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# Quaternionic Monopoles

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**Abstract:** We present the simplest non-abelian version of Seiberg–Witten theory Quaternionic monopoles These monopoles are associated with $Spin^h(4)$-structures on 4-manifolds and form finite-dimensional moduli spaces On a Kähler surface the quaternionic monopole equations decouple and lead to the projective vortex equation for holomorphic pairs This vortex equation comes from a moment map and gives rise to a new complex-geometric stability concept The moduli spaces of quaternionic monopoles on Kähler surfaces have two closed subspaces, both naturally isomorphic with moduli spaces of canonically stable holomorphic pairs These components intersect along a Donaldson instanton space and can be compactified with Seiberg–Witten moduli spaces This should provide a link between the two corresponding theories

## 0. Introduction

Recently, Seiberg and Witten [W] introduced new 4-manifold invariants, essentially by counting solutions of the monopole equations The new invariants have already found nice applications, like e g in the proof of the Thom conjecture [KM] or in a short proof of the Van de Ven conjecture [OT2] In this paper we introduce and study the simplest and the most natural non-abelian version of the Seiberg–Witten monopoles, the quaternionic monopoles

Let $(X, g)$ be an oriented Riemannian manifold of dimension 4 The structure group $SO(4)$ has as natural extension the quaternionic spinor group $Spin^h(4) = Spin(4)\times_{\mathbb{Z}_2} Sp(1)$

$$1 \to Sp(1) \to Spin^h(4) \to SO(4) \to 1$$

The projection onto the second factor $Sp(1) = SU(2)$ induces a "determinant map" $\delta \quad Spin^h(4) \to PU(2)$

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A $Spin^h(4)$-structure on $(X, g)$ consists of a $Spin^h(4)$-bundle over $X$ and an isomorphism of its $Sp(1)$-quotient with the (oriented) orthonormal frame bundle of $(X, g)$ Given a $Spin^h(4)$-structure on $X$, one has a one-one correspondence between $Spin^h$-connections projecting onto the Levi–Civita connection and $PU(2)$-connections in the associated "determinant" $PU(2)$-bundle The quaternionic monopole equations are

$$\begin{cases} \not{D}_A \Psi = 0 \\ \Gamma(F_{\hat{A}}^+) = (\Psi\bar{\Psi})_0 \end{cases},$$

where $A$ is a $PU(2)$-connection in the "determinant" of the $Spin^h(4)$-structure, $\not{D}_A$ is the induced Dirac operator, $\Gamma(F_{\hat{A}}^+)$ denotes Clifford multiplication by the self-dual part of the curvature, and $\Psi$ is a positive quaternionic half-spinor. The Dirac operator satisfies the crucial Weitzenböck formula (see Definition 2 2 for notations)

$$\not{D}_A^2 = \nabla_{\hat{A}}^* \nabla_{\hat{A}} + \Gamma(F_{\hat{A}}) + \frac{s}{4}\mathrm{id}$$

It can be used to show that the solutions of the quaternionic monopole equations are the absolute minima of a certain functional, just like in the $Spin^c(4)$-case [JPW]

The moduli space of quaternionic monopoles associated with a fixed $Spin^h(4)$-structure $\mathfrak{h}$ is a real analytic space of virtual dimension

$$m_{\mathfrak{h}} = -\frac{1}{2}(3p_1 + 3e + 4\sigma)$$

Here $p_1$ is the first Pontrjagin class of the determinant, $e$ and $\sigma$ denote the Euler characteristic and the signature of $X$ respectively

Note that $m_{\mathfrak{h}}$ is an even integer iff $X$ admits an almost complex structure

The moduli spaces of quaternionic monopoles contain the Donaldson instanton moduli spaces as well as the classical Seiberg–Witten moduli spaces, which suggests that they could provide a method of comparing the two theories (cf Sect 8) We study the analytic structure around the Donaldson moduli space

Much more can be said if the holonomy of $(X, g)$ reduces to $U(2)$, i e , if $(X, g)$ is a Kähler surface In this case we use the canonical $Spin^c(4)$-structure with $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$ and $\Sigma^- = \Lambda^{01}$ as spinor bundles The data of a $Spin^h(4)$-structure $\mathfrak{h}$ in $(X, g)$ is then equivalent to the data of a Hermitian 2-bundle $E$ with $\det E = \Lambda^{02}$ The determinant $\delta(\mathfrak{h})$ coincides with the $PU(2)$-bundle $P(E)$ associated with $E$ A positive spinor can be written as $\Psi = \varphi + \alpha$, where $\varphi \in A^0(E^\vee)$ and $\alpha \in A^{02}(E^\vee)$ are $E^\vee$-valued forms To give a $PU(2)$-connection in $P(E)$ means to give a $U(2)$-connection in $E$ inducing the Chern connection in $\Lambda^{02}$, or equivalently, a $U(2)$-connection $C$ in $E^\vee$ inducing the Chern connection in $K_X = \Lambda^{20}$ A pair $(C, \varphi + \alpha)$ solves the quaternionic monopole equation iff $C$ is a connection of type $(1,1)$, one of $\alpha$ or $\varphi$ vanishes, while the other is $\bar{\partial}_C$-holomorphic and a certain projective vortex equation is satisfied This shows that in the Kähler case the moduli space decomposes as a union of two Zariski closed subspaces intersecting along the Donaldson locus The two subspaces are interchanged by a natural real analytic involution whose fixed point set is precisely the Donaldson moduli space

The projective vortex equation comes from a moment map which corresponds to a new stability concept for pairs $(\mathcal{E}, \varphi)$ consisting of a holomorphic bundle $\mathcal{E}$ with canonical determinant $\det \mathcal{E} = \mathcal{K}_X$ and a holomorphic section $\varphi$ We call such a pair canonically stable iff either $\mathcal{E}$ is stable, or $\varphi \neq 0$ and the divisorial component $D_\varphi$ of the zero locus satisfies the inequality

$$c_1(\mathcal{O}_X(D_\varphi)^{\otimes 2} \otimes K_X^\vee) \cup [\omega_g] < 0$$

Our main result identifies the moduli spaces of irreducible quaternionic monopoles on a Kähler surface with the algebro-geometric moduli space of canonically stable pairs

In the algebraic case, moduli spaces of quaternionic monopoles can easily be computed using our main result (Theorem 7 3) and Lemma 5 5 The moduli spaces may have several components Every component contains a Zariski open subset which comes with a free $\mathbb{C}^*$-action Some components consist only of pairs $(\mathcal{E}, \varphi)$ with $\mathcal{E}$ polystable as a bundle, a component of this type can be obtained from the corresponding $\mathbb{C}^*$-space by adding a Donaldson moduli space at infinity In the other direction, the component is not compact, but has a *natural compactification* obtained by adding spaces associated with Seiberg–Witten moduli spaces The other components can be naturally compactified by using Seiberg–Witten moduli spaces in *both* directions

This compactification process (in the differential geometric context), as well as the necessary transversality results will be treated in [T], in the final section of the present work we state the main results concerning the compactification and we sketch some of the proofs

We like to point out that quaternionic monopoles are the simplest non-abelian examples of the general concept of $G$-monopoles, where $G$ is a compact Lie group with a central involution $\iota$ $G$-monopoles are associated with pairs consisting of a $Spin(4) \times_{\mathbb{Z}_2} G$ -structure on $(X, g)$ and a unitary representation $\sigma$ $G \to U(V)$ with $\sigma(\iota) = -\mathrm{id}_V$ Under a certain non-degeneracy condition on $\sigma$, the corresponding Weitzenböck formula gives an a priori $\mathcal{C}^0$-estimate for the spinor component of a $G$-monopole This estimation leads to Uhlenbeck-type results Details will appear in [T] [1]

## 1. $Spin^h$-Structures

The quaternionic spinor group is defined as

$$Spin^h = Spin \times_{\mathbb{Z}_2} Sp(1) = Spin \times_{\mathbb{Z}_2} SU(2)\,,$$

and fits in the exact sequences

$$1 \to Sp(1) \to Spin^h \xrightarrow{\pi} SO \to 1\,,$$

$$1 \to Spin \to Spin^h \xrightarrow{\delta} PU(2) \to 1 \qquad (1)$$

These can be combined in the sequence

$$1 \to \mathbb{Z}/2 \to Spin^h \xrightarrow{(\pi,\delta)} SO \times PU(2) \to 1 \qquad (2)$$

[1] After having completed our results we received a manuscript by Labastida and Marino [LM] in which related ideas are proposed from a physical point of view, and physical implications are discussed

In dimension 4, $Spin^h(4)$ has a simple description, coming from the splitting $Spin(4) = SU(2) \times SU(2)$

$$Spin^h(4) = \frac{SU(2) \times SU(2) \times SU(2)}{\mathbb{Z}/2}$$

with $\mathbb{Z}/2 = \langle(-\mathrm{id}, -\mathrm{id}, -\mathrm{id})\rangle$ There is another useful way to think of $Spin^h(4)$ let $G$ be the group

$$G := \{(a, b, c) \in U(2) \times U(2) \times U(2) \mid \det a = \det b = \det c\}$$

One has an obvious isomorphism $Spin^h(4) = G/S^1$, and a commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & SU(2) \times SU(2) \times SU(2) \arrow[r] \arrow[d] & Spin^h(4) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] & G \arrow[r] & Spin^h(4) \arrow[r] & 1 \end{tikzcd} \tag{3}$$

**Definition 1.1.** *Let $P$ be a principal $SO$-bundle over a space $X$ A $Spin^h$-structure in $P$ is a pair consisting of a $Spin^h$ bundle $P^h$ and an isomorphism $P \simeq P^h \times_\pi SO$ The $PU(2)$-bundle associated with a $Spin^h$-structure is the bundle $P^h \times_\delta PU(2)$*

**Lemma 1.2.** *A principal $SO$-bundle admits a $Spin^h$-structure iff there exists a $PU(2)$-bundle with the same second Stiefel–Whitney class*

*Proof* This follows from the cohomology sequence

$$\to H^1(X, \underline{Spin^h}) \to H^1(X, \underline{SO} \times \underline{PU(2)}) \xrightarrow{\beta} H^2(X, \mathbb{Z}/2)$$

associated to (2), since the connecting homomorphism $\beta$ is given by taking the sum of the second Stiefel–Whitney classes of the two factors. □

In this paper we will only use $Spin^h$-structures in $SO(4)$-bundles whose second Stiefel–Whitney class admit *integral* lifts Then we have

**Lemma 1.3.** *Let $P$ be a principal $SO(4)$-bundle whose second Stiefel–Whitney class $w_2(P)$ is the reduction of an integral class*

*Isomorphism classes of $Spin^h(4)$-structures in $P$ are in 1–1 correspondence with equivalence classes of triples consisting of a $Spin^c(4)$-structure $P^c/S^1 \simeq P$ in $P$, a $U(2)$-bundle $E$, and an isomorphism $\det P^c \simeq \det E$, where two triples are equivalent if they can be obtained from each other by tensoring with an $S^1$-bundle*

*Proof* The cohomology sequence associated with the second row in (3) shows that $Spin^h$-structures in bundles whose second Stiefel–Whitney classes admit integral lifts are given by $G$-structures modulo tensoring with $S^1$-bundles On the other hand, to give a $G$-structure in $P$ simply means to give a triple $(\Sigma^+, \Sigma^-, E)$ of $U(2)$-bundles together with isomorphisms

$$\det \Sigma^+ \simeq \det \Sigma^- \simeq \det E$$

This is equivalent to giving a triple consisiting of a $Spin^c(4)$-structure $P^c/S^1 \simeq P$ in $P$, a $U(2)$-bundle, and an isomorphism $\det P^c \simeq \det E$ $\square$

In the situation of this lemma, we get well defined vector bundles

$$\mathscr{H}^{\pm} = \Sigma^{\pm} \otimes E^{\vee}$$

depending only on the $Spin^h$-structure and not on the chosen $G$-lifting These spinor bundles have the following intrinsic interpretation identify $SU(2) \times_{\mathbb{Z}\,2} SU(2)$ with $SO(4)$, and denote by

$$\pi_{ij} \quad Spin^h(4) \to SO(4)$$

the projections of $Spin^h(4) = SU(2) \times SU(2) \times SU(2)/\mathbb{Z}/2$ onto the indicated factors ($\pi = \pi_{12}$) Using the inclusion $SO(4) \subset SU(4)$, we can form three $SU(4)$-vector bundles $P^h \times_{\pi_{ij}} \mathbb{C}^4$, $(i,j) \in \{(1,2),(1,3),(2,3)\}$

