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W-Geometry of the Toda Systems Associated with Non-Exceptional Simple Lie Algebras

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Abstract: The present paper describes the W-geometry of the Abelian finite non-periodic (conformal) Toda systems associated with the B, C and D series of the simple Lie algebras endowed with the canonical gradation. The principal tool here is a generalization of the classical Plücker embedding of the A-case to the flag manifolds associated with the fundamental representations of B_n, C_n and D_n, and a direct proof that the corresponding Kähler potentials satisfy the system of two-dimensional finite non-periodic (conformal) Toda equations It is shown that the W-geometry of the type mentioned above coincide with the differential geometry of special holomorphic (W) surfaces in target spaces which are submanifolds (quadrics) of CP^N with appropriate choices of N In addition, these W-surfaces are defined to satisfy quadratic holomorphic differential conditions that ensure consistency of the generalized Plücker embedding These conditions are automatically fulfilled when Toda equations hold

1. Introduction

A notion of W_φ-geometry of $\mathbf{CP}^N$-target manifolds associated with integrable systems, recently invented in [1] for the case of A_n-Abelian Toda system (see also [2]) seems to be a very important tool for solvable field theories as geometrical structures behind W-algebras, as well as for algebraic and differential geometries themselves In particular, such a geometrical picture should be rather essential in the gauge fields formulation of various models of the two-dimensional gravity, as well as their generalizations for higher dimensions On the same footing as W-algebras, being the algebras of the characteristic integrals – conserved currents – for the corresponding nonlinear systems, guarantee, under appropriate conditions, the integrability property for these systems and give their classification, a description of their W-geometry is equivalent, in a sense, to a classification scheme of

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the corresponding Kähler manifolds [3] It was shown in [1] that the Kähler potentials of the intrinsic metrics induced on the corresponding W-surfaces coincide with the A_n-Toda fields In what follows we prove that this fact takes place also for the W-surfaces associated with all other classical (non-exceptional) simple Lie algebras $\mathcal{G}$ and the corresponding $\mathcal{G}$-Toda fields, and can be realized explicitly (In fact, we have conjectured this statement already several years ago, but only now have a proof for that) We believe that this notion is relevant for a wide class of integrable dynamical systems as a geometrical counterpart of W-algebras.

For the readers who are not familiarized with the notions W-algebra and W-geometry, at least in the meaning which will be used in our paper, let us recall it in a few words

By A_n–W-geometry we mean the geometry of the $\mathbf{CP}^n$ W-surface of ref [1] which are two-dimensional manifolds Σ supplied with a complex structure, and an embedding into $\mathbf{CP}^n$ such that half of the coordinates X^A, $1 \leqq A \leqq n+1$, of the enveloping space holomorphically depends on a local coordinate z of Σ, $X^A = f^A(z)$, while the other half, $\bar{X}^A$, are anti-holomorphic functions $\bar{X}^A = \bar{f}^A(\bar{z})$ In other words, in the language of algebraic geometry, we speak here, on account of the appropriate reality condition, about holomorphic curves in the corresponding projective target space $\mathbf{CP}^n$ Note that we call them surface, instead, on account of their real dimension This is more appropriate for applications to conformal models and string theories The corresponding W-surface is called W_{A_n}-surface, it is related to the first fundamental representation of A_n, and there are associated surfaces related to the other fundamental representations of A_n. As shown in ref. [1], and as we shall recall below, the W_{A_n}- and associated surfaces correspond to the classical extrinsic geometry of the curves in $\mathbf{CP}^n$ having to deal with the Plücker image of the Grassmannians $\mathcal{G}r(n+1|k)$ in $\mathbf{CP}^{\binom{n+1}{k}-1}$.

Since the corresponding complex projective target space is defined as the quotient of the space $\mathbf{C}^{n+1}$ by the equivalence lifting (local rescaling of the coordinates), it immediately follows from the given definition for the W_{A_n}-surface, that these holomorphic (anti-holomorphic) functions are solutions of some homogeneous ordinary differential equation of the $(n+1)^{\text{th}}$ order,

$$\frac{\partial^{n+1} f^A}{\partial z^{n+1}} = \sum_{\alpha=1}^{n} W_\alpha \frac{\partial^\alpha f^A}{\partial z^\alpha}, \qquad \frac{\partial^{n+1} \bar{f}^A}{\partial \bar{z}^{n+1}} = \sum_{\alpha=1}^{n} \bar{W}_\alpha \frac{\partial^\alpha \bar{f}^A}{\partial \bar{z}^\alpha} \tag{1 1}$$

with nonzero[4] coefficients W_α ($\bar{W}_\alpha$) In writing these equations we made use of the fact that the Wronskians $\mathrm{Wr}[f(z)]$ and $\mathrm{Wr}[\bar{f}(\bar{z})]$ constructed with the functions $f^A(z)$ and $\bar{f}(\bar{z})$ do not vanish at regular points of Σ, and one can divide by them In other words, here we deal with the osculating hyperplanes to the generic W-surfaces. Note also in this context that the linear system of the Plücker quadrics which provides the decomposability property of the Plücker image, is automatically satisfied on the class of the solutions to (1 1)

One of the main points of this paper is to define some submanifolds in $\mathbf{CP}^N$, with appropriate N, which are target spaces for the $W_{\mathcal{G}}$-surfaces relevant for the other

[3] Some classification of the manifolds associated, in terms of the corresponding embedding problem (Gauss–Codazzi and Ricci equations), with the Toda systems, and an attempt of the geometrical formulation of the integrability criteria has been given in [3] However, it provides only some general links, in particular, with the diagonalisability of the corresponding 3rd fundamental form, and is too complicated for concrete conclusions

[4] In this paper, we do not consider singular points of W-surfaces This may be done straightforwardly

complex simple Lie algebras $\mathcal{G}$. These spaces are specified by quadratic conditions whose origin is as follows Now the set of the indices α in the sum in the r h s of (1 1), for which the coefficients $W_\alpha \neq 0$ and $\bar{W}_\alpha \neq 0$, coincides with the values of the exponents of the algebra $\mathcal{G}$ The vanishing of the corresponding coefficient functions in the series in the r h s in (1 1) leads to the set of the quadratic relations on the embedding functions $f^A(z)$ $(\bar{f}(\bar{z}))$ and their derivatives up to the $(n-1)^{\text{th}}$ order; and these local conditions on the functions ensures consistency of the generalized Plücker embedding described in the present paper It happens that the corresponding Kähler manifolds in W-geometry are ultimately related to the Toda fields, being described by the equations of the two-dimensional finite nonperiodic Toda system which, fortunately, are exactly solvable [4].

In accordance with [5], see also [6], the results of the Toda theory provide a realization of the W-algebras in terms of the polynomials constructed with the corresponding Toda fields, more exactly via their derivatives. By W-algebra we mean [5] an algebra with the defining relations

$$\{W_\gamma(z), W_\beta(z')\} = \sum_a \mathscr{P}^a_{\gamma,\beta}(W)\delta^{(a)}(z_1 - z'_1)$$

just for the coefficient functions entering (1 1), which realize the corresponding infinitesimal W-transformation of the functions f^A and $\bar{f}^A$ Here $\mathscr{P}^a_{\gamma,\beta}$ are polynomials of the W_γ's and their derivatives over spatial variable z_1 in two-dimensional space-time $\{\bar{z}, z\} = \{(z_0 \pm z_1)/2\}$ with the metric[5] $g_{11} = -g_{22} = 1$, the Poisson brackets are taken for equal time value, $z_0 = z'_0$ Moreover, such objects as the elements W_γ with values in the ring of gauge invariant differential polynomials, are quite well known in the integrable systems business, being in fact local characteristic integrals for the corresponding system of the partial differential equations, the existence of these integrals provides integrability of the system So, the theory of integrable systems is a natural place where Lie group-algebraic and differential and algebraic geometry aspects are intersected as the W-algebra & W-geometry

The relevant instrument for our description is a modification of the Plücker embedding for all classical series of the simple Lie algebras Recall that the standard Plücker embedding is formulated for the A_n-case, see e g. [7], and results in the infinitesimal and global Plücker formulas. Note that the statement which generalizes the infinitesimal Plücker formula (related to the canonical distribution) for an arbitrary simple Lie algebra $\mathcal{G}$, has been conjectured in [8] and then proved in [9], see also [10], without any connection with integrable systems, W-geometry, and all that In [11], using the relevant differential geometry setting, while without any reference to a coordinate representation of the corresponding flag manifolds associated with the Abelian Toda system, there was also obtained the generalized infinitesimal Plücker formula for an arbitrary simple Lie algebra $\mathcal{G}$. These flag manifolds are the quotient spaces G/P with P being the maximal nonsemisimple parabolic subgroups of G, $\mathcal{G} = \text{Lie } G$

As we will show, the relevant W-manifolds in our approach are related to the target manifolds of integrable systems gauged by a semi-direct product of a nilpotent and semisimple subgroups of Lie $\mathcal{G}$ In other words, the Kähler manifolds in question arise as parabolic spaces for a simple Lie group G whose coordinates satisfy some homogeneous equations quadratic in the coordinates Similar to those in the A_n-case, these coordinates are some minors of a matrix representative of the corresponding

[5] One may also define z as a complex variable Then $g_{11} = g_{22} = 1$

cosets, and they are submitted to homogeneous quadratic equations. However, only a part of these equations are quadrics of exactly the Plücker type,[6] while the other ones are caused by the specific features of such algebras as the orthogonal algebras In general, the Toda fields are related with (minors of) determinants Thus a natural tool is to perform skew products of representations, hence we shall use fermionic operators, see below. For A_n, one can obtain all finite dimensional irreducible representations by skew products of a finite number of copies of the first fundamental one As we will see, this is related with the fact that, since the Dynkin diagram is a simple line, the derivation of the Toda equation goes rather smoothly from the first root to the last For other algebras, the situation is more complicated Now, skew products of the first fundamental representation are not enough One should also include the last one for B_n, and the last two for D_n. These representations are of a different nature, and their highest weights have half integer components In the Dynkin diagram they correspond to non-generic points with branching where the derivation of Toda equations is much more subtle All these difficulties will be overcome in the ensuing discussion The study of the problem for the orthogonal algebras already contains[7] seemingly all peculiarities and "underwater stones" that are naturally absent in the case of A_n Thus we believe that our picture is truly general

To clarify the principal difference in the formulation of the problem in question for the simple Lie algebras other than A_n, we shall first recall some results for the A_n-case, mainly following[8] ref. [1], and complete them by some reasonings leading to the Plücker quadratic relations As already emphasized, and in distinction to the case of the Lie algebra A_n, a similar study of $W_{\mathcal{G}}$-geometry of the Toda systems for other simple Lie algebras is not so direct if one wishes to realize the program in the coordinate basis explicitly

Note that an important instrument of our consideration will be fermionic realizations of the elements of the classical Lie algebras, similar to the one which has been efficiently used in [1] for an investigation of the W-geometry of the A_n-Abelian Toda systems. The main advantage of this realization, apart from its technical simplicity, is that it allows to interpolate between different fundamental representations, and relate their basis vectors This is extremely suitable for a solution of the problem under consideration, where skew products of representations are the key

We give an explicit formulation of a relevant modification of the Plücker mapping for the manifolds associated with the fundamental representations of an arbitrary classical Lie algebra $\mathcal{G}$, and a direct proof that the corresponding Kähler potentials satisfy the system of partial differential equations of the Toda type. In general, we believe[9] that every integrable system is naturally associated with the corresponding Kähler manifold – that means with the relevant group G and its gauging – which in turn is determined by the invariance subgroup for the chosen representation space Here the manifold is defined by the gradation of the Lie algebra $\mathcal{G}$ and the grading spectrum of the corresponding component of the Maurer–Cartan 1-form which results in the nonlinear systems in question So, the algebraic counterparts of the

[6] Recall that just the system of the Plücker quadrics provides the condition of decomposability of a multivector in the corresponding complex projective space, and hence defines the Plücker image in it [7]

[7] The case of C_n is much simpler

[8] Part of this discussion already appeared in the preprint version of the second article of ref [1] It was removed from the printed version in order to shorten the article and satisfy the editor's request

[9] This was also hypothesized in the first article of ref [1]

given W-manifold are the Lie algebra, its gradation, and the grading spectrum of the connections. Note also that, as we have understood from discussions with M Kontsevich and Yu I Manin, our consideration of the nonlinear Toda type systems as holomorphic curves in the corresponding projective spaces, seems to be closely related to variations of the Hodge structures in the spirit of Ph A Griffiths.

