{BRST}}, b(z)\}$. Together this generates the twisted $N = 2$ superconformal algebra. For a string theory with a larger gauge algebra, one gets in this way some twisted extension of the $N = 2$ superconformal algebra. Once the presence of this twisted $N = 2$ structure is accepted, one might try to use it to *define* the string theory. This gives rise to a very algebraic and systematic approach to string theory. The obvious way to achieve such a systematics is through gauging Wess–Zumino–Witten models, which is also known as quantum Hamiltonian

[1] See however [7] for a potential counter-example

reduction. It is well known that the reduction of $sl(2|1)$ gives rise to the $N = 2$ superconformal algebra. Embeddings of $sl(2|1)$ in super Lie algebras yield then all extensions of the $N = 2$ algebra which can be obtained through Hamiltonian reduction. Once a particular embedding is given, the extended $N = 2$ superconformal algebra is uniquely determined. However, the particular (free field) realization (or quantum Miura transform) one obtains for this algebra is determined by a choice of a grading on the super Lie algebra. As we will see, only a very particular grading allows for a stringy interpretation. This approach to string theory has the great advantage that it is almost completely algebraic. The calculations are of an algorithmic nature, enabling one to obtain e.g. the explicit form of the BRST current in a straightforward way (compare this to the usual trial and error method). This program has been explicitly carried out for the non-critical W_n strings, based on a reduction of $sl(n|n-1)$ [4] and strings with N supersymmetries, based on a reduction of $osp(N+2|2)$ [5]. In the former case, only classical arguments were given, while in the latter, at least for $N = 1$ and 2, the full quantum structure was exhibited. As $N = 2$ superconformally invariant models exhibit a very rich structure, (for an extensive review, see [8]), the full classification of extended $N = 2$ algebras obtainable from Hamiltonian reduction is in itself an interesting result. In the present paper we start in the next section with the simplest example available: the bosonic string. We show that the BRST structure is indeed given by a twisted $N = 2$ algebra and we derive it from the reduction of $sl(2|1)$. This example exhibits already many of the complications which arise in the general case. Section 3 classifies all $sl(2|1)$ embeddings in super Lie algebras. In Sect. 4, a detailed study is made of the branching of the adjoint representation of a super Lie algebras into irreducible representations of the embedded $sl(2|1)$ algebra. This basically determines the field content of the extended $N = 2$ superconformal algebra. The results of Sect. 4 are applied in the next sections. In Sect. 5 we briefly discuss the reduction using the standard grading. This yields the so-called symmetric realizations of the extended $N = 2$ superconformal algebra. In Sect. 6 we determine, given some embedding of $sl(2|1)$ in a super Lie algebra, the grading which will yield a "stringy" reduction. We construct the gauged WZW model which describes the reduction. In the next section the model is quantized and the resulting string theory is discussed. We end with some conclusions and open problems. In the appendix we summarize some properties of WZW models.

2. A Simple Example

Before treating the general case, we illustrate the main ideas with the simplest example: the bosonic string. We give this example in considerable detail as it exhibits many of the complications which arise in the general case. The string consists of a matter, a gravity or Liouville, and a ghost sector. At this point, we do not make any assumptions about the particular structure of the matter sector. We just represent it by its energy-momentum tensor T_m which generates the Virasoro algebra with central charge c_m. The gravity sector is realized in terms of a Liouville field φ_L, $\partial\varphi_L(z_1)\partial\varphi_L(z_2) = -z_{12}^{-2}$, with energy-momentum tensor T_L:

$$T_L = -\frac{1}{2}\partial\varphi_L\partial\varphi_L + \sqrt{\frac{25-c_m}{12}}\partial^2\varphi_L\,, \tag{2.1}$$

which has central charge $c_L = 26 - c_m$. The energy-momentum tensor for the ghost system assumes the standard form:

$$T_{gh} = -2B\partial C - (\partial B)C\,, \tag{2.2}$$

and has central extension $c_{gh} = -26$. The total energy-momentum tensor $T = T_m + T_L + T_{gh}$ has central charge 0. The BRST current

$$J_{\mathrm{BRST}} = C\left(T_m + T_L + \frac{1}{2}T_{gh}\right) + \alpha\partial(C\partial\varphi_L) + \beta\partial^2 C\,, \tag{2.3}$$

with

$$\begin{aligned}\alpha &= -\frac{\sqrt{3}}{6}(\sqrt{1-c_m} + \sqrt{25-c_m})\,,\\ \beta &= -\frac{1}{12}(7 - c_m + \sqrt{(1-c_m)(25-c_m)})\end{aligned} \tag{2.4}$$

has only regular terms in its OPE with itself: $J_{\mathrm{BRST}}(z_1)J_{\mathrm{BRST}}(z_2) = \cdots$. Note that this is a stronger statement than nilpotency of the BRST charge $\mathcal{Q}^2_{\mathrm{BRST}} = 0$ with

$$\mathcal{Q}_{\mathrm{BRST}} = \frac{1}{2\pi i}\oint dz J_{\mathrm{BRST}}\,. \tag{2.5}$$

The total derivative terms in Eq. (2.3), which have no influence on the BRST operator, have precisely been added to achieve this [3]. Calling $G_+ \equiv J_{\mathrm{BRST}}$ and $G_- \equiv B$, one finds that the current algebra generated by T, G_+ and G_- closes, provided a $U(1)$ current U is introduced:

$$\begin{aligned}T(z_1)T(z_2) &= 2z_{12}^{-2}T(z_2) + z_{12}^{-1}\partial T(z_2)\,,\\ T(z_1)G_+(z_2) &= z_{12}^{-2}G_+(z_2) + z_{12}^{-1}\partial G_+(z_2)\,,\\ T(z_1)G_-(z_2) &= 2z_{12}^{-2}G_-(z_2) + z_{12}^{-1}\partial G_-(z_2)\,,\\ T(z_1)U(z_2) &= -\frac{c_{N=2}}{3}z_{12}^{-3} + z_{12}^{-2}U(z_2) + z_{12}^{-1}\partial U(z_2)\,,\\ G_+(z_1)G_-(z_2) &= \frac{c_{N=2}}{3}z_{12}^{-3} + z_{12}^{-2}U(z_2) + z_{12}^{-1}T(z_2)\,,\\ U(z_1)G_\pm(z_2) &= \pm z_{12}^{-1}G_\pm(z_2), \qquad U(z_1)U(z_2) = \frac{c_{N=2}}{3}z_{12}^{-2}\,,\end{aligned} \tag{2.6}$$

where U is a modification of the ghost number current:

$$U \equiv -BC - \alpha\partial\varphi_L\,, \tag{2.7}$$

and

$$c_{N=2} = 6\beta\,. \tag{2.8}$$

Upon untwisting $T_{N=2} = T - \frac{1}{2}\partial U$, one gets the standard $N = 2$ superconformal algebra with central extension $c_{N=2} = 6\beta$.

For the critical bosonic string, $c_m = 25$, we get in this way $c_{N=2} = 9$. Taking the Virasoro minimal models for the matter sector:

$$c_m = 1 - 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2\,, \tag{2.9}$$

one gets

$$c_{N=2} = 3\left(1 - 2\frac{p}{q}\right) . \tag{2.10}$$

In particular, taking $(p,q) = (1, k+2)$, we get $c_{N=2} = 3k/(k+2)$, i.e. the $N = 2$ minimal models, a fact which was heavily used in [4].

We now turn to the Hamiltonian reduction and show how to obtain the above from it. The super Lie algebra $sl(2|1)$ has a bosonic part generated by a $su(2) \oplus u(1)$ sector: $\{e_{\mp}, e_{=}, e_0, u_0\}$ with $[e_0, e_{\mp}] = +2e_{\mp}$, $[e_0, e_{=}] = -2e_{=}$ and $[e_{\mp}, e_{=}] = e_0$. The fermionic generators, $g_{\pm}$, $\bar{g}_{\pm}$ are $sl(2)$ doublets, while g ($\bar{g}$) has eigenvalue $+1$ (-1) under ad_{u_0}. The remaining commutation relations are easily derived from the 3×3 matrix representation[2] $e_{\mp} = e_{12}$, $e_{=} = e_{21}$, $e_0 = e_{11} - e_{22}$, $u_0 = -e_{11} - e_{22} - 2e_{33}$, $g_+ = e_{13}$, $g_- = e_{23}$, $\bar{g}_+ = e_{32}$ and $\bar{g}_- = e_{31}$. The WZW model, with action $\kappa S^-[g]$ on $sl(2|1)$ gives rise to affine currents $J = E^{\mp}e_{\mp} + E^0 e_0 + E^{=}e_{=} + U^0 u_0 + F^+ g_+ + F^- g_- + \bar{F}^+ \bar{g}_+ + \bar{F}^- \bar{g}_-$ which satisfy the OPE's[3]:

$$E^0(z_1)E^0(z_2) = \frac{\kappa}{8} z_{12}^{-2}, \qquad U^0(z_1)U^0(z_2) = -\frac{\kappa}{8} z_{12}^{-2} ,$$

$$E^0(z_1)E^{\mp}(z_2) = \frac{1}{2} z_{12}^{-1} E^{\mp}(z_2), \qquad E^0(z_1)E^{=}(z_2) = -\frac{1}{2} z_{12}^{-1} E^{=}(z_2) ,$$

$$E^{\mp}(z_1)E^{=}(z_2) = \frac{\kappa}{4} z_{12}^{-2} + z_{12}^{-1} E^0(z_2) ,$$

$$E^0(z_1)F^{\pm}(z_2) = \pm\frac{1}{4} z_{12}^{-1} F^{\pm}(z_2), \qquad E^0(z_1)\bar{F}^{\pm}(z_2) = \pm\frac{1}{4} z_{12}^{-1} \bar{F}^{\pm}(z_2) ,$$

$$U^0(z_1)F^{\pm}(z_2) = -\frac{1}{4} z_{12}^{-1} F^{\pm}(z_2), \qquad U^0(z_1)\bar{F}^{\pm}(z_2) = +\frac{1}{4} z_{12}^{-1} \bar{F}^{\pm}(z_2) ,$$

$$E^{\mp}(z_1)F^-(z_2) = +\frac{1}{2} z_{12}^{-1} F^+(z_2) , \qquad E^{=}(z_1)F^+(z_2) = +\frac{1}{2} z_{12}^{-1} F^-(z_2) ,$$

$$E^{\mp}(z_1)\bar{F}^-(z_2) = -\frac{1}{2} z_{12}^{-1} \bar{F}^+(z_2), \qquad E^{=}(z_1)\bar{F}^+(z_2) = -\frac{1}{2} z_{12}^{-1} \bar{F}^-(z_2) ,$$

$$F^+(z_1)\bar{F}^+(z_2) = \frac{1}{2} z_{12}^{-1} E^{\mp}(z_2), \qquad F^-(z_1)\bar{F}^-(z_2) = \frac{1}{2} z_{12}^{-1} E^{=}(z_2) ,$$

$$F^+(z_1)\bar{F}^-(z_2) = \frac{\kappa}{4} z_{12}^{-2} + \frac{1}{2} z_{12}^{-1} E^0(z_2) + \frac{1}{2} z_{12}^{-1} U^0(z_2) ,$$

$$F^-(z_1)\bar{F}^+(z_2) = +\frac{\kappa}{4} z_{12}^{-2} - \frac{1}{2} z_{12}^{-1} E^0(z_2) + \frac{1}{2} z_{12}^{-1} U^0(z_2) . \tag{2.11}$$

To perform the Hamiltonian reduction one has to introduce a grading and constrain the strictly negatively graded part of the current. This will give rise to a gauge symmetry generated by the strictly positively graded subalgebra of the super Lie algebra. Usually one takes the grading to be the one given by $\frac{1}{2}\mathrm{ad}_{e_0}$ [9],

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[2] e_{kl} is a matrix unit, i e $(e_{kl})_{rs} = \delta_{kr}\delta_{ls}$

[3] We use the metric $g_{ab} = -2\mathrm{str}(t_a t_b)$

which, in our case gives,

	$E^{\mp}$	E^0	$E^=$	U^0	F^+	F^-	$\bar{F}^+$	$\bar{F}^-$
grade	1	0	-1	0	1/2	$-1/2$	1/2	$-1/2$

The constraint one imposes on the affine current is then simply $\Pi_{<0}J = \frac{\kappa}{2}(e_= + \tau g_- + \bar{\tau}\bar{g}_-)$, where $\Pi_{<0}$ projects on the strictly negatively graded part of the Lie algebra and τ and $\bar{\tau}$ are auxiliary fields needed to obtain first class constraints[4]. These constraints follow from the action

$$\mathscr{S} = \kappa S^-[g] + \frac{1}{\pi x}\int \operatorname{str} A\left(J - \frac{\kappa}{2}e_= - \frac{\kappa}{2}[e_=, \tau]\right) - \frac{\kappa}{4\pi x}\int \operatorname{str}[e_=, \tau]\bar{\partial}\tau\,, \tag{2.12}$$

where

$$\begin{aligned} A &= A^{\mp}e_{\mp} + A^+g_+ + \bar{A}^+\bar{g}_+\,, \\ \tau &= \tau g_+ + \bar{\tau}\bar{g}_+\,. \end{aligned} \tag{2.13}$$

The action has a gauge invariance parametrized by $h = \exp\eta$, $\eta \in \Pi_{>0}sl(2|1)$ or $\eta = \eta^{\mp}e_{\mp} + \eta^+g_+ + \bar{\eta}^+\bar{g}_+$,

$$\begin{aligned} g &\to g' = hg\,, \\ A &\to A' = \bar{\partial}hh^{-1} + hAh^{-1}\,, \\ \tau &\to \tau' = \tau - \Pi_{\frac{1}{2}}\eta\,. \end{aligned} \tag{2.14}$$

Fixing this symmetry by putting $A = 0$, we get, upon introducing ghosts $c = c^{\mp}e_{\mp} + \gamma^+g_+ + \bar{\gamma}^+\bar{g}_+ \in \Pi_{>0}sl(2|1)$ and anti-ghosts $b = b^=e_= + \beta^-g_- + \bar{\beta}^-\bar{g}_- \in \Pi_{<0}sl(2|1)$, the gauge fixed action:

$$\mathscr{S}_{gf} = \kappa S^-[g] - \frac{\kappa}{4\pi x}\int \operatorname{str}[e_=, \tau]\bar{\partial}\tau + \frac{1}{2\pi x}\int \operatorname{str} b\bar{\partial}c\,, \tag{2.15}$$

and the BRST charge

$$\mathscr{Q}_{HR} = \frac{1}{4\pi i x}\oint \operatorname{str}\left\{c\left(J - \frac{\kappa}{2}e_= - \frac{\kappa}{2}[e_=, \tau] + \frac{1}{2}J_{gh}\right)\right\}, \tag{2.16}$$

where $J_{gh} = 1/2\{b, c\}$. Because of the constraints, the original $sl(2|1)$ affine symmetry of the WZW model breaks down to an $N = 2$ superconformal symmetry. The generators of the $N = 2$ superconformal algebra are precisely the generators of the cohomology $\mathscr{H}^*(\mathscr{A}, \mathscr{Q}_{HR})$, where $\mathscr{A}$ is the algebra generated by $\{b, \hat{J} = J + J_{gh}, \tau, c\}$ and all normal ordered products of these fields and their derivatives. In [10], the computation of this cohomology has been done in general.