Under the conditions of the previous lemma we have

$$\mathscr{H}^{+} = P^h \times_{\pi_{13}} \mathbb{C}^4, \qquad \mathscr{H}^{-} = P^h \times_{\pi_{23}} \mathbb{C}^4, \qquad \Sigma^{+} \otimes (\Sigma^{-})^{\vee} = P^h \times_{\pi} \mathbb{C}^4 .$$

The $PU(2)$-bundle $P^h \times_{\delta} PU(2)$ associated with the $Spin^h$-structure $P^c/S^1 \simeq P$ has in this case a very simple description it is the projectivization $P(E)$ of the $U(2)$-bundle $E$

## 2. The Quaternionic Monopole Equations

Let $(X,g)$ be an oriented Riemannian 4-manifold with orthonormal frame bundle $P$ The exact sequence (2) in the previous section shows two things first, isomorphism classes of $PU(2)$-bundles with second Stiefel–Whitney class equal to $w_2(P)$ are in 1–1 correspondence with orbits of $Spin^h(4)$-structures in $P$ under the action of $H^1(X,\mathbb{Z}/2)$, second, $Spin^h(4)$-connections in a $Spin^h(4)$-bundle $P^h$ which induce the Levi–Civita connection in $P$ correspond bijectively to connections in the associated $PU(2)$-bundle $P^h \times_{\delta} PU(2)$

Now it is well known that $w_2(P) = w_2(X)$ is always the reduction of an integral class [HH], so that we can think of a $Spin^h$-structure in $P$ as a triple $(\Sigma^{+},\Sigma^{-},E)$ of $U(2)$-bundles with isomorphisms $\det \Sigma^{+} \simeq \det \Sigma^{-} \simeq \det E$ modulo tensoring with unitary line bundles We denote the $Spin^h(4)$-connection corresponding to a connection $A \in \mathscr{A}(P(E))$ in the associated $PU(2)$-bundle by $\hat{A}$

*Remark 2 1* Given a fixed $U(1)$-connection $c$ in $\det E$, the elements in $\mathscr{A}(P(E))$ can be identified with those $U(2)$-connections in $E$, which induce the fixed connection $c$

Now view a $Spin^h(4)$-structure in $P$ as a $Spin^c(4)$-structure $P^c/S^1 \simeq P$ together with a $U(2)$-bundle $E$ and an isomorphism $\det P^c \simeq \det E$ Recall that the choice of $P^c/S^1 \simeq P$ induces an isomorphism

$$\gamma \quad \Lambda^1 \otimes \mathbb{C} \to (\Sigma^{+})^{\vee} \otimes \Sigma^{-} ,$$

which extends to a homomorphism

$$\Lambda^1 \otimes \mathbb{C} \to \mathrm{End}_0(\Sigma^{+} \oplus \Sigma^{-}) ,$$

mapping the bundle $\Lambda^1$ of real 1-forms into the bundle of trace-free skew-Hermitian endomorphisms The induced homomorphism

$$\Gamma \quad \Lambda^2 \otimes \mathbb{C} \to \mathrm{End}_0(\Sigma^+ \oplus \Sigma^-)$$

maps the subbundles $\Lambda^2_\pm \otimes \mathbb{C}$ isomorphically onto the bundles $\mathrm{End}_0(\Sigma^\pm)$, and identifies $\Lambda_\pm$ with the trace-free, skew-Hermitian endomorphisms ([H, OT1])

**Definition 2.2.** *Let $P^h \times_\pi SO(4) \simeq P$ be a $Spin^h(4)$-structure in $P$ with spinor bundle $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$ and associated $PU(2)$-bundle $P(E)$ Choose a connection $A \in \mathscr{A}(P(E))$, and let $\hat{A}$ be the corresponding $Spin^h(4)$-connection in $P^h$ The associated Dirac operator is defined as the composition*

$$\not{D}_A : A^0(\mathscr{H}) \xrightarrow{\nabla_{\hat{A}}} A^1(\mathscr{H}) \xrightarrow{\gamma} A^0(\mathscr{H}) ,$$

*where $\nabla_{\hat{A}}$ is the covariant derivative of $\hat{A}$ and $\gamma$ the Clifford multiplication*

Note that the restricted operators

$$\not{D}_A \quad A^0(\mathscr{H}^\pm) \to A^0(\mathscr{H}^\mp)$$

interchange the positive and negative half-spinors

Let $s$ be the scalar curvature of $(X, g)$

**Proposition 2.3.** *The Dirac operator $\not{D}_A \quad A^0(\mathscr{H}) \to A^0(\mathscr{H})$ is an elliptic, selfadjoint operator whose Laplacian satisfies the Weitzenböck formula*

$$\not{D}_A^2 = \nabla_{\hat{A}}^* \nabla_{\hat{A}} + \Gamma(F_A) + \frac{s}{4}\mathrm{id}_{\mathscr{H}} \tag{4}$$

*Proof* Choose a $Spin^c(4)$-structure $P^c/S^1 \simeq P$ and a $S^1$-connection $c$ in the unitary line bundle $\det P^c$ The connection $A \in \mathscr{A}(P(E))$ lifts to a unique $U(2)$-connection $C$ in the bundle $E^\vee$ which induces the dual connection of $c$ in $\det E^\vee = (\det P^c)^\vee$ In [OT1] we introduced the Dirac operator

$$\not{D}_{C,c} \quad A^0(\Sigma \otimes E^\vee) \to A^0(\Sigma \otimes E^\vee) ,$$

by construction it coincides with the operator $\not{D}_A \quad A^0(\mathscr{H}) \to A^0(\mathscr{H})$, and its Weitzenböck formula reads

$$\not{D}_{C,c}^2 = \nabla_{\hat{A}}^* \nabla_{\hat{A}} + \Gamma(F_{C,c}) + \frac{s}{4}\mathrm{id}_{\mathscr{H}} ,$$

where $F_{C,c} = F_C + \frac{1}{2}F_c \mathrm{id}_{E^\vee} \in A^2(\mathrm{End}\, E^\vee)$ Substituting

$$F_C = \frac{1}{2}\mathrm{Tr}(F_C)\mathrm{id}_{E^\vee} + F_A$$

and using $\frac{1}{2}\mathrm{Tr}(F_C) = -\frac{1}{2}F_c$ we get the Weitzenböck formula (4) for $\not{D}_A$ $\square$

Consider now a section $\Psi \in A^0(\mathscr{H}^{\pm})$ We denote by

$$(\Psi\bar{\Psi})_0 \in A^0(\mathrm{End}_0\, \Sigma^{\pm} \otimes \mathrm{End}_0\, E^{\vee})$$

the projection of $\Psi \otimes \bar{\Psi} \in A^0(\mathrm{End}\, \mathscr{H}^{\pm})$ onto the fourth summand in the decomposition

$$\mathrm{End}(\mathscr{H}^{\pm}) = \mathbb{C}\mathrm{id} \oplus \mathrm{End}_0\, \Sigma^{\pm} \otimes \mathrm{End}_0\, E^{\vee} \otimes (\mathrm{End}_0\, \Sigma^{\pm} \otimes \mathrm{End}_0\, E^{\vee}) .$$

$(\Psi\bar{\Psi})_0$ is a Hermitian endomorphism which is trace-free in both factors

**Definition 2.4.** *Choose a $Spin^h(4)$-structure in $P$ with spinor bundle $\mathscr{H}$ and associated $PU(2)$-bundle $P(E)$ The quaternionic monopole equations for a pair $(A, \Psi) \in \mathscr{A}(P(E)) \times A^0(\mathscr{H}^+)$ are*

$$\begin{cases} \not{D}_A\Psi &= 0 \\ \Gamma(F_A^+) &= (\Psi\bar{\Psi})_0 \end{cases} \tag{$SW^{\natural}$}$$

The following result is the analog of Witten's formula in the quaternionic case (see [W], Sect 3)

**Proposition 2.5.** *Let $\Psi \in A^0(\mathscr{H}^+)$ be a positive half-spinor, $A \in \mathscr{A}(P(E))$ a connection in $P(E)$ Then we have*

$$\|\not{D}_A\Psi\|^2 + \frac{1}{2}\|\Gamma(F_A^+) - (\Psi\bar{\Psi})_0\|^2 = \|\nabla_{\hat{A}}\Psi\|^2 + \frac{1}{2}\|F_A^+\|^2$$
$$+\frac{1}{2}\|(\Psi\bar{\Psi})_0\|^2 + \frac{1}{4}\int_X s|\Psi|^2 \tag{5}$$

*Proof* The pointwise inner product $(\Gamma(F_A)\Psi, \Psi)$ for a positive half-spinor $\Psi$ simplifies $(\Gamma(F_A)\Psi, \Psi) = (\Gamma(F_A^+)\Psi, \Psi) = (\Gamma(F_A^+), (\Psi\bar{\Psi})_0)$, since $\Gamma(F_A^-)$ vanishes on $A^0(\mathscr{H}^+)$, and since $\Gamma(F_A^+)$ is trace-free in both arguments.

Using the Weitzenböck formula (5), we find

$$(\not{D}_A\Psi, \Psi) = (\nabla_{\hat{A}}^*\nabla_{\hat{A}}\Psi, \Psi) + (\Gamma(F_A^+), (\Psi\bar{\Psi})_0) + \frac{s}{4}|\Psi|^2 , \tag{6}$$

which shows that

$$(\not{D}_A^2\Psi, \Psi) + \frac{1}{2}|\Gamma(F_A^+) - (\Psi\bar{\Psi})_0|^2 = (\nabla_{\hat{A}}^*\nabla_{\hat{A}}\Psi, \Psi) + \frac{1}{2}|F_A^+|^2 + \frac{1}{2}|(\Psi\bar{\Psi})_0|^2 + \frac{s}{4}|\Psi|^2$$

The identity (5) follows by integration over $X$

## 3. Moduli Spaces of Quaternionic Monopoles

We denote again by $P$ the orthonormal frame bundle of the oriented Riemannian 4-manifold $(X, g)$. Let $E$ be a $U(2)$-bundle with $w_2(P) \equiv \overline{c_1(E)} (\mathrm{mod}\, 2)$ and $\mathfrak{c}$ a $Spin^c(4)$-structure in $P$ with determinant $L := \det E$ Endow $(X, g)$ with the corresponding $Spin^h(4)$-structure (Lemma 1 3)

We fix an $S^1$-connection $c$ in $\det E^\vee$, and identify $\mathscr{A}(P(E))$ with the space $\mathscr{A}_c(E^\vee)$ of $U(2)$-connections in $E^\vee$ which induce the connection $c$ in $\det E^\vee$ Put

$$\mathscr{A} = \mathscr{A}_c(E^\vee) \times A^0(\mathscr{H}^+)$$

The natural gauge group is the group $\mathscr{G}$ consisting of unitary automorphisms of $E^\vee$ which induce the identity in $\det E^\vee$ $\mathscr{G}$ acts on $\mathscr{A}$ from the right in a natural way Let $\mathscr{A}^* \subset \mathscr{A}$ be the open subset of $\mathscr{A}$ consisting of pairs $(C, \Psi)$ whose stabilizer $\mathscr{G}_{(C,\Psi)}$ is contained in the center $\mathbb{Z}/2 = \{\pm \mathrm{id}_{E^\vee}\}$ of the gauge group.

*Remark 3 1* A pair $(C, \Psi)$ does not belong to $\mathscr{A}^*$ iff $C$ is a reducible connection and $\Psi = 0$

Indeed, the isotropy group of $\mathscr{G}$ acting only on the first factor $\mathscr{A}_c(E^\vee)$ is the centralizer of the holonomy of $C$ in $SU(2)$ The latter is $S^1$ or $SU(2)$ if $C$ is reducible, and $\mathbb{Z}/2$ in the irreducible case □

A pair belonging to $\mathscr{A}^*$ will be called irreducible Note that the stabilizer of *any* pair with vanishing second component $\Psi$ contains $\mathbb{Z}/2$.