2. Abelian Toda Systems, and W–A_n-Geometry

Consider a finite-dimensional complex simple Lie algebra $\mathcal{G} \equiv \mathcal{G}$ of rank n, with the following defining relations

$$[h_i, h_j] = 0, \qquad [h_i, E_{\pm j}] = \pm K^{\mathcal{G}}_{ji} E_{\pm j}, \qquad [E_{+i}, E_{-j}] = \delta_{ij} h_i \tag{2.1}$$

for its Cartan $\{h_i\}$ and Chevalley $\{E_{\pm i}\}$ elements, $1 \leqq i \leqq n$, and $K^{\mathcal{G}}$ being the Cartan matrix of $\mathcal{G}$ Let $\mathcal{G}$ be endowed with the canonical gradation,

$$\mathcal{G} = \bigoplus_{m \in \mathbf{Z}} \mathcal{G}_m, \ [\mathcal{G}_m, \mathcal{G}_l] \subset \mathcal{G}_{m+l} ,$$

for which $\mathcal{G}_0 = \{h_i\}$ is Abelian, and $\mathcal{G}_{\pm 1} = \{E_{\pm i}\}$ Then, in accordance with the group-algebraic approach [4],[10] the zero-curvature condition

$$[\partial/\partial \bar{z} + A_+, \partial/\partial z + A_-] = 0$$

for the connection components $A_\pm(z, \bar{z})$ taking values in the subspaces $\mathcal{G}_0 \oplus \mathcal{G}_{\pm 1}$, respectively, results in the partial differential equations describing the two-dimensional finite nonperiodic Toda system

$$\partial \bar{\partial} \Phi_i = -\exp \rho_i, \quad 1 \leqq i \leqq n, \qquad \rho_i = \sum_{j=1}^{n} K^{\mathcal{G}}_{ij} \Phi_j , \tag{2.2}$$

$\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$ The general solution to this system is written, in one of the equivalent forms, as follows Associated with each fundamental representation – with highest weight λ_k – there exists a Toda field Φ_k defined by

$$e^{-\Phi_k} = \langle \lambda_k | \bar{M}^{-1}(\bar{z}) M(z) | \lambda_k \rangle e^{-\bar{\xi}_k(\bar{z}) - \xi_k(z)} , \tag{2.3}$$

where

$$\frac{dM}{dz} = M \sum_{j=1}^{n} s_j(z) E_{-j}, \qquad \frac{d\bar{M}}{d\bar{z}} = \bar{M} \sum_{j=1}^{n} \bar{s}_j(z) E_j ,$$

$$\xi_k(z) = \sum_{j=1}^{n} ((K^{\mathcal{G}})^{-1})_{kj} \ln s_j(z), \qquad \bar{\xi}_k(\bar{z}) = \sum_{j=1}^{n} ((K^{\mathcal{G}})^{-1})_{kj} \ln \bar{s}_j(\bar{z}) \tag{2.4}$$

The n functions $s_j(z)$ and $\bar{s}_j(\bar{z})$ which are arbitary will be called screening functions, since they are the classical analogues of the Coulomb-gas operators They determine the general solution of the Goursat (boundary) value problem for (2 3) The matrix element $\langle \lambda_k | \bar{M}^{-1} M | \lambda_k \rangle$ in the r h s of (2 3), is in fact, the tau-function

[10] Here and in what follows, the results concerning the Toda theory are given following [4]

for system (2.2), associated with the highest weight vector matrix element of the k^{th} fundamental representation of $\mathscr{G}$ with the highest state $|\lambda_k\rangle$

For later use, we note that one can consider the same formulae as above, but for irreducible representations which are not fundamental By definition of the fundamental weights, any highest weight is of the form $\lambda = \sum_k \nu_k \lambda_k$, where ν_k are nonnegative integers Then the generalization of (2 3) and (2 4) is

$$e^{-\sum_k \nu_k \Phi_k} = e^{-\sum_{j=1}^{n} (\vec{\lambda} \cdot \vec{\gamma}_j) \ln s_j(z) - \sum_{j=1}^{n} (\vec{\lambda} \cdot \vec{\gamma}_j) \ln \bar{s}_j(\bar{z})} \langle \lambda | \bar{M}^{-1}(\bar{z}) M(z) | \lambda \rangle \tag{2.5}$$

The proof of this equality is carried out by taking the corresponding powers of (2 3) This automatically constructs the highest weight vector with highest weight λ, on which the action of M and $\bar{M}$ can be derived solely from the Lie group theory.

For the case of the A_n-Toda system, all the Toda fields $\exp(-\Phi_j)$, $1 \leqq j \leqq n$, are expressed via the first one, $\exp(-\Phi_1)$ which can be written as

$$\exp(-\Phi_1) = \sum_A f^A(z) \cdot \bar{f}^A(\bar{z}) \tag{2 6}$$

Here the functions $f^A(z)$ and $\bar{f}^{\bar{A}}(\bar{z})$ satisfy the conditions which can be expressed in terms of the Wronskians constructed with these functions,

$$\mathrm{Wr}[f(z)] = 1, \qquad \mathrm{Wr}[\bar{f}(\bar{z})] = 1\,, \tag{2 7}$$

and are formulated via the independent (chiral) screening functions $s_i(z)$ and $\bar{s}_i(\bar{z})$, $1 \leqq i \leqq n$, entering the general solution (2.3) as the nested integrals (4.9), in our A_n-case with $i, = s$ All other Toda fields $\exp(-\Phi_j)$, $j > 1$, are written in terms of $\exp(-\Phi_1)$ by the formulas

$$\exp(-\Phi_j) = \Delta_j\,, \tag{2 8}$$

and hence

$$\exp(-\Phi_j) = \sum_j \det_j(f) \cdot \det_j(\bar{f}) \tag{2 9}$$

Here the sum runs over all the j^{th} order minors $\det_j(f)$ and $\det_j(\bar{f})$ constituted by the first j rows of the matrices $(\partial)^B f^A$ and $(\bar{\partial})^B \bar{f}^A$, respectively, Δ_j is the j^{th} order principal minor of the matrix $\partial^A \bar{\partial}^B \exp(-\Phi_1)$ Recall that such minors satisfies very important relation

$$\partial\bar{\partial} \log \Delta_j = \frac{\Delta_{j+1} \cdot \Delta_{j-1}}{\Delta_j^2}\,, \tag{2 10}$$

which is used in what follows

To have a more precise picture of what we are going to do in the general case, let us reproduce here some basic steps leading to the W_{A_n}-geometry We mainly follow (see footnote 4) the paper [1], but supply some additional formulas needed for understanding the quadratic relations of the Plücker type which are absent there First introduce the relevant notations, see e.g. [12, 13]. Let $\vec{e}_p$ be the set of orthonormal vectors in a $n+1$ Euclidean space $\vec{e}_p \cdot \vec{e}_q = \delta_{pq}$, $1 \leqq p, q \leqq n+1$, which parametrize the positive and negative roots $\pm(\vec{e}_p - \vec{e}_q)$, $1 \leqq p < q \leqq n+1$, of A_n The vectors $\vec{\pi}_i = \vec{e}_i - \vec{e}_{i+1}$ are a set of simple roots. Denote by $\vec{\lambda}_i$, $1 \leqq i \leqq n$,

the fundamental weights of A_n, $\vec{\lambda}_i$ is equal to $\sum_{j=1}^{l} \vec{e}_j - i \sum_{j=1}^{n+1} \vec{e}_j/(n+1)$. The corresponding highest weight state $|\lambda_l\rangle$ is defined by the conditions

$$h_j|\lambda_l\rangle = \delta_{lj}|\lambda_l\rangle, \qquad E_{+j}|\lambda_l\rangle = 0 \quad \text{for all } 1 \leqq j \leqq n\,, \tag{2 11}$$

and normalization $\langle\lambda_l|\lambda_l\rangle = 1$; moreover, $E_{-j}|\lambda_i\rangle = 0$ for all $j \neq i$ In accordance with this definition, the whole representation space of the i^{th} fundamental representation consists of all the vectors $|A_p \quad A_1\rangle_l \equiv E_{-A_p} \cdot E_{-A_1}|\lambda_i\rangle$, $1 \leqq A \leqq n$, $0 \leqq p \leqq N_i - 1$, $N_i = \binom{n+1}{l}$, with nonzero norm In what follows we use the fermionic realization of the elements of A_n, in which the Cartan and Chevalley generators are written as

$$h_l = \flat_l^+ \flat_l - \flat_{l+1}^+ \flat_{l+1}, \qquad E_i = \flat_{l+1}^+ \flat_i, \qquad E_{-i} = \flat_l^+ \flat_{l+1}, \quad 1 \leqq i \leqq n+1 \tag{2 12}$$

Here $\flat_p$ and $\flat_p^+$ are fermionic operators satisfying the standard anticommutation relations

$$[\flat_p, \flat_q]_+ = [\flat_p^+, \flat_q^+]_+ = 0, \qquad [\flat_p, \flat_q^+]_+ = \delta_{pq}\,,$$

and there exists a vacuum state $|0\rangle$, such that $\flat_p|0\rangle = 0$ for all $1 \leqq p \leqq n+1$, and, correspondingly, for the dual state $\langle 0|\flat_p^+ = 0$ In this, the i^{th} particle state, which is the highest weight state of the i^{th} fundamental representation of A_n, is obtained from the vacuum (cyclic) vector by the action of the raising operators, namely

$$|\lambda_l\rangle = \flat_l^+ \flat_{l-1}^+ \quad \flat_1^+|0\rangle, \qquad \text{and, respectively,} \quad \langle\lambda_i| = \langle 0|\flat_1 \quad \flat_{l-1}\flat_l$$

Now, let us consider the coset space $\mathscr{C}^{[i]}$ associated with the i^{th} fundamental representation of A_n It is quite clear that here the vector space of A_n is splitted into the semi-direct sum $\mathscr{G} = \mathscr{G}_{\|}^{[i]} \oplus \mathscr{G}_{\perp}^{[i]}$ of the subalgebra $\mathscr{G}_{\|}^{[i]}$ which is the stabiliser of the highest weight state $|\lambda_i\rangle$,

$$\mathscr{G}_{\|}^{[i]} = \{h_j,\ j \neq i,\ E_{-\vec{e}_p + \vec{e}_q},\ q \leqq p \leqq i \text{ and } q \geqq i+1,\ E_{\vec{e}_p - \vec{e}_q},\ p < q\}\,, \tag{2 13}$$

and the complement

$$\mathscr{G}_{\perp}^{[i]} = \{h_l,\ E_{-\vec{e}_p + \vec{e}_q},\ q \leqq i, \text{ and } p \geqq i+1\} \tag{2 14}$$

In these notations the coset $\mathscr{C}^{[i]}$ is constructed by exponentiation of the linear span of $\mathscr{G}_{\perp}^{[i]}$, namely

$$\mathscr{C}^{[i]} \quad e^{\Omega_i}|\lambda_l\rangle = \sum_{A_1 < A_2 < \quad < A_l} \Lambda^{[l]}_{A_1, \ldots, A_l} \flat_{A_l}^+ \quad \flat_{A_1}^+|0\rangle\,, \tag{2 15}$$

where

$$\Omega_l = \kappa_l(\flat_l^+ \flat_l - \flat_{l+1}^+ \flat_{l+1}) + \sum_{1 \leqq q \leqq l,\ l+1 \leqq p \leqq n+1} x_{pq}^{[i]} \flat_p^+ \flat_q \tag{2 16}$$

Every finite dimensional irreducible representation of A_l (and hence all the fundamental ones) is contained in a skew product of the finite number of copies of the 1^{st} fundamental irreducible representation This is why we may obtain all fundamental representations in this way In practice, the calculation of the coordinates

$\Lambda^{[i]}_{\alpha_1,\ldots,\alpha_i}$ goes in two steps. First one may introduce quantities noted $X^{[p],\alpha A_p}$, with $1 \leqq p \leqq n$ defined by

$$e^{\Omega_i} b^+_\alpha e^{-\Omega_i} = \sum_{A=1}^{n+1} X^{[i],\alpha,A} b^+_A \quad \text{for } A \leqq i\,, \tag{2 17}$$

and (2 15) gives

$$\sum_{B_1<B_2<\ldots<B_i} \Lambda^{[i]}_{B_1,\ldots,B_i} b^+_{B_i} \ldots b^+_{B_1}|0\rangle = \sum_{A_1,\ldots,A_i} X^{[i]i,A_i} \ldots X^{[i]1,A_1} b^+_{A_i} \ldots b^+_{A_1}|0\rangle \tag{2 18}$$

Thus the coordinates $\Lambda^{[i]}_{A_1,\ldots,A_i}$ of the i^{th} fundamental-representation space are the antisymmetrized combinations of the products $X^{[i]i,A_i}X^{[i]i-1,A_{i-1}}\ldots X^{[i]1,A_1}$ However, for the sake of brevity, we conventionally call the quantities $X^{[i]\alpha,A}$ coordinates also.