[4] We use a slightly confusing notation in Eq (2 13), as τ denotes both a Lie algebra valued field and one of its components. The context should make it clear what is meant by τ

Applying the results and methods from [10], one obtains

$$\begin{aligned}
T_{N=2} &= \frac{2\kappa}{\kappa+1}\left(\hat{E}^{+} + \hat{\bar{F}}^{+}\tau + \hat{F}^{+}\bar{\tau} - \frac{2}{\kappa}\hat{U}^{0}\hat{U}^{0} + \frac{2}{\kappa}\hat{E}^{0}\hat{E}^{0}\right. \\
&\quad \left. - \frac{\kappa+1}{\kappa}\partial\hat{E}^{0} - \frac{\kappa+1}{4}(\tau\partial\bar{\tau} - \partial\tau\bar{\tau})\right), \\
G_{+} &= \sqrt{\frac{4\kappa}{\kappa+1}}\left(\hat{F}^{+} - \tau(\hat{E}^{0} + \hat{U}^{0}) + \frac{\kappa+1}{2}\partial\tau\right), \\
G_{-} &= \sqrt{\frac{4\kappa}{\kappa+1}}\left(\hat{\bar{F}}^{+} - \bar{\tau}(\hat{E}^{0} - \hat{U}^{0}) + \frac{\kappa+1}{2}\partial\bar{\tau}\right), \\
U &= -4\left(\hat{U}^{0} - \frac{\kappa}{4}\tau\bar{\tau}\right),
\end{aligned} \tag{2.17}$$

which satisfies the $N = 2$ superconformal algebra with $c_{N=2} = -3(1 + 2\kappa)$. The algebra $\mathscr{A}$ has a natural double grading,

$$\mathscr{A} = \bigoplus_{\substack{m,n\in\frac{1}{2}\mathbf{Z} \\ m+n\in\mathbf{Z}}} \mathscr{A}_{(m,n)}, \tag{2.18}$$

where in (m, n), m is the canonical grading used in the reduction and $m + n$ is the ghost number. The auxiliary fields τ are assigned grading $(0, 0)$. In [10] it was proven that the map $X \to \Pi_{(0,0)}X$, where $X = T_{N=2}$, $G_{\pm}$ or U, is an algebra isomorphism. This is the so-called Miura transform. Performing this map, we get the standard free field realization of the $N = 2$ superconformal algebra:

$$\begin{aligned}
T_{N=2} &= \partial\varphi\partial\bar{\varphi} - \frac{\sqrt{\kappa+1}}{2}\partial^{2}(\varphi + \bar{\varphi}) - \frac{1}{2}(\psi\partial\bar{\psi} - \partial\psi\bar{\psi}), \\
G_{+} &= -\psi\partial\varphi + \sqrt{\kappa+1}\,\partial\psi, \\
G_{-} &= -\bar{\psi}\partial\varphi + \sqrt{\kappa+1}\,\partial\bar{\psi}, \\
U &= \psi\bar{\psi} - \sqrt{\kappa+1}(\partial\varphi - \partial\bar{\varphi}),
\end{aligned} \tag{2.19}$$

where $\partial\varphi(z_1)\partial\bar{\varphi}(z_2) = z_{12}^{-2}$, $\psi(z_1)\bar{\psi}(z_2) = z_{12}^{-1}$ and we introduced some simple rescalings:

$$\begin{aligned}
\partial\varphi &= \frac{2}{\sqrt{\kappa+1}}(\hat{E}^{0} + \hat{U}^{0}), \quad \psi = \sqrt{\kappa}\tau, \\
\partial\bar{\varphi} &= \frac{2}{\sqrt{\kappa+1}}(\hat{E}^{0} - \hat{U}^{0}), \quad \bar{\psi} = \sqrt{\kappa}\bar{\tau}.
\end{aligned} \tag{2.20}$$

This provides us with a realization where G_{+} and G_{-} are treated on the same footing. This is the so-called symmetric realization of the $N = 2$ algebra: both G_{+} and G_{-} are given by composite operators. In order to obtain a stringy interpretation of the reduction, we want to identify G_{+} with the BRST current and G_- with the Virasoro anti-ghost, so a single field instead of a composite. To achieve this we consider a different grading [11], namely according to the eigenvalues of $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$. We obtain for the gradings of the various currents:

	$E^{\mp}$	E^0	$E^{=}$	U^0	F^+	F^-	$\bar{F}^+$	$\bar{F}^-$
grade	1	0	-1	0	3/2	1/2	$-1/2$	$-3/2$

We follow quite the same procedure as before, but whenever we refer to the grading on the Lie algebra, we always imply it to be the grading induced by $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$. Again we constrain the strictly negatively graded part of the algebra:

$$\Pi_{<0} J = \frac{\kappa}{2}(e_{=} + \psi \bar{g}_+) \,. \tag{2.21}$$

The appearance of the auxiliary field ψ is understood as follows. The current $\bar{F}^+$ is a highest $sl(2)$ weight and will become the leading term of a conformal current [10]. But by the same token, $\bar{F}^+$ has a negative grading, so it has to be constrained. Thus we need to constrain it in a non-singular way, i.e. by putting it equal to an auxiliary field which is inert under the gauge transformations. The action which reproduces the constraints is easily obtained:

$$\mathscr{S} = \kappa S^-[g] + \frac{1}{\pi x}\int \mathrm{str}\, A\left(J - \frac{\kappa}{2}e_{=} - \frac{\kappa}{2}\Psi\right) + \frac{\kappa}{2\pi x}\int \mathrm{str}\, \Psi \bar{\partial} \bar{\Psi} \,, \tag{2.22}$$

where

$$\begin{aligned} A &= A^{\mp} e_{\mp} + A^+ g_+ + A^- g_- \,, \\ \Psi &= \psi \bar{g}_+, \qquad \bar{\Psi} = \bar{\psi} g_- \,. \end{aligned} \tag{2.23}$$

The gauge invariance is parametrized by $h = \exp \eta$, $\eta \in \Pi_{>0} sl(2|1)$ or $\eta = \eta^{\mp} e_{\mp} + \eta^+ g_+ + \eta^- g_-$,

$$\begin{aligned} g &\to g' = hg \,, \\ A &\to A' = \bar{\partial} h h^{-1} + hAh^{-1} \,, \\ \bar{\Psi} &\to \bar{\Psi}' = \bar{\Psi} + \eta^- g_- \,. \end{aligned} \tag{2.24}$$

One immediately sees that the combined requirements of gauge invariance and the existence of a non-degenerate highest weight gauge, requires the introduction of the field $\bar{\psi}$ conjugate to ψ, even if it was not needed for the constraints. As before, the gauge choice is $A = 0$. Introducing ghosts $c = c^{\mp} e_{\mp} + \gamma^+ g_+ + \gamma^- g_- \in \Pi_{>0} sl(2|1)$ and anti-ghosts $b = b^{=} e_{=} + \bar{\beta}^+ \bar{g}_+ + \bar{\beta}^- \bar{g}_- \in \Pi_{<0} sl(2|1)$, we get the gauge fixed action:

$$\mathscr{S}_{gf} = \kappa S^-[g] + \frac{\kappa}{2\pi x}\int \mathrm{str}\, \Psi \bar{\partial} \bar{\Psi} + \frac{1}{2\pi x}\int \mathrm{str}\, b \bar{\partial} c \,, \tag{2.25}$$

and the BRST charge

$$\mathscr{Q}_{HR} = \frac{1}{4\pi i x}\oint \mathrm{str}\left\{c\left(J - \frac{\kappa}{2}e_{=} - \frac{\kappa}{2}\Psi + \frac{1}{2}J_{gh}\right)\right\} \,. \tag{2.26}$$

Again, the affine symmetry of the WZW model breaks down to an $N = 2$ superconformal symmetry whose generators are the generators of the cohomology $\mathscr{H}^*(\mathscr{A}, \mathscr{Q}_{HR})$, where $\mathscr{A}$ is the algebra generated by $\{b, \hat{J} = J + J_{gh}, \psi, \bar{\psi}, c\}$ and all normal ordered products of these fields and their derivatives. We are now in the position to use exactly the same methods, i.e. spectral sequence techniques, to solve the cohomology, provided we consider the double grading (m, n), where m is induced by the action of $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$ and $m + n$ is the ghost number. We assign grading $(0,0)$ to the auxiliary fields ψ and $\bar{\psi}$. Applying the methods developed in [10], one arrives at

$$
\begin{aligned}
T_{N=2} &= \frac{2\kappa}{\kappa+1}\Big(\hat{E}^{+} + \psi\hat{F}^{-} - \frac{2}{\kappa}\hat{U}^0\hat{U}^0 + \frac{2}{\kappa}\hat{E}^0\hat{E}^0 \\
&\quad -\partial\hat{E}^0 - \frac{1}{\kappa}\partial\hat{U}^0 - \frac{\kappa+1}{4}(3\psi\partial\bar{\psi} + \partial\psi\bar{\psi})\Big)\,, \\
G_+ &= \frac{2\kappa^2}{1+\kappa}\Big(\hat{F}^{+} + \hat{E}^{+}\bar{\psi} - \frac{2}{\kappa}(\hat{E}^0 + \hat{U}^0)\hat{F}^{-} + \hat{F}^{-}\bar{\psi}\psi + \partial\hat{F}^{-} \\
&\quad -\frac{(\kappa+1)(2\kappa+1)}{4\kappa}\partial^2\bar{\psi} + \frac{2}{\kappa}(\hat{E}^0\hat{E}^0 - \hat{U}^0\hat{U}^0)\bar{\psi} \\
&\quad -\partial(\hat{E}^0 - \hat{U}^0)\bar{\psi} - \frac{1+\kappa}{2}\psi\partial\bar{\psi}\bar{\psi} + \frac{2(1+\kappa)}{\kappa}\partial\bar{\psi}\hat{U}^0\Big)\,, \\
G_- &= \psi\,, \\
U &= -4\left(\hat{U}^0 + \frac{\kappa}{4}\psi\bar{\psi}\right)\,,
\end{aligned}
\tag{2.27}
$$

which satisfies the $N = 2$ superconformal algebra with $c_{N=2} = -3(1 + 2\kappa)$. Again, the Miura transform is given by the algebra isomorphism $X \to \Pi_{(0,0)}X$, where X stands for the conformal currents. Together with the OPE's $\hat{E}_0(z_1)\hat{E}_0(z_2) = -\hat{U}_0(z_1)\hat{U}_0(z_2) = (\kappa+1)/8z_{12}^{-2}$ and $\psi(z_1)\bar{\psi}(z_2) = 1/\kappa z_{12}^{-1}$ gives the desired realization of the $N = 2$ algebra. Indeed, identifying $B \equiv \psi$, $C \equiv \kappa\bar{\psi}$, $\partial\varphi_L \equiv \sqrt{8/(\kappa+1)}\hat{U}_0$ and $\partial\varphi_m \equiv i\sqrt{8/(\kappa+1)}\hat{E}_0$, and

$$
T_m = -\frac{1}{2}\partial\varphi_m\partial\varphi_m + i\frac{\kappa}{\sqrt{2(\kappa+1)}}\partial^2\varphi_m\,,
\tag{2.28}
$$

precisely reproduces, upon twisting, the non-critical string theory discussed at the beginning of this section with

$$
c_m = 1 - 6\left(\sqrt{\kappa+1} - \frac{1}{\sqrt{\kappa+1}}\right)^2 .
\tag{2.29}
$$

It is interesting to note that if we take for the matter sector a reduction of the $sl(2)$ WZW model, one gets for c in terms of the level $\kappa_{sl(2)}$ of the underlying $sl(2)$ WZW model precisely the previous expression for c_m, provided one identifies $\kappa + 1$ with $\kappa_{sl(2)} + 2$. The shift can be understood as an additional ghostcontribution to the $sl(2)$ central charge. This is not a stand alone case. This program has been carried out in a few other cases as well: strings with N supersymmetries have the

Knizhnik–Bershadsky $SO(N+2)$ superconformal symmetry and are obtained from the reduction of $osp(N+2|2)$ [5]. Classically, it was shown in [4] that W_n strings have an $N = 2\, W_n$ algebra and they are obtained from a reduction of $sl(n|n-1)$. We will now turn to a detailed investigation of this construction. In the next sections we analyze the embeddings of $sl(2|1)$ in supergroups after which we come back to the construction of string theories.

3. Classification of $sl(1|2)$ Embeddings into Lie Superalgebras

Let $\mathcal{G}$ be a Lie superalgebra. We want to classify its $sl(1|2)$ sub-superalgebras. As $sl(1|2)$ admits an $osp(1|2)$ (principal) sub-superalgebra, the classification of $sl(1|2)$ embeddings is a subclass of $osp(1|2)$ ones. Let us first recall the two fundamental theorems concerning the classification of $osp(1|2)$ sub-superalgebra [12, 13]:

3.1. $osp(1|2)$ embeddings: A reminder
Principal $osp(1|2)$ embeddings in superalgebras. The only simple superalgebras that admit a principal $osp(1|2)$-embedding are the $sl(n|n \pm 1)$, $osp(2n|2n \pm 1)$, $osp(2n|2n)$, $osp(2n|2n+2)$, or $D(2,1;\alpha)$ superalgebras. They all admit a totally fermionic simple roots system[5].

We recall that the principal $osp(1|2)$-subalgebra of a superalgebra $\mathcal{G}$ possesses as a simple fermionic root generator the sum of all the $\mathcal{G}$ simple fermionic root generators. It is maximal in $\mathcal{G}$ (i.e. the only superalgebra that contains the principal $osp(1|2)$ in $\mathcal{G}$ is $\mathcal{G}$ itself). On the opposite, a superalgebra $\mathcal{H}$ is regular in $\mathcal{G}$ when its root generators are root generators for $\mathcal{G}$: the regular $osp(1|2)$ in $\mathcal{G}$ is the "smallest" subalgebra of $\mathcal{G}$.

Classification of $osp(1|2)$ embeddings. Any $osp(1|2)$-embedding in a simple Lie superalgebra $\mathcal{G}$ can be considered as the principal $osp(1|2)$ of a regular $\mathcal{G}$-sub-superalgebra (of the type given just above), up to the following exceptions:

- *For $\mathcal{G} = osp(2n \pm 2|2n)$ with $n \geqq 2$, besides the principal embeddings described above, there exists also $osp(1|2)$ sub-superalgebras associated to the singular embeddings $osp(2k \pm 1|2k) \oplus osp(2n-2k \pm 1|2n-2k)$ with $1 \leqq k \leqq [\frac{n-1}{2}]$.*
- *For $\mathcal{G} = osp(2n|2n)$ with $n \geqq 2$, besides the principal embeddings described above, there exists also $osp(1|2)$ sub-superalgebras associated to the singular embeddings $osp(2k \pm 1|2k) \oplus osp(2n-2k \mp 1|2n-2k)$ with $1 \leqq k \leqq [\frac{n-2}{2}]$.*

3.2. $sl(1|2)$ embeddings. As already mentioned, any $sl(1|2)$ sub-superalgebra provide an $osp(1|2)$-embedding. Hence, the classification of $sl(1|2)$-embeddings will also be associated to some of the above $\mathcal{G}$-sub-superalgebra(s). Let us be precise, we have:

Theorem 1. *Let $\mathcal{G}$ be a superalgebra. Any $sl(1|2)$ embedding into $\mathcal{G}$ can be seen as the principal sub-superalgebra of a (sum of) regular $sl(p|p \pm 1)$ sub-superalgebra(s) of $\mathcal{G}$, except in the case of $osp(m|2n)$ $(m > 1)$, $F(4)$ and $D(2,1;\alpha)$ where the (sum of) regular $osp(2|2)$ has also to be considered*[6].

[5] This necessary condition is almost sufficient, since only the $sl(n|n)$ superalgebra has a totally fermionic simple roots system while not admitting a principal $osp(1|2)$

[6] We make a distinction between the $osp(2|2)$ and $sl(1|2)$ superalgebras: these two superalgebras are isomorphic, but not the associated supergroups The smallest non-trivial representation for $osp(2|2)$ is 4-dimensional, while it is 3-dimensional for $sl(1|2)$

This theorem will be proved using some lemmas that we introduce now.

Lemma 1. *The principal $osp(1|2)$ sub-superalgebra of $sl(n|n \pm 1)$ can be "enlarged" to a principal $sl(1|2)$ sub-superalgebra.*

Proof. It is obvious from the construction of the principal $osp(1|2)$ sub-superalgebra, as it has being presented in [12].

Using this lemma, one can immediately obtain a classification of $sl(1|2)$ sub-superalgebra into $sl(m|n)$ superalgebras:

Lemma 2. *For $\mathscr{G} = sl(m|n)$, the $osp(1|2)$ embeddings classify the $sl(1|2)$ ones.*

Proof. One already knows that in $sl(m|n)$, the $osp(1|2)$ embeddings are associated to (sums of) $sl(p|p \pm 1)$ sub-superalgebras. But from Lemma 1, we know that these superalgebras admit a principal $sl(1|2)$. As two different $osp(1|2)$ sub-superalgebras cannot be principal in the same $sl(1|2)$ sub-superalgebra, we deduce that each $osp(1|2)$ is associated to a different $sl(1|2)$.

Lemma 3. *Let $\mathscr{G} = osp(2n \pm 2|2n)$ or $osp(2n|2n)$. We consider an $osp(1|2)$-embedding classified by a* **singular** *sub-superalgebra in $\mathscr{G}$ (see classification of $osp(1|2)$-embeddings). Then, there is no $sl(1|2)$ in $\mathscr{G}$ that contains the $osp(1|2)$ under consideration.*

Proof. The proof relies on the decomposition of the adjoint of $\mathscr{G}$ into $osp(1|2)$ representations R_j. It has been given in reference [13] and in each case[7] it is easy to see that there is no $R_{1/2}$ representation in the $\mathscr{G}$-adjoint. As an $sl(1|2)$ decomposes into $R_1 \oplus R_{1/2}$ under its $osp(1|2)$ sub-superalgebra (see below), Lemma 3 is clear.

Lemma 4. *Let $\mathscr{H}$ be a* **regular** *sub-superalgebra of $\mathscr{G}$ defining an $osp(1|2)$ embedding in $\mathscr{G}$. Let $\{\alpha_i\}$ be the set of fermionic simple roots of $\mathscr{H}$, and $\{\beta_j\}$ the set of simple roots of $\mathscr{H}_0$, the bosonic part of $\mathscr{H}$. We suppose that the principal $osp(1|2)$ sub-superalgebra of $\mathscr{H}$ can be enlarged in g to a $sl(1|2)$ sub-superalgebra, and we denote by B the $sl(1|2)$-Cartan generator that commutes with $sl(2) \subset sl(1|2)$.*

Then, $[B, E_{\pm\beta_j}] = 0$ and $[B, F_{\pm\alpha_i}] = \pm b_i F_{\pm\alpha_i}$ with $b_i \neq 0$.

Proof. $\mathscr{H}$ is regular in $\mathscr{G}$, thus the root-generators of $\mathscr{H}$ are root-generators (i.e. eigen-vector under any Cartan generator) of $\mathscr{G}$. As B is a Cartan generator, we deduce that

$$[B, F_{\pm\alpha_i}] = \pm b_i F_{\pm\alpha_i} \,. \tag{3.1}$$

The same argument for the bosonic part $\mathscr{H}_0$ leads to

$$[B, E_{\pm\beta_j}] = \pm y_j E_{\pm\beta_j} \,. \tag{3.2}$$

Now, denoting by $E_\pm$ and $F_\pm$ the root-generators of $osp(1|2)$ and by $F_\pm^\pm$ the fermionic roots of $sl(1|2)$, we have

$$\begin{gathered} F_\pm = F_\pm^+ + F_\pm^- \,, \\ [B, E_\pm] = 0 \quad \text{and} \quad [B, F_\pm] = F_\pm^+ - F_\pm^- \,, \end{gathered} \tag{3.3}$$

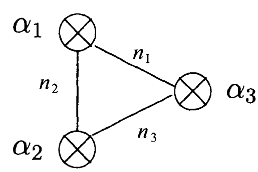
[7] Be careful of a misprint about the boundary values for k in reference [13]

But we can choose $E_+ = \sum_j E_{\beta_j}$ and $F_+ = \sum_i F_{\alpha_i}$, where the sums run over all the simple roots of $\mathscr{H}_0$ and $\mathscr{H}$ respectively. Applying the formulae (3.1–3.2) to (3.3) leads to $y_j = 0$ and $b_i \neq 0$.