From now on we also assume that $\mathscr{A}$ and $\mathscr{G}$ are completed with respect to suitable Sobolev norms $L^2_k$, such that $\mathscr{G}$ becomes a Hilbert Lie group acting smoothly on $\mathscr{A}$ Let $\mathscr{B} := \mathscr{A}/\mathscr{G}$, $\mathscr{B}^* := \mathscr{A}^*/\mathscr{G}$ be the indicated quotients, and denote the orbit-map $[\ ]\ \mathscr{A} \to \mathscr{B}$ by $\pi$

An element in $(A, \Psi) \in \mathscr{A}$ will be called *strongly irreducible* if its stabilizer is trivial or, equivalently, if $\Psi \neq 0$ Let $\mathscr{A}^{**} \subset \mathscr{A}^*$ be the subset of strongly irreducible pairs, and put $\mathscr{B}^{**} = \mathscr{A}^{**}/\mathscr{G}$

**Proposition 3.2.** *$\mathscr{B}$ is a Hausdorff space $\mathscr{B}^{**} \subset \mathscr{B}$ is open and has the structure of a differentiable Hilbert manifold The map $\mathscr{A}^{**} \to \mathscr{B}^{**}$ is a differentiable principal $\mathscr{G}$-bundle*

*Proof* Standard, cf [DK, FU]

Fix a point $p = (C, \Psi) \in \mathscr{A}$. The differential of the map $\mathscr{G} \to \mathscr{A}$ given by the action of $\mathscr{G}$ on $p$ is the map

$$\begin{aligned} D^0_p : A^0(su(E^\vee)) &\to A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \\ f &\mapsto (D_C(f), -f\Psi) \end{aligned}$$

Setting

$$N_p(\varepsilon) = \{\beta \in A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \mid D^{0\,*}_p \beta = 0,\ ||\beta|| < \varepsilon\},$$

for $\varepsilon > 0$ sufficiently small, one obtains local slices for the action of $\mathscr{G}$ on $\mathscr{A}^{**}$ and charts $\pi|_{N_p(\varepsilon)} : N_p(\varepsilon) \to \mathscr{B}^{**}$ for $\mathscr{B}^{**}$ □

Note that the curvature $F_A$ of a connection in $P(E)$ equals the trace-free part $F^0_C$ of the curvature of the corresponding connection $C \in \mathscr{A}_c(E^\vee)$

Using the identification $\mathscr{A}(P(E)) = \mathscr{A}_c(E^\vee)$, we can rewrite the quaternionic monopole equations in terms of pairs $(C, \Psi) \in \mathscr{A}$. Let $\mathscr{A}^{SW^{\mathfrak{h}}} \subset \mathscr{A}$ be the space of solutions

**Definition 3.3.** *Fix a $Spin^h$-structure in $P$ The moduli space of quaternionic monopoles is the quotient $\mathscr{M} = \mathscr{A}^{SW^{\mathfrak{h}}}/\mathscr{G}$ We denote by $\mathscr{M}^{**} = (\mathscr{A}^{SW^{\mathfrak{h}}} \cap \mathscr{A}^{**})/\mathscr{G}$, $\mathscr{M}^{*} = (\mathscr{A}^{SW^{\mathfrak{h}}} \cap \mathscr{A}^{*})/\mathscr{G}$ the subspaces of (strongly) irreducible monopoles*

The tangent space to $\mathscr{A}^{SW^{\mathfrak{h}}}$ at $p = (C, \Psi) \in \mathscr{A}$ is the kernel of the operator

$$D_p^1 \quad A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \to A^0(su(\Sigma^+) \otimes su(E^\vee)) \oplus A^0(\Sigma^- \otimes E^\vee)$$

defined by

$$D_p^1((\alpha, \psi)) = (\Gamma(D_C^+(\alpha)) - [(\psi\bar{\Psi})_0 + (\Psi\bar{\psi})_0],\ \not{D}_{C,c}\psi + \gamma(\alpha)\Psi)\,,$$

where we consider $\gamma(\alpha)$ as map $\gamma(\alpha) \quad \Sigma^+ \to \Sigma^- \otimes su(E^\vee)$ Clearly $D_p^1 \circ D_p^0 = 0$, since the monopole equations are gauge invariant

Using the isomorphism $\Gamma^{-1} \quad A^0(su(\Sigma^+)) \to A_+^2$, we can consider $D_p^1$ as an operator $D_p^1 \quad A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \to A_+^2(su(E^\vee)) \oplus A^0(\Sigma^- \otimes E^\vee)$

Let $\sigma(X)$ and $e(X)$ be the signature and the topological Euler characteristic of the oriented manifold $X$

**Proposition 3.4.** *For a solution $p = (C, \Psi) \in \mathscr{A}^{SW^{\mathfrak{h}}}$, the complex*

$$0 \to A^0 su(E^\vee) \xrightarrow{D_p^0} A^1 su(E^\vee) \oplus A^0\mathscr{H}^+ \xrightarrow{D_p^1} A_+^2 su(E^\vee) \oplus A^0\mathscr{H}^- \to 0 \qquad (\mathscr{C}_p)$$

*is elliptic and its index is*

$$\frac{3}{2}(4c_2(E^\vee) - c_1(E^\vee)^2) - \frac{1}{2}(3e(X) + 4\sigma(X)) \qquad (7)$$

*Proof* The complex $\mathscr{C}_p$ has the same symbol sequence as

$$0 \to A^0 su(E^\vee) \xrightarrow{(D_C,0)} A^1 su(E^\vee) \oplus A^0\mathscr{H}^+ \xrightarrow{(D_C^+,\ \not{D}_{C,c})} A_+^2 su(E^\vee) \oplus A^0\mathscr{H}^- \to 0\,,$$

which is an elliptic complex with index

$$2(4c_2(E^\vee) - c_1(E^\vee)^2) - \frac{3}{2}(\sigma(X) + e(X)) + index\ \not{D}_{C,c}\,.$$

The latter term is

$$index\ \not{D}_{C,c} = 2[ch(E^\vee)e^{\frac{1}{2}c_1(E)}\hat{A}(X)]_4 = -2c_2(E^\vee) + \frac{1}{2}c_1(E^\vee)^2 - \frac{1}{2}\sigma(X)\,. \qquad \square$$

*Remark 3 5* The integer in (7) is always an even number if $X$ admits almost complex structures

Our next step is to endow the spaces $\mathcal{M}^{**}$ ($\mathcal{M}^{*}$) with the structure of a real analytic space (orbifold).

In the first case (compare with [FU, DK, OT1, LT]), we have an analytic map $\sigma: \mathcal{A} \to A^2_+(su(E^\vee)) \oplus A^0(\mathcal{H}^-)$ defined by

$$\sigma(C, \Psi) = ((F^0_C)^+ - \Gamma^{-1}(\Psi\bar{\Psi})_0,\ \not{D}_{C,c}\Psi),$$

which gives rise to a section $\tilde{\sigma}$ in the bundle $\mathcal{A}^{**} \times_{\mathcal{G}} (A^2_+(su(E^\vee)) \oplus A^0(\mathcal{H}^-))$. We endow $\mathcal{M}^{**}$ with a real analytic structure by identifying it with the vanishing locus $Z(\tilde{\sigma})$ of $\tilde{\sigma}$, regarded as a subspace of the Hilbert manifold $\mathcal{B}^{**}$ (in Douady's sense) ([M, LT]).

Now fix a point $p = (C, \Psi) \in \mathcal{A}^*$. We put

$$S_p(\varepsilon) = \{p+\beta \mid \beta \in A^1 su(E^\vee) \oplus A^0\mathcal{H}^+,\ D^0_p D^{0^*}_p \beta + D^{1^*}_p \sigma(p+\beta) = 0,\ ||\beta|| < \varepsilon\}$$

**Claim 3.6.** *For sufficiently small $\varepsilon > 0$, $S_p(\varepsilon)$ is a finite dimensional submanifold of $\mathcal{A}$ which is contained in the slice $N_p(\varepsilon)$ and whose tangent space at $p$ is the first harmonic space $\mathbb{H}^1_p$ of the deformation complex $\mathcal{C}_p$.*

To prove this claim, we consider the map

$$s_p: A^1(su(E^\vee)) \oplus A^0(\mathcal{H}^+) \to \mathrm{im}(D^0_p) \oplus \mathrm{im}(D^1_p)^*$$

given by the left-hand terms in the equations defining $S_p(\varepsilon)$. The derivative of $s_p$ at 0 is the first Laplacian

$$\Delta^1_p: A^1(su(E^\vee)) \oplus A^0(\mathcal{H}^+) \to \mathrm{im}(D^0_p) \oplus \mathrm{im}(D^1_p)^*$$

associated with the elliptic complex $\mathcal{C}_p$, hence $s_p$ is a submersion in 0. This proves the claim. □

The intersection $\mathcal{A}^{SW^b} \cap N_p(\varepsilon) = Z(\sigma) \cap N_p(\varepsilon)$ of the space of solutions with the standard slice through $p$ is contained in $S_p(\varepsilon)$ and can be identified with the finite dimensional model

$$Z(\sigma) \cap N_p(\varepsilon) = Z(\sigma|_{S_p(\varepsilon)}).$$

If $p \in \mathcal{A}^{**}$ is strongly irreducible, then the map

$$\pi|_{Z(\sigma|_{S_p(\varepsilon)})}: Z(\sigma|_{S_p(\varepsilon)}) \to \mathcal{M}^{**}$$

is a local parametrization of $\mathcal{M}^*$ at $p$, hence $Z(\sigma|_{S_p(\varepsilon)})$ is a local model for the moduli space around $p$.

If $p \in \mathcal{A}^* \setminus \mathcal{A}^{**}$ is irreducible but not strongly irreducible, then necessarily $\Psi = 0$, and the isotropy group $\mathcal{G}_p = \mathbb{Z}/2$ acts on $S_p(\varepsilon)$. Since $\sigma$ is $\mathbb{Z}/2$-equivariant, we obtain an induced action on $Z(\sigma|_{S_p(\varepsilon)})$. In this case $\pi|_{Z(\sigma|_{S_p(\varepsilon)})}$ induces a homeomorphism of the quotient $Z(\sigma|_{S_p(\varepsilon)})/\mathbb{Z}/2$ with an open neighbourhood of $p$ in $\mathcal{M}^*$, and $\mathcal{M}^*$ becomes an orbifold at $p$, if we use the map

$$\pi|_{Z(\sigma|_{S_p})}: Z(\sigma|_{S_p(\varepsilon)}) \to \mathcal{M}^*$$

as an orbifold chart.

*Remark 3.7.* Using a real analytic isomorphism which identifies the germ of $S_p(\varepsilon)$ at $p$ with the germ of $\mathbb{H}^1_p = T_p(S_p(\varepsilon))$ at 0, we obtain a local model of Kuranishi-type for $\mathcal{M}^*$ at $p$.

*Remark 3.8.* The points in $\mathcal{D}^* := \mathcal{M}^* \setminus \mathcal{M}^{**}$ have the form $[(C,0)]$, where $C$ is irreducible and projectively anti-self-dual, i.e., $(F^0_C)^+ = 0$. There is a natural finite map

$$\mathcal{D}^* \longrightarrow \mathcal{M}(P(E^\vee))$$

into the Donaldson moduli space of $PU(2)$-instantons in $P(E^\vee)$, which maps $\mathcal{D}^*$ isomorphically onto $\mathcal{M}(P(E^\vee))^*$ if $H^1(X,\mathbb{Z}/2) = 0$. In general $\mathcal{D}^*$ and $\mathcal{M}(P(E^\vee))^*$ cannot be identified. The difference comes from the fact that our gauge group is $SU(E^\vee)$, whereas the $PU(2)$-instantons are classified modulo $PU(E^\vee)$.

For simplicity we shall however refer to $\mathcal{D}^*$ as Donaldson instanton moduli space.