The explicit relation for the coordinates ξ_i, $x^{[i]}_{pq}$ of the coset $\mathscr{C}^{[i]}$ and the coordinates $X^{[i]\alpha,A}$ defined on the i^{th} fundamental representation space, is obtained by the straightforward computation using the simple formula

$$e^{\Omega_i} b^+_A e^{-\Omega_i} = \sum_{B=A}^{n+1} u^{[i]}_{BA} b^+_B \quad \text{for } A \leqq i\,, \tag{2 19}$$

where

$$u^{[i]}_{ii} = e^{\kappa_i}, \qquad u^{[i]}_{i+1\,i} = \frac{\sinh \kappa_i}{\kappa_i} x^{[i]}_{i+1,i}, \qquad u^{[i]}_{pi} = \frac{e^{\kappa_i}-1}{\kappa_i} x^{[i]}_{p,i}, \quad i+2 \leqq p \leqq n+1\,, \tag{2 20}$$

and for $A < i$, $A < B \leqq i$, and $i+2 \leqq p \leqq n+1$,

$$u^{[i]}_{AA} = 1, \qquad u^{[i]}_{BA} = 0, \qquad u^{[i]}_{i+1\,A} = \frac{1-e^{-\kappa_i}}{\kappa_i} x^{[i]}_{i+1,A}, \quad u^{[i]}_{pA} = x^{[i]}_{p,A}\,. \tag{2 21}$$

Then we come to the following parametrization of the coordinates $X^{[i]\alpha,A}$ entering (2 15),

$$X^{[i]i,A} = u^{[i]}_{Ai} \quad \text{for } A \geqq i, \qquad X^{[i]i,A} = 0 \quad \text{for } A \leqq i-1\,; \tag{2.22}$$

and for $\alpha < i$,

$$X^{[i]\alpha,A} = u^{[i]}_{A\alpha} \quad \text{for } A \geqq i+1,\ X^{[i]\alpha,\alpha} = 1\,, \tag{2.23}$$

$$X^{[i]\alpha,A} = 0 \quad \text{for } 1 \leqq A \leqq i-1,\ A \neq \alpha$$

With these coordinates, representation (2 15) leads to the corresponding Kähler potential $\mathscr{K}^{[i]}$ of the manifold $\mathscr{C}^{[i]}$,

$$\mathscr{K}^{[i]} = \log \|\Lambda^{[i]}\|^2 \equiv \log\,[\langle 0|\bar{X}^{[i]1} \ldots \bar{X}^{[i]i} X^{[i]i} \ldots X^{[i]1}|0\rangle] \tag{2 24}$$

Here

$$X^{[i]\alpha} \equiv \sum_A X^{[i]\alpha,A} b^+_A,\ \bar{X}^{[i]\alpha} \equiv \sum_A \bar{X}^{[i]\alpha,A} b_A,\ \Lambda^{[i]} \equiv \sum_{A_1<\ldots<A_i} \Lambda^{[i]}_{A_1,\ldots,A_i} b^+_{A_i} \ldots b^+_{A_1}|0\rangle\,.$$

Note that, in fact, formula (2.24), on account of the aforementioned identification of the Kähler potentials $\mathscr{K}^{[i]}$ with the Toda fields Φ_i, which is discussed below, gives a different representation of the tau-function for (2 2) than those from [4] Namely,

it is more adequate to the skew-product structure of the fundamental representation space

The relation (2.18) between the coordinates $\Lambda^{[i]}_{A_1,\ \ldots,A_i}$ of the i^{th} fundamental representation space, and the parameters $X^{[i]\alpha,A}$ immediately gives the quadratic relations

$$\sum_{\omega} \delta_{\omega}\, \Lambda^{[i]}_{A_1,\ \ldots,A_{i-1}A_{\omega(i)}}\, \Lambda^{[i]}_{A_{\omega(i+1)},\ \ldots,A_{\omega(2i)}} = 0, \quad 1 \leqq A_1 < \quad < A_{2i} \leqq n+1\,, \tag{2.25}$$

that defines Plücker quadrics, and the same is true for $\bar{\Lambda}^{[i]}_{A_1,\ \ldots,A_i}$ Here the sum runs over all inequivalent permutations ω of the integers $i, i+1, \ldots, 2i$, on account of the antisymmetricity of $\Lambda^{[i]}_{A_1,\ \ldots,A_i}$ under permutations of the indices $A_1, \quad, A_i$, and δ_{ω} is the parity of the permutation ω. Note that the system of quadrics (2.25) comes, in accordance with the contraction procedure given in [7], from equating to zero the skew-product of the i-vector $\hat{\Lambda}^{[i]}$, $\Lambda^{[i]} = \hat{\Lambda}^{[i]}|0\rangle$, and the vector $\flat^{(i-1)}\hat{\Lambda}^{[i]}$, which in turn is obtained from $\hat{\Lambda}^{[i]}$ under the action of $(i-1)$ annihilation operators $\flat$.

Indeed, one can be convinced that the metric which arises from such a Kähler potential (2.24), constructed with the coordinates $X^{[j]\gamma,A}$ and $\bar{X}^{[j]\alpha,A}$, is invariant under their transformation by an arbitrary $j \times j$ matrix from $GL(j, \mathbf{C})$, as well as relations (2.25) In particular, for the case of the 1^{st} fundamental representation of A_n, the corresponding complex projective target space is defined as the quotient of the space $\mathbf{C}^{n+1}$ by the equivalence rescaling $\mathbf{X} \sim \mathbf{Y}$, if $X^A = Y^A \rho(Y)$, and $\bar{X}^{\bar{A}} = \bar{Y}^{\bar{A}} \bar{\rho}(\bar{Y})$, with arbitrary chiral functions $\rho(Y)$ and $\bar{\rho}(\bar{Y})$, the metric, invariant under this rescaling, is the Fubini–Study metric corresponding to the Kähler potential $\mathcal{K}^{[1]} = \ln \sum_{A=1}^{n+1} X^A \bar{X}^A$

On the W_{A_n}-surface $\mathcal{K}^{[1]} = -\Phi_1$, and on the associated surfaces the Kähler potentials also coincide, up to the sign, with the corresponding A_n-Toda fields which are given by the r.h.s. of expressions (2.9). So, as we have already mentioned, a relevant object for the description of the W_{A_n}-geometry of $\mathbf{C}^n$ – target manifolds with a positive curvature form is the Plücker embeddings of the Grassmannians $\mathcal{G}r(n+1|k)$ in the projective spaces, $\mathcal{G}r(n+1|k) \Rightarrow \mathbf{CP}^{\binom{n+1}{k}-1}$. In this, we identify $X^{[1]1,A}$ and $\bar{X}^{[1]1,A}$ with the embedding functions in (2.6) on account of condition (2.7), and, in general, putting

$$X^{[i]\alpha,A} = \partial^{\alpha-1} f^A, \qquad \bar{X}^{[i]\gamma,A} = \bar{\partial}^{\alpha-1} \bar{f}^A\,, \tag{2.26}$$

leads to the aforementioned equalities between the Kähler potentials and the A_n-Toda fields [1] (up to a minus sign) Then, the known Plücker representation for the pseudo-metrics $d\mathcal{S}^2_k$ on Σ, specified by the corresponding Kähler potentials, see e.g. [7], is written in terms of these fields as

$$\begin{aligned} d\mathcal{S}^2_k &= \frac{i}{2} \exp 2(-\Phi_{k-1} + 2\Phi_k - \Phi_{k+1})\, dz\, d\bar{z} \\ &\equiv \frac{i}{2} \exp 2 \left(\sum_{j=1}^{n} K^{A_n}_{kj} \Phi_j \right) dz\, d\bar{z}\,, \end{aligned} \tag{2.27}$$

where $K^{A_n}_{ij}$ is the Cartan matrix of the algebra A_n

In distinction to the case of the Lie algebra A_n, a similar study of $W_{\mathcal{G}}$-geometry of the Toda systems for other simple Lie algebras requires a modification of the standard Plücker embedding The point is that the number of independent screening functions $s_i(z)$ and $\bar{s}_i(\bar{z})$ determining the general solution (2 3) of the corresponding Toda systems is always $2n$, while the number of functions $f^A(z)$ and $\bar{f}^{\bar{A}}(\bar{z})$ is, in general, much more, even for the representations of $\mathcal{G}$ of minimal dimension For example, the relations $\Phi_j = \Phi_{n-j+1}$ for the Toda fields of A_n leads for $n = 2s$ to the solutions corresponding to the series B_s, for $n = (2s-1)$ – to those of C_s, and in terms of the screening functions this equating means that $s_i(z) = s_{n-i+1}(z)$ and $\bar{s}_i(\bar{z}) = \bar{s}_{n-i+1}(\bar{z})$ Moreover, due to the known relation between the orders of the functionally independent characteristic integrals for the Toda systems, in other words the W-elements, and those of the Casimir operators of the corresponding Lie algebras $\mathcal{G}$, the mentioned difference is also quite natural.

3. The Target Spaces as Group-Orbits of Fundamental Weights: The Case of D_n

3 1 Fermionic realizations Let us first recall some properties of the Lie algebra D_n, see e g. [12, 13], and its fermionic realization The roots are of the form

$$\vec{\alpha} = \pm\vec{e}_i \pm \vec{e}_j, \quad \text{with } 1 \leqq i < j \leqq n\,, \tag{3 1}$$

in the n-dimensional space spanned by the orthonormal vectors $\vec{e}_i$ The elements of D_n can be realized using $2n$ fermionic operators $\flat_{\pm j}$, $j = 1, \quad , n$, which satisfy the relations

$$[\flat_l, \flat_m]_+ = [\flat_l^+, \flat_m^+]_+ = 0, \qquad [\flat_l, \flat_m^+]_+ = \delta_{l,m}, \quad -n \leqq l, m \leqq n\,, \tag{3 2}$$

as

$$E_{\vec{e}_i \mp \vec{e}_j} = \flat_i^+ \flat_{\pm j} - \flat_{\mp j}^+ \flat_{-i}\,, \qquad E_{-\vec{e}_i \pm \vec{e}_j} = E^+_{\vec{e}_i \mp \vec{e}_j} = \flat^+_{\pm j}\flat_i - \flat^+_{-i}\flat_{\mp j}\,, \tag{3.3}$$

and for the Cartan generators

$$\begin{aligned} &h_i = H_i - H_{i+1}, \quad i = 1, \ . \ , n-1, \\ &h_n = H_{n-1} + H_n; \quad H_i = \flat_i^+ \flat_i - \flat_{-i}^+ \flat_{-i} \end{aligned} \tag{3 4}$$

A set of simple positive roots is $\vec{\pi}_i = \vec{e}_i - \vec{e}_{i+1}$, $i = 1, \ldots, n-1$, and $\vec{\pi}_n = \vec{e}_{n-1} + \vec{e}_n$ Let $E_{\pm i}$ be the corresponding Chevalley elements One has

$$E_i = \flat_i^+ \flat_{i+1} - \flat^+_{-i-1}\flat_{-i}, \qquad E_{-i} = \flat^+_{i+1}\flat_i - \flat^+_{-i}\flat_{-i-1} \tag{3 5}$$

for $i = 1, \quad , n-1$, and

$$E_n = \flat^+_{n-1}\flat_{-n} - \flat^+_n \flat_{-n+1}, \qquad E_{-n} = \flat^+_{-n}\flat_{n-1} - \flat^+_{-n+1}\flat_n \tag{3 6}$$

The fundamental weights are

$$\vec{\lambda}_j = \sum_1^j \vec{e}_k, \quad k \leqq n-2, \quad \vec{\lambda}_{n-1} = \frac{1}{2}\left(\sum_1^{n-1}\vec{e}_k - \vec{e}_n\right), \quad \vec{\lambda}_n = \sum_1^n \frac{\vec{e}_k}{2}\,. \tag{3 7}$$

As is well-known [12], the first $n-2$ fundamental representations are of the same nature – in contrast with the last two ones. Their weight vectors have integer components as Eqs. (3 7) show They are immediately realized in the Fock space of the fermionic operators just introduced as follows The state

$$|\lambda_p\rangle \equiv \flat_p^+ \quad \flat_1^+|0\rangle, \quad 1 \leqq p \leqq n-2\,, \tag{3 8}$$

satisfy the highest weight condition (2 11) The state $|0\rangle$ is the usual vacuum state with zero occupation numbers, such that

$$\flat_l|0\rangle, \quad -n \leqq l \leqq n \tag{3.9}$$

How can we get the last two representations? The trick is to introduce the operators

$$\begin{aligned} c_l &\equiv \frac{1}{\sqrt{2}}(\flat_l + \flat_{-l}^+), \qquad c_l^+ \equiv \frac{1}{\sqrt{2}}(\flat_l^+ + \flat_{-l})\,, \\ d_l &\equiv \frac{1}{i\sqrt{2}}(\flat_l - \flat_{-l}^+), \qquad d_l^+ \equiv \frac{1}{i\sqrt{2}}(\flat_l^+ - \flat_{-l}), \quad 1 \leqq l \leqq n\,; \end{aligned} \tag{3 10}$$

which satisfy, according to (3 2),

$$\begin{aligned} [c_l, c_m]_+ &= [c_l^+, c_m^+]_+ = 0, \qquad [c_l, c_m^+]_+ = \delta_{l,m}\,, \\ [d_l, d_m]_+ &= [d_l^+, d_m^+]_+ = 0, \qquad [d_l, d_m^+]_+ = \delta_{l,m}\,, \\ [c_l, d_m]_+ &= [c_l^+, d_m^+]_+ = 0, \qquad [c_l, d_m^+]_+ = 0 \end{aligned} \tag{3.11}$$

These new operators give us two other realizations of the algebra D_n by introducing

$$\begin{aligned} E^{(1,2)}_{\vec{e}_i+\vec{e}_j} &= c_i^+ c_j^+, \qquad E^{(1,2)}_{-\vec{e}_i-\vec{e}_j} = c_j c_i, \\ E^{(1,2)}_{\vec{e}_i-\vec{e}_j} &= c_i^+ c_j, \qquad E^{(1,2)}_{-\vec{e}_i+\vec{e}_j} = c_j^+ c_i, \\ h_l^{(1,2)} &= H_l^{(1,2)} - H_{l+1}^{(1,2)}, h_n^{(1,2)} = H_{n-1}^{(1,2)} + H_n^{(1,2)}, \\ H_l^{(1,2)} &= c_l^+ c_l - \frac{1}{2}\,, \end{aligned} \tag{3 12}$$

$$\begin{aligned} \widetilde{E}^{(1,2)}_{\vec{e}_i+\vec{e}_j} &= d_i^+ d_j^+, \qquad \widetilde{E}^{(1,2)}_{-\vec{e}_i-\vec{e}_j} = d_j d_i, \\ \widetilde{E}^{(1,2)}_{\vec{e}_i-\vec{e}_j} &= d_i^+ d_j, \qquad \widetilde{E}^{(1,2)}_{-\vec{e}_i+\vec{e}_j} = d_j^+ d_i, \\ \widetilde{h}_l^{(1,2)} &= \widetilde{H}_l^{(1,2)} - \widetilde{H}_{l+1}^{(1,2)}, \qquad \widetilde{h}_n^{(1,2)} = \widetilde{H}_{n-1}^{(1,2)} + \widetilde{H}_n^{(1,2)}, \\ \widetilde{H}_l^{(1,2)} &= d_l^+ d_l - \frac{1}{2} \end{aligned} \tag{3.13}$$