Lemma 5. *Let $\mathscr{H}$ be a $\mathscr{G}$-regular sub-superalgebra which possesses a principal $osp(1|2)$. If the $osp(1|2)$ can be enlarged (in $\mathscr{G}$) to a $sl(1|2)$ sub-superalgebra, then the totally fermionic simple root basis of $\mathscr{H}$ does not contain an $osp(1|2)$-type root ("black root").*

Proof. We prove this lemma *ad absurdum*. Let α_0 be a "black" simple root of $\mathscr{H}$. Then, $2\alpha_0$ is also a (bosonic) root (in $\mathscr{H}$). From Lemma 4, we have $[B, E_{2\alpha_0}] = 0$ and $[B, F_{\alpha_0}] = b_0 F_{\alpha_0}$ with $b_0 \neq 0$. These two commutation relations are clearly incompatible, as it can be seen from the Jacobi identity $(B, F_{\alpha_0}, F_{\alpha_0})$. Thus the black root α_0 does not exist if the $sl(1|2)$-generator B does.

Lemma 6. *Under the same conditions as in Lemma 5, the totally fermionic Dynkin diagram of $\mathscr{H}$ does not contain any "triangle" of roots:*



where $n_i \neq 0$ $(i = 1, 2, 3)$ represent the non-vanishing numbers of lines.

Proof. The existence of the triangle implies that $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$ and $\alpha_3 + \alpha_1$ are all bosonic simple roots (for $\mathscr{H}_0$). Then, Lemma 4, with the help of the Jacobi identities $(B, F_{\alpha_i}, F_{\alpha_i})$, $i = 1, 2, 3$, constrains the B-eigenvalues b_1, b_2 and b_3 of F_{α_1}, F_{α_2} and F_{α_3} to satisfy $b_1 + b_2 = 0$, $b_2 + b_3 = 0$ and $b_3 + b_1 = 0$. Again, it is incompatible with $b_i \neq 0$.

Note that the proof relies only on the fact that the sum of any couple of roots in the triangle is a root: thus, this lemma also excludes $D(2, 1; \alpha)$ as a candidate for $sl(1|2)$ classification[8].

Gathering Lemmas 5 and 6, we see that for any superalgebra $\mathscr{G}$, the *regular* sub-superalgebras classifying $sl(1|2)$ embeddings possess a totally fermionic Dynkin diagram without any black root, nor triangle. Looking at the list given at the beginning of this section, it is clear that only the (sums of) superalgebras $sl(n|n \pm 1)$ and $osp(2|2)$ obey these rules. The $osp(2|2)$ sub-superalgebra is isomorphic to $sl(1|2)$ but gives a different decomposition of the fundamental representation (see below). The only cases where $osp(2|2)$ appears are[9] $\mathscr{G} = osp(m|2n)$ (with $m > 1$), $G(3)$, $F(4)$ and $D(2, 1; \alpha)$. Lemma 1 ensures that the other sub-superalgebras can indeed be associated to a $sl(1|2)$ sub-superalgebra. Moreover, the only case (including exceptional superalgebras) where singular embeddings are required (in the classification

[8] We recall that although there exists a $N = 4$ superconformal algebra based on $D(2, 1; \alpha)$, it is obtained from the Hamiltonian reduction of $D(2, 1; \alpha)$ w r t to *regular* $osp(2|2)$ subalgebra, and not a (possible) principal $sl(1|2)$-embedding

[9] Note that, for $G(3)$, the classification of $osp(1|2)$ embeddings done in [13], Table 15, mention the regular $osp(1|2) = B(0, 1)$ and not the $osp(1|2)$ principal in $osp(2|2)$ because these two embeddings are equivalent

of $osp(1|2)$ sub-superalgebras) have been treated in Lemma 3. So Theorem 1 is proved for all the simple Lie superalgebras. It extends trivially to sums of simple Lie superalgebras.

4. Decomposition of Lie Superalgebras w.r.t. $sl(1|2)$

4.1. Summary on $sl(1|2)$ representations. Denoting by $E_\pm$ and H the $sl(2)$ generators and B the $gl(1)$ generator that commutes with $sl(2)$ in $sl(1|2)$, the states of an $sl(1|2)$-irreducible representation will be classified according to their H-"isospin" j and "baryon number" b. One can distinguish:

- Typical representations (b,j) with $b \neq \pm j$

For $j \geqq 1$, they are made with the $(2j+1)$ states of the $sl(2)$-representation $\mathscr{D}_j$ with a B-eigenvalue b, together with the states of two $\mathscr{D}_{j-1/2}$ representations of B-eigenvalue $b \pm \frac{1}{2}$ respectively, and finally the states of the $\mathscr{D}_{j-1}$-representation with B-eigenvalue b. Reducing the representation (b,j) w.r.t. its $sl(2) \times gl(1)$ subalgebra, we will note:

$$(b, j) = |b,j\rangle \oplus \left|b - \frac{1}{2}, j - \frac{1}{2}\right\rangle \oplus \left|b + \frac{1}{2}, j - \frac{1}{2}\right\rangle \oplus \left|b, j - 1\right\rangle . \tag{4.1}$$

For $j = \frac{1}{2}$, the representation $(b, \frac{1}{2})$ reads

$$\left(b, \frac{1}{2}\right) = \left|b, \frac{1}{2}\right\rangle \oplus \left|b - \frac{1}{2}, 0\right\rangle \oplus \left|b + \frac{1}{2}, 0\right\rangle . \tag{4.2}$$

The dimension of a typical representation (b,j) is $8j$.

- Atypical representations $(b = j,j)$ and $(b = -j,j)$ with $j \geqq 0$

For $j \neq 0$, and using the same notations as above, we have

$$(\pm j,j) = |\pm j,j\rangle \oplus \left|\pm\left(j + \frac{1}{2}\right), j - \frac{1}{2}\right\rangle . \tag{4.3}$$

The $j = 0$ atypical representation is just the trivial representation.

The dimension of the atypical representations $(\pm j,j)$ is $4j + 1$.

We want to emphasize that the sign of the $U(1)$ charge in an atypical representation has no real meaning: the two representations $(\pm j,j)$ are related by an outer automorphism of the $sl(1|2)$ algebra. We will come back later to this point that has some consequence in the decomposition of the adjoint representation w.r.t. $sl(1|2)$.

Note that if one decomposes the $sl(1|2)$-representations with respect to the $osp(1|2)$-sub-superalgebra of $sl(1|2)$, the typical representation (b,j) corresponds to the sum of two $osp(1|2)$-representations $R_j \oplus R_{j-1/2}$, while the atypical representation $(\pm j,j)$ is just a R_j representation. Let us also remark that the two $sl(1|2)$-Casimir operators are zero in an atypical representation.

Finally, we want to stress that the product of two $sl(1|2)$-irreducible representations is *not* always completely reducible.

4.2. Products of $sl(1|2)$-representations. Our aim is to decompose the adjoint representation of a simple Lie superalgebra $\mathscr{G}$ into representations of $sl(1|2)$ considered as a sub-superalgebra of $\mathcal{G}$. Following the techniques used for the decomposition of $\mathcal{G}$ into $osp(1|2)$-representations, we will start by decomposing the $\mathcal{G}$-fundamental representation. Then, performing the product of this fundamental representation by its contragredient, we will obtain the desired decomposition. One will be helped by the following formulae [14]:

$$\begin{aligned}(\pm j,j)\times(\pm k,k) &= (\pm(j+k),j+k)\bigoplus_{l=|j-k|+\frac{1}{2}}^{j+k-\frac{1}{2}}\left(\pm\left(j+k+\frac{1}{2}\right),l\right)\\ &= (\pm(j+k),j+k)\oplus\left(\pm\left(j+k+\frac{1}{2}\right),j+k-\frac{1}{2}\right)\\ &\quad\oplus\left(\pm\left(j+k+\frac{1}{2}\right),j+k-\frac{3}{2}\right)\\ &\quad\oplus\cdots\oplus\left(\pm\left(j+k+\frac{1}{2}\right),|j-k|+\frac{3}{2}\right)\\ &\quad\oplus\left(\pm\left(j+k+\frac{1}{2}\right),|j-k|+\frac{1}{2}\right)\quad\forall\, j,k\geqq 0\,.\end{aligned}\tag{4.4}$$

If we consider only products of atypical representations we already know that they are decomposable into a sum of irreducible $sl(1|2)$-representations (since they are of the type $S_\pm$ introduced in [14]). Thus, we can focus on the $sl(2)\oplus gl(1)$ part to deduce information about $sl(1|2)$-representations. Then, using the $sl(2)\times gl(1)$ decomposition given above, it is also possible to compute:

$$(j,j)\times(-k,k)=(j-k,j+k)\oplus(j-k,j+k-1)\oplus\cdots\oplus(j-k,|j-k|)\,.\tag{4.5}$$

As examples, we have

$$(j,j)\times(-j,j)=(0,2j)\oplus(0,2j-1)\oplus(0,2j-2)\oplus\cdots\oplus(0,0)\,,$$

$$\left(\pm\frac{1}{2},\frac{1}{2}\right)\times\left(\pm\frac{1}{2},\frac{1}{2}\right)=(\pm 1,1)\oplus\left(\pm\frac{3}{2},\frac{1}{2}\right)$$

while

$$\left(\frac{1}{2},\frac{1}{2}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)=(0,1)\oplus(0,0)\,.\tag{4.6}$$

Considering indecomposable products, we have for instance [14] $(0,\frac{1}{2})\times(0,\frac{1}{2})$: we will come back extensively on this point in Sect. 4.5.

We will also need to select the (anti)symmetric part of the product of representations. For brevity, we will note $[(b,j)]^2_S$ and $[(b,j)]^2_A$ the symmetric and antisymmetric part of the product $(b,j)\times(b,j)$. Using the rules given in [12] for $sl(2)\oplus gl(1)$ representations, it is easy to deduce:

For $m\in\mathbb{N}$

$$[(m,m)\oplus(-m,m)]^2_{\mathrm{A}}=\bigoplus_{j=0}^{2m}(0,j)\bigoplus_{j=1}^{m}(2m+\tfrac{1}{2},2j-\tfrac{1}{2})\bigoplus_{j=1}^{m}(-(2m+\tfrac{1}{2}),2j-\tfrac{1}{2})\,,$$

$$[(m+\tfrac{1}{2},m+\tfrac{1}{2})\oplus(-(m+\tfrac{1}{2}),m+\tfrac{1}{2})]^2_{\mathrm{A}} = \bigoplus_{j=0}^{2m+1}(0,j)\bigoplus_{j=0}^{m}(2m+\tfrac{3}{2},2j+\tfrac{1}{2})$$

$$\bigoplus_{j=0}^{m}(-(2m+\tfrac{3}{2}),2j+\tfrac{1}{2})$$

$$[(m,m)\oplus(-m,m)]^2_{\mathrm{S}} = \bigoplus_{j=0}^{2m}(0,j)\bigoplus_{j=0}^{m-1}(2m+\tfrac{1}{2},2j+\tfrac{1}{2})\bigoplus_{j=0}^{m-1}(-(2m+\tfrac{1}{2}),2j+\tfrac{1}{2})$$

$$\oplus(2m,2m)\oplus(-2m,2m)\,,$$

$$[(m+\tfrac{1}{2},m+\tfrac{1}{2})\oplus(-(m+\tfrac{1}{2}),m+\tfrac{1}{2})]^2_{\mathrm{S}}$$

$$= \bigoplus_{j=0}^{2m+1}(0,j)\bigoplus_{j=1}^{m}(2m+\tfrac{3}{2},2j-\tfrac{1}{2})\bigoplus_{j=1}^{m}(-(2m+\tfrac{3}{2}),2j-\tfrac{1}{2})$$

$$\oplus(2m+1,2m+1)\oplus(-(2m+1),2m+1)\,,$$

together with:

For $j \in \mathbb{N}/2$,

$$[n(\pm j,j)\times n(\pm j,j)]_{\mathrm{A}} = \frac{n(n+1)}{2}[(\pm j,j)\times(\pm j,j)]_{\mathrm{A}}$$

$$\oplus\frac{n(n-1)}{2}[(\pm j,j)\times(\pm j,j)]_{\mathrm{S}}$$

$$[n(\pm j,j)\times n(\pm j,j)]_{\mathrm{S}} = \frac{n(n+1)}{2}[(\pm j,j)\times(\pm j,j)]_{\mathrm{S}}$$

$$\oplus\frac{n(n-1)}{2}[(\pm j,j)\times(\pm j,j)]_{\mathrm{A}}\,.$$

We will also use:

For $j_1,j_2 \in \mathbb{Z}/2$

$$[(j_1,|j_1|)\oplus(j_2,|j_2|)]^2_{\mathrm{A}} = [(j_1,|j_1|)\times(j_1,|j_1|)]_{\mathrm{A}}\oplus[(j_2,|j_2|)\times(j_2,|j_2|)]_{\mathrm{A}}$$

$$\oplus(j_1,|j_1|)\times(j_2,|j_2|)$$

$$[(j_1,|j_1|)\oplus(j_2,|j_2|)]^2_{\mathrm{S}} = [(j_1,|j_1|)\times(j_1,|j_1|)]_{\mathrm{S}}\oplus[(j_2,|j_2|)\times(j_2,|j_2|)]_{\mathrm{S}}$$

$$\oplus(j_1,|j_1|)\times(j_2,|j_2|) \tag{4.7}$$

4.3. Superalgebras fundamental representations. The techniques for the decomposition of the fundamental representations is the same as in [13]. We start with a superalgebra $\mathscr{G}$ that we want to reduce w.r.t. a given $sl(1|2)$-subalgebra, defined through its principal embedding into a $\mathscr{G}$-subalgebra $\mathscr{H}$. Decomposing $\mathscr{H}$ into its simple parts $\mathscr{H}_i : \mathscr{H} = \bigoplus_i \mathscr{H}_i$, we associate to each type of $\mathscr{H}_i$ and each type of superalgebra $\mathscr{G}$, a $sl(1|2)$-representation. Then, we sum these different $sl(1|2)$-representations, and eventually complete this sum by trivial representations, in such a way that the dimension of the $\mathscr{G}$-fundamental is recovered.

For $\mathcal{G} = sl(m|n)$ superalgebras, we will get a $(\pm\frac{p}{2}, \frac{p}{2})$ atypical representation for each $sl(p+1|p)$ sub-superalgebra and a $(\pm\frac{p}{2}, \frac{p}{2})^\pi$ representation for each $sl(p|p+1)$ sub-superalgebra. We use the superscript π to distinguish the $(\pm\frac{p}{2}, \frac{p}{2})$ representations coming from these two types of superalgebras. When the trivial representation occurs, it will be denoted $(0,0)$ if it is in the $sl(m)$ fundamental representation, and $(0,0)^\pi$ if it is in the $sl(n)$ one. This superscript π will have deep consequences on the spin structure of the resulting adjoint decomposition. We repeat that the sign of the $U(1)$ charge is meaningless (see Sect. 4.4)

For $\mathcal{G} = osp(m|2n)$ superalgebras, we get a sum $(\frac{p}{2}, \frac{p}{2}) \oplus (-\frac{p}{2}, \frac{p}{2})$ of atypical representations (here the sign of the $U(1)$ charge has been fixed by the reality condition) for each $sl(p+1|p)$ sub-superalgebra and a sum $(\frac{p}{2}, \frac{p}{2})^\pi \oplus (-\frac{p}{2}, \frac{p}{2})^\pi$ of representations for each $sl(p|p+1)$ sub-superalgebra. Again, the trivial representation will be denoted $(0,0)$ if it is in the $so(m)$ fundamental representation, and $(0,0)^\pi$ if it stands in the $sp(2n)$ one.

4.4. $sl(1|2)$-decomposition of the adjoint representation. The rules given in Sect. 4.2 do not indicate the statistics of the representation. In other words, when we obtain a representation (b,j) in the adjoint $\mathcal{G}$, we have not yet specified whether the $\mathcal{D}_j$, $\mathcal{D}_{j-1/2}$ and $\mathcal{D}_{j-1}$ representation are associated to commuting or anti-commuting generators of $\mathcal{G}$. A natural statistics associate (anti-)commuting generators to (half-) integers j, but there are some cases where it is the opposite. To distinguish these two (very different) cases, we will note *in the adjoint representation* with a prime $(b,j)'$ the representation with "unusual" statistics, keeping the form (b,j) for the representations with usual statistics.

Then, the rules to distinguish the two kinds of representations are the same as the ones given for $osp(1|2)$ representations, i.e.

$$(b_1,j_1) \times (b_2,j_2) = \begin{cases} \bigoplus_{b_3,j_3}(b_3,j_3) & \text{if } j_1+j_2 \in \mathbb{Z} \\ \bigoplus_{b_3,j_3}(b_3,j_3)' & \text{if } j_1+j_2 \in \frac{1}{2}+\mathbb{Z} \end{cases}, \tag{4.8}$$

$$(b_1,j_1)^\pi \times (b_2,j_2)^\pi = \begin{cases} \bigoplus_{b_3,j_3}(b_3,j_3) & \text{if } j_1+j_2 \in \mathbb{Z} \\ \bigoplus_{b_3,j_3}(b_3,j_3)' & \text{if } j_1+j_2 \in \frac{1}{2}+\mathbb{Z} \end{cases}, \tag{4.9}$$

$$(b_1,j_1) \times (b_2,j_2)^\pi = \begin{cases} \bigoplus_{b_3,j_3}(b_3,j_3)' & \text{if } j_1+j_2 \in \mathbb{Z} \\ \bigoplus_{b_3,j_3}(b_3,j_3) & \text{if } j_1+j_2 \in \frac{1}{2}+\mathbb{Z} \end{cases}. \tag{4.10}$$

Using the rules given above, it is now easy to get the decomposition of the adjoint from the product $F \times \bar{F}$, where F is the fundamental representation (decomposed into $sl(1|2)$ representations):

For $sl(m|n)$ superalgebras, the decomposition of the adjoint representation will be $(F \times \bar{F}) - (0,0)$ when $m \neq n$, and $(F \times \bar{F}) - 2(0,0)$ when $m = n$, the rules for the product being given in Sect. 4.2, with the property $\overline{(p,p)} = (-p,p)$.