Concluding, we get

**Proposition 3.9.** *$\mathcal{M}^{**}$ is a real analytic space. $\mathcal{M}^*$ is a real analytic orbifold, and the points in $\mathcal{M}^* \setminus \mathcal{M}^{**}$ have neighbourhoods modeled on $\mathbb{Z}/2$-quotients. $\mathcal{M}^* \setminus \mathcal{M}^{**}$ can be identified as a set with the Donaldson moduli space $\mathcal{D}^*$ of irreducible projectively anti-self-dual connections in $E^\vee$ with fixed determinant $c$.*

The local structure of the moduli space $\mathcal{M}$ in reducible points, which correspond to pairs formed by a reducible instanton and a trivial spinor, can also be described using the method above (compare with [DK]).

Let $\mathcal{M}^{SW} \subset \mathcal{M}$ be the subspace of $\mathcal{M}$ consisting of all orbits of the form $(C,\Psi)\cdot SU(E^\vee)$, where $C$ is a reducible connection and $\Psi$ belongs to one of the summands. Let $L := \det \Sigma^\pm = \det E$. It is easy to see that

$$\mathcal{M}^{SW} \simeq \bigcup_{\substack{S \text{ summand} \\ \text{of } E^\vee}} \mathcal{M}^{SW}_{L\otimes S^{\otimes 2}},$$

where $\mathcal{M}^{SW}_M$ denotes the rank-1 Seiberg–Witten moduli space associated to a $Spin^c(4)$-structure of determinant $M$.

The fact that the moduli spaces of quaternionic monopoles contain Donaldson moduli spaces as well as Seiberg–Witten moduli spaces suggests that they can be used for comparing the invariants given by the two theories.

## 4. Quaternionic Monopoles on Kähler Surfaces

Let $(X,g)$ be a Kähler surface with canonical $Spin^c(4)$-structure, in this case $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$, and $\Sigma^- = \Lambda^{01}$. A $Spin^h(4)$-structure in the frame bundle is given by a unitary vector bundle $E$ together with an isomorphism $\det E \simeq \Lambda^{02}$. A $Spin^h(4)$-connection $\hat{A}$ corresponds to a $PU(2)$-connection $A$ in the associated bundle $P(E)$, or alternatively, to a unitary connection $C$ in $E^\vee$ which induces a fixed $S^1$-connection $c$ in $\Lambda^{20}$. Recall that the curvature $F_A$ of $A$ equals the trace-free component $F^0_C$ of $F_C$.

If we choose $c$ to be the Chern connection in the canonical bundle $\Lambda^{20} = K_X$, then the $Spin^h(4)$-connection in $\mathcal{H} = \Sigma \otimes E^\vee$ is simply the tensor product of the canonical connection in $\Sigma = \Sigma^+ \oplus \Sigma^-$ and the connection $C$.

A positive quaternionic spinor $\Psi \in A^0(\mathscr{H}^+)$ can be written as $\Psi = \varphi + \alpha$, with $\varphi \in A^0(E^\vee)$, and $\alpha \in A^{02}(E^\vee)$

**Proposition 4.1.** *Let $C$ be a unitary connection in $E^\vee$ inducing the Chern connection $c$ in $\det E^\vee = K_X$ A pair $(C, \varphi + \alpha)$ solves the quaternionic monopole equations if and only if $F_C$ is of type $(1,1)$ and one of the following conditions holds.*

$$1\quad \alpha = 0, \quad \bar{\partial}_C \varphi = 0 \quad and \quad i\Lambda_g F_C^0 + \frac{1}{2}(\varphi \otimes \bar{\varphi})_0 = 0\,,$$

$$2\quad \varphi = 0, \quad \partial_C \alpha = 0 \quad and \quad i\Lambda_g F_C^0 - \frac{1}{2} * (\alpha \otimes \bar{\alpha})_0 = 0 \tag{8}$$

*Proof* Using the notation in the proof of the Weitzenböck formula, we have $F_{C,c} = \frac{1}{2}(\operatorname{Tr} F_C + F_c)\mathrm{id}_{E^\vee} + F_A = F_A = F_C^0 \in A^2(su(E^\vee))$ By Proposition 2 6 of [OT1] the quaternionic Seiberg–Witten equations become

$$\begin{cases} F_A^{20} &= -\frac{1}{2}(\varphi \otimes \bar{\alpha})_0 \\ F_A^{02} &= \frac{1}{2}(\alpha \otimes \bar{\varphi})_0 \\ i\Lambda_g F_A &= -\frac{1}{2}\left[(\varphi \otimes \bar{\varphi})_0 - *(\alpha \otimes \bar{\alpha})_0\right] \\ \bar{\partial}_C \varphi &= i\Lambda_g \partial_C \alpha \end{cases}$$

Note that the right-hand side of formula (5) is invariant under Witten's transformation $(C, \varphi + \alpha) \mapsto (C, \varphi - \alpha)$ Therefore, every solution satisfies $F_A^{20} = F_A^{02} = 0$, and $(\varphi \otimes \bar{\alpha})_0 = (\alpha \otimes \bar{\varphi})_0 = 0$ Elementary computations show that this can happen only if $\varphi = 0$ or $\alpha = 0$. On the other hand, since the Chern connection in $K_X$ is integrable, we also get $F_C^{20} = F_C^{02} = 0$ □

*Remark 4 2* The second case in this proposition reduces to the first· in fact, if $\varphi = 0$ and $\alpha \in A^{02}(E^\vee)$ satisfies $i\Lambda_g \partial \alpha = 0$, we set $\psi = \bar{\alpha} \in A^{20}(\bar{E}^\vee) = A^0(\Lambda^{20} \otimes E) = A^0(E^\vee)$, and we obtain $\bar{\partial}_C \psi = \overline{\partial_C \bar{\psi}} = \overline{\partial_C \alpha} = 0$ Here we used the fact that $\Lambda_g \;\; \Lambda^{12} \to \Lambda^{01}$ is an isomorphism, the adjoint of the Lefschetz isomorphism $\wedge \omega_g$ [LT]. A simple calculation in coordinates gives $- * (\alpha \otimes \bar{\alpha})_0 = (\bar{\alpha} \otimes \bar{\bar{\alpha}})_0 = (\psi \otimes \bar{\psi})_0$

## 5. Stability

Let $(X, g)$ be a compact Kähler manifold of arbitrary dimension, $E$ a differentiable vector bundle, and let $\mathscr{L}$ be a fixed holomorphic line bundle, whose underlying differentiable line bundle is $L = \det E$

**Definition 5.1.** *A holomorphic pair of type $(E, \mathscr{L})$ is a pair $(\mathscr{E}, \varphi)$ consisting of a holomorphic bundle $\mathscr{E}$ and a section $\varphi \in H^0(X, \mathscr{E})$ such that the underlying differentiable bundle of $\mathscr{E}$ is $E$ and $\det \mathscr{E} = \mathscr{L}$*

Note that the determinant of the holomorphic bundle $\mathscr{E}$ is fixed, not only its isomorphism type

Two pairs $(\mathscr{E}_i, \varphi_i)$, $i = 1, 2$ of the same type are isomorphic if there exists an isomorphism $f \;\; \mathscr{E}_1 \to \mathscr{E}_2$ with $f^*(\varphi_2) = \varphi_1$ and $\det f = \mathrm{id}_{\mathscr{L}}$

In other words, $(\mathscr{E}_i, \varphi_i)$ are isomorphic iff there exists a complex gauge transformation $f \in SL(E)$ with $f^*(\varphi_2) = \varphi_1$ such that $f$ is holomorphic as a map $f \;\; \mathscr{E}_1 \to \mathscr{E}_2$

**Definition 5.2.** *A holomorphic pair $(\mathcal{E},\varphi)$ is simple if any automorphism of it is of the form $f=\varepsilon\,\mathrm{id}_{\mathcal{E}}$, where $\varepsilon^{\mathrm{rk}\,\mathcal{E}}=1$ A pair $(\mathcal{E},\varphi)$ is strongly simple if its only automorphism is* $\mathrm{id}_{\mathcal{E}}$

Note that a simple pair $(\mathcal{E},\varphi)$ with $\varphi\neq 0$ is strongly simple, whereas a pair $(\mathcal{E},0)$ is simple iff $\mathcal{E}$ is a simple bundle

Note also that $(\mathcal{E},\varphi)$ is simple iff any trace-free holomorphic endomorphism $f$ of $\mathcal{E}$ with $f(\varphi)=0$ vanishes

For a nontrivial torsion free sheaf $\mathcal{F}$ on $X$, we denote by $\mu_g(\mathcal{F})$ its slope with respect to the Kähler metric $g$ Given a holomorphic bundle $\mathcal{E}$ over $X$ and a holomorphic section $\varphi\in H^0(X,\mathcal{E})$, we let $\mathcal{S}(\mathcal{E})$ be the set of reflexive subsheaves $\mathcal{F}\subset\mathcal{E}$ with $0<\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{E})$, and we define

$$\mathcal{S}_\varphi(\mathcal{E})=\{\mathcal{F}\in\mathcal{S}(\mathcal{E})\,|\,\varphi\in H^0(X,\mathcal{F})\}$$

Recall the following stability concepts [B2]

**Definition 5.3.**

1 *$\mathcal{E}$ is $\varphi$-stable if*

$$\max\Big(\mu_g(\mathcal{E}),\sup_{\mathcal{F}'\in\mathcal{S}(\mathcal{E})}\mu_g(\mathcal{F}')\Big)<\inf_{\mathcal{F}\in\mathcal{S}_\varphi(\mathcal{E})}\mu_g(\mathcal{E}/\mathcal{F})$$

2 *Let $\lambda\in\mathbb{R}$ be a real parameter The pair $(\mathcal{E},\varphi)$ is $\lambda$-stable iff*

$$\max\Big(\mu_g(\mathcal{E}),\sup_{\mathcal{F}'\in\mathcal{S}(\mathcal{E})}\mu_g(\mathcal{F}')\Big)<\lambda<\inf_{\mathcal{F}\in\mathcal{S}_\varphi(\mathcal{E})}\mu_g(\mathcal{E}/\mathcal{F})\,.$$

3 *$(\mathcal{E},\varphi)$ is called $\lambda$-polystable if $\mathcal{E}$ splits holomorphically as $\mathcal{E}=\mathcal{E}'\oplus\mathcal{E}''$, such that $\varphi\in H^0(X,\mathcal{E}')$, $(\mathcal{E}',\varphi)$ is a $\lambda$-stable pair, and $\mathcal{E}''$ is a polystable vector bundle of slope $\lambda$*

From now on we restrict ourselves to the case $\mathrm{rk}(\mathcal{E})=2$.