For later use we note that a straightforward calculation gives

$$E_{\pm\vec{e}_i\pm\vec{e}_j} = E^{(1,2)}_{\pm\vec{e}_i\pm\vec{e}_j} + \widetilde{E}^{(1,2)}_{\pm\vec{e}_i\pm\vec{e}_j}, \qquad H_i = H_i^{(1,2)} + \widetilde{H}_i^{(1,2)}\,. \tag{3 14}$$

After this Bogolubov type transformation, the new vacuum state noted $|0\rangle^{(1\,2)}$ which is annihilated by the operators c_l and d_l, is given by

$$|0\rangle^{(1\,2)} = b^+_{-n} \quad b^+_{-1}|0\rangle, \qquad c_l|0\rangle^{(1\,2)} = d_l|0\rangle^{(1\,2)} = 0\,. \tag{3 15}$$

Using either of the above two sets we can construct the last two highest weight vectors Consider, for instance the c-oscillators One easily verifies that the states

$$|\lambda_{n-1}\rangle = c^+_{n-1} \quad \cdot c^+_1|0\rangle^{(1\,2)}, \qquad |\lambda_n\rangle = c^+_n \quad \cdot c^+_1|0\rangle^{(1\,2)}\,, \tag{3 16}$$

satisfy the highest-weight equations

$$E^{(1\,2)}_l|\lambda_p\rangle = 0, \qquad h^{(1\,2)}_l|\lambda_p\rangle = \delta_{l,\,p}, \quad p = n-1,\, n \tag{3 17}$$

The generators (3 3), (3 4) commute with the fermionic number[11]

$$\mathcal{N}_F \equiv \sum_{l=-n}^{n} b^+_l b_l\,, \tag{3 18}$$

and a representation with weight λ_p, with $p \leqq n-2$ is realized in the space with a fixed number (p) of fermions Thus we call it a bosonic representation On the contrary, the operators of the realization (3.12) or (3.13) do not commute with $\mathcal{N}_F$ We call them fermionic representations

The Fock space we are considering allows us to realize every fundamental representation in the same Hilbert space This is instrumental for the coming discussions since the Toda equations and the corresponding infinitesimal Plücker formulae do in fact connect these different representations, so that they will be most naturally understood in the Fock space of the b operators Moreover, this Fock space contains additional highest weight vectors which will be very useful as well First, the fermionic fundamental representations are realized twice, since we may also use formulae similar to (3 17), obtained after replacing c- by d-oscillators We shall denote these states by $|\tilde{\lambda}_{n-1}\rangle$ and $|\tilde{\lambda}_n\rangle$ (c f e g [12]) Second, there are other states analogous to (3 8) They are given by $b^+_{n-1} \quad b^+_1|0\rangle$, $b^+_n \quad b^+_1|0\rangle$, and $b^+_{-n}b^+_{n-1} \quad b^+_1|0\rangle$ These are highest-weight states since it follows from (3.5), and (3 6) that they are annihilated by E_i, for $i = 1, \ldots, n$. The corresponding highest weights are given by

$$\begin{aligned}
h_l b^+_{n-1} \quad b^+_1|0\rangle &= (\delta_{l,n-1} + \delta_{l,n})b^+_{n-1} \quad \cdot \quad b^+_1|0\rangle\,,\\
h_l b^+_n \quad b^+_1|0\rangle &= 2\delta_{l,n} b^+_n \quad b^+_1|0\rangle\,,\\
h_l b^+_{-n}b^+_{n-1} \quad b^+_1|0\rangle &= 2\delta_{l,n-1} b^+_n \quad b^+_1|0\rangle
\end{aligned}$$

Comparing with (3 7), one sees that their weights are $\lambda_{n-1} + \lambda_n$, $2\lambda_n$, and $2\lambda_{n-1}$, respectively Thus we write

$$\begin{aligned}
|\lambda_{n-1} + \lambda_n\rangle &= b^+_{n-1} \quad b^+_1|0\rangle, \qquad |2\lambda_n\rangle = b^+_n \quad b^+_1|0\rangle\,,\\
|2\lambda_{n-1}\rangle &= b^+_{-n}b^+_{n-1} \quad b^+_1|0\rangle
\end{aligned} \tag{3 19}$$

[11] In this formula and in the following, summations from $-n$ to n do not include zero

The representations generated by these highest-weight states are irreducible, but not fundamental We shall see how they fit in the general scheme of Toda solutions, where they come out naturally in the fermionic derivation of Toda solutions Last we not – this will be useful later on – that there is another realization of the fundamental highest weight vectors of the bosonic type Indeed, it is easy to see that

$$[H_l,\ \flat_k \flat_{-k}] = 0\,, \tag{3.20}$$

so that there is another highest weight state

$$|\lambda_{-p}\rangle \equiv \flat^+_{-p-1}\flat^+_{-p-2} \qquad \flat^+_{-n}\flat^+_n \qquad \flat^+_1|0\rangle \tag{3 21}$$

with the same highest weight as $|\lambda_p\rangle$, that is $\lambda_p = \lambda_{-p}$ From the viewpoint of the fermionic operators, a transition from $|\lambda_{-p}\rangle$ to $|\lambda_p\rangle$ is equivalent to the exchange of $\flat_j$ with $\flat^+_{-j}$ Indeed, we have

$$\begin{aligned} \flat^+_j|\lambda_p\rangle &= 0, \quad 1 \leqq j \leqq p, \qquad \flat_j|\lambda_p\rangle = 0,\ j > p, \ \text{ or } j \leqq -1, \\ \flat_{-j}|\lambda_{-p}\rangle &= 0, \quad 1 \leqq j \leqq p, \qquad \flat^+_{-j}|\lambda_{-p}\rangle = 0,\ j > p, \ \text{ or } j \leqq -1 \end{aligned} \tag{3 22}$$

3 2 The target spaces associated with bosonic representations For any given highest-weight vector $|\lambda_p\rangle$, we split the Lie algebra D_n into two parts.

$$D_n = \mathscr{G}^{[p]}_{\|} \uplus \mathscr{G}^{[p]}_{\perp} \tag{3 23}$$

The one called $\mathscr{G}^{[p]}_{\|}$ leaves $|\lambda_p\rangle$ invariant, it forms a Lie algebra The symbol $\mathscr{G}^{[p]}_{\perp}$ denotes the orthogonal complement The corresponding coset, denoted $\mathscr{C}^{[p]}$, is generated by exponentiating its linear span The mathematical properties of these cosets are re-derived in Appendix A using the present fermionic realization Next we describe the geometrical properties of these cosets

3 2 1 The coset space associated with λ_1 Following the method just described, and according to Appendix A, this space is parametrized by[12]

$$\begin{aligned} e^{\kappa_1 h_1} e^{\Omega_1} \flat^+_1|0\rangle, \quad \text{with } \Omega_1 &= \sum_{k=2}^{n} (x_k E_{-\vec{e}_1+\vec{e}_k} + x_{-k} E_{-\vec{e}_1-\vec{e}_k})\,, \\ \langle 0|\flat_1 e^{-\bar{\kappa}_1 h_1} e^{-\bar{\Omega}_1}, \quad \text{with } \bar{\Omega}_1 &= \sum_{k=2}^{n} (\bar{x}_k E_{\vec{e}_1+\vec{e}_k} + \bar{x}_{-k} E_{\vec{e}_1-\vec{e}_k})\,, \end{aligned} \tag{3 24}$$

where κ_1, $\bar{\kappa}_1$ and $x_{\pm k}$, $\bar{x}_{\pm k}$ are group parameters that will give a special parametrization of the coset After some computations, one derives that

$$e^{\Omega_1} \flat^+_1 e^{-\Omega_1} = \flat^+_1 + \sum_{k=2}^{n} \sum_{\varepsilon=\pm 1} x_{\varepsilon k} \flat^+_{\varepsilon k} - \left(\sum_{k=2}^{n} x_k x_{-k} \right) \flat^+_{-1}\,, \tag{3 25}$$

[12] For D_n we exponentiate the Cartan generator h_p separately so that the explicit formulae do not become too complicated (see the appendix)

with similar equations for the $\bar{\kappa}$ and $\bar{x}$ coordinates It is convenient to write

$$e^{\kappa_1 h_1} e^{\Omega_1} \flat_1^+ |0\rangle = \sum_{-n \leqq A \leqq n,\, A \neq 0} X^A \flat_A^+ |0\rangle ,$$

$$\langle 0| \flat_1 e^{-\bar{\kappa}_1 h_1} e^{-\bar{\Omega}_1} = \sum_{-n \leqq A \leqq n,\, A \neq 0} \langle 0| \flat_A \bar{X}^A ; \tag{3 26}$$

where

$$X^1 = e^{\kappa_1}, \qquad X^{-1} = -e^{-\kappa_1} \sum_{k=2}^{n} x_k x_{-k}, \qquad X^{\pm 2} = e^{\mp \kappa_1} x_{\pm 2},$$

$$X^{\pm k} = x_{\pm k},\ k > 2 ,$$

$$\bar{X}^1 = e^{\bar{\kappa}_1}, \qquad \bar{X}^{-1} = -e^{-\bar{\kappa}_1} \sum_{k=2}^{n} \bar{x}_k \bar{x}_{-k}, \qquad \bar{X}^{\pm 2} = e^{\mp \bar{\kappa}_1} \bar{x}_{\pm 2},$$

$$\bar{X}^{\pm k} = \bar{x}_{\pm k},\ k > 2 . \tag{3.27}$$

The functions X^A, $\bar{X}^A$ satisfy the quadratic equations

$$\sum_A X^A X^{-A} = 0, \qquad \sum_A \bar{X}^A \bar{X}^{-A} = 0 \tag{3 28}$$

In this equation and in the following, the sums over A run from $-n$ to n, with 0 excluded. It is convenient to introduce the following notation·

$$X \equiv \sum_A X^A \flat_A^+, \qquad \bar{X} \equiv \sum_A \bar{X}^A \flat_A . \tag{3 29}$$

It is natural to define a Kähler metric on $\mathscr{C}^{[1]}$ derived from the Kähler potential

$$\mathscr{K}^{(1)}(X, \bar{X}) \equiv \ln[\langle 0| \flat_1 e^{-\bar{\Omega}_1} e^{-\bar{\kappa}_1 h_1} e^{\kappa_1 h_1} e^{\Omega_1} \flat_1^+ |0\rangle] = \ln[\langle 0| \bar{X} X |0\rangle] \tag{3.30}$$

which has an obvious group invariance Together with condition (3.28), this completes the definition of the manifold $\mathscr{C}^{[1]}$. It may be understood as a submanifold of CP^{2n-1} Indeed, the Kähler potential coincides with the one of Fubini–Study, and the quadratic constraints are invariant under the rescalings

$$X^A \to X^A \rho(X), \qquad \bar{X}^A \to \bar{X}^A \bar{\rho}(\bar{X}) , \tag{3 31}$$

that leave the points of CP^{2n-1} invariant. The manifold $\mathscr{C}^{[1]}$ is thus a quadric in CP^{2n-1}

Choose coordinates such that $X^1 = \bar{X}^1 = 1$. Then we can solve the constraint (3 28), obtaining

$$X^{-1} = -\sum_{A=2}^{n} X^A X^{-A}, \qquad \bar{X}^{-1} = -\sum_{A=2}^{n} \bar{X}^A \bar{X}^{-A} , \tag{3 32}$$

and the Kähler potential becomes

$$\mathscr{K}^{(1)}(X, \bar{X}) = \ln \left[1 + \sum_{A=2}^{n} (X^A \bar{X}^A + X^{-A} \bar{X}^{-A}) + \left(\sum_{B=2}^{n} X^B X^{-B} \right) \left(\sum_{\bar{B}=2}^{n} \bar{X}^{\bar{B}} \bar{X}^{-\bar{B}} \right) \right] \tag{3.33}$$

This is the equivalent of the Fubini–Study metric for the present case

3 2 2 The coset space associated with generic bosonic fundamental highest weight The discussion is very close to the above. The manifold $\mathscr{C}^{[p]}$ is parametrized by

$$e^{\kappa_p h_p} e^{\Omega_p} |\lambda_p\rangle = \sum_{A_1, \cdots, A_p} X^{[p]\,p, A_p} \quad X^{[p]\,1, A_1} \flat^+_{A_p} \quad \flat^+_{A_1} |0\rangle ,$$

$$\langle \lambda_p | e^{-\bar{\Omega}_p} e^{-\bar{\kappa}_p h_p} = \sum_{A_1, \cdots, A_p} \langle 0 | \flat_{A_1} \quad \flat_{A_p} \bar{X}^{[p]\,p, A_p} \quad \bar{X}^{[p]\,1, A_1} , \tag{3 34}$$

where we have let

$$\begin{aligned} \sum_A X^{[p]\,\alpha, A} \flat^+_A &\equiv X^{[p]\,\alpha} = e^{\kappa_p h_p} e^{\Omega_p} \flat^+_\gamma e^{-\kappa_p h_p} e^{-\Omega_p} , \\ \sum_A \bar{X}^{[p]\,\alpha, A} \flat^+_A &\equiv \bar{X}^{[p]\,\gamma} = e^{-\bar{\kappa}_p h_p} e^{-\bar{\Omega}_p} \flat_\alpha e^{\bar{\kappa}_p h_p} e^{\bar{\Omega}_p} \end{aligned} \tag{3 35}$$

The natural Kähler potential

$$\mathscr{K}^{[p]} \equiv \ln[\langle \lambda_p | e^{-\bar{\Omega}_p} e^{-\bar{\kappa}_p h_p} e^{\kappa_p h_p} e^{\Omega_p} | \lambda_p \rangle] \tag{3 36}$$

takes the form of Kobayashi–Nomizu

$$\mathscr{K}^{[p]}(X^{[p]}, \bar{X}^{[p]}) = \ln[\langle 0 | \bar{X}^{[p]1} \quad \bar{X}^{[p]p-1} \bar{X}^{[p]p} X^{[p]p} X^{[p]p-1} \cdots \quad X^{[p]1} | 0 \rangle] . \tag{3.37}$$