Note that the choice between $(p_1,p_2) \oplus (p_2,p_2)$ or $(p_1,p_1) \oplus (-p_2,p_2)$ leads to a very different adjoint decomposition. However, these decompositions are equivalent. Indeed, the change $(p_2,p_2) \to (-p_2,p_2)$ correspond to the following changes (algebra isomorphism) in the $sl(1|2)$-generators: $Y \to -Y; F_{+-} \to F_{++}$; $F_{++} \to F_{+-}$; $F_{--} \to -F_{-+}$; $F_{-+} \to -F_{--}$; $E_\pm \to E_\pm$; $H \to H$. This is clearly just a choice of normalisation and thus corresponds to equivalent $sl(1|2)$-subalgebras.

Note that in the series $sl(2)$, $osp(1|2)$, $sl(1|2)$, the $sl(1|2)$ superalgebra is the first (super)algebra possessing an outer automorphism: this explains why these "multiple adjoint decompositions" have not being encountered when studying $osp(1|2)$ and $sl(2)$ decompositions of (super) algebras. In the next section, we show explicitly on an example how things are going on. It should be clear to the reader that "multiple $sl(1|2)$-decomposition" will occur only where more than one non-trivial atypical representation is involved in the fundamental representation.

For $osp(m|2n)$ algebras, the adjoint decomposition will be given by the antisymmetric product of (p, p) representation, plus the symmetric product of $(p, p)^\pi$, plus once the products of the (p, p)'s by the $(q, q)^\pi$'s representations.

For the exceptional superalgebras $G(3)$, $F(4)$ and $D(2, 1; \alpha)$, a direct calculation has been done. Note that for $D(2, 1; \alpha)$, the decomposition of the three $sl(1/2)$ embeddings are ulated through the transformations $\alpha \rightarrow 1/\alpha, -(\alpha + 1), -\alpha/\alpha + 1$ respectively, which are indeed outer automorphisms of the superalgebra.

4.5. Case of $osp(2|2)$ *subalgebras and indecomposable products.* Up to now, we have studied the embeddings of $sl(1|2)$ into superalgebras. However, $sl(1|2)$ is isomorphic to another superalgebra, namely $osp(2|2)$. As they are isomorphic, one could think that one has not to distinguish them. However, already at the level of representations, it is clear that these two superalgebras are distinct, since, for instance, $osp(2|2)$ has only real representations, while $sl(1|2)$ has complex ones. This distinction appears also here when decomposing the fundamental of an $osp(m|2n)$ superalgebra w.r.t. $osp(2|2)$. If the sub-superalgebra is an $osp(2|2)$ superalgebra (instead of a $sl(1|2)$ one) we will get one $(0, \frac{1}{2})^\pi$ representation in the fundamental. Note that the distinction between the two isomorphic superalgebras $osp(2|2)$ and $sl(1|2)$ is of the same type as the one introduced in [13] to distinguish between the $sl(2)$-decompositon coming from the algebras A_1 and C_1 (in symplectic algebras), D_2 and $2A_1$, or D_3 and A_3 (in orthogonal algebras).

Thus, considering the decomposition of $osp(m|2n)$ superalgebras, we have to add the cases where one or several $osp(2|2)$ appear. For each $osp(2|2)$ subalgebra, we will have a $(0, \frac{1}{2})^\pi$ representation in the fundamental of $osp(m|2n)$. Then, the decomposition of the adjoint representation will be obtained with the same rules as given in the previous section.

However, note that the product of two $(0, \frac{1}{2})$ representations is not completely reducible. More precisely, the symmetric part of $(0, \frac{1}{2}) \times (0, \frac{1}{2})$ contains a $(0, 1)$ representation, while the antisymmetric part is non-fully reducible: from the $sl(2) \oplus gl(1)$ decomposition this part *looks* like $(\frac{1}{2}, \frac{1}{2}) \oplus (-\frac{1}{2}, \frac{1}{2}) \oplus 2(0, 0)$, but one verifies that one of the two $D_0(0)$ generators is obtained from both the "$(\frac{1}{2}, \frac{1}{2})$" and "$(-\frac{1}{2}, \frac{1}{2})$" parts by application of negative root generators. Thus, apart from a $(0, 0)$ representation, the antisymmetric part of this product is non-fully reducible. Below, we will keep the notation $[(0, \frac{1}{2})]^2_A$ for the indecomposable part (plus a trivial representation),

$$[(0, \tfrac{1}{2})]^2_{\mathrm{S}} = (0, 1) \quad \text{while } (0, \tfrac{1}{2}) \times (0, \tfrac{1}{2}) = (0, 1) \oplus [(0, \tfrac{1}{2})]^2_{\mathbf{A}} \, . \tag{4.11}$$

Therefore, when one decomposes $osp(m|2n)$ superalgebras w.r.t. the diagonal of several regular $osp(2|2)$ subalgebras, one will get an *non-fully reducible part.* Although this fact seems quite intriguing, one has to recall that most of the representations of Lie superalgebras are *non-fully reducible* [14]. Fortunately, the product of $(0, \frac{1}{2})^\pi$

with an atypical representation is reducible:

$$(\pm j, j) \times \left(0, \frac{1}{2}\right) = \left(\pm j, j + \frac{1}{2}\right) \oplus \left(\pm j, j - \frac{1}{2}\right) \oplus \left(\pm j + \frac{1}{2}, j\right) \oplus \left(\pm j - \frac{1}{2}, j\right). \tag{4.12}$$

As an example, let us consider the reduction of $osp(4|4)$ w.r.t. 2 $osp(2|2)$. The fundamental reads

$$\underline{4} = 2(0, \tfrac{1}{2})^{\pi}, \tag{4.13}$$

while the adjoint is

$$[2(0, \tfrac{1}{2})]^2_{\mathrm{S}} = \frac{2.3}{2} \left[\left(0, \frac{1}{2}\right)\right]^2_{\mathrm{S}} \oplus \frac{2.1}{2} \left[\left(0, \frac{1}{2}\right)\right]^2_{\mathrm{A}} = 3(0,1) \oplus \left[\left(0, \frac{1}{2}\right)\right]^2_{\mathrm{A}}. \tag{4.14}$$

We have explicitly checked that the decomposition is really the one given in (4.14).

4.6. Example. Let us treat one of the simplest examples where the problem of multiple $sl(2|1)$ decomposition occurs, namely the reduction $sl(4|3)$ w.r.t. the diagonal $sl(2|1)$ in $2sl(2|1)$.

We use the notation $e_{i,j}$ for the matrix basis $(e_{i,j})_{kl} = \delta_{ik}\delta_{jl}$. In the fundamental representation, an element of $sl(4|3)$ will be represented by an 7×7 supertrace-less matrix. We define as the first $sl(2|1)$ subalgebra:

$$\begin{aligned}
F^{(1)}_{++} &= e_{5,2}; \qquad F^{(1)}_{+-} = e_{1,5}; \qquad F^{(1)}_{-+} = e_{5,1}; \qquad F^{(1)}_{--} = e_{2,5}, \\
H^{(1)} &= \frac{1}{2}(e_{1,1} - e_{2,2}); \qquad E^{(1)}_{+} = e_{1,2}; \qquad E^{(1)}_{-} = e_{2,1}, \\
Y^{(1)} &= \frac{1}{2}(e_{1,1} + e_{2,2} + 2e_{5,5}),
\end{aligned} \tag{4.15}$$

and for the second algebra we take:

$$\begin{aligned}
F^{(2)}_{++} &= e_{7,4}; \qquad F^{(2)}_{+-} = e_{3,7}; \qquad F^{(2)}_{-+} = e_{7,3}; \qquad F^{(2)}_{--} = e_{4,7}, \\
H^{(2)} &= \frac{1}{2}(e_{3,3} - e_{4,4}); \qquad E^{(2)}_{+} = e_{3,4}; \qquad E^{(2)}_{-} = e_{4,3}, \\
Y^{(2)} &= \frac{1}{2}(e_{3,3} + e_{4,4} + 2e_{7,7}).
\end{aligned} \tag{4.16}$$

Then, the generators of the diagonal $sl(2|1)$ superalgebra are defined by[10] $F = F^{(1)} - F^{(2)}$ and $B = B^{(1)} + B^{(2)}$, where $F^{(i)}(B^{(i)})$, $i = 1, 2$ are the fermionic (bosonic) generators of the two $sl(2|1)$ (4.15) and (4.16). Besides the two highest weights built on the bosonic roots of the $sl(1|2)$ subalgebras $E_{+}(0,1) = e_{1,2} + e_{3,4}$ and $W_1(0,1) =$

[10] The minus sign is for later convenience

$e_{3,4} - e_{1,2}$, the highest weights for the diagonal $sl(2|1)$ are

$$\begin{aligned} W_2(0,1) &= e_{1,4}; \qquad W_3(0,1) = e_{3,2}\,, \\ W_1\left(\frac{1}{2},\frac{1}{2}\right) &= e_{3,6}; \qquad W_2\left(\frac{1}{2},\frac{1}{2}\right) = e_{1,6}\,, \\ W_1\left(-\frac{1}{2},\frac{1}{2}\right) &= e_{6,2}; \qquad W_2\left(-\frac{1}{2},\frac{1}{2}\right) = e_{6,4}\,, \\ W_1(0,0) &= e_{2,4} + e_{1,3} - e_{5,7}; \qquad W_2(0,0) = e_{3,1} + e_{4,2} - e_{7,5}\,, \\ W_3(0,0) &= e_{1,1} + e_{2,2} + e_{5,5} + e_{6,6}\,; \\ W_4(0,0) &= e_{3,3} + e_{4,4} + e_{6,6} + e_{7,7}\,. \end{aligned}$$

We have quoted in parenthesis the eigenvalues (b, j) of the generator w.r.t. Y and H respectively. This corresponds to the decomposition $4(0,1) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus 4(0,0)$. It is easily obtained using the decomposition of the fundamental obtained from two regular $sl(2|1)$-subalgebras in $sl(4|3)$: $\underline{7} = 2(\frac{1}{2},\frac{1}{2}) \oplus (0,0)$ and computing $7 \times \bar{7} - \underline{1}$.

Now, instead of starting from (4.16) as the second $sl(2|1)$, we could have chosen:

$$\begin{aligned} F^{(2)}_{+-} &= e_{7,4}; \qquad F^{(2)}_{++} = e_{3,7}; \qquad F^{(2)}_{--} = -e_{7,3}; \qquad F^{(2)}_{-+} = -e_{4,7}\,, \\ H^{(2)} &= \frac{1}{2}(e_{3,3} - e_{4,4}); \qquad E^{(2)}_{+} = e_{3,4}; \qquad E^{(2)}_{-} = e_{4,3}\,, \\ Y^{(2)} &= -\frac{1}{2}(e_{3,3} + e_{4,4} + 2e_{7,7})\,. \end{aligned} \tag{4.17}$$

Comparing (4.16) with (4.17), it is clear that the two $sl(2|1)$ are isomorphic. However, apart from $E_+(0,1) = e_{1,2} + e_{3,4}$ and $W(0,1) = e_{3,4} - e_{1,2}$, the highest weights of the diagonal $sl(2|1)$ become

$$\begin{aligned} W(1,1) &= e_{1,4}; \qquad W(-1,1) = e_{3,2}\,, \\ W_1\left(\frac{1}{2},\frac{1}{2}\right) &= e_{6,4}; \qquad W_2\left(\frac{1}{2},\frac{1}{2}\right) = e_{1,6}\,, \\ W_1\left(-\frac{1}{2},\frac{1}{2}\right) &= e_{6,2}; \qquad W_2\left(-\frac{1}{2},\frac{1}{2}\right) = e_{3,6}\,, \\ W\left(\frac{3}{2},\frac{1}{2}\right) &= e_{5,4} + e_{1,7}; \qquad W\left(-\frac{3}{2},\frac{1}{2}\right) = e_{7,2} + e_{3,5}\,, \\ W_1(0,0) &= e_{1,1} + e_{2,2} + e_{5,5} + e_{6,6}\,, \\ W_2(0,0) &= e_{3,3} + e_{4,4} + e_{6,6} + e_{7,7}\,, \end{aligned}$$

which gives the decomposition $2(0,1) \oplus (1,1) \oplus (-1,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)$. As announced, this decomposition is obtained from the fundamental $\underline{7} = (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (0,0)^{\pi}$.

Thus, we see that starting from $2sl(2|1)$ in $sl(4|3)$, one can take as the decomposition of the fundamental either $\underline{7} = 2(\frac{1}{2},\frac{1}{2}) \oplus (0,0)$, or $\underline{7} = (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus$

$(0,0)^\pi$, depending on the normalization one chooses. The corresponding adjoint decomposition will be different but equivalent to the first one.

To conclude, let us add that doing the folding of $sl(4|3)$ to get[11] $osp(3|4)$, we have to apply the rules

$$\begin{aligned} e_{i,j} &\equiv (-)^{i+j+1}\ e_{5-j,5-i} \quad \text{for } i,j = 1,2,3,4\,, \\ e_{i,j} &\equiv (-)^{i+j+1}\ e_{12-j,12-i} \quad \text{for } i,j = 5,6,7\,, \\ e_{i,j} &\equiv (-)^{i+j}\ e_{12-j,5-i} \quad \text{for } i = 1,2,3,4;\ j = 5,6,7\,. \end{aligned}$$

These rules are clearly incompatible with the first choice of normalisation (4.16), since they impose $Y = Y^{(1)} + Y^{(2)} \equiv 0$. On the contrary, the second choice of normalisation (4.17) survive the folding procedure. Thus, the decomposition of the $osp(3|4)$ fundamental must be $\underline{7} = (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (0,0)^\pi$. This is an illustration of the "sign fixing" of atypical representations that occurs in the $osp(m|2n)$ superalgebras. Moreover, looking at the $sl(2|1)$-representation, one realises that we have the identities (written below for the highest weights, but true for any generator of the corresponding $sl(2|1)$-representation):

$$\begin{gathered} W(0,1) \equiv 0; \qquad W\left(\frac{3}{2},\frac{1}{2}\right) \equiv 0; \qquad W\left(-\frac{3}{2},\frac{1}{2}\right) \equiv 0\,, \\ W_1\left(\frac{1}{2},\frac{1}{2}\right) \equiv W_2\left(\frac{1}{2},\frac{1}{2}\right); \qquad W_1\left(-\frac{1}{2},\frac{1}{2}\right) \equiv W_2\left(-\frac{1}{2},\frac{1}{2}\right)\;; \\ W_1(0,0) \equiv W_2(0,0)\,. \end{gathered}$$

From these identities, one deduces that $osp(3|4)$ decomposes as $(0,1) \oplus (1,1) \oplus (-1,1) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)$. Once more, this result is easily obtained from the product

$$[(\tfrac{1}{2},\tfrac{1}{2}) \oplus (-\tfrac{1}{2},\tfrac{1}{2})]^2_{\mathrm{S}} \oplus [(0,0)]^2_{\mathrm{A}} \oplus [(\tfrac{1}{2},\tfrac{1}{2}) \oplus (-\tfrac{1}{2},\tfrac{1}{2})] \times (0,0)\,.$$

5. The Standard Reduction

Consider an embedding of $sl(2|1)$ in some super Lie algebra $\mathcal{G}$. We normalize the $sl(2)$ subalgebra of $sl(2|1)$ such that $[e_0, e_{\neq}] = +2e_{\neq}, [e_0, e_=] = -2e_=$ and $[e_{\neq}, e_=] = e_0$. The standard grading is nothing but the $sl(2)$ grading, i.e. given by $\frac{1}{2}\mathrm{ad}_{e_0}$. The action

$$\mathcal{S}_0 = \kappa S^-[g] + \frac{1}{\pi x} \int \mathrm{str}\, A\left(J - \frac{\kappa}{2}e_= - \frac{\kappa}{2}[e_=,\tau]\right) - \frac{\kappa}{4\pi x} \int \mathrm{str}[e_=,\tau]\bar{\partial}\tau\,, \tag{5.1}$$

where

$$\begin{aligned} A &\in \Pi_{>0}\,\mathcal{G}\,, \\ \tau &\in \Pi_{\frac{1}{2}}\,\mathcal{G} \end{aligned} \tag{5.2}$$