**Definition 5.4.**

1 *A holomorphic pair $(\mathcal{E},\varphi)$ of type $(E,\mathcal{L})$ is called stable if one of the following conditions is satisfied·*

i) *$\mathcal{E}$ is $\varphi$-stable*

ii) *$\varphi\neq 0$ and $\mathcal{E}$ splits in direct sum of line bundles $\mathcal{E}=\mathcal{E}'\oplus\mathcal{E}''$, such that $\varphi\in H^0(\mathcal{E}')$ and the pair $(\mathcal{E}',\varphi)$ is $\mu_g(E)$-stable*

2. *A holomorphic pair $(\mathcal{E},\varphi)$ of type $(E,\mathcal{L})$ is called polystable if it is stable, or $\varphi=0$ and $\mathcal{E}$ is a polystable bundle*

Note that there is *no* parameter $\lambda$ in the stability concept for holomorphic pairs of a fixed type The conditions depend only on the metric $g$ and on the slope $\mu_g(E)$ of the underlying differentiable bundle $E$

**Lemma 5.5.** *Let $(\mathscr{E},\varphi)$ be a holomorphic pair of type $(E,\mathscr{L})$ with $\varphi \neq 0$ There exists a uniquely determined effective divisor $D = D_\varphi$ and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(D) \arrow[r, "\hat{\varphi}"] & \mathscr{E} \arrow[r] & \mathscr{L}(-D)\otimes J_Z \arrow[r] & 0\,, \\ & \mathcal{O}_X \arrow[u, "D"] \arrow[ur, "\varphi"'] & & & \end{tikzcd} \tag{9}$$

*with a local complete intersection $Z \subset X$ of codimension 2 The pair $(\mathscr{E},\varphi)$ is stable if and only if $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$*

*Proof* $D = D_\varphi$ is the divisorial component of the zero locus $Z(\varphi)$ of $\varphi$ which is defined by the ideal $\mathrm{im}(\varphi^\vee\ \mathscr{E}^\vee \to \mathcal{O}_X)$, $\hat{\varphi}$ is the induced map and $Z := Z(\hat{\varphi})$ The set $\mathscr{S}_\varphi(\mathscr{E})$ consists precisely of the line bundles $\mathscr{F} \subset \mathcal{O}_X(D)$, so that

$$\inf_{\mathscr{F}\in\mathscr{S}_\varphi(\mathscr{E})} \mu_g(\mathscr{E}/\mathscr{F}) = 2\mu_g(E) - \mu_g(\mathcal{O}_X(D))\,.$$

Suppose $(\mathscr{E},\varphi)$ is stable If $\mathscr{E}$ is $\varphi$-stable, we have $\mu_g(E) < 2\mu_g(E) - \mu_g(\mathcal{O}_X(D))$, which gives the required inequality If $\mathscr{E}$ is not $\varphi$-stable, then $Z = \emptyset$, the extension (9) splits, and the pair $(\mathcal{O}_X(D),\varphi)$ is $\mu_g(E)$-stable, i e $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$

Conversely, suppose $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$, and assume first that the extension (9) does not split In this case $\mathscr{E}$ is $\varphi$-stable in fact, if $\mathscr{F}' \subset \mathscr{E}$ is an arbitrary line bundle, either $\mathscr{F}' \subset \mathcal{O}_X(D)$, or the induced map $\mathscr{F}' \subset \mathscr{E} \to \mathscr{J}_Z \otimes \mathscr{L}(-D)$ is non-trivial But then $\mathscr{F}' \simeq \mathscr{L}\otimes\mathcal{O}_X(-D-\Delta)$ for an effective divisor $\Delta$ containing $Z$, and we find

$$\mu_g(\mathscr{F}') = 2\mu_g(E) - \mu_g(D) - \mu_g(\Delta) \leqq 2\mu_g(E) - \mu_g(\mathcal{O}_X(D))$$

Furthermore, strict inequality holds, unless $Z = \emptyset$ and the extension (9) splits, which it does not by assumption.

In the case of a split extension, we only have to notice that a pair $(\mathscr{E}',\varphi)$ is $\lambda$-stable for any parameter $\lambda > \mu_g(\mathscr{E}')$ [B1] $\square$

*Remark 5 6* Consider a pair $(\mathscr{E},\varphi)$ of type $(E,\mathscr{L})$ with $\varphi \neq 0$ and associated extension (9) $\mathscr{E}$ is $\varphi$-stable iff $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$, and the extension does not split

Indeed, if the extension splits, then $\mathscr{E}$ is not $\varphi$-stable, since

$$\mu_g(\mathscr{L}(-D)) = \inf_{\mathscr{F}\in\mathscr{S}_\varphi(\mathscr{E})} \mu_g(\mathscr{E}/\mathscr{F})\,.$$

## 6. The Projective Vortex Equation

Let $E$ be a differentiable vector bundle over a compact Kähler manifold $(X,g)$ We fix a holomorphic line bundle $\mathscr{L}$ and a Hermitian metric $l$ in $\mathscr{L}$ Let $(\mathscr{E},\varphi)$ be a holomorphic pair of type $(E,\mathscr{L})$

**Definition 6.1.** *A Hermitian metric in $\mathscr{E}$ with $\det h = l$ is a solution of the projective vortex equation iff the trace free part $F_h^0$ of the curvature $F_h$ satisfies the equation*

$$i\Lambda_g F_h^0 + \frac{1}{2}(\varphi\bar{\varphi}^h)_0 = 0 \tag{V}$$

**Theorem 6.2.** *Let $(\mathcal{E}, \varphi)$ be a holomorphic pair of type $(E, \mathcal{L})$ with* $\mathrm{rk}(\mathcal{E}) = 2$ *Fix a Hermitian metric $l$ in $\mathcal{L}$*

*The pair $(\mathcal{E}, \varphi)$ is polystable iff $\mathcal{E}$ admits a Hermitian metric $h$ with* $\det h = l$ *which is a solution of the projective vortex equation If $(\mathcal{E}, \varphi)$ is stable, then the metric $h$ is unique*

*Proof* Suppose first that $h$ is a solution of the projective vortex equation $(V)$ Then we have

$$iAF_h + \frac{1}{2}(\varphi\bar{\varphi}^h) = \frac{1}{2}\left(iA\mathrm{Tr}F_h + \frac{1}{2}|\varphi|^2\right)\mathrm{id}_E \ ,$$

i e $h$ satisfies the weak vortex equation $(V_t)$ associated to the real function $t = \frac{1}{2}(2iA\mathrm{Tr}\,F_h + |\varphi|^2)$ Therefore, by [OT1], the pair $(\mathcal{E}, \varphi)$ is $\lambda$-polystable for the parameter $\lambda = \frac{(n-1)!}{4\pi}\int_X t vol_g = \mu_g(\mathcal{E}) + \frac{(n-1)!}{8\pi}||\varphi||^2$

Let $A$ be the Chern connection of $h$, and denote by $\mathcal{E}'$ the minimal $A$-invariant subbundle which contains $\varphi$ If $\mathcal{E}' = \mathcal{E}$, then $\mathcal{E}$ is $\varphi$-stable and the pair $(\mathcal{E}, \varphi)$ is stable

If $\mathcal{E}' = 0$, hence $\varphi = 0$, then $h$ is a weak Hermitian–Einstein metric, $\mathcal{E}$ is a polystable bundle, and the pair $(\mathcal{E}, \varphi)$ is polystable by definition.

In the remaining case $\mathcal{E}'$ is a line bundle and $\varphi \neq 0$. Let $\mathcal{E}'' = \mathcal{E}'^{\perp}$ be the orthogonal complement of $\mathcal{E}'$, and let $h'$ and $h''$ be the induced metrics in $\mathcal{E}'$ and $\mathcal{E}''$ We put $s = iA_g\mathrm{Tr}F_h$ Then, since $h = h' \oplus h''$, the projective vortex equation can be rewritten as

$$\begin{cases} iAF_{h'} + \frac{1}{2}(\varphi\bar{\varphi}^{h'}) = \frac{1}{2}(s + \frac{1}{2}|\varphi|^2_{h'})\mathrm{id}_{\mathcal{E}'} \\ iAF_{h''} = \frac{1}{2}(s + \frac{1}{2}|\varphi|^2_{h'})\mathrm{id}_{\mathcal{E}''} \end{cases}$$

The first of these equations is equivalent to

$$iAF_{h'} + \frac{1}{4}(\varphi\bar{\varphi}^{h'}) = \frac{s}{2}\mathrm{id}_{\mathcal{E}'} \ ,$$

which implies that $(\mathcal{E}', \frac{\varphi}{\sqrt{2}})$ is $\mu_g(\mathcal{E})$-stable by [OT1].

Conversely, suppose first that $(\mathcal{E}, \varphi)$ is stable We have to consider two cases

*Case* 1 $\mathcal{E}$ is $\varphi$-stable Using Bradlow's existence theorem, we obtain Hermitian metrics in $\mathcal{E}$ satisfying the usual vortex equations associated with suitable chosen $\lambda$, and, of course these metrics all satisfy the equation $(V)$ The problem is, however, to find a solution with an a priori given determinant $l$.

In order to achieve this stronger result, Bradlow's proof has to modified slightly at some points

One starts by fixing a background metric $k$ such that $\det k = l$. Denote by $S_0(k)$ the space of trace-free $k$-Hermitian endomorphisms of $E$, and let $\mathcal{M}et(l)$ be the space of Hermitian metrics in $E$ with $\det h = l$. On

$$\mathcal{M}et(l)^p_2 = \{ke^s |\ s \in L^p_2(S_0(k))\}$$

we define the functional $M_\varphi$ $\mathcal{M}et(l)^p_2 \to \mathbb{R}$ by

$$M_\varphi(h) := M_D(k, h) + ||\varphi||^2_h - ||\varphi||^2_k$$

Here $M_D$ is the Donaldson functional, which is known to satisfy the identity $\frac{d}{dt}$ $M_D(k,h(t))=2\int_X \mathrm{Tr}[h^{-1}(t)h(t)i\Lambda_g F_h]$ for any smooth path of metrics $h(t)$ [Do, Ko] Since $h^{-1}(t)h(t)$ is trace-free for a path in $\mathcal{M}et(l)$, we obtain

$$\frac{d}{dt}\Big| M_D(k,h(t)) = 2\int_X \mathrm{Tr}[h^{-1}h(t)i\Lambda_g F_h^0]$$

Similarly, for a path of the form $h(t) = he^{ts}$, with $s \in S_0(h)$, we get

$$\frac{d}{dt}_{|t=0}||\varphi||^2_{h_t} = \frac{d}{dt}_{|t=0}\langle e^{ts}\varphi,\varphi\rangle_h = \left\langle \frac{d}{dt}_{|t=0} e^{ts}\varphi,\varphi\right\rangle_h = \langle s,\varphi\bar\varphi^h\rangle_h = \int_X \mathrm{Tr}[s(\varphi\bar\varphi^h)_0]$$

This means that, putting $m_\varphi(h) := i\Lambda F_h^0 + \frac{1}{2}(\varphi\bar\varphi^h)_0$, we always have

$$\frac{d}{dt}_{|t=0} M_\varphi(he^{ts}) = 2\int_X \mathrm{Tr}[s\, m_\varphi(he^{ts})]\ ,$$

so that solving the projective vortex equation is equivalent to finding a critical point of the functional $M_\varphi$ (compare with Lemma 3 3 [B2])

**Claim 6.3.** *Suppose $(\mathscr{E},\varphi)$ is simple Choose $B > 0$ and put*

$$\mathcal{M}et(l)^p_2(B) := \{h \in \mathcal{M}et(l)^p_2|\ ||m_\varphi(h)||_{L^p} \leqq B\}$$

*Then any $h \in \mathcal{M}et(l)^p_2(B)$ which minimizes $M_\varphi$ on $\mathcal{M}et(l)^p_2(B)$ is a weak solution of the projective vortex equation*

The essential point is the injectivity of the operator $s \mapsto \Lambda'_h(s) + \frac{1}{2}[(\varphi\bar\varphi)s]_0$ acting on $L^p_2 S_0(h)$ But from

$$\left\langle \Lambda'_h s + \frac{1}{2}[\varphi\bar\varphi^h)s]_0,\ s\right\rangle_h = ||\bar\partial_h(s)||^2_h + ||s\varphi||^2_h$$

we see that this operator is injective on trace-free endomorphisms if $(\mathscr{E},\varphi)$ is simple □

Now we can follow again Bradlow's proof if $\mathscr{E}$ is $\varphi$-stable, then there exist positive constants $C_1$, $C_2$ such that for all $s \in L^p_2 S_0(k)$ with $ke^s \in \mathcal{M}et(l)^p_2(B)$ the following "main estimate" holds

$$\sup|s| \leqq C_1 M_\varphi(ke^s) + C_2$$

This follows by applying Proposition 3 2 of [B2] to an arbitrary $\tau \in \mathbb{R}$ with

$$\max\left(\mu_g(\mathscr{E}),\ \sup_{\mathscr{F}'\in\mathscr{S}(\mathscr{E})}\mu_g(\mathscr{F}')\right) < \frac{(n-1)!\tau Vol_g(X)}{4\pi} < \inf_{\mathscr{F}\in\mathscr{S}_\varphi(\mathscr{E})}\mu_g(\mathscr{E}/\mathscr{F})\ ,$$

since Bradlow's functional $\mathcal{M}_{\varphi,\tau}$ coincides on $\mathcal{M}et(l)$ with $M_\varphi$.