The precise connection between the coordinates X, $\bar{X}$ and the group parameters is given in the appendix, where it is also shown that the coordinates X, and $\bar{X}$ satisfy the quadratic relations,

$$\begin{cases} \sum_A X^{[p]\,\alpha, A} X^{[p]\,\beta, -A} = 0, \\ \sum_A \bar{X}^{[p]\,\alpha, A} \bar{X}^{[p]\,\beta, -A} = 0, \end{cases} \quad 1 \leqq \alpha \leqq p, \ 1 \leqq \beta \leqq p \tag{3.38}$$

The origin of these relations is that there are less group parameters than coordinates X and $\bar{X}$ A compact proof of these identities goes as follows[13] Due to the special form of the generators of D_n, which is displayed in (3.34)–(3 4), there exists a symmetry which we call charge conjugation, and denote by a superscript c It is defined by

$$(\flat^+_{A_k} \quad \flat^+_{A_1} \flat_{B_l} \quad \flat_{B_1})^c = (\flat^+_{-A_k} \quad \flat^+_{-A_1} \flat_{-B_l} \quad \flat_{-B_1})^+ , \tag{3 39}$$

and relations (3.34)–(3.4) show that

$$E^c_{\pm \vec{e}_i \mp \vec{e}_j} = -E_{\pm \vec{e}_i \mp \vec{e}_j}, \qquad h^c_i = -h_i \tag{3 40}$$

The origin of this charge conjugation is the orthogonality of D_n It transforms the first line of (3 35) into

$$\sum_A X^{[p]\,\alpha, A} \flat^+_{-A} = e^{\kappa_p h_p} e^{\Omega_p} \flat_{-\alpha} e^{-\kappa_p h_p} e^{-\Omega_p} ,$$

so that

$$\langle 0 | \flat_{-\gamma} \flat^+_\beta | 0 \rangle = \langle 0 | e^{\kappa_p h_p} e^{\Omega_p} \flat_{-\alpha} e^{-\Omega_p} e^{-\kappa_p h_p} e^{\kappa_p h_p} e^{\Omega_p} \flat^+_\beta e^{\Omega_p} | 0 \rangle = \sum_A X^{[p]\,\alpha, A} X^{[p]\,\beta, -A}$$

[13] We treat the coordinates X The coordinates $\bar{X}$ could obviously be discussed similarly

Since, obviously $\langle 0|\flat_{-\alpha}\flat^{+}_{\beta}|0\rangle = 0$, relations (3 38) follows This fact completes[14] the definition of the manifold $\mathscr{C}^{[p]}$ Next we show that it is a submanifold of the usual Grassmannian manifold $Gr(2n-1|p)$. In general, $Gr(2n-1|p)$ is the set of $(2n-1)\times p$ matrices $\mathscr{F}_p$ with the equivalence relation $\mathscr{F}_p \sim \rho\mathscr{F}_p$, where ρ is an arbitrary $p\times p$ matrix, that is the generalization of (3 31), which corresponds to $p=1$ The geometrical meaning of this equivalence is well known· given $\mathscr{F}_p$, one defines hyperplanes in CP^{2n-1} by equations of the form $Z^{A}(t)=\sum_{\alpha}\mathscr{F}_{p}^{\alpha,A}t_{\alpha}$ The equivalence relations is equivalent to linear transformations of the parameters t_{α} Thus the Grassmannian describes the geometrical hyperplane which should not depend upon their parametrizations In our case the coordinates are $X^{[p]\alpha,A}$, and $\bar{X}^{[p]\alpha,\bar{A}}$ It is well known that the metric derived from the Kähler potential (3 37) is invariant under the transformation

$$X^{[p]\alpha,A}\to\rho(X^{[p]})^{\alpha}_{\beta}X^{[p]\beta,A},\qquad \bar{X}^{[p]\alpha,\bar{A}}\to\bar{\rho}(\bar{X}^{[p]})^{\alpha}_{\beta}\bar{X}^{[p]\beta,\bar{A}} \tag{3.41}$$

Moreover, it is easy to see that, if $X^{[p]\alpha,A}$ satisfies condition (3 38), this is also true for $\sum_{\beta}\rho^{\alpha}_{\beta}X^{[p]\beta,A}$. Thus these conditions define a quadric in the Grassmannian $Gr(2n-1|p)$

3 2 3 The three additional coset spaces It is obvious that the previous description of the cosets extends to the representations with the highest vectors $|\lambda_{n-1}+\lambda_n\rangle$, $|2\lambda_n\rangle$, and $|2\lambda_{n-1}\rangle$ (see definitions (3 19)) without any problem The first two cases are direct extensions of the formulae given for $|\lambda_p\rangle$ with $p\leqq n-1$ The last one is obtained from the calculation for $|2\lambda_{n-1}\rangle$, by exchanging everywhere $\flat_n$ with $\flat_{-n}$ Some details are given in the appendix Always using similar notations, we introduce

$$X^{[\lambda]l}=e^{\kappa_l h_l}e^{\Omega_l}\flat_l^{+}e^{-\Omega_l}e^{-\kappa_l h_l},\bar{X}^{[\lambda]l}=e^{\bar{\kappa}_l h_l}e^{\bar{\Omega}_l}\flat_l e^{-\bar{\Omega}_l}e^{-\bar{\kappa}_l h_l}, \tag{3 42}$$

$$\mathscr{K}^{[\lambda]}(X^{[\lambda]},\bar{X}^{[\lambda]})=\ln[\langle 0|\bar{X}^{[\lambda]1}\quad\bar{X}^{[\lambda]p_l}X^{[\lambda]p_l}\quad X^{[\lambda]1}|0\rangle]\,. \tag{3.43}$$

The same reasoning as above shows that one has the quadratic conditions

$$\sum_{A}X^{[\lambda]pA}X^{[\lambda]q-A}=0 \tag{3.44}$$

These Kähler potentials take the Kobayashi–Nomizu form. They will appear naturally in connection with the explicit solution of the D_n-Toda equations The formulae just given define manifolds $\mathscr{C}^{[\lambda]}$ for $\lambda=\lambda_{n-1}+\lambda_n$, $\lambda=2\lambda_n$, and $\lambda=2\lambda_{n-1}$.

3 3 The case of the two fermionic fundamental representations As already recalled, these two fundamental representations are of a completely different nature [12] While the $n-2$ first ones have dimensions $\binom{2n}{p}$, $p\leqq n-2$, they have dimension 2^{n-1} We make use of the realization (3 12), the highest weight states are given by (3 16) This being established, the discussion proceeds exactly as before

[14] Obviously, the coordinates $X^{[p]\alpha,A}$ and $\bar{X}^{[p]\alpha,A}$ satisfy, in addition, Plücker type relations similar to (2 25) We shall not dwell on this aspect

The coordinates are given by

$$e^{\kappa_p h_p^{(1\,2)}} e^{\Omega_p^{(1\,2)}} |\lambda_p\rangle = \check{X}^{[p]\,p} \check{X}^{[p]\,p-1} \quad \check{X}^{[p]\,1} |0\rangle ,$$

$$\check{X}^{[p]\,\gamma} \equiv e^{\kappa_p h_p^{(1\,2)}} e^{\Omega_p^{(1\,2)}} c_\gamma^+ e^{-\Omega_p^{(1\,2)}} e^{-\kappa_p h_p^{(1\,2)}} = \sum_{A>0} (X^{[p]\,\gamma,A} c_A^+ + X^{[p]\,\gamma,-A} c_A) ,$$

$$\langle \lambda_p | e^{-\bar{\kappa}_p h_p^{(1\,2)}} e^{-\bar{\Omega}_p^{(1\,2)}} = \langle 0 | \check{\bar{X}}^{[p]\,p} \check{\bar{X}}^{[p]\,p-1} \quad \check{\bar{X}}^{[p]\,1} ,$$

$$\check{\bar{X}}^{[p]\,\gamma} \equiv e^{\bar{\kappa}_p h_p^{(1\,2)}} e^{\bar{\Omega}_p^{(1\,2)}} c_\gamma e^{-\bar{\Omega}_p^{(1\,2)}} e^{-\bar{\kappa}_p h_p^{(1\,2)}} = \sum_{A>0} (\bar{X}^{[p]\,\gamma,A} c_A + \bar{X}^{[p]\,\gamma,-A} c_A^+) , \qquad (3\,45)$$

for $\alpha \leqq p$, $p = n-1$, and $p = n$ As before there are again quadratic constraints which may be derived by writing

$$e^{\kappa_p h_p^{(1\,2)}} e^{\Omega_p^{(1\,2)}} [c_\alpha^+, c_\beta^+]_+ e^{-\Omega_p^{(1\,2)}} e^{-\kappa_p h_p^{(1\,2)}} = 0 = [\check{X}^{[p]\,\gamma}, \check{X}^{[p]\,\beta}] .$$

This gives again relations (3.38), now with $p = n-1$ and $p = n$ The natural Kähler potential, that is (3 36) for $p = n-1$, n, is given by

$$\mathcal{K}^{[p]}(X^{[p]}, \bar{X}^{[p]}) = \ln[\langle 0 | \check{\bar{X}}^{[p]1} \cdot \check{\bar{X}}^{[p]p-1} \check{\bar{X}}^{[p]p} \check{X}^{[p]p} \check{X}^{[p]p-1} \quad \check{X}^{[p]1} |0\rangle] . \quad (3\,46)$$

A priori it is different from the Kobayashi–Nomizu form, since the operators $\check{X}$, and $\check{\bar{X}}$ involve both creation and annihilation operators. We shall spell out the connection below This completes[15] the definition of the manifolds $\mathcal{C}^{[p]}$, for $p = n-1$, n

3 4 Connection between Kähler potentials The last three coset spaces just discussed are not associated with fundamental representations We now show that they can be re-expressed in terms of the potential associated with the last two fundamental highest weights This, of course, is due to the fact that their highest weights are linear combinations of λ_{n-1} and λ_n The present fermionic method gives a quick derivation of this fact Indeed, we already mentioned that the fermionic fundamental representations are realized twice, once in terms of the c-oscillators (3.12), and once in terms of the d-oscillators (3 13) Using formulas (3 10), one sees that

$$c_n^+ \quad c_1^+ d_n^+ \quad d_1^+ |0\rangle^{(1/2)} = i^n (-1)^{n(n+1)/2} |2\lambda_n\rangle ,$$

$$c_{n-1}^+ \quad c_1^+ d_n^+ \quad d_1^+ |0\rangle^{(1\,2)} = i^n (-1)^{n(n+1)\,2} / \sqrt{2} (|\lambda_{n-1} + \lambda_n\rangle + |\lambda_{-(n-1)} + \lambda_{-n}\rangle) ,$$

$$c_{n-1}^+ \quad c_1^+ d_{n-1}^+ \quad d_1^+ |0\rangle^{(1\,2)} = i^{n-1} (-1)^{n(n-1)/2} |2\lambda_{n-1}\rangle , \qquad (3.47)$$

where the state $|\lambda_{-(n-1)} + \lambda_{-n}\rangle$ is defined by the obvious generalization of (3 21) The formulae just written are clearly consistent with (3 14) Now, we may re-derive the expressions (3 43) of the Kähler potentials $\mathcal{K}^{[\lambda]}$ with $\lambda = \lambda_{n-1} + \lambda_n$, $\lambda = 2\lambda_n$, and $\lambda = 2\lambda_{n-1}$, using the l.h s of the last equations together with (3.13). In this way, the calculations involving the c and d operators become completely separated Each of them is entirely specified by the group properties of the fermionic fundamental representations which do not depend upon of the realization chosen Moreover, the

[15] Here also there are additional quadratic relations similar to (2 25)

dimensions of the manifolds involved coincide, so that there are natural mappings between them. It is then easy to conclude that the Kähler potentials are related by

$$\begin{aligned}
\mathscr{K}^{[2\nu_n]}(X^{[n]},\bar{X}^{[n]}) &= 2\mathscr{K}^{[n]}(X^{[n]},\bar{X}^{[n]}),\\
\mathscr{K}^{[\nu_{n-1}+\nu_{n-1}]}(X^{[n]},\bar{X}^{[n]}) &= \mathscr{K}^{[n]}(X^{[n]},\bar{X}^{[n]})+\mathscr{K}^{[n-1]}(X^{[n]},\bar{X}^{[n]}),\\
\mathscr{K}^{[2\nu_{n-1}]}(X^{[n-1]},\bar{X}^{[n-1]}) &= 2\mathscr{K}^{[n-1]}(X^{[n-1]},\bar{X}^{[n-1]})
\end{aligned} \tag{3 48}$$

These relations will be important later on

4. Generalized Plücker Embeddings for D_n

4 1 Definitions Let us introduce the following definitions which will be motivated by the forthcoming discussions.

Definition 1. *D_n-W-surfaces The W surfaces associated with the Lie algebra D_n are two dimensional surfaces $\Sigma^{[1]}$ in $\mathscr{C}^{[1]}$ defined by the equations*

$$X^{[1]A}=f^A(z),\qquad \bar{X}^{[1]A}=\bar{f}^A(\bar{z}), \tag{4.1}$$

where $f^A(z)$ and $\bar{f}^A(\bar{z})$ satisfy the quadratic differential relations

$$\begin{aligned}
&\sum_{A>0} f^{(a)A}(z)\,f^{(b)-A}(z)=\delta_{a,n-1}\delta_{b,n-1},\\
&\sum_{A>0} \bar{f}^{(a)A}(\bar{z})\,\bar{f}^{(b)-A}(\bar{z})=\delta_{a,n-1}\delta_{b,n-1},\quad \textit{for}\ \ 0\leqq a,b\leqq n-1
\end{aligned} \tag{4 2}$$

In the last formula, and hereafter, upper indices in between round parentheses denote the order of derivatives in z or $\bar{z}$ For $a=b=0$ the conditions just written are, of course, necessary for $\Sigma^{[1]}$ to be a submanifold of $\mathscr{C}^{[1]}$ (see (3 28)) The additional conditions will be needed for consistency with the following.