[11] Be careful that the $sp(4)$ subalgebra is in the upper left block instead of lower right block

has a gauge invariance:

$$\begin{aligned} \delta g &= \eta g\,, \\ \delta A &= \bar{\partial}\eta + [\eta, A]\,, \\ \delta\tau &= -\Pi_{\frac{1}{2}}\,\eta\,, \end{aligned} \tag{5.3}$$

and $\eta \in \Pi_{>0}\mathscr{G}$. Gauge invariance requires the introduction of the τ field[12]. The constraints and the resulting gauge invariance breaks the original chiral affine symmetry of the WZW model to some extension of the $N = 2$ superconformal algebra. The generators are precisely the gauge invariant polynomials in $\Pi_{\geqq 0}J$ and τ and their derivatives. In order to quantize the model, we take $A = 0$ as a gauge choice. Upon the introduction of the ghosts $c \in \Pi_{>0}\mathscr{G}$ and anti-ghosts $b \in \Pi_{<0}\mathscr{G}$, we get the gauge fixed action:

$$\mathscr{S}_{\mathrm{gf}} = \kappa S^{-}[g] + \frac{\kappa}{4\pi x}\int \mathrm{str}[\tau, e_{=}]\bar{\partial}\tau + \frac{1}{2\pi x}\int \mathrm{str}\, b\bar{\partial}c\,, \tag{5.4}$$

and the BRST charge $\mathscr{Q}_{HR}$:

$$\mathscr{Q}_{HR} = \frac{1}{4\pi i x}\oint \mathrm{str}\left\{c\left(J - \frac{\kappa}{2}e_{=} - \frac{\kappa}{2}[e_{=}, \tau] + \frac{1}{2}J^{\mathrm{gh}}\right)\right\}\,, \tag{5.5}$$

where $J^{\mathrm{gh}} = \frac{1}{2}\{b, c\}$. The generators of the extended $N = 2$ superconformal algebra are now the generators of the cohomology of $\mathscr{Q}_{HR}$ computed on the algebra $\mathscr{A}$ which is generated by $\{b, \hat{J}, \tau, c\}$, with $\hat{J} = J + J^{\mathrm{gh}}$, and it consists of all regularized products of the generating fields and their derivatives modulo the usual relations between different orderings, derivatives, etc. The calculation of this cohomology proceeds along exactly the same lines as in [10]. To make this paper selfcontained, we summarize the results. The subcomplex $\mathscr{A}^{(1)}$ generated by $\{b, \Pi_{<0}\hat{J}\}$ has a trivial cohomology $H^*(\mathscr{A}^{(1)}; \mathscr{Q}) = \mathbb{C}$. The only field with negative ghost number is b. This implies that the full cohomology $H^*(\mathscr{A}; \mathscr{Q}_{HR})$ is equal to the cohomology $H^*(\hat{\mathscr{A}}; \mathscr{Q}_{HR})$, where we introduced the reduced complex $\hat{\mathscr{A}}$ generated by $\{\Pi_{\geqq 0}\hat{J}, \tau, c\}$. The OPE's close on $\hat{\mathscr{A}}$. The underlying $sl(2)$ grading implies the existence of a double grading on $\hat{\mathscr{A}}$:

$$\hat{\mathscr{A}} = \bigoplus_{\substack{m,n \in \frac{1}{2}\mathbb{Z} \\ m+n \in \mathbb{Z}}} \hat{\mathscr{A}}_{(m,n)}\,, \tag{5.6}$$

where

$$X \in \hat{\mathscr{A}}_{(m,n)} \Leftrightarrow [e_0, X] = 2mX \quad \text{and} \quad m + n = \text{ghostnumber}(X)\,. \tag{5.7}$$

We assign to τ grading $(0, 0)$. The BRST operator itself decomposes into three parts, each of definite grading: $\mathscr{Q}_{HR} = \mathscr{Q}_{(1,0)} + \mathscr{Q}_{(1/2,1/2)} + \mathscr{Q}_{(0,1)}$ with

$$\begin{aligned} \mathscr{Q}_{(1,0)} &= -\frac{\kappa}{8\pi i x}\oint \mathrm{str}\,\mathrm{c}e_{=} \\ \mathscr{Q}_{(1/2,1/2)} &= -\frac{\kappa}{8\pi i x}\oint \mathrm{str}\,\mathrm{c}[e_{=}, \tau]\,, \end{aligned} \tag{5.8}$$

[12] In a Hamiltonian treatment, they are needed to obtain first class constraints

and from $\mathcal{Q}_{HR}^2 = 0$ one gets immediately

$$\begin{aligned}\mathcal{Q}_{(1,0)}^2 &= \mathcal{Q}_{(0,1)}^2 = \{\mathcal{Q}_{(1,0)}, \mathcal{Q}_{(1/2,1/2)}\} = \{\mathcal{Q}_{(0,1)}, \mathcal{Q}_{(1/2,1/2)}\} \\ &= \mathcal{Q}_{(1/2,1/2)}^2 + \{\mathcal{Q}_{(0,1)}, \mathcal{Q}_{(1,0)}\} = 0 \,. \end{aligned} \tag{5.9}$$

The filtration $\hat{\mathscr{A}}^m, m \in \frac{1}{2}\mathbb{Z}$ of $\hat{\mathscr{A}}$:

$$\hat{\mathscr{A}}^m = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \bigoplus_{l \geqq m} \hat{\mathscr{A}}_{(k,l)} \,, \tag{5.10}$$

leads to a spectral sequence (E_r, d_r), $r \geqq 1$, converging to $H^*(\hat{\mathscr{A}}; \mathcal{Q})$. Each term in the sequence is given by the cohomology of the previous term with a derivation that represents the effective action of the BRST operator at that level: $E_r = H^*(E_{r-1}; d_{r-1})$. The first term in the sequence is then $E_0 = \hat{\mathscr{A}}$, $d_0 = \mathcal{Q}_{(1,0)}$. One readily computes E_1:

$$E_1 \simeq \hat{\mathscr{A}}[\, \Pi_{\ker ad_{e_+}} \hat{J}] \otimes \hat{\mathscr{A}}[\tau] \otimes \hat{\mathscr{A}}[\Pi_{\frac{1}{2}}\, c] \,, \tag{5.11}$$

the subsequent term has $d_1 = \mathcal{Q}_{(1/2,1/2)}$ and one gets:

$$E_2 \simeq \hat{\mathscr{A}} \left[\Pi_{\ker ad_{e_+}} \left(\hat{J} + \frac{\kappa}{4}[\tau, [e_-, \tau]]\right)\right] \,. \tag{5.12}$$

After this the spectral sequence collapses, and we get

$$H^*(\mathscr{A}; \mathcal{Q}_{HR}) \simeq E_2 = H^*(H^*(\hat{\mathscr{A}}, \mathcal{Q}_{(1,0)}), \mathcal{Q}_{(1/2,1/2)}) \,. \tag{5.13}$$

Of course this is only an isomorphism of the cohomologies as vectorspaces. In order to get the generators of $H^*(\hat{\mathscr{A}}; \mathcal{Q}_{HR})$ we use a generalized tic-tac-toe construction, which determines the currents up to a scale factor. Indeed if $X_{(j,-j)}$ belongs to $\hat{J} + \frac{\kappa}{4}[\tau, [e_-, \tau]]$ and has fixed grading $(j, -j)$ we obtain the full generator $\hat{X}_j$

$$\hat{X}_j = \sum_{2m=0}^{j} X_{(m,-m)} \,, \tag{5.14}$$

where $X_{(m,-m)}$ are recursively determined from

$$\{\mathcal{Q}_{(0,1)}, X_{(n,-n)}\} + \{\mathcal{Q}_{(1/2,1/2)}, X_{(n-1/2,-n+1/2)}\} + \{\mathcal{Q}_{(1,0)}, X_{(n-1,-n+1)}\} = 0 \,. \tag{5.15}$$

In this way we get as many BRST invariant currents as there are $sl(2)$ irreps present in the decomposition of the adjoint representation. The OPE's of these currents close: by construction, the OPE of two generators of $H^*(\hat{\mathscr{A}}; \mathcal{Q}_{HR})$ closes modulo BRST exact terms. However, as we found that the cohomology is only non-trivial in the ghost number zero sector and as we computed our cohomology on the reduced complex $\hat{\mathscr{A}}$ which has no negative ghost number currents we get that the OPE's of the generators of $H^*(\hat{\mathscr{A}}; \mathcal{Q}_{HR})$ close. The quantum Miura transformation follows for free from this construction. The map $\hat{X}_j \to X_{(0,0)}$ is obviously an algebra homomorphism. In fact it is also an algebra isomorphism. To see this, one only has to show that for each $\hat{X}_j$, $X_{(0,0)}$ is non-vanishing. Consider for this the mirror of the spectral sequence, i.e. the one which follows from the filtration

$$\hat{\mathscr{A}}'^m \equiv \bigoplus_{l \in \frac{1}{2}\mathbb{Z}} \bigoplus_{k \geqq m} \hat{\mathscr{A}}_{(k,l)} \,. \tag{5.16}$$

A small computation teaches us that $E_1 = H^*(\hat{\mathscr{A}}; Q_{(0,1)})$ is non-vanishing only for grades $(\frac{m}{2}, \frac{m}{2})$, $m \geqq 0$. We already know that E_∞ is non-trivial only at ghost number 0. Combining these statements shows that $X_{(0,0)}$ is always non-trivial. Till now we only used the underlying $sl(2)$ embedding in $\mathscr{G}$. However the full $sl(2|1)$ embedding is relevant in that it guarantees that the resulting algebra is an extension of the $N=2$ superconformal algebra. So what remains to be shown is that the resulting conformal algebra has an $N=2$ superconformal subalgebra. That this is the case follows immediately from the lemma [15]:

Lemma 7. *If the conformal algebra $\mathscr{V}_1$ is obtained from the Hamiltonian reduction based on the algebra homomorphism $i_1 : sl(2) \to \mathscr{G}_1$ and $\mathscr{V}_2$ from $i_2 : sl(2) \to \mathscr{G}_2$, then $\mathscr{V}_1 \subseteq \mathscr{V}_2$ if any of the following two cases are satisfied:*

1. *$\mathscr{G}_2 = \mathscr{G}_1 \oplus \mathscr{G}'$ and $i_2|_{\mathscr{G}_1} = i_1$.*
2. *There is an algebra homomorphism $j : \mathscr{G}_1 \to \mathscr{G}_2$, such that $i_2 = j \circ i_1$.*

Obviously 2. is satisfied here. It remains for us to determine the central extension of the algebra. One takes the energy-momentum tensor, improved by a BRST exact piece as to get it in a Sugawara-like form:

$$\begin{aligned}\hat{T}^{\mathrm{IMP}} \equiv\ & \frac{1}{x(\kappa+\tilde{h})}\operatorname{str} JJ - \frac{1}{8xy}\operatorname{str} e_0\partial J - \frac{\kappa}{4x}\operatorname{str}([\tau, e_=]\partial\tau) \\ & + \frac{1}{4x}\operatorname{str} b[e_0, \partial c] - \frac{1}{2x}\operatorname{str} b\partial c + \frac{1}{4x}\operatorname{str}\partial b[e_0, c]\,, \end{aligned} \tag{5.17}$$

where y is the index of embedding and one finds

$$c = \frac{1}{2}c_{\mathrm{crit}} - \frac{(d_B - d_F)\tilde{h}}{\kappa+\tilde{h}} - 6y(\kappa+\tilde{h})\,, \tag{5.18}$$

where c_{crit} is the expected critical charge of the string, i.e. the value of c for which the conformal anomaly cancels:

$$c_{\mathrm{crit}} = \sum_{j,\alpha_j} (-)^{(\alpha_j)}(12j^2 + 12j + 2)\,, \tag{5.19}$$

where the sum runs over all $sl(2)$ representations $\underline{2j+1}$ occurring in the decomposition of the adjoint representation of $\mathscr{G}$, α_j is the multiplicity of a representation $\underline{2j+1}$ and the phase $(-)^{(\alpha_j)}$ is $+1, -1$ resp., if the representation has a bosonic, fermionic resp., nature. Finally y is the index of embedding. A particularly useful expression for it is given by

$$y = \frac{1}{3\tilde{h}}\sum_{j,\alpha_j} (-)^{(\alpha_j)} j(j+1)(2j+1)\,. \tag{5.20}$$

Summarizing, we arrive at the following picture. One starts with an embedding of $sl(2|1)$ in some super Lie algebra $\mathscr{G}$. The adjoint representation of $\mathscr{G}$ decomposes into irreps of $sl(2|1)$ as

$$\operatorname{adjoint}(\mathscr{G}) = \bigoplus_{j\in\frac{1}{2}\mathbb{N},\, b\in\frac{1}{2}\mathbb{Z}} n_{(b,j)}(b,j)\,, \tag{5.21}$$

with $n_{(b,j)} \in \mathbb{N}$, the multiplicities. Performing the reduction in the way we discussed above yields an extension of the $N = 2$ superconformal algebra with central charge given in Eq. (5.18). The embedded $sl(2|1)$ then gives rise to the $N = 2$ superconformal algebra. For the remainder we get that every $sl(2|1)$ irrep (b,j), $b \neq \pm j$, gives rise to a set of 4 conformal currents (Z, H_+, H_-, Y) which form a primary, unconstrained $N = 2$ multiplet of conformal dimension j and charge $2b$. The OPE's of the untwisted $N = 2$ generators $T_{N=2}$, $G_\pm$ and U with these currents follow from $N = 2$ representaiton theory:

$$\begin{aligned}
T_{N=2}(z_1)Z(z_2) &= hz_{12}^{-2}Z(z_2) + z_{12}^{-1}\partial Z(z_2)\,, \\
T_{N=2}(z_1)H_\pm(z_2) &= \left(h + \frac{1}{2}\right) z_{12}^{-2}H_\pm(z_2) + z_{12}^{-1}\partial H_\pm(z_2)\,, \\
T_{N=2}(z_1)Y(z_2) &= \frac{q}{2}z_{12}^{-3}Z(z_2) + (h+1)z_{12}^{-2}Y(z_2) + z_{12}^{-1}\partial Y(z_2)\,. \\
G_\pm(z_1)Y(z_2) &= \left(h + \frac{q}{2} + \frac{1}{2}\right) z_{12}^{-2}H_\pm(z_2) + \frac{1}{2}z_{12}^{-1}\partial H_\pm(z_2)\,, \\
G_+(z_1)H_-(z_2) &= \left(h + \frac{q}{2}\right) z_{12}^{-2}Z(z_2) + z_{12}^{-1}\left(Y(z_2) + \frac{1}{2}\partial Z(z_2)\right)\,, \\
G_-(z_1)H_+(z_2) &= -\left(h - \frac{q}{2}\right) z_{12}^{-2}Z(z_2) + z_{12}^{-1}\left(Y(z_2) - \frac{1}{2}\partial Z(z_2)\right)\,, \\
G_\pm(z_1)Z(z_2) &= \mp z_{12}^{-1}H_\pm(z_2)\,, \\
U(z_1)Z(z_2) = qz_{12}^{-1}Z(z_2), &\qquad U(z_1)H_\pm(z_2) = (q \pm 1)z_{12}^{-1}H_\pm(z_2)\,, \\
U(z_1)Y(z_2) &= hz_{12}^{-2}Z(z_2) + qz_{12}^{-1}Y(z_2)\,,
\end{aligned} \tag{5.22}$$

where $j = h$ and $q = 2b$.

For a typical $sl(2|1)$ representations (j,j) or $(-j,j)$, we get 2 conformal currents which form a primary chiral or anti-chiral $N = 2$ multiplet of conformal dimension j and charge $2j$ or $-2j$. E.g. for (j,j), we get currents Z and H_-, whose OPE's follow from Eq. (5.22) by putting $b = j$ (or $q = 2h$) and setting $Y = \frac{1}{2}\partial Z$ and $H_+ = 0$. Similar statements hold for the $(-j,j)$ case, where $q = -2h$ and one puts $H_- = 0$ and $Y = -\frac{1}{2}\partial Z$.

One more subtlety has to be mentioned here. In the case that $j = 0$ or $j = 1/2$, the conformal multiplets are not complete. Indeed in the first case, 2 fields of conformal dimension 1/2 and one of conformal dimension 0 (which only appears through its derivative in the algebra) are lacking, while in the second case one scalar is missing. This is due to the fact that neither scalars nor dimension 1/2 fermions can be generated through Hamiltonian reduction. By redoing the previous analysis in $N = 1$ superspace, see e.g. [16], one does generate dimension 1/2 fermions, but the scalars are still missing. However, we can repair this situation by reversing the Goddard–Schwimmer scheme [17]. They showed that dimension 1/2 and 0 fields can always be decoupled from the conformal algebra. This transformation turns out to be invertible and so can be applied here.

Finally, we did not take into account in Eq. (5.21) that for multiple $osp(2|2)$ embeddings, the adjoint is not fully reducible in terms of $sl(2|1)$ representations. Presumably, this will give rise to a new type of $N = 2$ representations. The study of those will be reported elsewhere.

6. The General Construction

6.1. Some general considerations. We leave the standard reduction behind us and introduce a different grading which will allow for a stringy interpretation of the reduction. We choose an embedding of $sl(2|1)$ in some super Lie algebra $\mathscr{G}$. The adjoint representation of $\mathscr{G}$ decomposes into a number of irreducible $sl(2|1)$ representations, see Eq. (5.21). From the example in Sect. 2, we expect $2b$ to be identified with the ghost number of the resulting currents. So it is quite natural to focus on the case where only $sl(2|1)$ representations with $b = 0$ occur. One finds that this happens for

1. $sl(m|n)$
   A principal embedding of $sl(2|1)$ in $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$, which in its turn is regularly embedded in $sl(m|n)$ with $p, q \geqq 0$, $j \in \frac{1}{2}\mathbb{N}$, $m = p(2j+1) + 2qj$ and $n = 2pj + q(2j+1)$.
2. $osp(m|2n)$
   The diagonal embedding of $osp(2|2)$ in $k\; osp(2|2)$ which in its turn sits regularly embedded in $osp(m|2n)$. However, only for $k = 1$ is the adjoint representation fully reducible. So we only take $k = 1$ into account.