It remains to be shown that the existence of this main estimate implies the existence of a solution of the projective vortex equation

The main estimate implies that for any $c > 0$, the set

$$\{s \in L_2^p S_0(k) | ke^s \in \mathcal{M}et(l)_2^p(B), \quad M_\varphi(ke^s) < c\}$$

is bounded in $L_2^p$ Let $(s_i)$ be a sequence in $L_2^p S_0(k)$ such that $ke^{s_i} \in \mathcal{M}et(l)_2^p(B)$ is a minimizing sequence for $M_\varphi$, and let $s$ be weak limit Then $h := ke^s$ is a weak solution of the projective vortex equation, which is smooth by elliptic regularity [B2]

Finally, we have to treat

*Case* 2 $\varphi \neq 0$, $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, with $\varphi \in H^0(\mathcal{E}')$, and the pair $(\mathcal{E}', \varphi)$ is $\mu_g(E)$-stable

We wish to find metrics $h'$ and $h''$ in $\mathcal{E}'$ and $\mathcal{E}''$, such that for $s = i\Lambda F_l$ the following equations are satisfied·

$$\begin{cases} h' \cdot h'' = l \\ i\Lambda F_{h'} + \frac{1}{4}(\varphi\bar{\varphi}^{h'}) = \frac{1}{2} s \mathrm{id}_{\mathcal{E}'} \\ i\Lambda F_{h''} = \frac{1}{2}(s + \frac{1}{2}|\varphi|^2_{h'})\mathrm{id}_{\mathcal{E}''} . \end{cases}$$

Since the pair $(\mathcal{E}', \frac{1}{\sqrt{2}}\varphi)$ is $\mu_g(E)$-stable, there exists by [OT1] a unique Hermitian metric $h'$ in $\mathcal{E}'$ solving the second of these equations With this solution the third equation can be rewritten as

$$i\Lambda_g F_{h''} = s - i\Lambda_g F_{h'}$$

Since $\int_X (s - i\Lambda_g F_{h'}) = \deg(\mathcal{E}'')$, we can solve this weak Hermitian–Einstein equation by a metric $h''$, which is unique up to constant rescaling The product $h' \cdot h''$ is a metric in $\mathcal{E}' \otimes \mathcal{E}'' = \mathcal{L}$ which has the same mean curvature $s$ as $l$, and therefore differs from $l$ by a constant factor We can now simply rescale $h''$ by the inverse of this constant, and we get a pair of metrics satisfying the three equations above □

## 7. Moduli Spaces of Pairs

Let $E$ be a differentiable vector bundle of rank $r$ over a Kähler manifold $(X, g)$, and let $\mathcal{L}$ be a holomorphic line bundle whose underlying differentiable bundle is $L = \det E$

**Proposition 7.1.** *There exists a possibly non-Hausdorff complex analytic orbifold $\mathcal{M}^s(E, \mathcal{L})$ parametrizing isomorphism classes of simple holomorphic pairs of type $(E, \mathcal{L})$ The open subset $\mathcal{M}^{ss}(E, \mathcal{L}) \subset \mathcal{M}^s(E, \mathcal{L})$ consisting of strongly simple pairs is a complex analytic space, and the points in $\mathcal{M}^s(E, \mathcal{L}) \setminus \mathcal{M}^{ss}(E, \mathcal{L})$ have neighbourhoods modeled on $\mathbb{Z}/r$-quotients*

*Proof* Since we use the same method as in the proof of Proposition 3 9, we only sketch the main ideas.

Let $\bar{\lambda}$ be the semiconnection defining the holomorphic structure of $\mathcal{L}$, and put $\bar{\mathcal{A}} = \bar{\mathcal{A}}_{\bar{\lambda}}(E) \times A^0(E)$, where $\bar{\mathcal{A}}_{\bar{\lambda}}(E)$ denotes the affine space of semiconnections in $E$ inducing $\bar{\lambda}$ in $L = \det E$ The complex gauge group $SL(E)$ acts on $\bar{\mathcal{A}}$, and we write $\bar{\mathcal{A}}^s$ ($\bar{\mathcal{A}}^{ss}$) for the open subset of pairs whose stabilizer is contained in the center $\mathbb{Z}/r$ of $SL(E)$ (is trivial) After suitable Sobolev completions, $\bar{\mathcal{A}}^{ss}$ becomes the total space of a holomorphic Hilbert principal $SL(E)$-bundle over $\bar{\mathcal{B}}^{ss} := \bar{\mathcal{A}}^{ss}/SL(E)$

A point $(\bar{\delta}, \varphi) \in \bar{\mathcal{A}}$ defines a pair of type $(E, \mathcal{L})$ iff it is integrable, i e iff it satisfies the following equations.

$$\begin{cases} F^{02}_{\bar{\delta}} = 0 \\ \bar{\delta}\varphi = 0 \end{cases} \tag{10}$$

Here $F^{02}_{\bar{\delta}} = \bar{\delta}^2$ is a $(0, 2)$-form with values in the bundle $\mathrm{End}_0(E)$ of trace-free endomorphisms Moreover, isomorphy of pairs of type $(E, \mathcal{L})$ corresponds to equivalence modulo the action of the complex gauge group $SL(E)$

Let $\bar{\sigma}$ be the map $\bar{\mathcal{A}} \to A^{02}(\mathrm{End}_0(E)) \oplus A^{01}(E)$ sending a pair $(\bar{\delta}, \varphi)$ to the left-hand sides of (10) We endow the sets $\mathcal{M}^{ss}_X(E, \mathcal{L}) = Z(\sigma) \cap \bar{\mathcal{A}}^{ss}/SL(E)$ ( $\mathcal{M}^s_X(E, \mathcal{L}) = Z(\sigma) \cap \bar{\mathcal{A}}^s/SL(E)$) with the structure of a complex analytic space (orbifold) as follows

$\mathcal{M}^{ss}_X(E, \mathcal{L})$ is defined to be the vanishing locus of the section $\tilde{\bar{\sigma}}$ in the Hilbert vector bundle $\bar{\mathcal{A}}^{ss} \times_{SL(E)} \left(A^{02}\mathrm{End}_0(E) \oplus A^{01}E\right)$ over $\bar{\mathcal{B}}^{ss}$ which is defined by $\bar{\sigma}$

To define the orbifold structure in $\mathcal{M}^s_X(E, \mathcal{L})$, we use local models derived from a deformation complex

Let $\bar{p} = (\bar{\delta}, \varphi) \in \bar{\mathcal{A}}$ an integrable point The associated deformation complex $\bar{\mathcal{D}}_{\bar{p}}$ is the cone over the evaluation map $ev^*_\varphi$

$$ev^q_\varphi \quad A^{0q}(\mathrm{End}_0(E)) \to A^{0q}(E)\,,$$

and has the form

$$\begin{aligned} 0 \to A^0(\mathrm{End}_0(E)) &\xrightarrow{\bar{D}^0_{\bar{p}}} A^{01}(\mathrm{End}_0(E)) \oplus A^0(E) \xrightarrow{\bar{D}^1_{\bar{p}}} \\ &\xrightarrow{\bar{D}^1_{\bar{p}}} A^{02}(\mathrm{End}_0(E)) \oplus A^{01}(E) \xrightarrow{\bar{D}^2_{\bar{p}}} \end{aligned} \tag{$\bar{\mathcal{D}}_{\bar{p}}$}$$

(compare with [OT1] Sect 4) We define

$$\begin{aligned} \bar{S}_{\bar{p}}(\varepsilon) \;=\; \{&\bar{p} + \beta \mid \beta \in A^{01}\mathrm{End}_0(E) \oplus A^0E,\ \bar{D}^0_{\bar{p}}\bar{D}^{0\,*}_{\bar{p}}(\beta) \\ &+\bar{D}^{1\,*}_{\bar{p}}(\bar{\sigma}(\bar{p} + \beta)) = 0, ||\beta|| < \varepsilon\} \end{aligned}$$

The same arguments as in the proof of Proposition 3 9 show that for sufficiently small $\varepsilon > 0$, $\bar{S}_{\bar{p}}(\varepsilon)$ is a submanifold of $\bar{\mathcal{A}}$, whose tangent space in $\bar{p}$ coincides with the first harmonic space $\mathbb{H}^1_{\bar{p}}$ of the elliptic complex $(\bar{\mathcal{D}}_{\bar{p}})$ Therefore, we get a local finite dimensional model $Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})$ for the intersection $Z(\bar{\sigma}) \cap \bar{N}_{\bar{p}}(\varepsilon)$ of the integrable locus with the standard slice

$$\bar{N}_{\bar{p}}(\varepsilon) \;=\; \{\bar{p} + \beta \mid \beta \in A^{01}(\mathrm{End}_0(E)) \oplus A^0(E),\ \bar{D}^{0\,*}_{\bar{p}}(\beta) = 0,\ \ ||\beta|| < \varepsilon\}$$

through $\bar{p}$ The restriction

$$\bar{\pi}|_{Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})} \quad Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)}) \longrightarrow \mathcal{M}^s_X(E, \mathcal{L})$$

of the orbit map is étale if $[\bar{p}] \in \mathcal{M}_X^{ss}(E, \mathcal{L})$, and induces an open injection

$$Z(\bar{\sigma}|_{\bar{S}_{\bar{p}(\varepsilon)}})/\mathbb{Z}/r \longrightarrow \mathcal{M}_X^s(E, \mathcal{L})$$

if $[\bar{p}] \in \mathcal{M}_X^s(E, \mathcal{L}) \setminus \mathcal{M}_X^{ss}(E, \mathcal{L})$ We define the orbifold structure of $\mathcal{M}_X^s(E, \mathcal{L})$ by taking the maps $\bar{\pi}|_{Z(\bar{\sigma}|_{\bar{S}_{\bar{p}(\varepsilon)}})}$ as orbifold-charts □

Our next purpose is to compare the two types of moduli spaces constructed in this paper Let $(X, g)$ be a Kähler surface endowed with the canonical $Spin^c$-structure $\mathfrak{c}$ Let $E$ be a $U(2)$ bundle with $\det E = K_X$, and denote by $\mathcal{M}^*(E)$ the moduli space of irreducible quaternionic monopoles associated to the $Spin^h(4)$-structure defined by $(\mathfrak{c}, E^\vee)$ (Lemma 1 3)

It follows from Proposition 4 1 that $\mathcal{M}^*(E)$ has a decomposition

$$\mathcal{M}^*(E) = \mathcal{M}^*(E)_{\alpha=0} \cup \mathcal{M}^*(E)_{\varphi=0},$$

where $\mathcal{M}^*(E)_{\alpha=0}$ ($\mathcal{M}^*(E)_{\varphi=0}$) is the Zariski closed subspace of $\mathcal{M}^*(E)$ cut out by the equation $\alpha = 0$ ($\varphi = 0$) The intersection

$$\mathcal{M}^*(E)_{\alpha=0} \cap \mathcal{M}^*(E)_{\varphi=0}$$

is the Donaldson instanton moduli space $\mathcal{D}^*$ of irreducible projectively anti-self-dual connections in $E$, inducing the Chern connection in $\mathcal{K}_X$. Put $\mathcal{A}_{\alpha=0} = \mathcal{A}_c(E) \times A^0(E)$, where $c$ is the Chern connection in $K_X$

**Proposition 7.2.** *The affine isomorphism $\mathcal{A}_{\alpha=0} \ni (C, \varphi) \mapsto (\bar{\partial}_C, \varphi) \in \bar{\mathcal{A}}$ induces a natural real analytic open embedding*

$$J \quad \mathcal{M}^*(E)_{\alpha=0} \hookrightarrow \mathcal{M}^s(E, \mathcal{K}_X)$$

*whose image is the suborbifold of stable pairs of type $(E, \mathcal{K}_X)$*

*Proof* Standard arguments (cf [OT1]) show that $J$ is an étale map which induces natural identifications of the local models

A point $[(\bar{\delta}, \varphi)]$ lies in the image of $J$ iff the $SL(E)$-orbit of $(\bar{\delta}, \varphi)$ intersects the zero locus of the map

$$m \quad \bar{\mathcal{A}} \to A^0(su(E)), \quad (\bar{\partial}_C, \varphi) \mapsto \Lambda_g F_C^0 - \frac{i}{2}(\varphi\bar{\varphi})_0$$