Definition 2. *Associated surfaces Given any D_n-W-surfaces, in the sense of Definition 1, it is convenient to introduce a family of surfaces $\Sigma^{[p]}$ in $\mathscr{C}^{[p]}$, $p=2,\ \ ,n$ defined by the equations*

$$\begin{aligned}
X^{[p]A} &= f^{(p-1)A}(z)-\delta_{p,n}f_{\|}^{[n-1]A}(z),\\
\bar{X}^{[p]A} &= \bar{f}^{(p-1)A}(\bar{z})-\delta_{p,n}\bar{f}_{\|}^{[n-1]A}(\bar{z}),
\end{aligned} \tag{4 3}$$

where

$$\begin{aligned}
f_{\|}^{[n-1]A} &= \sum_q (\Theta^{-1})_{n-1\,q}f^{(q)A},\quad A>0,\qquad f_{\|}^{[n-1]A}=0,\quad A<0,\\
\bar{f}_{\|}^{[n-1]A} &= \sum_q (\bar{\Theta}^{-1})_{n-1\,q}\bar{f}^{(q)A},\quad A>0,\qquad \bar{f}_{\|}^{[n-1]A}=0,\quad A<0,
\end{aligned} \tag{4 4}$$

$$\Theta_{pq}\equiv\sum_{A>0}f^{(p)A}f^{(q)-A},\qquad \bar{\Theta}_{pq}\equiv\sum_{A>0}\bar{f}^{(p)A}\bar{f}^{(q)-A},\quad 0\leqq p,q\leqq n-1. \tag{4 5}$$

The definition of $f_{\|}$ and $\bar{f}_{\|}$ is such that

$$\sum_A f_{\|}^{[n-1]A} f^{[p]-A} = \sum_A \bar{f}_{\|}^{[n-1]A} \bar{f}^{[p]-A} = \delta_{p,n-1} \tag{4 6}$$

In view of relations (4.2), it follows that (4 3) are compatible with conditions (3 38), and $\Sigma^{[p]} \in \mathscr{C}^{[p]}$, as the definition claims As usual, the geometrical interpretation of (4 3) should be that a point of $\Sigma^{[p]}$ represents the osculating hyperplane with contact of order $p-1$ at the point $X^{[1]A} = f^A(z)$, $\bar{X}^{[1]A} = \bar{f}^A(\bar{z})$ of $\Sigma^{[1]}$ Conditions (4 2) precisely ensure that $\Sigma^{[p]}$ has such a contact with the quadric of equations $\sum_A X^A X^{-A} = 0$, $\sum_A \bar{X}^A \bar{X}^{-A} = 0$, which defines $\Sigma^{[1]}$ as a submanifold of CP^{2n-1}. Thus we shall consider the definition just given as the one of the generalized Plücker embedding associated with D_n. The above definition makes sense at generic points of the W-surface where Θ and $\bar{\Theta}$ are invertible matrices

4 2 Plücker embedding from Toda dynamics The main aim of the present subsection is the derivation of the following

Theorem 1. *Associated with any solution of the D_n-Toda equations, there exist a D_n–W-surface and a family of associated surfaces as introduced by Definitions* 1 *and* 2, *where f and $\bar{f}$ are given by*

$$f^A = e^{-\xi_1} F^A, \qquad \bar{f}^A = e^{-\bar{\xi}_1} \bar{F}^A, \tag{4.7}$$

$$F^1 = 1, \qquad F^2 = (1), \qquad F^k = (1,2, \quad ,k-1), \quad k \leqq n,$$

$$F^{-n} = (1,2, \quad ,n-2,n),$$

$$F^{-n+1} = -(1,2, \quad ,n-2,n-1,n) - (1,2, \quad ,n-2,n,n-1)$$

$$F^{-l} = (-1)^{n-l}[(1,2, \quad ,n-2,n-1,n,n-2, \quad ,l) + (1,2, \quad ,n-2,n,n-1,n-2, \quad ,l)], \quad l < n-1 \tag{4 8}$$

The last equation uses the following compact notation for the repeated integrals over screening functions

$$(i_1, i_2, \ldots, i_r) \equiv \int_{z_0}^{z} dx_1 s_{i_1}(x_1) \int_{z_0}^{x_1} dx_2 s_{i_2}(x_2) \cdots \int_{z_0}^{x_{r-1}} dx_r s_{i_r}(x_r), \tag{4 9}$$

and the anti-holomorphic parts are given by similar expressions

Proof. We have to show that the functions f^A defined by formulas (4 7) obey conditions (4 2) Using the explicit realization (3 4), (3 5), it is easy to verify that

$$M(z) \flat_1^+ M(z)^{-1} = \sum_A F^A(z) \flat_A^+ \equiv F(z), \tag{4 10}$$

where F^A is given by (4 8) According to (2 4) one has,

$$d(M(z) \flat_k^+ M^{-1}(z))/dz = M(z)[L, \flat_k^+] M^{-1}(z), \qquad L = \sum_l s_l E_{-l} \tag{4 11}$$

Together with (3.5) this gives for $1 \leqq k \leqq n-1$,

$$M(z)\flat_k^+ M^{-1}(z) = D_k F\,, \tag{4 12}$$

where we introduced the notation

$$D_k \equiv \frac{1}{s_{k-1}}\frac{d}{dz}\frac{1}{s_{k-2}}\frac{d}{dz} \quad \frac{1}{s_1}\frac{d}{dz}, \quad k \geqq 2;\ D_1 = 1 \tag{4 13}$$

By the charge conjugation (3 39), the differential equation for $M(z)$ becomes $dM^c/dz = -(\sum_{j=1}^n s_j(z)E_{-j})M^c$, so that $M^c = M^{-1}$ This was expected since we are dealing with the orthogonal algebra After charge conjugation, (4 12) becomes

$$M(z)\flat_{-k}M(z)^{-1} = D_k F^c, \quad \text{where } F^c \equiv \sum_A F^A \flat_{-A} \tag{4.14}$$

The method for deriving quadratic relations is to consider

$$\langle 0|M(z)\flat_{-k}M^{-1}(z)M(z)\flat_l^+ M^{-1}(z)|0\rangle = \langle 0|\flat_{-k}\flat_l^+|0\rangle = 0\,.$$

One re-expresses the l.h s. using (4 12) and (4.14) This gives

$$\sum_A D_k F^A(z)\, D_l F^{-A}(z) = 0, \quad 1 \leqq k < l \leqq n-1\,. \tag{4 15}$$

It follows from (4.7) that

$$\begin{aligned} D_k F^A &= e^{\xi_k - \xi_{k-1}} f^{(k-1)A} + \text{ lower order derivatives}, \quad k \leqq n-2\,; \\ D_{n-1}F^A &= e^{\xi_n + \xi_{n-1} - \xi_{n-2}} f^{(n-2)A} + \text{ lower order derivatives} \end{aligned} \tag{4.16}$$

Combining the last two formulae, one concludes

$$\sum_A f^{(a)A}(z)\, f^{(b)-A}(z) = 0 \quad \text{for } 0 \leqq a, b \leqq n-2 \tag{4.17}$$

The case $k = n-1$ is different, since (4.11) gives

$$\begin{aligned} D_n F &= M(z)\flat_n^+ M^{-1}(z) + \frac{s_n}{s_{n-1}} M(z)\flat_{-n}^+ M^{-1}(z)\,, \\ D_n F^c &= M(z)\flat_{-n}M^{-1}(z) + \frac{s_n}{s_{n-1}} M(z)\flat_n M^{-1}(z)\,, \\ \langle 0|D_k F^c D_n F|0\rangle &= 2\delta_{n,k}\, s_n/s_{n-1} \end{aligned} \tag{4 18}$$

Making use of (4.16), together with the equation

$$D_n F^A = e^{\xi_n - \xi_{n-1}} f^{(n-1)A} + \text{ lower order derivatives}\,, \tag{4 19}$$

one obtains

$$\sum_{A>0} f^{(a)A}(z)\, f^{(n-1)-A}(z) = \delta_{a,n-1}, \quad 0 \leqq k \leqq n-1$$

This completes the proof that (4 7) and (4 8) define embedding functions f that obey conditions (4.2). The case of $\bar{f}$ is similar □

A direct consequence of the fermionic method we are using is the

Corollary 1. *Kähler potentials from Toda fields The intrinsic metric of the surface* $\Sigma^{[p]}$ *defined by the above theorem, is derivable from the Kähler potential equal to* $-\Phi_p$, $p = 1, \quad , n$

Proof Consider, first, the representation with highest-weight vector $\flat_p^+ \quad \flat_1^+|0\rangle$ for $1 \leqq p \leqq n-2$ Making use of (4 12), one concludes that

$$M(z)\flat_p^+ \quad \flat_1^+|0\rangle = \prod_{r=1}^{p-1} s_r^{p-r} \sum_{A_{p-1}} F^{(p-1)A_{p-1}} \flat^+_{A_{p-1}} \quad \sum_{A_1} F^{A_1} \flat^+_{A_1}|0\rangle\,. \tag{4 20}$$

It is well-known that the inverse of the Cartan matrix is expressible in terms of the fundamental weights; $(K^{(D_n)-1})_{ij} = \vec{\lambda}_i, \vec{\lambda}_j$. Using (3.7), and substituting (4.7) for f^A, one finds finally that

$$\begin{aligned} e^{-\xi_p} M(z)\flat_p^+ \quad \flat_1^+|0\rangle &= f^{(p-1)} \quad f^{(1)} f|0\rangle\,, \\ e^{-\bar{\xi}_p}\langle 0|\flat_1 \quad \flat_p \bar{M}^{-1}(\bar{z}) &= \langle 0|\bar{f}\,\bar{f}^{(1)} \quad \bar{f}^{(p-1)} \end{aligned} \tag{4.21}$$

According to (2 3), (2.4), and (3.37), this gives the desired relation

$$\mathscr{K}^{[p]}(f, . \quad , f^{(p-1)}, \bar{f}, \quad , \bar{f}^{(p-1)}) = -\Phi_p(z, \bar{z}) \tag{4 22}$$

Next we consider the associated surfaces in $\mathscr{C}^{[p]}$, for $p \geqq n-2$ Of course, this part makes use of the fermionic realization in terms of the c-fermionic operators (see (3 12)) A calculation similar to the one that leads to (4 10) gives

$$M(z)c_1^+ M(z)^{-1} = \sum_{A>0} (F^A(z)\, c_A^+ + F^{-A}(z)\, c_A) \equiv \check{F}(z) \tag{4 23}$$

Since one has,

$$d(M(z)c_k^+ M^{-1}(z))/dz = M(z)[\mathscr{E}, c_k^+]M^{-1}(z), \qquad \mathscr{E} = \sum_i s_i E_{-i},$$

it follows that, for $k \leqq n-2$,

$$M(z)c_k^+ M^{-1}(z) = D_k \check{F} \tag{4.24}$$

In agreement with (4 7), we let

$$\check{f}(z) \equiv e^{-\xi_1}\check{F}(z)\,, \tag{4 25}$$

obtaining

$$D_k \check{F} = e^{\xi_k - \xi_{k-1}} \check{f}^{(k-1)} + \text{ lower derivative terms,} \quad \text{for } k \leqq n-2\,, \tag{4.26}$$

and, thereof,

$$M(z)c_k^+ \quad c_1^+|0\rangle^{(1/2)} = e^{\xi_k} \check{f}^{(k-1)} \quad \check{f}^{(1)} \check{f}|0\rangle^{(1\,2)}, \quad \text{for } k \leqq n-2 \tag{4 27}$$

So far, this is much like what was discussed in the previous case For $k = n-1$, the calculation is again similar, but the expression of λ_{n-1} is different, and one

finds

$$M(z)c^{+}_{n-1}M^{-1}(z) = D_{n-1}\check{F} = e^{\xi_n+\xi_{n-1}-\xi_{n-2}}\check{f}^{(n-2)} + \text{lower derivative terms},$$

$$M(z)c^{+}_{n-1} \quad c^{+}_{1}|0\rangle^{(1\,2)} = e^{\xi_n+\xi_{n-1}}\check{f}^{(n-2)} \quad \check{f}^{(1)}\check{f}|0\rangle^{(1\,2)}. \tag{4 28}$$

The orthogonality conditions (4 2) can be re-derived using the c-operators They come out very simply from the obvious relations

$$[M(z)c^{+}_{k}M^{-1}(z),\ M(z)c^{+}_{l}M^{-1}(z)]_{+} = 0, \tag{4.29}$$

and from the counterpart of (4 18), that is,

$$d(M(z)c^{+}_{n-1}M^{-1}(z))/dz = s_{n-1}Mc^{+}_{n}M^{-1} + s_nMc_nM^{-1}.$$

This gives the equation

$$Mc^{+}_{n}M^{-1} = D_n\check{F} - \frac{s_n}{s_{n-1}}Mc_nM^{-1} \tag{4.30}$$

that will be useful below. Next consider the case of the associated surface in $\mathscr{C}^{[n-1]}$ The embedding is very similar to the case $p \leqq n-2$, since one only makes use of conditions (4 2) for $p \leqq n-2$, and $q \leqq n-2$, which are homogeneous. Note, however, that formula (4 28) involves the factor $\exp(\xi_n + \xi_{n-1})$, instead of the factor $\exp(\xi_{n-1})$ that would be the direct generalization of the bosonic representation case Thus one finds

$$\mathscr{K}^{[n-1]}(f, \ldots, f^{(n-2)}, \bar{f}, \ldots, \bar{f}^{(n-2)}) = -(\Phi_{n-1}(z,\bar{z}) + \xi_n + \bar{\xi}_n) \tag{4 31}$$