To show this, one uses the results of Sect. 4 from which it followed that the $sl(2|1)$ decomposition of the adjoint of $\mathscr{G}$ follows from the products $(0, 1/2) \otimes (0, 1/2)$, $(0, 1/2) \otimes (\pm j, j)$, $(j, j) \otimes (\pm k, k)$ and $(-j, j) \otimes (\pm k, k)$. Excluding the cases where non-fully reducible representations occur, we find

$$
\begin{aligned}
\left.\left(0,\frac{1}{2}\right) \otimes \left(0,\frac{1}{2}\right)\right|_S &= (0,1)\,,\\
\left(0,\frac{1}{2}\right) \otimes (\pm j,j) &= \left(\pm j, j+\frac{1}{2}\right) \oplus \left(\pm\left(j+\frac{1}{2}\right), j\right),\\
(j,j) \otimes (k,k) &= \bigoplus_{l=|j-k|+\frac{1}{2}}^{j+k-\frac{1}{2}} \left(j+k+\frac{1}{2}, l\right) \oplus (j+k, j+k)\,,\\
(j,j) \otimes (-k,k) &= \bigoplus_{l=|j-k|}^{j+k} (j-k, l)\,.
\end{aligned}
\tag{6.1}
$$

From this it follows that $b = 0$ if only $(0, 1/2) \otimes (0, 1/2)$ and $(j, j) \otimes (-j, j)$ occur. For the fundamental representation of $sl(m|n)$ this implies that only the regular embedding of $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$ should be considered for which the fundamental representation decomposes as

$$
\underline{m+n} = p(j,j) \oplus q(j,j)^{\pi}\,. \tag{6.2}
$$

From the last equation we get that $m = p(2j+1) + q2j$ and $n = q(2j+1) + p2j$. A similar analysis applies to the case of $osp(m|2n)$. The exceptional algebras $\mathscr{G} = G(3)$, $F(4)$ or $D(2,1,\alpha)$ follow from direct inspection. For both $G(3)$ and $F(4)$, one finds always $sl(2|1)$ representations with $b \neq 0$. We take a grading given by[13] $\frac{1}{2}\mathrm{ad}_{e_0} + l\mathrm{ad}_{u_0}$, where l is some positive integer which will be determined now. An $sl(2|1)$ irrep $(0,j)$ decomposes into $u(1) \oplus sl(2)$ irreps as $(0,j) = |0,j\rangle \oplus |-\frac{1}{2},j-\frac{1}{2}\rangle \oplus |\frac{1}{2},j-\frac{1}{2}\rangle \oplus |0,j-1\rangle$. After the reduction we expect that for each $sl(2)$ irrep. there will be one conformal current. The conformal current associated to the $|0,j\rangle$, with conformal dimension $j+1$, should be such that, after twisting, we can identify it with a total (including matter, ghost and gravity contributions) symmetry current of the string theory. Furthermore we want to identify the anti-ghost corresponding to this symmetry with the current which is associated to the $|-\frac{1}{2},j-\frac{1}{2}\rangle$ irrep. In order to achieve this we want the highest weight of $|-\frac{1}{2},j-\frac{1}{2}\rangle$ to be of negative grading so that its constraint puts it equal to an auxiliary field. We can achieve this by choosing l in the grading as $l > j_{\max} - \frac{1}{2}$, where $j_{\max}$ is the largest value of j appearing in the decomposition of the adjoint of $\mathscr{G}$. For[14] $osp(2|2) \hookrightarrow osp(m|2n)$ one gets $l = 1$ and for $sl(2|1) \rightarrow p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1) \hookrightarrow sl(p(2j+1)+2qj|2pj+q(2j+1))$ we get that $l = 2j$. For generic[15] values of j, we get that the $sl(2|1)$ irrep $(0,j)$ is $8j$ dimensional. If j in $(0,j)$ is an integer then one finds that $4j-1$ elements of the representation have a strict positive grading. This gives us both the dimension of the gauge group and the number of constraints. Subtracting this from the original number of affine currents, we are left with 2 currents. However we still have to introduce auxiliary fields of the τ and the $\psi, \bar{\psi}$ type. No fields of the τ type are needed as $\Pi_{-1/2}\,\Pi_{\mathrm{im}\,\mathrm{ad}e_=}\mathscr{G} = \emptyset$. We only have one affine current which is both a highest $sl(2)$ weight and negatively graded, so one set of $\psi, \bar{\psi}$ fields has to be introduced. In total this leaves us 4 currents, the correct number of degrees of freedom. If j is a half integer we get a slightly different counting. The gauge group has now dimension $4j$. Again we have to introduce one set of $\psi, \bar{\psi}$ fields to account for the negatively graded highest $sl(2)$ weight state and we also have that $\Pi_{-1/2}\,\Pi_{\mathrm{im}\,\mathrm{ad}e_=}\mathscr{G}$ contains two elements, requiring the introduction of two τ fields. Doing the counting gives us that again, as it should be, 4 currents are left. We thus come to the following picture. Performing the reduction with the above grading will yield for each $sl(2|1)$ irrep 4 conformal currents which form a standard $N = 2$ unconstrained multiplet. After twisting, we identify the current associated with $|\frac{1}{2},\frac{1}{2}\rangle$ component of the embedded $sl(2|1)$ itself with the BRST current. From standard $N = 2$ representation theory, it follows that after twisting, each total current of the string theory, associated with the $|0,j\rangle$ component of $(0,j)$ is the BRST transform of the corresponding anti-ghost, which is associated to the $|-\frac{1}{2},j-\frac{1}{2}\rangle$ component of $(0,j)$. We briefly return to the general case where also representations with $b \neq 0$ occur. One expects that the $b = 0$ subsector will have a direct "stringy" interpretation. However, in order that the BRST structure closes one needs here the introduction of extra $N = 2$ multiplets with a non-vanishing ghost number. It remains obscure how elementary objects with ghost number greater than one can ever arise in a string theory. So, for the moment we do not discuss the $b \neq 0$ case. We will come back to this case in a future publication.

[13] Our normalizations are such that for a highest weight state of an $sl(2|1)$ irrep, $t_{(b,j)}$, $[e_0, t_{(b,j)}] = 2jt_{(b,j)}$ and $[u_0, t_{(b,j)}] = 2bt_{(b,j)}$

[14] Here and elsewhere, we use the symbol $\rightarrow$ for a principal embedding, and $\hookrightarrow$ for a regular embedding

[15] We do the counting here for generic representations $(0,j)$, the case $(0,0)$ has to be done separately We leave it to the reader

6.2. The invariant action. From now on, we only use the grading discussed above and furthermore, we only consider the cases where $sl(2|1)$ representations occur which are fully reducible and which have $b = 0$. The action

$$\mathscr{S}_0 = \kappa S^-[g] + \frac{1}{\pi x}\int \mathrm{str}\, A\left(J - \frac{\kappa}{2}e_= - \frac{\kappa}{2}[e_=, \tau]\right) - \frac{\kappa}{4\pi x}\int \mathrm{str}[e_=, \tau]\bar{\partial}\tau\,, \tag{6.3}$$

where

$$\begin{aligned} A &\in \Pi_{>0}\mathscr{G}\,, \\ \tau &\in \Pi_{\frac{1}{2}}\,\Pi_{\mathrm{im\,ad}_{e_{\mp}}}\mathscr{G} \end{aligned} \tag{6.4}$$

is gauge invariant:

$$\begin{aligned} \delta g &= \eta g\,, \\ \delta A &= \bar{\partial}\eta + [\eta, A]\,, \\ \delta\tau &= -\Pi_{\frac{1}{2}}\,\Pi_{\mathrm{im\,ad}_{e_{\mp}}}\eta\,, \end{aligned} \tag{6.5}$$

and $\eta \in \Pi_{>0}\mathscr{G}$. However, as we just discussed (and in the example in Sect. 2), we have to face the possibility that constraints of the form

$$\Pi_{<0}\,\Pi_{\mathrm{ker\,ad}_{e_{\mp}}}J = 0\,, \tag{6.6}$$

arise. In order to avoid this we introduce an extra term in the action of the form

$$-\frac{\kappa}{2\pi x}\int \mathrm{str}\, A\Psi\,, \tag{6.7}$$

where $\Psi \in \Pi_{<0}\,\Pi_{\mathrm{ker\,ad}_{e_{\mp}}}\mathscr{G}$. Of course the action is now not invariant anymore. Obviously one gets non-invariance terms of the form $\int \mathrm{str}(\bar{\partial}\eta\Psi)$. These can easily be cancelled by adding a new field $\bar{\Psi}$, where $\bar{\Psi} \in \Pi_{>0}\,\Pi_{\mathrm{ker\,ad}_{e_=}}\mathscr{G}$ which transforms as

$$\delta\bar{\Psi} = \Pi_{\mathrm{ker\,ad}_{e_=}}\eta\,. \tag{6.8}$$

We modify the action to $\mathscr{S}_0 + \mathscr{S}_1$, where $\mathscr{S}_0$ is given in Eq. (6.3) and

$$\mathscr{S}_1 = -\frac{\kappa}{2\pi x}\int \mathrm{str}\, A\Psi + \frac{\kappa}{2\pi x}\int \mathrm{str}\,\Psi\bar{\partial}\bar{\Psi}\,. \tag{6.9}$$

The resulting action is still not quite invariant. Indeed varying $\mathscr{S}_0 + \mathscr{S}_1$ under Eqs. (6.5) and (6.8), yields

$$\delta(\mathscr{S}_0 + \mathscr{S}_1) = \frac{\kappa}{2\pi x}\int \mathrm{str}\, A[\eta, \Psi]\,. \tag{6.10}$$

We can further rewrite this using $\eta = \Pi_{\mathrm{ker\,ad}_{e_=}}\eta + \Pi_{\mathrm{im\,ad}_{e_{\mp}}}\eta$ and $A = \Pi_{\mathrm{ker\,ad}_{e_=}}A + \Pi_{\mathrm{im\,ad}_{e_{\mp}}}A$. The terms proportional to $\Pi_{\mathrm{ker\,ad}_{e_=}}A$ can be cancelled by modifying the transformation rule of Ψ while terms proportional to $\Pi_{\mathrm{ker\,ad}_{e_=}}\eta$ are cancelled by adding extra terms proportional to $\bar{\Psi}$ to the action. However, this will leave us with, among others, a term proportional to

$$\int \mathrm{str}(\,\Pi_{\mathrm{im\,ad}_{e_{\mp}}}A[\Pi_{\mathrm{im\,ad}_{e_{\mp}}}\eta, \Psi]), \tag{6.11}$$

which cannot be cancelled without the introduction of new fields. Introducing new fields would disrupt the balance of degrees of freedom which could only be restored by the introduction of a larger gauge symmetry. We will get around this problem by modifying the definition of Ψ which will be allowed to have a component in the image of $\mathrm{ad}_{e_=}$. We will do this in such a way that Ψ still has the same number of degrees of freedom as if it belonged exclusively to the kernel of $\mathrm{ad}_{e_{\neq}}$ and such that the highest weight gauge remains non-singular, i.e. $\Pi_{<0}\,\Pi_{\ker \mathrm{ad}_{e_{\neq}}} J \neq 0$. We split $\Pi_{>0}\mathscr{G}$ in two parts: $\Pi_{>0}\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1$ and similarly for $\Pi_{<0}\mathscr{G} : \Pi_{<0}\mathscr{G} = \bar{\mathscr{G}}_0 \oplus \bar{\mathscr{G}}_1$ such that $\bar{\mathscr{G}}_0 \perp \mathscr{G}_1$ and $\bar{\mathscr{G}}_1 \perp \mathscr{G}_0$. The action

$$\mathscr{S}_1 = -\frac{\kappa}{2\pi x}\int \operatorname{str} A\Psi + \frac{\kappa}{2\pi x}\int \operatorname{str} \Psi\bar{\partial}\bar{\Psi} - \frac{\kappa}{2\pi x}\int \operatorname{str} A[\bar{\Psi},\Psi]\,, \tag{6.12}$$

where $\Psi \in \bar{\mathscr{G}}_0$ and $\bar{\Psi} \in \mathscr{G}_0$ is invariant under

$$\begin{aligned} \delta A &= \bar{\partial}\eta + [\eta, A]\,, \\ \delta\bar{\Psi} &= \Pi_{\mathscr{G}_0}(\eta + [\eta,\bar{\Psi}])\,, \\ \delta\Psi &= \Pi_{\bar{\mathscr{G}}_0}[\eta,\Psi]\,, \end{aligned} \tag{6.13}$$

provided the conditions

$$\begin{aligned} &[\mathscr{G}_0,\mathscr{G}_0] \subseteq \mathscr{G}_1\,, \\ &[\mathscr{G}_1,\mathscr{G}_1] \subseteq \mathscr{G}_1\,, \\ &[\mathscr{G}_0,[\mathscr{G}_0,\mathscr{G}_0]] \subseteq \mathscr{G}_1\,, \\ &[\mathscr{G}_0,[\mathscr{G}_0,\mathscr{G}_1]] \subseteq \mathscr{G}_1\,, \end{aligned} \tag{6.14}$$

and similarly for $\bar{\mathscr{G}}_0$ and $\bar{\mathscr{G}}_1$, are satisfied. The full gauge invariant action used for the reduction is then simply $\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1$, where $\mathscr{S}_0$ is given in Eq. (6.3) and $\mathscr{S}_1$ in Eq. (6.12). It remains now for us to determine $\mathscr{G}_0$ as a function of the choice of $\mathscr{G}$ and the embedding.

1. $osp(2|2)$ as a regular subalgebra[16] of $osp(m|2n)$.
The adjoint representation of $\mathscr{G}$ decomposes into $(0,0), (0,1/2)$ and $(0,1)$ representations of $sl(2|1)$. The grading we consider is given by the action of $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$. It turns out that the conditions (6.14) are satisfied provided one chooses:

$$\begin{aligned} \mathscr{G}_0 &= \Pi_{\ker \mathrm{ad}_{e_=}}\,\Pi_{>0}\,\mathscr{G}\,, \\ \mathscr{G}_1 &= \Pi_{\mathrm{im}\, \mathrm{ad}_{e_{\neq}}}\,\Pi_{>0}\,\mathscr{G}\,, \end{aligned} \tag{6.15}$$

and similarly $\bar{\mathscr{G}}_0 = \Pi_{\ker \mathrm{ad}_{e_{\neq}}}\,\Pi_{<0}\,\mathscr{G}$ and $\bar{\mathscr{G}}_1 = \Pi_{\mathrm{im}\,\mathrm{ad}_{e_=}}\,\Pi_{<0}\,\mathscr{G}$. In order to show this, it is sufficient to realize that the elements of $\mathscr{G}_0$ have charge $b = 1/2$ and their grading is either $1/2$ or 1. This already implies that $[\mathscr{G}_0,\mathscr{G}_0] = 0$. So condition one, three and four are satisfied. Furthermore, all strict positively graded elements of $\mathscr{G}$ have either charge $b = 1/2$ or $b = 0$. Finally the only charged elements with grades $1/2$ or 1 belong necessarily to $\mathscr{G}_0$. From this the third condition follows.

[16]Again we discard the case where a sum of $osp(2|2)$'s is embedded In that case we have to deal with $sl(2|1)$ representations which are not fully reducible, a pathology we want to avoid

2. $sl(2|1)$ as a principal subalgebra of $sl(2j+1|2j)$ or $sl(2j|2j+1)$.
One can verify that already for $j = 3/2$, the second of the conditions (6.14), gets violated if one chooses $\mathscr{G}_0 = \Pi_{\ker \mathrm{ad}_{e_=}} \Pi_{>0}\mathscr{G}$. However, if we choose

$$\mathscr{G}_0 = \Pi_{\ker \mathrm{ad}_{e'_=}} \Pi_{\ker \mathrm{ad}_{e_=}} \Pi_{>0}\mathscr{G}\,, \tag{6.16}$$

where $e'_=$ is $e_=$ restricted to the $sl(2j+1)$ subalgebra of $sl(2j+1|2j)$ or $sl(2j|2j+1)$, we find that conditions (6.14) are satisfied. Indeed, take first the case of $sl(2j+1|2j)$ and parametrize it by $(4j+1)\times(4j+1)$ matrices where the first $2j+1$ rows and columns are of a bosonic nature, while the last $2j$ rows and columns are fermionic. With this, one gets

$$\begin{aligned}
e_0 &= \sum_{p=1}^{2j+1} 2(j-p+1)e_{p,p} + \sum_{p=1}^{2j} 2\left(j+\frac{1}{2}-p\right) e_{2j+1+p,2j+1+p}\,,\\
e_= &= \sum_{p=1}^{2j} e_{p+1,p} + \sum_{p=1}^{2j-1} e_{2j+p+2,2j+p+1}\,,\\
e'_= &= \sum_{p=1}^{2j} e_{p+1,p}\,,\\
u_0 &= -2j\sum_{p=1}^{2j+1} e_{p,p} - (2j+1)\sum_{p=1}^{2j} e_{2j+1+p,2j+1+p}\,,
\end{aligned}\tag{6.17}$$

where $e_{r,s}$ is a $(4j+1)\times(4j+1)$ matrix unit: $(e_{r,s})_{kl} = \delta_{r,k}\delta_{s,l}$. According to the previous discussion, we choose the grading given by the action of $\frac{1}{2}\mathrm{ad}_{e_0} + 2j\mathrm{ad}_{u_0}$ and we find that $\Pi_{>0}\mathscr{G}$ is generated by $e_{k,l}$ with $k < l$. $\mathscr{G}_0$ is generated by $\{e_{2j+1,2j+1+p}, p \in \{1,2,\ldots,2j\}\}$, and we have e.g. $\bar{\Psi} = \sum_{p=1}^{p=2j} \bar{\Psi}^p e_{2j+1,2j+1+p}$. With this one verifies that the conditions (6.14) are satisfied. In particular, we have again that $[\mathscr{G}_0, \mathscr{G}_0] = 0$. The reduction of the adjoint representation is

$$\mathrm{adjoint}(sl(2j+1|2j)) = \bigoplus_{p=1}^{2j} (0,p)\,. \tag{6.18}$$

The discussion for $\mathscr{G} = sl(2j|2j+1)$ is completely analogous. We represent $\mathscr{G}$ by $(4j+1)\times(4j+1)$ matrices of which the first $2j$ rows and columns are bosonic, while the last $2j+1$ rows and columns are fermionic. We have

$$\begin{aligned}
e_0 &= \sum_{p=1}^{2j} 2\left(j-p+\frac{1}{2}\right) e_{p,p} + \sum_{p=1}^{2j+1} 2(j-p+1)e_{2j+p,2j+p}\,,\\
e_= &= \sum_{p=1}^{2j-1} e_{p+1,p} + \sum_{p=1}^{2j} e_{2j+p+1,2j+p}\,,\\
e'_= &= \sum_{p=1}^{2j} e_{2j+p+1,2j+p}\,,\\
u_0 &= -(2j+1)\sum_{p=1}^{2j} e_{p,p} - 2j\sum_{p=1}^{2j+1} e_{2j+p,2j+p}\,.
\end{aligned}\tag{6.19}$$

$\Pi_{>0}\mathscr{G}$ is now generated by $e_{k,l}$ with $1 \leqq k < l \leqq 2j$ or with $2j+1 \leqq k < l \leqq 4j+1$ or $2j+1 \leqq k \leqq 4j+1$ and $1 \leqq l \leqq 2j$. $\mathscr{G}_0$ is generated by $\{e_{4j+1,p}, p \in \{1,2,\ldots,2j\}\}$ and the adjoint representation decomposes as in Eq. (6.18).