Let $(\mathcal{E}, \varphi)$ be the holomorphic pair of type $(E, \mathcal{K}_X)$ defined by $(\bar{\delta}, \varphi)$ We can reformulate the condition above in the following way $[(\mathcal{E}, \varphi)]$ lies in the image of $J$ iff there exists a Hermitian metric $h$ in $\mathcal{E}$ inducing the Kähler metric in $\mathcal{K}_X = \det \mathcal{E}$ which satisfies the projective vortex equation $(V)$ But we know already that this holds iff $(\mathcal{E}, \varphi)$ is stable Moreover, the unicity of the solution of the projective vortex equation is equivalent to the injectivity of $J$ □

Using the remark after Proposition 4 1, we can now state the main result of this paper

**Theorem 7.3.** *Let $(X, g)$ be a Kähler surface with canonical bundle $\mathcal{K}_X$, and let $E$ be a $U(2)$-bundle with $\det E = K_X$. Consider the $Spin^h$-structure associated with the canonical $Spin^c(4)$-structure and the $U(2)$-bundle $E^\vee$. The corresponding moduli space of irreducible quaternionic monopoles is a union of two Zariski closed subspaces. Each of these subspaces is naturally isomorphic as a real analytic orbifold to the moduli space of stable pairs of type $(E, \mathcal{K}_X)$. There exists a real analytic involution on the quaternionic moduli space which interchanges these two closed subspaces. The fixed point set of this involution is the Donaldson moduli space of instantons in $E$ with fixed determinant, modulo the gauge group $SU(E)$. The closure of the complement of the Donaldson moduli space intersects the moduli space of instantons in the Brill–Noether locus.*

*The union $\mathcal{M}^{SW}$ of all rank 1-Seiberg–Witten moduli spaces associated with splittings $E = E' \oplus E''$ corresponds to the subspace of stable pairs of type ii).*

## 8. Compactification, transversality, and applications

In this final section we indicate the main steps in proving the existence of a natural Uhlenbeck-type compactification of the moduli spaces of quaternionic monopoles, full details will appear in the Habilitationsschrift [T] of the second author.

Let $(X, g)$ be a closed oriented Riemannian 4-manifold endowed with a $Spin^h(4)$-structure $\mathfrak{h} = (P^h, P^h/Sp(1) \xrightarrow{\simeq} P)$. We denote the associated $PU(2)$-bundle $P^h \times_\delta PU(2)$ by $\delta(\mathfrak{h})$.

An ideal monopole of type $\mathfrak{h}$ is a pair $([A', \Psi'], \{x_1, \ldots, x_l\})$ consisting of a monopole $[A', \Psi'] \in \mathcal{M}^g_X(\mathfrak{h}')$ for a $Spin^h(4)$-structure $\mathfrak{h}'$ and an element $\{x_1, \ldots, x_l\} \in S^l X$ in the $l^{\text{th}}$ symmetric product of $X$ with

$$l = l(\mathfrak{h}') = \frac{1}{4}(p_1(\delta(\mathfrak{h}')) - p_1(\delta(\mathfrak{h})))$$

The set of ideal monopoles of type $\mathfrak{h}$ is

$$I\mathcal{M}^g_X(\mathfrak{h}) = \coprod_{\mathfrak{h}'} \mathcal{M}^g_X(\mathfrak{h}') \times S^{l(\mathfrak{h}')} X ,$$

the union being over all isomorphism classes of $Spin^h(4)$-structures $\mathfrak{h}'$ with $p_1(\delta(\mathfrak{h}')) \geqq p_1(\delta(\mathfrak{h}))$.

**Theorem 8.1.** *There exists a metric topology on $I\mathcal{M}^g_X(\mathfrak{h})$ such that the moduli space $\mathcal{M}^g_X(\mathfrak{h}) \subset I\mathcal{M}^g_X(\mathfrak{h})$ becomes an open subspace with compact closure $\overline{\mathcal{M}^g_X(\mathfrak{h})}$.*

The proof of this theorem is long and technical, it is based on

a) Local estimates for quaternionic monopoles in terms of the curvature of the connection component.
b) A regularity theorem for $L^2_1$-solutions.
c) A removable singularities theorem.

The techniques of the proofs are similar to the ones which have been developed to prove the analogous results in the instanton case [DK], however, since the monopole equations are not conformally invariant, several new difficulties arise.

In the present paper we state the main technical results and sketch the proof of the fundamental removable singularities theorem, for a complete proof we refer the reader to [T]

a) Let $E_0$ be the trivial $SU(2)$-vector bundle over the closed standard 4-ball $\bar{B}$ and put $\Sigma_0^\pm = \bar{B} \times \mathbb{H}^\pm$, where $\mathbb{H}^\pm$ are copies of the quaternionic field $\mathbb{H}$ A *Clifford map* is an $\mathbb{R}$-linear isomorphism

$$\gamma \quad \Lambda^1_{\bar{B}} \to \mathrm{Hom}_{\mathbb{H}}(\Sigma_0^+, \Sigma_0^-)$$

The choice of a Clifford map defines a metric $g_\gamma$ on $\bar{B}$ such that $\gamma$ becomes the Clifford multiplication of a $Spin(4)$-structure in $(\bar{B}, g_\gamma)$ Let $\gamma_0$ be a Clifford map inducing the standard flat metric $g_0$ on $\bar{B}$

**Theorem 8.2** (*Estimates in a Coulomb gauge*) *There exist a positive constant $\varepsilon > 0$ and a neighbourhood $N$ of $\gamma_0$ in the $\mathscr{C}^3$-topology with the following properties For any Clifford map $\gamma \in N$, any interior domain $D \Subset B$, and any positive integer $l$, there is a constant $C_{\gamma,D,l}$ such that for each solution $(A, \Psi)$ of the quaternionic monopole equations for the triple $(\bar{B}, E_0, \gamma)$ satisfying*

$$\text{(i) } d^*_{g_0} A = 0,$$

$$\text{(ii) } \|A\|_{L^4} \leqq \varepsilon, \quad \|\Psi\|_{L^4} \leqq \varepsilon$$

*the following estimates hold*

$$\|A\|_{L^2_l(D)} \leqq C_{\gamma,D,l}(\|A\|_{L^4} + \|\Psi\|_{L^4}),$$

$$\|\Psi\|_{L^2_l(D)} \leqq C_{\gamma,D,l}(\|A\|_{L^4} + \|\Psi\|_{L^4})$$

The proof can be reduced to a problem on the (closed) sphere $S^4$ endowed with a metric $\hat{g}_0$ of non-negative sectional curvature, such that the upper hemisphere is isometric to the flat ball $(\bar{B}, g_0)$ Let $\hat{\gamma}_0$ be a fixed extension of $\gamma_0$ to a Clifford map on $(S^4, \hat{g}_0)$ Each $\gamma \in N$ can be extended to a Clifford map $\hat{\gamma}$ which is close to $\hat{\gamma}_0$ and gives rise to two first order elliptic operators

$$\not{D}_{\hat{\gamma}} \quad A^0(\Sigma^+ \otimes E^\vee) \to A^0(\Sigma^- \otimes E^\vee),$$

$$\delta_{\hat{\gamma}} = d^*_{\hat{\gamma}} + d^+_{\hat{\gamma}} \quad A^1(\mathfrak{su}(2)) \to A^0(\mathfrak{su}(2)) \oplus A^2_+(\mathfrak{su}(2)).$$

Here $\Sigma^\pm$ are the half-spinor bundles of the standard $Spin(4)$-structure, $E$ is the trivial $SU(2)$-bundle, and $A^2_+$ denotes the space of 2-forms which are self-dual with respect to the metric associated with $\hat{\gamma}$

Essential points needed in the proof of Theorem 8 2 are the injectivity of the operators $\not{D}_{\hat{\gamma}_0}$ and $\delta_{\hat{\gamma}_0}$, which follows from corresponding Weitzenböck formulae, elliptic estimates, and standard bootstrapping arguments □

Combining Theorem 8 2 with the Gauge fixing theorem of K Uhlenbeck (Theorem 2 3 7 in [DK]) one gets two important consequences

**Corollary 8.3** (*Estimates in terms of the curvature*) *There exist a positive constant $\varepsilon$ and a neighbourhood $N$ of $\gamma_0$ in the $\mathscr{C}^3$-topology with the following properties For any $\gamma \in N$, any interior domain $D \Subset B$, and any positive integer $l$, there is a positive constant $C_{\gamma,D,l}$ such that any solution $(A, \Psi)$ of the quaternionic monopole equations for the triple $(B, E_0, \gamma)$ satisfying $\|F_A\|_{L^2} \leqq \varepsilon$, is gauge equivalent to a* *solution* $(\tilde{A}, \tilde{\Psi})$ *for which the following estimates hold*

$$\|\tilde{A}\|_{L^2_l(D)} \leqq C_{\gamma,D,l}\|F_A\|_{L^2}, \qquad \|\tilde{\Psi}\|_{L^2_l(D)} \leqq C_{\gamma,D,l}\|F_A\|_{L^2}^{\frac{1}{2}}$$

The following global-compactness result is the analogon of Proposition (4.4.9) in [DK]

**Corollary 8.4.** *Let* $(\Omega, g)$ *be an oriented Riemannian manifold endowed with a* $Spin^h(4)$*-structure, and let* $(A_n, \Psi_n)_{n\in\mathbb{N}}$ *be a sequence of solutions of the quaternionic monopole equations If every point* $x \in \Omega$ *admits a geodesic ball* $B_x$ *with*

$$\int_{B_x} |F_{A_n}|^2 \leqq \varepsilon^2$$

*for all large enough* $n$, *then there is a subsequence* $(n_\nu)_{\nu\in\mathbb{N}}$ *and gauge transformations* $u_\nu \in \mathscr{G}$ *such that* $u_\nu^*(A_{n_\nu}, \Psi_{n_\nu})$ *converges in the* $\mathscr{C}^\infty$*-topology on* $\Omega$

b) The next important step is the following delicate regularity result for $L^4$-small approximate $L^2_1$-solutions

**Proposition 8.5** (*Regularity of* $L^2_1$*-solutions*) *Let* $\hat{\gamma}$ *be a Clifford map on* $S^4$ *which is sufficiently* $\mathscr{C}^1$*-close to* $\hat{\gamma}_0$ *and let* $\hat{g}$ *be the associated metric*

*There are positive constants* $a, b, c, d$, *depending on* $\hat{\gamma}$, *such that any pair* $(A, \Psi) \in L^2_1(\Lambda^1\mathfrak{su}(2)) \times L^2_1(\Sigma^+ \otimes E^\vee)$ *satisfying*

(i) $d^*_{\hat{g}}(A) = 0$,
(ii) $\|A\|_{L^4} \leqq a$, $\|\Psi\|_{L^4} \leqq b$,
(iii) $\|\not{D}_{\hat{\gamma},A}\Psi\|_{L^2} \leqq c$, $\|\Gamma_{\hat{\gamma}}(F_A^+) - (\Psi\bar{\Psi})_0\|_{L^2} \leqq d$,

*for which* $\not{D}_{\hat{\gamma},A}\Psi$ *and* $(F_A^+) - (\Psi\bar{\Psi})_0$ *are smooth, is also regular Moreover, there exist positive constants* $e$, $f$, *depending on* $\hat{\gamma}$, *such that the following estimates hold*

$$\|A\|_{L^2_1} \leqq e(\|\not{D}_{\hat{\gamma},A}\Psi\|_{L^2} + \|\Gamma_{\hat{\gamma}}(F_A^+) - (\Psi\bar{\Psi})_0\|_{L^2}),$$

$$\|\Psi\|_{L^2_1} \leqq f\|\not{D}_{\hat{\gamma},A}\Psi\|_{L^2}$$

The analogous statement in the instanton case is Proposition (4 4 13) in [DK], the proof of Theorem 8 5, similar to the one in [DK], also uses the continuity method, but some of the arguments are more difficult [T].

c) Now put $B^\bullet := B \setminus \{0\}$, $S^\bullet := S^4 \setminus \{0\}$

**Theorem 8.6** (*Removable singularities*) *Let* $\gamma$ *be a Clifford map on the ball Let* $(A_0, \Psi_0)$ *be a solution of the quaternionic monopole equations for the triple* $(B^\bullet, E_0|_{B^\bullet}, \gamma|_{B^\bullet})$ *with* $\int_{B^\bullet} |F_{A_0}|^2 < \infty$ *Then there exists a solution* $(A, \Psi)$ *of the monopole equations for the triple* $(B, E_0, \gamma)$ *and an isomorphism* $u : E_0|_{B^\bullet} \to E_0|_{B^\bullet}$ *such that* $u^*((A, \Psi)|_{B^\bullet}) = (A_0, \Psi_0)$

*Proof* It suffices to prove the theorem for a Clifford map which is close to the standard one $\gamma_0$ Indeed, this can be achieved by constant rescalings of the metric, under which the monopole equations are invariant If $\gamma$ is close to $\gamma_0$, then it extends to a Clifford map $\hat{\gamma}$ on $S^4$ which is close to $\hat{\gamma}_0$.