Finally let us discuss the associated surface in $\mathscr{C}^{[n]}$ According to (4 30),

$$[M(z)c^{+}_{k}M^{-1}(z),\ M(z)c^{+}_{n}M^{-1}(z)]_{+} = 0$$

$$= \left[D_n\check{F} - \frac{s_n}{s_{n-1}}Mc_nM^{-1}, D_n\check{F} - \frac{s_n}{s_{n-1}}Mc_nM^{-1}\right]_{+}$$

Thus, by keeping the second term, one arrives at an homogeneous relation Comparing with (4 6), one concludes that

$$Mc_nM^{-1} \equiv \check{f}^{[n-1]}_{\|} = \sum_{A>0} f^{[n-1]A}_{\|}c_A \tag{4.32}$$

The fact that the last formula involves only annihilation operators is a direct consequence of the explicit realization (3 12) It is easily seen that the second term of (4 30) drops out when one computes the generalization of (4 27) for $k = n$ One gets

$$Mc^{+}_{n} \quad c^{+}_{1}|0\rangle^{(1\,2)} = e^{2\xi_n}\check{f}^{(n-1)} \quad \check{f}^{(1)}\check{f}|0\rangle^{(1\,2)} \tag{4.33}$$

Thus we have

$$\mathscr{K}^{[n]}(f, \ldots, f^{(n-2)}, f^{(n-1)} - f^{[n-1]}_{\|}, \bar{f}, \ldots, \bar{f}^{(n-2)}\bar{f}^{(n-1)} - \bar{f}^{[n-1]}_{\|})$$

$$= \mathscr{K}^{[n]}(f, \ldots, f^{(n-1)}, \bar{f}, \ldots, \bar{f}^{(n-1)}) = -(\Phi_n(z,\bar{z}) + \xi_n + \bar{\xi}_n) \tag{4 34}$$

The outcome of the preceding discussion is that the Kähler potentials of $\Sigma^{[p]}$ coincide with $-\Phi_p$, up to an irrelevant re-definition – that do not change the Riemannian metric This terminates the proof □

4 3 Toda fields from D_n–W-surfaces In this subsection we establish the following converse to Theorem 1 and Corollary 1

Theorem 2. *Toda solution from Plücker embeddings The Kähler potentials of any D_n–W-surface introduced by Definition 1 and of its associated surfaces introduced by Definition 2, may be written as*

$$\begin{aligned}
\mathscr{K}^{[p]}(f, \quad , f^{(p-1)}, \bar{f}, \quad , \bar{f}^{(p-1)}) &= -\Phi_p(z,\bar{z}), \quad p \leqq n-1\,, \\
\mathscr{K}^{[n-1]}(f, . \quad , f^{(n-1)}, \bar{f}, \quad , \bar{f}^{(n-1)}) &= -(\Phi_{n-1}(z,\bar{z}) + \xi_n + \bar{\xi}_n)\,, \\
\mathscr{K}^{[n]}(f, \ . \ , f^{(n-2)}, \bar{f}, .. \ , \bar{f}^{(n-2)}) &= -(\Phi_n(z,\bar{z}) + \xi_n + \bar{\xi}_n)\,,
\end{aligned} \tag{4 35}$$

where Φ_p are solutions of the D_n-Toda equations

Proof At this point it is useful to recall the expression of the Cartan matrix, which is the same as for A_n, except in the following lower right 3×3 corner

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{4 36}$$

First we re-derive the Toda equation for $p \leqq n-3$ directly from the fermionic expressions obtained by substituting (4 1) in the Kähler potential (3 37), that is,

$$e^{-\Phi_p} = \langle 0|\, \bar{f} \quad \bar{f}^{(p-1)} f^{(p-1)} \quad f|0\rangle\,. \tag{4 37}$$

By explicit computations, one finds

$$\begin{aligned}
e^{-2\Phi_p}\partial\bar{\partial}\Phi_p = {} & \langle 0|\, \bar{f} \quad \bar{f}^{(p-2)}\bar{f}^{(p)} f^{(p-1)} f^{(p-2)} \cdot \ f|0\rangle \\
& \times \langle 0|\, \bar{f} \quad \bar{f}^{(p-2)}\bar{f}^{(p-1)} f^{(p)} f^{(p-2)} \cdot f|0\rangle \\
& - \langle 0|\, \bar{f} \quad \bar{f}^{(p-2)}\bar{f}^{(p)} f^{(p)} f^{(p-2)} \quad f|0\rangle \\
& \times \langle 0|\, \bar{f} \quad \bar{f}^{(p-1)} f^{(p-1)} \quad f|0\rangle\,,
\end{aligned}$$

and, thereof, applying Wick's theorem,

$$\partial\bar{\partial}\Phi_p = -e^{2\Phi_p}\langle 0|\, \bar{f} \cdot \quad \bar{f}^{(p)} f^{(p)} \quad f|0\rangle\langle 0|\, \bar{f} \cdot \quad \bar{f}^{(p-2)} f^{(p-2)} \quad f|0\rangle$$

According to the form of the Cartan matrix for D_n, this coincides with Toda equations for $p \leqq n-3$ Consider, now the case $p = n-2$ Clearly the derivation just recalled works in the same way, but now gives fermionic expressions that are generalizations of expression (4 37) Thus we introduce

$$\begin{aligned}
\Delta_{n-1} &\equiv \langle 0|\, \bar{f} \quad \bar{f}^{(n-2)} f^{(n-2)} \quad f|0\rangle\,, \\
\Delta_n &\equiv \langle 0|\, \bar{f} \quad \bar{f}^{(n-1)} f^{(n-1)} \quad f|0\rangle\,.
\end{aligned} \tag{4 38}$$

Now we show how they are related with the additional bosonic coset spaces discussed in Subsect 3 2.3 Since $\mathscr{C}^{[n-1]}$, and $\mathscr{C}^{[\nu_{n-1}+\nu_n]}$ (resp. $\mathscr{C}^{[n]}$, and $\mathscr{C}^{[2\nu_n]}$) have the same dimension, formulas (4 3) taken for $p = n-1$, and $p = n$, also define

associated surfaces in $\mathscr{C}^{[\lambda_{n-1}+\lambda_n]}$, and $\mathscr{C}^{[2\lambda_n]}$. First, extending the preceding derivation, one immediately sees that

$$\Delta_{n-1} = e^{-(\xi_{n-1}+\xi_n+\bar{\xi}_{n-1}+\bar{\xi}_n)}\langle\lambda_{n-1}+\lambda_n|\bar{M}^{-1}M|\lambda_{n-1}+\lambda_n\rangle \equiv e^{-\Phi_{\lambda_n+\lambda_{n-1}}} . \tag{4.39}$$

On the other hand, and making use of (4.30), one concludes that

$$\Delta_n = e^{-2(\xi_{n-1}+\bar{\xi}_{n-1})}\langle 2\lambda_{n-1}|\bar{M}^{-1}M|2\lambda_{n-1}\rangle + e^{-2(\xi_n+\bar{\xi}_n)}\langle 2\lambda_n|\bar{M}^{-1}M|2\lambda_n\rangle ,$$

$$\Delta_n = e^{-\Phi_{2\lambda_{n-1}}} + e^{-\Phi_{2\lambda_n}} \tag{4.40}$$

Combining the D_n-Toda equations with the relations satisfied by Δ_{n-1} and Δ_n (thanks to Wick's theorem), we find that we should have

$$\partial\bar{\partial}\Phi_{n-2} = e^{2\Phi_{n-2}-\Phi_{n-3}-\Phi_{n-1}-\Phi_n} = e^{2\Phi_{n-2}-\Phi_{n-3}-\Phi_{\lambda_n+\lambda_{n-1}}} ,$$

so that

$$\Phi_{\lambda_n+\lambda_{n-1}} = \Phi_{n-1} + \Phi_n \tag{4.41}$$

Moreover,

$$\partial\bar{\partial}\Phi_{\lambda_n+\lambda_{n-1}} = \partial\bar{\partial}(\Phi_{n-1}+\Phi_n) = e^{-\Phi_{n-2}}(e^{2\Phi_{n-1}}+e^{2\Phi_n}) = e^{2\Phi_{\lambda_n+\lambda_{n-1}}-\Phi_{n-2}-\Phi_{2\lambda_n}} ,$$

so that

$$e^{-\Phi_{2\lambda_n}} = e^{-2\Phi_{n-1}} + e^{-2\Phi_n} . \tag{4.42}$$

Expressions (4.41) and (4.42) are immediate consequences of (4.39) and (4.40), in view of the relationship (3.48) between Kähler potentials □

As a preparation for the coming subsection, let us note that, due to the connection between Kähler potentials and Toda fields just established, it follows from the Toda equations that the intrinsic metric tensor $g^{[p]}_{z\bar{z}}$ of $\Sigma^{[p]}$ is given by

$$g^{[p]}_{z\bar{z}} \equiv -\partial\bar{\partial}\Phi_p = \exp\left(\sum_j K^{(D_n)}_{pj}\,\Phi_j\right) \tag{4.43}$$

4.4 Infinitesimal Plücker formulae Extending the discussion of [1], we next show that the connection with Toda dynamics immediately leads to the

Theorem 3. Infinitesimal Plücker formulae. *At the regular points of the embedding, the family of scalar curvatures are related by*

$$R^{(p)}_{z\bar{z}}\sqrt{g^{(p)}_{z\bar{z}}} = \sum_q K^{(D_n)}_{pq} g^{(q)}_{z\bar{z}} \tag{4.44}$$

Proof This is derived by computing the curvature

$$R^{(k)}_{z\bar{z}}\sqrt{g^{(k)}_{z\bar{z}}} \equiv -\partial\bar{\partial}\ln g^{(k)}_{z\bar{z}} = -\partial\bar{\partial}\ln\left(\exp\left(\sum_j K^{(D_n)}_{pj}\,\Phi_j\right)\right) . \qquad \square \tag{4.45}$$

5. General Formulation

The consideration of the W-geometry of the Toda systems associated with the algebras C_n and B_n follows exactly the same direction as the D_n-case, and is even simpler Before discussing the main steps of the construction for an arbitrary simple classical Lie algebra $\mathcal{G}$, let us recall briefly some information about these two series, see e g [12], and their fermionic realizations

For the algebra C_n the roots are of the form $\vec{\alpha} = \pm 2\vec{e}_p$, $1 \leqq p \leqq n$, $\pm\vec{e}_p \pm \vec{e}_q$, $p < q$, and the elements of C_n can be realized using $2n$ fermionic operators $\flat_{\pm p}$. The simple (positive) roots are $\vec{\pi}_i = \vec{e}_i - \vec{e}_{i+1}$, $1 \leqq i \leqq n-1$, and $\vec{\pi}_n = 2\vec{e}_n$, the corresponding fundamental weights are $\vec{\lambda}_i = \sum_{1 \leqq j \leqq i} \vec{e}_j$, $1 \leqq i \leqq n$, all n fundamental representations are of the same nature, and have the dimensions $\binom{2n}{i} - \binom{2n}{i-2}$ Their weight vectors have integer components, and are realized in the Fock space with the highest weight states $|\lambda_i\rangle \equiv \flat_i^+ \cdot \quad \flat_1^+|0\rangle$, $1 \leqq i \leqq n$, satisfying (2 11) with the cyclic vacuum vector $|0\rangle$, $\flat_p|0\rangle = 0$ for all $p = 1, \quad , n$

For the algebra B_n the roots are $\vec{\alpha} = \pm\vec{e}_p$, $1 \leqq p \leqq n$, $\pm\vec{e}_p \pm \vec{e}_q$, $1 \leqq p < q \leqq n$, and the root vectors corresponding to these roots are realized in terms of $2n+1$ fermionic operators $\flat_{\pm p}$, $1 \leqq p \leqq n$, and $\flat_0$, the simple roots are $\vec{\pi}_i = \vec{e}_i - \vec{e}_{i+1}$, $1 \leqq i \leqq n-1$, and $\vec{\pi}_n = \vec{e}_n$, the corresponding fundamental weights are $\vec{\lambda}_i = \sum_{1 \leqq j \leqq i} \vec{e}_j$, $1 \leqq i \leqq n-1$, and $\vec{\lambda}_n = \frac{1}{2}\sum_{1 \leqq j \leqq n} \vec{e}_j$. Here only the first $n-1$ fundamental representations have the weight vectors with the integer components, and the highest weight states $|\lambda_i\rangle \equiv \flat_i^+ \quad \flat_1^+|0\rangle$, $1 \leqq i \leqq n-1$, have the dimensions $\binom{2n+1}{i}$, while the last one is spinorial, its dimension is 2^n All the reasonings given in the previous section for the D_n-case work precisely in the same way, with the relevant minor modification, here, besides $|\lambda_i\rangle$, $1 \leqq i \leqq n-1$, there are two other highest states $\flat_n^+ \quad \flat_1^+|0\rangle$ and $\flat_{-1}^+\flat_{n-1}^+ \quad \flat_1^+|0\rangle$.