3. $sl(2|1)$ as a principal subalgebra of $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$. This case follows immediately from the previous one. The grading is still given by the action of $\frac{1}{2}\mathrm{ad}_{e_0} + 2j\mathrm{ad}_{u_0}$ and $\mathscr{G}_0$ is taken as in Eq. (6.16), but now $e'_=$ is $e_=$ restricted to the $p\,sl(2j+1) \oplus q\,sl(2j+1)$ subalgebra of $\mathscr{G}$. The adjoint of $\mathscr{G}$ decomposes as

$$\mathrm{adjoint}(p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)) = (p+q)\bigoplus_{r=1}^{2j}(0,r)\,. \qquad (6.20)$$

4. $sl(2|1)$ as a principal subalgebra of $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$, which in its turn is regularly embedded in $sl(p(2j+1)+2qj|2pj+q(2j+1))$.

Again, we choose the action of $\frac{1}{2}\mathrm{ad}_{e_0} + 2j\mathrm{ad}_{u_0}$ as the grading. The choice of $\mathscr{G}_0$ is exactly as in the previous case, i.e. one takes Eq. (6.16), with $e'_=$ as $e_=$ restricted to the $p\,sl(2j+1) \oplus q\,sl(2j+1)$ subalgebra of the regularly in $\mathscr{G}$ embedded $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$ algebra. To show this, one only has to show this to be true for the part of $\mathscr{G}$ which does not belong to the regularly embedded $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$ subalgebra. The $p\,sl(2j+1|2j) \oplus q\,sl(2j \mid 2j+1)$ subalgebra follows from the previous one. Those generators fall (using a slightly abusive notation) either in a $((\pm j,j), (\mp j,j))$ of an $sl(2j+1|2j) \otimes sl(2j+1|2j)$ subalgebra, while being a scalar for the other factors of $p\,sl(2j+1|2j) \oplus q\,sl(2j|2j+1)$, or in a $((\pm j,j),(\mp j,j)^{\pi})$ representation of an $sl(2j+1|2j) \oplus sl(2j|2j+1)$ subalgebra. An explicit parametrization of these two subcases, as we did under case 2, quickly shows that the choice of $\mathscr{G}_0$ is consistent with Eqs. (6.14). The adjoint representation decomposes as

$$\begin{aligned}&\mathrm{adjoint}(sl(p(2j+1)+2qj|2pj+q(2j+1)))\\ &= (p+q)\bigoplus_{r=1}^{2j}(0,r) \oplus (p(p-1)+q(q-1))\bigoplus_{r=0}^{2j}(0,r) \oplus 2pq\bigoplus_{r=0}^{2j}(0,r)'\,. \qquad (6.21)\end{aligned}$$

7. Quantizing the Model

Our starting point is the action $\mathscr{S}_0 + \mathscr{S}_1$, where $\mathscr{S}_0$ is given in Eq. (6.3) and $\mathscr{S}_1$ in Eq. (6.12). The gauge invariance Eq. (6.13) is fixed by the choice $A = 0$. We introduce ghosts $c \in \Pi_{>0}\mathscr{G}$ and anti-ghosts $b \in \Pi_{<0}\mathscr{G}$ and obtain the gauge fixed action:

$$\mathscr{S}_{\mathrm{gf}} = \kappa S^{-}[g] + \frac{\kappa}{4\pi x}\int \mathrm{str}[\tau, e_=]\bar{\partial}\tau + \frac{\kappa}{2\pi x}\int \mathrm{str}\,\Psi\bar{\partial}\bar{\Psi} + \frac{1}{2\pi x}\int \mathrm{str}\,b\bar{\partial}c\,, \qquad (7.1)$$

with the BRST charge $\mathscr{Q}_{HR}$:

$$\mathscr{Q}_{HR} = \frac{1}{4\pi i x}\oint \mathrm{str}\left\{c\left(J - \frac{\kappa}{2}e_= - \frac{\kappa}{2}[e_=,\tau] - \frac{\kappa}{2}\Psi - \frac{\kappa}{2}[\bar{\Psi},\Psi] + \frac{1}{2}J^{\mathrm{gh}}\right)\right\}, \qquad (7.2)$$

where $J^{\mathrm{gh}} = \frac{1}{2}\{b,c\}$. The conformal currents are obtained as the generators of the cohomology $H^*(\mathscr{A},\mathscr{Q}_{HR})$, where $\mathscr{A}$ is the algebra generated by $\{b,\hat{J} =$

$J + J^{\mathrm{gh}}, \tau, \Psi, \bar{\Psi}, c\}$. Every field has a double grading (m, n), where m is the grading used in the reduction and $m + n$ is the ghost number of the field. The fields τ, Ψ and $\bar{\Psi}$ have grading $(0, 0)$. Again the BRST operator decomposes into pieces of definite grading. A novel feature which appears here is the fact that in general there will be more than 3 pieces! A similar situation occurred in the construction of topological strings from Hamiltonian reduction [6]. We decompose $\mathscr{Q}_{HR}$ as $\mathscr{Q}_{HR} = \mathscr{Q}^{\Psi} + \mathscr{Q}_{(1,0)} + \mathscr{Q}_{(1/2,1/2)} + \mathscr{Q}_{(0,1)}$, where $\mathscr{Q}^{\Psi}$ is the Ψ dependent part of $\mathscr{Q}_{HR}$ and the remainder of the decomposition is as in Eq. (5.8). If Ψ has gradings $-1/2, -1, \ldots, -n_{\max}$, then $\mathscr{Q}^{\Psi}$ decomposes as

$$\mathscr{Q}^{\Psi} = \sum_{p=1}^{2n_{\max}-1} \mathscr{Q}^{\Psi}_{(\frac{1}{2}+\frac{p}{2},\frac{1}{2}-\frac{p}{2})} \,. \tag{7.3}$$

The nontrivial action of $\mathscr{Q}_{HR}$ on various fields is tabulated below:

$$\begin{aligned}
\mathscr{Q}^{\Psi}: \quad & b \to -\frac{\kappa}{2}(\Psi + \Pi_{<0}[\bar{\Psi}, \Psi]) \\
& \Psi \to \frac{1}{2}\,\Pi_{\bar{\mathscr{G}}_0}[c, \Psi] \\
& \bar{\Psi} \to \frac{1}{2}\,\Pi_{\mathscr{G}_0}(c + [c, \bar{\Psi}]) \\
& \hat{J} \to \frac{\kappa}{4}[c, \Psi + [\bar{\Psi}, \Psi]] \\
\mathscr{Q}_{(1,0)}: \quad & b \to -\frac{\kappa}{2} e_{=} \\
& \hat{J} \to -\frac{\kappa}{4}[e_{=}, c] \\
\mathscr{Q}_{(\frac{1}{2},\frac{1}{2})}: \quad & b \to -\frac{\kappa}{2}[e_{=}, \tau] \\
& \tau \to -\frac{1}{2}\,\Pi_{\frac{1}{2}}\,\Pi_{\mathrm{im\,ad}_{e_{+}}}\, c \\
& \hat{J} \to -\frac{\kappa}{4}[[e_{=}, \tau], c] \\
\mathscr{Q}_{(0,1)}: \quad & b \to \Pi_{<0}\hat{J} \\
& \hat{J} \to \frac{1}{2}\vec{[}c, \Pi_{\geqq 0}\hat{J}] + \frac{\kappa}{4}\partial c - \frac{1}{2}[\Pi_{<0}(t^A), [\Pi_{>0}(t_A), \partial c]] \\
& c \to \frac{1}{2} cc \,,
\end{aligned} \tag{7.4}$$

where $\vec{[}A, B]$ stands for

$$\vec{[}X, Y] = (-)^{(AB)}(X^A Y^B) f_{AB}^{C} t_C \,, \tag{7.5}$$

with $(X^A Y^B)$, a regularized product. Using Eq. (7.4), one shows that the cohomology $H^*(\mathscr{A}, \mathscr{Q}_{HR})$ is isomorphic to $H^*(\hat{\mathscr{A}}, \mathscr{Q}_{HR})$, where $\hat{\mathscr{A}}$ is a reduced complex generated by $\{\Pi_{\geq 0}\hat{J}, \tau, \Psi, \bar{\Psi}, c\}$. The double grading introduced before, carries over to the reduced complex:

$$\hat{\mathscr{A}} = \bigoplus_{p \in \frac{1}{2}\mathbb{N}} \bigoplus_{m \in \mathbb{N}} \hat{\mathscr{A}}_{(p, -p+m)} \,. \tag{7.6}$$

We introduce a filtration, which in the case of $n_{\max} \leqq 1$ is given by

$$\hat{\mathscr{A}}^m = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} \bigoplus_{l\geqq m} \hat{\mathscr{A}}_{(k,l)}\,, \tag{7.7}$$

otherwise

$$\hat{\mathscr{A}}^m = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} \bigoplus_{l\geqq \frac{1-n_{\max}}{n_{\max}}k+m} \hat{\mathscr{A}}_{(k,l)}\,. \tag{7.8}$$

The rest is now quite standard. One sets up a spectral sequence $(E_r, d_r), r \geqq 1$, which in the first case collapses after two steps: $E_2 = E_\infty$, and in the second case after $2n_{\max}$ steps: $E_{2n_{\max}} = E_\infty$. The actual computation is quite involved. As an example we explicitly compute the sequence for the case of a regular $osp(2|2)$ embedding. In that case one deals only with $(0,0), (0,1/2)$ and $(0,1)$ irreps of $sl(2|1)$. As explained, we have that the grading is given by the action of $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$, and the choice for $\mathscr{G}_0$ and $\mathscr{G}_1$ is given in Eq. (6.15). From this one gets that $\mathscr{G}_0$ has elements of grade 1/2 and 1 only and so it follows that $\mathscr{Q}_{HR} = \mathscr{Q}_{(1,0)} + \mathscr{Q}_{(1/2,1/2)} + \mathscr{Q}_{(0,1)}$, where we absorbed the Ψ dependent parts of appropriate grading in $\mathscr{Q}_{(1,0)}$ and $\mathscr{Q}_{(1/2,1/2)}$. Using Eq. (7.4) one finds:

$$\begin{aligned} E_1 &= H^*(E_0; d_0) = H^*(\hat{\mathscr{A}} : \mathscr{Q}_{(1,0)}) = \mathscr{A}(\Pi_{\frac{1}{2}}\, c) \otimes \mathscr{A}(\Psi) \otimes \mathscr{A}(\Pi_{\frac{1}{2}}\, \bar{\Psi}) \otimes \mathscr{A}(\tau) \\ &\otimes \mathscr{A}\left(\Pi_{\ker \mathrm{ad}_{e_{\neq}}}\, \Pi_{\geqq 0}\left(\hat{J} - \frac{\kappa}{2}[\bar{\Psi}, \Pi_{-1}\,\Psi] + [L \circ \Pi_{\geqq 0}\hat{J}, \Pi_{-1}\Psi]\right)\right), \end{aligned} \tag{7.9}$$

where L is defined by

$$L \circ \mathrm{ad}_{e_=} = \Pi_{\mathrm{im}\, \mathrm{ad}_{e_{\neq}}}\,, \qquad \mathrm{ad}_{e_=} \circ L = \Pi_{\mathrm{im}\, \mathrm{ad}_{e_=}}\,. \tag{7.10}$$

Subsequently we get

$$\begin{aligned} E_2 &= E_\infty = H^*(E_1; d_1) = H^*(E_1; \mathscr{Q}_{(\frac{1}{2},\frac{1}{2})}) \\ &= \mathscr{A}\left(\Pi_{\ker \mathrm{ad}_{e_{\neq}}}\, \Pi_{\geqq 0}\left(\hat{J} + \frac{\kappa}{4}[\tau, [e_=, \tau]] - \frac{\kappa}{2}[\Pi_1\, \bar{\Psi}, \Pi_{-1}\, \Psi]\right.\right. \\ &\quad \left.\left. - \frac{\kappa}{2}[\Pi_{\frac{1}{2}}\, \bar{\Psi}, \Pi_{-\frac{1}{2}}\, \Psi] + [L \circ \Pi_{\geqq 0}\hat{J}, \Pi_{-1}\Psi]\right)\right) \\ &\quad \otimes \mathscr{A}(\Pi_{\ker \mathrm{ad}_{e_{\neq}}}\, \Pi_{<0}\,(\Psi + [\tau, \Psi])). \end{aligned} \tag{7.11}$$

The full generators are then obtained by a generalized tic-tac-toe construction. Finally for $(0,1/2)$ and $(0,0)$ multiplets, we introduce scalars and fermions to complete the superconformal representations through a reversed Goddard–Schwimmer mechanism as was explained in Sect. 5.

However, one more point remains to be clarified. As advertised before, we would like to identify the generators (which are already BRST invariant) $\Pi_{\ker \mathrm{ad}_{e_{\neq}}}\, \Pi_{<0}$ ($\Psi + [\tau, \Psi]+$ contributions arising from the inverse Goddard–Schwimmer mechanism) with the anti-ghosts. An anti-ghost is a simple field, while the previous expression contains composite terms albeit of a very simple nature. This problem was solved in [5] through the introduction of a similarity transformation generated by $S = \exp R$ with

$$R \propto \frac{1}{2\pi i} \oint (\bar{\Psi}[\tau, \Psi] + \text{contributions arising from the inverse Goddard–Schwimmer mechanism}) . \tag{7.12}$$

8. Discussion

In this paper we classified all possible embeddings of $sl(2|1)$ into super Lie algebras. This classification is equivalent to the classification of all extended $N = 2$ superconformal algebras which can be obtained from Hamiltonian reduction. While a specific embedding fixes the conformal algebra, including the value of the central charge, completely, the particular realization of that algebra is only determined once a grading on the super Lie algebra has been chosen. The canonical grading, which is just the $sl(2)$ grading inherited from the embedding, yields the standard or symmetric realizations. Twisting or modifying the grading by adding a multiple of the $u(1)$ charge to the $sl(2)$ grading results in realizations which allow for a stringy interpretation. After the reduction we are left with the $N = 2$ superconformal currents together with a set of $N = 2$ multiplets each of which generically contains four currents which yields some extension of the $N = 2$ superconformal algebra. The currents fall generically (we will see that chiral and antichiral multiplets do not have to be considered) into unconstrained $N = 2$ multiplets each containing four currents, say $Y(x)$, $H_+(x)$, $H_-(x)$ and $Z(x)$ of conformal dimensions $h + 1$, $h + 1/2$, $h + 1/2$ and h. Twisting amounts to replacing $Y(x)$ by $X(x) \equiv Y(x) + \frac{1}{2}\partial Z(x)$. The OPE's of the twisted $N = 2$ subalgebra itself were given in Eq. (2.6). The OPE's of $T = T_{N=2} + \frac{1}{2}\partial U$, $G_\pm$ and U with $X(x), H_+(x)$, $H_-(x)$ and $Z(x)$ follow immediately from Eq. (5.22). We give the most significant ones:

$$\begin{aligned}
T(z_1)X(z_2) &= \left(h + 1 - \frac{q}{2}\right) z_{12}^{-2} X(z_2) + z_{12}^{-1} \partial X(z_2) , \\
T(z_1)H_-(z_2) &= \left(h + 1 - \frac{q}{2}\right) z_{12}^{-2} H_-(z_2) + z_{12}^{-1} \partial H_-(z_2) , \\
G_+(z_1)H_-(z_2) &= \left(h + \frac{q}{2}\right) z_{12}^{-2} Z(z_2) + z_{12}^{-1} X(z_2) , \\
U(z_1)X(z_2) &= \left(h + \frac{q}{2}\right) z_{12}^{-2} Z(z_2) + q z_{12}^{-1} X(z_2) ,
\end{aligned} \tag{8.1}$$

If now, G_- and all fields of the H_- type are realized as single fields, something which was achieved in this paper, then we can view the above system as a string theory. Indeed $\mathcal{Q}$,

$$\mathcal{Q} = \frac{1}{2\pi i} \oint dz \, G_+(z) , \tag{8.2}$$

is the BRST charge, satisfying $\mathcal{Q}^2 = 0$. The G_- current and all currents of the H_- type are the anti-ghosts. The total symmetry currents (matter + gravity + ghosts) are the energy-momentum tensor $T = T_{N=2} + \frac{1}{2}\partial U$ and the currents of the X type. One notices that w.r.t. the twisted energy-momentum tensor X has become primary, this in contrast with the situation of Y vs. $T_{N=2}$. One verifies from Eq. (8.1) that they are indeed the BRST transform of the corresponding antighosts:

$$T = [\mathcal{Q}, G_-]_+ \qquad X = [\mathcal{Q}, H_-]_\pm \,. \tag{8.3}$$

However, one more restriction follows. We saw that U has, in leading order, the interpretation of a ghost number current. So we should consider those reductions where in Eq. (8.1) only $q = 0$ appear. This is equivalent to restricting to those $sl(2|1)$ embeddings where in the decomposition of the adjoint representation, only $sl(2|1)$ irreps (b, j) with $b = 0$ occur. They were classified in Sect. 6. The question which obviously arises is: given an $sl(2|1)$ embedding, which string theory do we describe? It is straightforward to see that for the two main cases we obtain the following pattern:

I. $sl(2|1) \to p\, sl(2j+1|2j) \oplus q\, sl(2j|2j+1) \hookrightarrow sl(p(2j+1)+2qj|2pj+q(2j+1))$.
The matter sector of the string theory corresponds to the reduction:

$$sl(2) \to p\, sl(2j+1) \oplus q\, sl(2j+1) \hookrightarrow sl\,(p(2j+1)|q(2j+1))\,. \tag{8.4}$$

II. $osp(2|2) \hookrightarrow osp(m|2n)$.
The matter sector is now given by the reduction

$$sp(2) \hookrightarrow sp(2n) \hookrightarrow osp(m-2|2n)\,. \tag{8.5}$$

A very interesting open question remains: which string theories arise from the reduction of $D(2,1,\alpha)$? One would expect $N = 2$ strings. However in [5] it was shown that the standard $N = 2$ strings arise from the reduction of $osp(4|2)$ which turns out to be isomorphic to $D(2,1,\alpha)$ with $\alpha = 1$. Presumably, one will get a new type of $N = 2$ string. In view of the very particular properties of $N = 2$ strings [18], this case definitely needs further investigation. Work in this direction is in progress [19].