Now the technique which was developed in [DK] for the instanton case can be adapted to the situation at hand

Consider a small positive number $R$. In a first step we cut off the solution $(A_0, \Psi_0)$ towards the boundary of $B$ to get an approximate solution $(A_R, \Psi_R)$ on the punctured sphere $S^\bullet$ with $L^2$-small curvature, whose restriction to $B_{\frac{R}{2}} \setminus \{0\}$ is a solution, and which vanishes outside $B_R$. Replacing the pair $(A_R, \Psi_R)$ by a gauge equivalent one if necessary, we can estimate the $L^4$-norm of the connection component in terms of the $L^2$-norm of its curvature (use the analogon of Proposition (4.4.10) in [DK] for the quaternionic monopole equations).

In a second step we keep $R$ fixed and apply the same procedure for any $r$ with $0 < r \ll R$ towards the origin: we cut off with a suitable function supported in $S^4 \setminus B_r$ and then apply a gauge transformation in order to bring the connection into the Coulomb gauge and to control its $L^4$-norm in terms of the $L^2$-norm of the curvature (use the Gauge fixing theorem on the sphere, Proposition (2.3.13) in [DK]). The $L^2_1$-estimates in the Regularity theorem give a uniform bound for the $L^2_1$-norms of the obtained family $(A_{R,r}, \Psi_{R,r})_r$ of approximate solutions. Hence, for a suitable sequence $r_i \to 0$, the sequence $(A_{R,r_i}, \Psi_{R,r_i})_{i\in\mathbb{N}}$ converges weakly in $L^2_1$ to a pair $(A, \Psi)$ which is a weak $L^2_1$-solution of the quaternionic monopole equations in a neighbourhood of 0. This pair must be smooth by the Regularity theorem. Indeed, the associated sections $\not{D}_{\gamma,A}\Psi$ and $\Gamma_\gamma(F_A^+) - (\Psi\bar{\Psi})_0$ vanish in a neighbourhood of the origin and are smooth away from the origin. □

Now consider again a compact Riemannian manifold $(X, g)$. The Weitzenböck formula provides an a priori $\mathscr{C}^0$-bound for the spinor component on the space of solutions of the monopole equations. Using the decomposition $|F_A|^2 = |F_A^-|^2 + |F_A^+|^2$ and the second monopole equation, it follows that there exists a positive constant $C$ such that for every quaternionic monopole $(A, \Psi)$ the following pointwise a priori estimate holds:

$$|F_A|^2 \leqq (|F_A^-|^2 - |F_A^+|^2) + C$$

The integral of the first term on the right is a topological invariant $k(\mathfrak{h})$ of the $Spin^h(4)$-structure. If now $(A_n, \Psi_n)_{n\in\mathbb{N}}$ is a sequence of quaternionic monopoles, the sequence of measures associated with $|F_{A_n}|^2 dvol_g$ is uniformly bounded, hence it has a subsequence converging to a measure whose total volume is bounded by $k(\mathfrak{h}) + C\, Vol_g$. Using Corollary 8.4 and the same arguments as in [DK], Sect. 4.4.3 one shows:

**Theorem 8.7.** *For every sequence $(A_n, \Psi_n)_{n\in\mathbb{N}}$ of solutions $(A_n, \Psi_n)$ of the quaternionic monopole equations associated with the $Spin^h(4)$-structure $\mathfrak{h}$ there exists a $Spin^h(4)$-structure $\mathfrak{h}'$ and a pair $((A', \Psi'), \{x_1, \dots, x_l\})$ consisting of a solution of the quaternionic monopole equation for $\mathfrak{h}'$ and a set $\{x_1, \dots, x_l\} \subset X$ such that the following holds: There is a subsequence $(n_\nu)_{\nu\in\mathbb{N}}$ and isomorphisms $P'^h|_{X\setminus\{x_1,\dots,x_l\}} \xrightarrow{u_\nu} P^h|_{X\setminus\{x_1,\dots,x_l\}}$ over the frame bundle $P|_{X\setminus\{x_1,\dots,x_l\}}$ such that the sequence $u_\nu^*((A_{n_\nu}, \Psi_{n_\nu})|_{X\setminus\{x_1,\dots,x_l\}})$ converges to $(A', \Psi')|_{X\setminus\{x_1,\dots,x_l\}}$.*

From here, Theorem 1.1 follows immediately, as in the instanton case (see [DK], Sect. 4.4.1).

The moduli spaces $\mathscr{M}_X^q(\mathfrak{h})$ are in general not smooth, e.g. they have $\mathbb{Z}/2$-orbifold singularities along the Brill–Noether locus ($\Psi = 0$), and the addition of ideal solutions usually introduces further singularities. On the other hand, the $S^1$-action $(\zeta, [A, \Psi]) \mapsto [A, \zeta^{\frac{1}{2}}\Psi]$ extends naturally to the Uhlenbeck compactification, and this action is free away from the union of the compactified Brill–Noether locus and the subspace of ideal abelian solutions

The singularities in the points $[A,0]$ with $A$ irreducible can be removed by performing a *real blow up* (in the sense of [GS]) of the subspace $(\Psi = 0)$ in the direction of the spinor component of the configuration space. one replaces the spinor-component $\Psi$ by a pair $(\Phi, t)$ consisting of a spinor $\Phi$ with $\|\Phi\|_{L^2} = 1$ and a real number $t$ The new configuration space is *weakly contractible* and the induced action of the gauge group $\mathcal{G}$ becomes *free*, therefore, the $\mu$-classes of Donaldson can easily be defined on its $\mathcal{G}$-quotient

We introduce modified quaternionic monopole equations by

$$\begin{cases} \not{D}_A\Phi = 0 \\ \Gamma(F_A^+) = t^2(\Phi\bar{\Phi})_0 \end{cases} \qquad (\hat{SW}^{\mathfrak{h}})$$

and we denote by $\hat{\mathcal{M}}^g_X(\mathfrak{h})$ and $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h})}$ the corresponding moduli space of solutions and its Uhlenbeck compactification

The induced $S^1$-action on $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h})}$ is free away from the *abelian locus* $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h})}_{ab}$ consisting of (possibly ideal) solutions $[A, (\Phi, t), \{x_1, \dots, x_l\}]$ with $A$ reducible and $t\Phi$ contained in an $A$-parallel summand

The following theorem, for which we refer to [T], is the basis of all further developments

**Theorem 8.8.** *There exists a natural $\mathcal{G} \times S^1$-equivariant perturbation of the modified quaternionic monopole equations $(\hat{SW}^{\mathfrak{h}})$ such that for any sufficiently small generic perturbation-parameter $\sigma$ the corresponding moduli space of solutions $\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)$ has the following properties*

i) *The complement $\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)^* := \hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma) \setminus \hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)_{ab}$ of the abelian locus is smooth of the expected dimension*

ii) *$\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)$ has a natural $S^1$-equivariant Uhlenbeck compactification $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)}$*

iii) *The $S^1$-action on $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)}$ is free away from the subspace of (ideal) abelian solutions*

iv) *The closed subspace $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)} \setminus \hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)^*$ admits an $S^1$-invariant neighbourhood which can be explicitely described*

v) *The equation $t = 0$ defines an $S^1$-invariant subspace in $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)}$ which fibers over a deformation $\overline{\mathcal{BM}^g_X(\mathfrak{h}, \sigma)}$ of the compactified Brill–Noether locus inside the compactified Donaldson moduli-space*

We believe that Theorem 8 8 can be used in the following two directions

a) To relate the Seiberg–Witten invariants to $Spin^c$-polynomials [PT] On request of the referee we describe briefly our approach to this problem

Let $\mathcal{P}^g_X(\mathfrak{h}, \sigma)$ ($\overline{\mathcal{P}^g_X(\mathfrak{h}, \sigma)}$) be the $S^1$-quotient of the subspace $t = 0$ in $\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)$ ($\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h}, \sigma)}$) This space fibers over the corresponding deformation of the (compactified) Brill–Noether locus with complex projective spaces as fibres Under the assumption that $\overline{\mathcal{P}^g_X(\mathfrak{h}, \sigma)}$ contains no abelian solutions, Donaldson's $\mu$-classes descend to $\mathcal{P}^g_X(\mathfrak{h}, \sigma)$, can be extended to $\overline{\mathcal{P}^g_X(\mathfrak{h}, \sigma)}$, and associated $Spin^c$-polynomials can be defined To relate these polynomials to Seiberg–Witten classes, one first

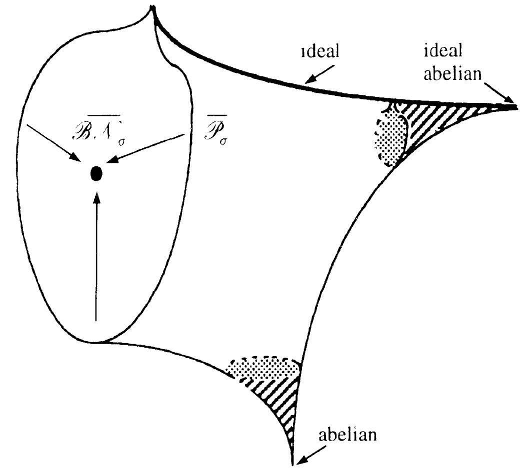


**Fig. 1.**

forms the $S^1$-quotient of $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)}$ and then orients the (smooth) complement of the abelian and the ideal locus (Theorem 8 8 i)) The restrictions of the $\mu$-classes to $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)} \setminus \overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)_{ab}}$ descend to the $S^1$-quotient $[\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)} \setminus \overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)_{ab}}]/S^1$ The subspace $\mathcal{P}^g_X(\mathfrak{h},\sigma)$ is a smooth hypersurface provided the Brill–Noether locus contains no reducible connection Consider now a standard neighbourhood $\mathcal{U}$ (Theorem 8 8 iv)) of $\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)_{ab}}/S^1$ in $[\overline{\hat{\mathcal{M}}^g_X(\mathfrak{h},\sigma)}/S^1] \cap \{t \geqq 0\}$ and triangulate its complement to obtain a chain with boundary $\overline{\mathcal{P}}^g_X(\mathfrak{h},\sigma) \cup \partial\mathcal{U}$ Since $\partial\mathcal{U}$ fibres in a natural way over a disjoint union of perturbed Seiberg–Witten moduli spaces, one should be able to get an explicit formula relating Seiberg–Witten invariants and $Spin^c$-polynomials [2]

b) To define invariants for differentiable 4-manifolds

Here it seems to be more appropriate to perform a *complex blow up* of the locus $(\Psi = 0)$ in the spinor direction of the configuration space The new configuration space will not be weakly contractible, the stabilizers (under the action of the gauge group) of the non-abelian points in the locus $(\Psi = 0)$ will still be $\mathbb{Z}/2$, and $S^1$ will act trivially on this subspace

However, by a careful examination of the local models of these points, one can prove that, for a generic perturbation $\sigma$, the obtained moduli space $\tilde{\mathcal{M}}^g_X(\mathfrak{h},\sigma)$ is smooth also along the non-abelian part of the blown up (perturbed) Brill–Noether locus The next step is now the definition of universal classes on the Uhlenbeck compactification $\overline{\tilde{\mathcal{M}}^g_X(\mathfrak{h},\sigma)}$

We plan to come back to this project in [OT6]

[2] We are aware of the fact that Pidstrigach and Tyurin have independently proposed a similar program which aims at proving the equivalence of Seiberg–Witten invariants and Donaldson invariants

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