Explicitly worked out examples of the general method developed in this paper can be found in the literature. In [4] e.g., classical W_n strings were obtained from a reduction based on $sl(2|1) \to sl(n|n-1)$. The quantum structure can now also be obtained following the strategy developed in the previous section. In [5], N-extended superstrings were obtained from the reduction $osp(2|2) \hookrightarrow osp(N+2|2)$. One can now wonder whether, in the case of an embedding where the adjoint representation decomposes into $sl(2|1)$ irreps (b, j), where b is not necessarily zero, some stringy interpretation can still be given. Similarly, another point of interest is the occurrence of non-fully reducible representations in the decomposition of the adjoint representations for certain $osp(2|2)$ embeddings. An immediate consequence of this is that there must exist non-fully reducible $N = 2$ superconformal representations. One might wonder whether such representations might provide clues to the open problem of finding an off-shell description of certain $N = 2$ non-linear σ-models [20]. These questions are presently under study and the results will be reported elsewhere. Finally, a most interesting point would be to push the present work further and address questions such as "What is the spectrum of these string theories?" Using the recent results in [21] it should be possible to obtain at least the partition function explicitly.

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A. Wess–Zumino–Witten Models

We briefly review WZW models. Given a super Lie algebra with generators $\{t_a; a \in \{1,\dots,d_B+d_F\}\}$, where d_B (d_F) is the number of bosonic (fermionic) generators, we denote the (anti)commutation relations by

$$[t_a,t_b] = t_a t_b - (-)^{(a)(b)} t_b t_a = f^c_{\ ab} t_c \ , \tag{A.1}$$

where for $t_a, (a) = 0$ (1) when t_a is bosonic (fermionic). We always use that $Xt_a = (-)^{(X)(a)} t_a X$, where X is not Lie algebra valued. The adjoint representation is

$$[t_a]^c_b \equiv f^c_{\ ba} \ . \tag{A.2}$$

The Killing metric g_{ab} is defined by

$$f^d_{\ ca} f^c_{\ db} (-)^{(c)} = -\tilde{h} g_{ab} \ , \tag{A.3}$$

with $\tilde{h}$, the dual Coxeter number. This is fine for Lie algebras, but for super algebras the dual Coxeter number might vanish. More generally we have then

$$\mathrm{str}(t_a t_b) \equiv [t_a]^\beta_\alpha [t_b]^\alpha_\beta (-)^{(\alpha)} \equiv -x g_{ab} \ , \tag{A.4}$$

where x is the index of the representation. In the adjoint representation one has $x = \tilde{h}$. A contraction runs from upper left to lower right, e.g. $A^a B_a$. Raising and lowering indices happen according to this convention (implying $g^{ac} g_{bc} = \delta^a_b$):

$$A^a = g^{ab} A_b, \qquad A_a = A^b g_{ba} \ . \tag{A.5}$$

We tabulate some properties of the (super) Lie algebras which appear in this paper:

algebra	bosonic subalgebra	d_B	d_F	$\tilde{h}$
$osp(m\|2n)$	$so(n) \oplus sp(2n)$	$\frac{1}{2}m(m-1) + n(2n+1)$	$2mn$	$\frac{1}{2}m - n - 1$
$D(2,1,\alpha)$	$sl(2) \oplus sl(2) \oplus sl(2)$	9	8	0
$sl(m\|m)$	$sl(m) \oplus sl(m)$	$2(m^2-1)$	$2m^2$	0
$sl(m\|n)$ $m \neq n$	$sl(m) + sl(n) + gl(1)$	$m^2 + n^2 - 1$	$2mn$	$m - n$
$g(3)$	$g_2 \oplus sp(2)$	17	14	$\frac{1}{4}$
$f(4)$	$so(7) \oplus sp(2)$	21	16	$\frac{1}{2}$

The WZW action $\kappa S^+[g]$ is given by

$$\kappa S^+[g] = \frac{\kappa}{4\pi x}\int d^2z\,\mathrm{str}\{\partial g^{-1}\bar{\partial}g\} + \frac{\kappa}{12\pi x}\int d^3z\,\varepsilon^{\alpha\beta\gamma}\,\mathrm{str}\{g,_{\alpha}g^{-1}g,_{\beta}g^{-1}g,_{\gamma}g^{-1}\}\,, \tag{A.6}$$

and satisfies the Polyakov–Wiegman identity,

$$S^+[hg] = S^+[h] + S^+[g] - \frac{1}{2\pi x}\int \mathrm{str}(h^{-1}\partial h\bar{\partial}gg^{-1})\,. \tag{A.7}$$

The functional $S^-[g]$ is defined by

$$S^-[g] = S^+[g^{-1}]\,. \tag{A.8}$$

Using the equations of motion

$$\begin{aligned}\delta S^+[g] &= \frac{1}{2\pi x}\int \mathrm{str}\{\bar{\partial}(g^{-1}\partial g)g^{-1}\delta g\}\\ &= \frac{1}{2\pi x}\int \mathrm{str}\{\partial(\bar{\partial}gg^{-1})\delta gg^{-1}\}\,,\end{aligned} \tag{A.9}$$

which are solved by putting $g \equiv g(\bar{z})g(z)$, where $\partial g(\bar{z}) = \bar{\partial}g(z) = 0$, one gets the conserved affine currents

$$\begin{aligned}J_z &= -\frac{\kappa}{2}g^{-1}\partial g\,,\\ J_{\bar{z}} &= \frac{\kappa}{2}\bar{\partial}gg^{-1}\,,\end{aligned} \tag{A.10}$$

which generate the affine symmetries

$$\begin{aligned}\delta J_z^a &= -\frac{\kappa}{2}\partial\eta^a - (-)^{(b)(c)}f^a_{\ bc}\eta^b J_z^c\,,\\ \delta J_{\bar{z}}^a &= \frac{\kappa}{2}\bar{\partial}\bar{\eta}^a + (-)^{(b)(c)}f^a_{\ bc}\bar{\eta}^b J_{\bar{z}}^c\,,\end{aligned} \tag{A.11}$$

where

$$\bar{\partial}\eta^a = \partial\bar{\eta}^a = 0\,. \tag{A.12}$$

From

$$\delta J_z^a(z) = \frac{1}{2\pi i}\oint_z dw\,J_z^b(w)\eta_b(w)J_z^a(z)\,, \tag{A.13}$$

we get the OPE of an affine Lie algebra of level κ:

$$J_z^a(z)J_z^b(w) = -\frac{\kappa}{2}g^{ab}(z-w)^{-2} + (z-w)^{-1}(-)^{(c)}f^{ab}_{\ \ c}J_z^c(w) + \cdots\,, \tag{A.14}$$

and similarly for $J_{\bar{z}}$. The Sugawara construction for the energy-momentum tensor is given by

$$T = \frac{1}{x(\kappa + \tilde{h})} \operatorname{str} J_z J_z \,, \tag{A.15}$$

and it satisfies the Virasoro algebra with the central extension given by:

$$c = \frac{\kappa(d_B - d_F)}{\kappa + \tilde{h}} \,. \tag{A.16}$$

B. $N = 2$ Super $\mathscr{W}$-Algebras from Lie Superalgebras of Rank up to 4

Table 1. $sl(m|n)$ superalgebras up to rank 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Fundamental of $\mathscr{G}$	Adjoint of $\mathscr{G}$
$sl(1\|2)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi$	$(0,1)$
$sl(1\|3)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi \oplus (0,0)^\pi$	$(0,1) \oplus (\frac{1}{2},\frac{1}{2})' \oplus (-\frac{1}{2},\frac{1}{2})' \oplus (0,0)$
$sl(2\|2)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi \oplus (0,0)$	$(0,1) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2})$
$sl(1\|4)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi \oplus 2(0,0)^\pi$	$(0,1) \oplus 2(\frac{1}{2},\frac{1}{2})' \oplus 2(-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0)$
$sl(2\|3)$	$sl(2\|3)$	$(1,1)^\pi$	$(0,2) \oplus (0,1)$
	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi \oplus (0,0) \oplus (0,0)^\pi$	$(0,1) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (\frac{1}{2},\frac{1}{2})'$ $\oplus (-\frac{1}{2},\frac{1}{2})' \oplus 2(0,0) \oplus 2(0,0)'$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)^\pi$	$(0,1) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus 4(0,0)$

Table 2. $osp(2m+1|2n)$ superalgebras of rank < 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Fundamental of $\mathscr{G}$	Adjoint of $\mathscr{G}$
$osp(1\|2n)$	$\emptyset$	–	–
$osp(3\|2)$	$osp(2\|2)$	$(0,\frac{1}{2})^\pi \oplus (0,0)$	$(0,1) \oplus (0,\frac{1}{2})$
$osp(3\|4)$	$osp(2\|2)$	$(0,\frac{1}{2})^\pi \oplus (0,0) \oplus 2(0,0)^\pi$	$(0,1) \oplus (0,\frac{1}{2}) \oplus 2(0,\frac{1}{2})'$ $\oplus 3(0,0) \oplus 2(0,0)'$
	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^\pi \oplus (-\frac{1}{2},\frac{1}{2})^\pi \oplus (0,0)$	$(0,1) \oplus (1,1) \oplus (-1,1)$ $\oplus (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (0,0)$
$osp(5\|2)$	$osp(2\|2)$	$(0,\frac{1}{2})^\pi \oplus 3(0,0)$	$(0,1) \oplus 3(0,\frac{1}{2}) \oplus 3(0,0)$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (0,0)$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2})$ $\oplus (\frac{1}{2},\frac{1}{2})' \oplus (-\frac{1}{2},\frac{1}{2})' \oplus (0,0)$

Table 3. $osp(2m+1|2n)$ superalgebras of rank 4

$\mathcal{G}$	SSA in $\mathcal{G}$	Fundamental of $\mathcal{G}$	Adjoint of $\mathcal{G}$
$osp(3\|6)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^{\pi} \oplus (-\frac{1}{2},\frac{1}{2})^{\pi} \oplus (0,0) \oplus 2(0,0)^{\pi}$	$(0,1) \oplus (1,1) \oplus (-1,1) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},\frac{1}{2})' \oplus 2(-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0) \oplus 2(0,0)'$
	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 4(0,0)^{\pi} \oplus (0,0)$	$(0,1) \oplus (0,\frac{1}{2}) \oplus 4(0,\frac{1}{2})' \oplus 10(0,0) \oplus 4(0,0)'$
$osp(5\|4)$	$2\ osp(2\|2)$	$2(0,\frac{1}{2})^{\pi} \oplus (0,0)$	$3(0,1) \oplus 2(0,\frac{1}{2}) \oplus [(0,\frac{1}{2})]^2_A$
	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^{\pi} \oplus (-\frac{1}{2},\frac{1}{2})^{\pi} \oplus 3(0,0)$	$(0,1) \oplus (1,1) \oplus (-1,1) \oplus 3(\frac{1}{2},\frac{1}{2})' \oplus 3(-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0)$
	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 3(0,0) \oplus 2(0,0)^{\pi}$	$(0,1) \oplus 3(0,\frac{1}{2}) \oplus 2(0,\frac{1}{2})' \oplus 6(0,0) \oplus 6(0,0)'$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus (0,0) \oplus 2(0,0)^{\pi}$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus (\frac{1}{2},\frac{1}{2})' \oplus (-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0) \oplus 2(0,0)'$
$osp(7\|2)$	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 5(0,0)$	$(0,1) \oplus 5(0,\frac{1}{2}) \oplus 10(0,0)$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2})^{\pi} \oplus 3(0,0)$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus 3(\frac{1}{2},\frac{1}{2})' \oplus 3(-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0)$

Table 4. $osp(2m|2n)$ superalgebras up to rank 4

$\mathcal{G}$	SSA in $\mathcal{G}$	Fundamental of $\mathcal{G}$	Adjoint of $\mathcal{G}$
$osp(4\|2)$	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 2(0,0)$	$(0,1) \oplus 2(0,\frac{1}{2}) \oplus (0,0)$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2})$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus (0,0)$
$osp(4\|4)$	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 2(0,0) \oplus 2(0,0)^{\pi}$	$(0,1) \oplus 2(0,\frac{1}{2}) \oplus 2(0,\frac{1}{2})' \oplus 4(0,0) \oplus 4(0,0)'$
	$2\,osp(2\|2)$	$2(0,\frac{1}{2})^{\pi}$	$3(0,1) \oplus [(0,\frac{1}{2})]^2_{\mathbf{A}}$
	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^{\pi} \oplus (-\frac{1}{2},\frac{1}{2})^{\pi} \oplus 2(0,0)$	$(0,1) \oplus (1,1) \oplus (-1,1) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)^{\pi}$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},\frac{1}{2}) \oplus 2(-\frac{1}{2},\frac{1}{2}) \oplus 4(0,0)$
$osp(6\|2)$	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 4(0,0)$	$(0,1) \oplus 4(0,\frac{1}{2}) \oplus 6(0,0)$
	$sl(2\|1)$	$(\frac{1}{2},\frac{1}{2}) \oplus (-\frac{1}{2},\frac{1}{2}) \oplus 2(0,0)$	$(0,1) \oplus (\frac{3}{2},\frac{1}{2}) \oplus (-\frac{3}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},\frac{1}{2})' \oplus 2(-\frac{1}{2},\frac{1}{2})' \oplus 2(0,0)$
$osp(2\|4)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^{\pi} \oplus (-\frac{1}{2},\frac{1}{2})^{\pi}$	$(0,1) \oplus (1,1) \oplus (-1,1) \oplus (0,0)$
	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 2(0,0)^{\pi}$	$(0,1) \oplus 2(0,\frac{1}{2})' \oplus 3(0,0)$
$osp(2\|6)$	$sl(1\|2)$	$(\frac{1}{2},\frac{1}{2})^{\pi} \oplus (-\frac{1}{2},\frac{1}{2})^{\pi} \oplus 2(0,0)^{\pi}$	$(0,1) \oplus (1,1) \oplus (-1,1) \oplus 2(\frac{1}{2},\frac{1}{2})' \oplus 2(-\frac{1}{2},\frac{1}{2})' \oplus 4(0,0)$
	$osp(2\|2)$	$(0,\frac{1}{2})^{\pi} \oplus 4(0,0)^{\pi}$	$(0,1) \oplus 4(0,\frac{1}{2})' \oplus 10(0,0)$

Table 5. The exceptional superalgebras

$\mathcal{G}$	SSA in $\mathcal{G}$	$sl(1\|2)$ decomposition
$G(3)$	$sl(2\|1)$	$(0,1)\oplus(\frac{5}{6},\frac{1}{2})\oplus(-\frac{5}{6},\frac{1}{2})\oplus(\frac{1}{6},\frac{1}{2})'\oplus(-\frac{1}{6},\frac{1}{2})'\oplus(\frac{1}{2},\frac{1}{2})'$ $\oplus(-\frac{1}{2},\frac{1}{2})'\oplus(0,0)$
	$sl(2\|1)'$	$(0,1)\oplus(\frac{7}{2},\frac{1}{2})\oplus(-\frac{7}{2},\frac{1}{2})\oplus(\frac{3}{2},\frac{3}{2})'\oplus(-\frac{3}{2},\frac{3}{2})'\oplus(0,0)$
	$osp(2\|2)$	$(0,1)\oplus 2(\frac{1}{4},\frac{1}{2})\oplus 2(-\frac{1}{4},\frac{1}{2})\oplus(0,\frac{1}{2})$
$F(4)$	$sl(2\|1)$	$(0,1)\oplus 3(\frac{1}{6},\frac{1}{2})\oplus 3(-\frac{1}{6},\frac{1}{2})\oplus 8(0,0)$
	$sl(1\|2)$	$(0,1)\oplus(1,\frac{1}{2})\oplus(-1,\frac{1}{2})\oplus 4(0,0)\oplus 2(\frac{1}{2},\frac{1}{2})'\oplus 2(-\frac{1}{2},\frac{1}{2})'\oplus 2(0,\frac{1}{2})'$
	$osp(2\|2)$	$(0,1)\oplus 2(1,1)\oplus 2(-1,1)\oplus(\frac{5}{2},\frac{1}{2})\oplus(-\frac{5}{2},\frac{1}{2})\oplus 4(0,0)$
$D(2,1;\alpha)$	$osp(2\|2)$	$(0,1)\oplus(\pm\frac{1}{2}(1+2\alpha),\frac{1}{2})\oplus(0,0)$
	$sl(2\|1)$	$(0,1)\oplus(\pm\frac{1}{2}(2+\alpha)/\alpha,\frac{1}{2})\oplus(0,0)$
	$sl(2\|1)'$	$(0,1)\oplus\left(\pm\frac{1}{2}\frac{1-\alpha}{1+\alpha},\frac{1}{2}\right)\oplus(0,0)$

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