With these words and some algebra, one arrives at the analogous conclusions as for the D_n-case, concerning the relation between the Kähler potentials of the corresponding $\mathcal{C}^{[i]}$-manifolds and the Toda fields satisfying (2.2) with K being the Cartan matrix of the algebra C_n or B_n, etc Let us only mention that the reconstruction formulas (2 8) and (2.9) take place for all $1 \leqq j \leqq n$ for C_n series, for B_n it is valid for $j \leqq n-1$, while $e^{-2\Phi_n} = \Delta_n$.

It is natural to decouple the construction in two steps First, let us parametrize the cosets of G for all representations of $\mathcal{G} =$ Lie G with the highest weights λ_p, $p = 1, \quad , n$, by exponentiating the linear span of the quotient $\mathcal{G}/\mathcal{G}_{\|}^{[p]}$ As we have already said, for all simple non-exceptional Lie algebras we use the fermionic realization of their elements, and the number of the creation $\flat_A^+$ (annihilation $\flat_A$) operators is equal to the dimension of the Euclidean space whose coordinates parametrize the positive and negative roots of $\mathcal{G}$. Namely, for nonexceptional representations of $\mathcal{G}$ we have

$$\mathcal{C}^{[p]} \quad e^{\Omega_p}|\lambda_p\rangle = e^{\sum_a \mathcal{F}_a^{[p]} \mathcal{V}_a^{[p]}}|\lambda_p\rangle \equiv e^{\sum_a \mathcal{F}_a^{[p]} \mathcal{V}_a^{[p]}} \flat_p^+ \quad \flat_1^+|0\rangle \tag{5.1}$$

$$\equiv \sum_{A_1, \ldots, A_p} X^{[p]p, A_p} X^{[p]p-1, A_{p-1}} \quad X^{[p]1, A_1} \flat_{A_p}^+ \cdot \flat_{A_1}^+|0\rangle \tag{5 2}$$

Here $|0\rangle$ is the vacuum cyclic vector, by action on which of the creation operators $\flat_p^+ \quad \flat_1^+$ one obtains the highest weight λ_p state $|\lambda_p\rangle$, Ω_p is expanded over the elements $\mathcal{F}_a^{[p]}$ of $\mathcal{G}_\perp^{[p]}$, the series in (5 2) gives a decomposition over all vectors

$b^+_{A_p} \quad b^+_{A_1}|0\rangle \equiv |A_p \quad A_1\rangle$ of the p^{th} fundamental representation space. For the case of the exceptional fundamental representations (the last one for B_n, and the last two ones for D_n), the meaning of the vacuum vector $|0\rangle$ and a fermionic realization of the elements of the algebra is different than those for nonexceptional representations, and formula (5 2) is modified, see above

The space $\mathscr{G}^{[p]}_{\perp}$, as an algebraic manifold, is parametrized by independent coordinates $x^{[p]}_a$, $1 \leqq a \leqq \dim \mathscr{G}^{[p]}_{\perp} \equiv N_p$, in the space $\tilde{\mathscr{G}}^{[p]}_{\perp}$, dual to the space $\mathscr{G}^{[p]}_{\perp}$, with the following elements· the Cartan generator h_p, and the root vectors corresponding to the root string $\alpha^{[p]}_1, \ldots, \alpha^{[p]}_{N_p}$ containing the simple root $\pi_p \equiv \alpha^{[p]}_1$. At the same time, the coset $\mathscr{C}^{[p]} = G/G^{[p]}_{\|}$, as a group manifold, is parametrized by the coordinates $X^{[p]\chi, A}$ in the space dual to the space $G/G^{[p]}_{\|}$ corresponding, in addition to those of $\mathscr{G}^{[p]}_{\perp}$, the double highest weight ω_p of the p^{th} fundamental representation, and all the differences $2\omega_p - \alpha^{[p]}_i$ which do not coincide with the roots from the root string defined above. (Of course, for the series A_n the set of the elements of $\mathscr{G}^{[p]}_{\perp}$ and those in the r h s of (5.1) are in one-to-one correspondence.) By this reason, already on this step, one comes to the homogeneous quadratic relations for the coordinates $X^{[p]\chi, A}$, so to deal only with the independent coordinates of the cosets Thus we arrive at a realization of the cosets in terms of the coordinates which satisfy the relations corresponding to some algebraic curves and surfaces However, on the different cosets (for different values of p), the coordinates $X^{[p]\chi, A}$ and $\bar{X}^{[p]\chi, A}$ clearly are different, and are not connected yet. And, of course, they satisfy their own quadratic relations also separately, the origin of the relations has been explained above Finally, define the Kähler potential $\mathscr{K}^{[p]}(X^{[p]}, \bar{X}^{[p]})$ of a $\mathscr{C}^{[p]}$ in accordance with (5 2) as an appropriate scalar product in the space of the p^{th} fundamental representation, and recall again that, up to now, the potentials for different manifolds $\mathscr{C}^{[p]}$ are defined independently

The given reasonings clarify the origin of the quadratic relations from the purely Lie algebraic point of view At the same time, in the differential geometry language, the necessity of these relations for the case of an arbitrary simple Lie algebra $\mathscr{G}$ is still the decomposability of a matrix representative of the modified Plücker image of $\mathscr{C}^{[p]}$ (for the corresponding algebra $\mathscr{G}$) in the relevant subspace of the projective space Here, of course, one takes into account the specific structure of the representation space vectors for this or that simple Lie algebra However, it seems to us clearer to formulate the relations in question not for the right coordinates $\Lambda^{[p]}_{A_1, \ldots, A_p}$ of the manifold, but directly in terms of the $X^{[p]\chi, A}$'s, as it has been done in the previous section for the D_n-case, the same is for the series B_n and C_n

In fact, the homogeneous space $\mathscr{C}^{[p]}$ is a flag manifold (or a parabolic space); and since we deal with G being a connected complex algebraic group, the algebraic manifold $\mathscr{C}^{[p]}$ naturally is a projective and simply connected manifold The set of the flag manifolds $\mathscr{C}^{[p]}$ which we consider here, realizes the cosets associated with the fundamental representations of $\mathscr{G}$, and is defined by the corresponding parabolic subalgebras of $\mathscr{G}$, or, up to a local isomorphism, by its **Z**-gradations The relevant reconstruction procedure looks as follows, see e.g. [14] Up to a transformation from the inner automorphisms group Int $\mathscr{G}$ = Ad (G), a **Z**-gradation of $\mathscr{G}$ can be given by the element H from the Cartan subalgebra $\mathscr{H}$ of $\mathscr{G}$, namely, $\mathscr{G}_m = \{\mathscr{F} \in \mathscr{G} \cdot [H, \mathscr{F}] = m\mathscr{F}\}$, such that $\pi_i(h) \equiv m_i$ are nonnegative integers for

all $1 \leqq i \leqq n$ It is clear that

$$\mathscr{G}_m|_{m\in\mathbf{Z}} = \delta_{m0}\mathscr{H} \oplus \bigoplus_{\alpha\in\Delta_m} \mathscr{G}_\alpha ,$$

with

$$\Delta_m \equiv \Delta_m(\pi_{i_1}, \quad , \pi_{i_s}) = \left\{ \alpha = \sum_{1\leqq i\leqq n} q_i\pi_i \in \Delta \quad \sum_{1\leqq i\leqq n} q_i m_i = m \right\},$$

and, moreover, these subspaces $\mathscr{G}_\alpha, \alpha \in \Delta_m$, are invariant with respect to $\mathscr{G}_0$ Here by $\pi_{i_1}, \quad , \pi_{i_s}$ we denote such simple roots which correspond to nonzero values of m_i In accordance with this gradation of $\mathscr{G}$, $\mathscr{G} = \bigoplus_{-n\leqq m\leqq n} \mathscr{G}_m$, the subalgebra $\mathscr{P}^+ = \bigoplus_{0\leqq m\leqq n} \mathscr{G}_m$, and the opposite to it (under the reflection $\alpha \to -\alpha$) $\mathscr{P}^- = \bigoplus_{0\leqq m\leqq n} \mathscr{G}_{-m}$, are the Lie algebras of parabolic subgroups $P^\pm$ of the Lie group G For the case $s = 1$ these subgroups are the maximal nonsemisimple subgroups of G, and just this case corresponds to the flag manifolds $\mathscr{C}^{[p]}$, $p = i_1$, which realize the cosets we are looking for

On the second step, let us now identify the Kähler potentials $\mathscr{K}^{[p]}(X^{[p]}, \bar{X}^{[p]})$ with the Toda fields satisfying the equations of motion, just by setting

$$e^{\Omega_p}|\lambda_p\rangle = Me^{-\sum_{ij} h_i(k^{-1})_{ij} \log s_j}|\lambda_p\rangle ,$$

cf (2.3), on the corresponding $W_\mathscr{G}$ – and associated surfaces Here arises the first nontrivial point With this identification, the coordinates $x_a^{[p]}$ for different values of p are not already independent, and are constructed in terms of the same screening functions s_j So, one should get convinced in the following two statements

i) The functions $f^A = X^{[1]1,A}$ entering $\mathscr{C}^{[1]}$ do satisfy the corresponding quadratic relations, in other words these relations do not contradict the nested structure (4 9) of f^A

ii) The functions $f^{(p-1)A} = X^{[p]\alpha,A}$, $p > 1$, entering $\mathscr{C}^{[p]}$, lead to the Toda fields Φ_p determined by formulas (2 3) via the screening functions, and satisfy the same quadratic relations as above

In the previous section we have proved these statements for the D_n-case by a direct verification, for B_n and C_n series it can be shown in a similar way So, the relations in question are identically satisfied on the class of the solutions to the Toda system, when the coordinates $X^{[p]\alpha,A}$ are expressed via the screening functions as the nested integrals (4.9), i e. on the corresponding W-surfaces

Of course, our discussion of the quadratic relations concerns only a part of the problem We have the cosets $\mathscr{C}^{[p]}$ which, in general, are submanifolds of the projective spaces The holomorphic (antiholomorphic) blocks entering the Kähler potentials under their identification with the Toda fields $\Phi_i(z, \bar{z})$ given by formula (2 3), are related to the nested integral structure of the nilpotent elements M ($\bar{M}$) written in terms of the screening functions,

$$M = \sum_{m=1}^{\infty} \sum_{1\leqq i_1, \ , i_m\leqq n} (i_1, \ \ldots, i_m) E_{-i_m} \quad E_{-i_1} , \tag{5 3}$$

where $(i_1, \quad , i_m)$ is the compact notation (4 9) for the corresponding repeated integrals

Just this representation automatically takes into account the fact that the functions f^A are not independent, and are parametrized by exactly n number of independent screening functions s_i The embedding functions $f^A(z)$ ($\bar{f}^A(\bar{z})$) which define the corresponding $W_{\mathcal{G}}$-and associated surfaces just satisfy the necessary quadratic relations thanks to their nested structure One can move in the opposite direction and observe that the identification of the Toda fields with the Kähler potentials for the associated surfaces in $\mathcal{C}^{[p]}$ gives that the embedding functions $f^A(z)$ ($\bar{f}^A(\bar{z})$) and their derivatives of the corresponding order, coincide with the coordinates $X^{[p]\alpha A}$, cf with (2 26), and provides the necessary relations And, moreover, the Kähler potentials of the manifolds $\mathcal{C}^{[p]}$ satisfy the system of partial differential equations (2 2)

Finally, the second part of formula (2 27) for the case of a simple Lie algebra $\mathcal{G}$ endowed with the principal gradation, takes, on account of the equations of motion (2 2), the form

$$d\mathcal{S}_k^2 = \frac{i}{2}\exp 2\left(\sum_{j=1}^{n} K_{kj}\Phi_j\right) dz\, d\bar{z} \equiv \frac{i}{2}\exp 2\rho_j\, dz\, d\bar{z}\,. \tag{5 4}$$

The curvature form of the pseudo-metrics $d\mathcal{S}^2$ appears as

$$-i\partial\bar{\partial}\rho_k = \sum_j K_{kj} d\mathcal{S}_j^2 \tag{5 5}$$

Then we naturally come to the following concerning a generalization for an arbitrary simple Lie algebra $\mathcal{G}$ of the global Plücker formula

Conjecture 1. *Global Plücker formula*[16] *For an arbitrary simple Lie algebra* $\mathcal{G}$, *with degrees* d_k, *on a W-surface of genus* g, *with total ramification numbers* β_k, *one has*

$$2g - 2 - \beta_k + 2\sum_{j=1}^{n} K_{kj} d_j = 0 \tag{5 6}$$

In accordance with an interpretation given in [1], W-surfaces for the case of A_n are instantons of the associated nonlinear σ-model, and in turn are described by the solutions of the cylindrically symmetric self-dual Yang–Mills equations, for which the action coincides, up to an inessential numerical factor, with the topological charge (or Pontryagin index, or instanton number) Q_k of this configuration The same reasonings work also for the cylindrically symmetric self-dual fields associated with an arbitrary simple Lie algebra $\mathcal{G}$ which, in accordance with [4], satisfy the Toda system of equations (2 2) Here there is also an explicit expression for the topological charge density, which provides, with the help of the Gauss–Bonnet formula, a bridge between the infinitesimal (5.5) and the global (5 6) Plücker type formulas In other words, formula (5.6) gives a relation between the genus of a $W_{\mathcal{G}}$-manifold and its topological characteristics $Q_k = d_k$ Moreover, since the cylindrically symmetric instantons for $\mathcal{G}$ constitute a subclass of $2r$-parametric solutions of (2 2) regular on the one-point compactification of $\mathbf{R}^4$ and with finite action (or topological charge), a justification of these requirements by imposing the corresponding boundary conditions on the Toda fields, leads to the evident relation between the ramification indices β_k and the degrees m_k of the singularities of the functions $\exp 2\rho_j$ in the

[16] See note added in proof

r.h.s of (5.4), $\beta_i = \sum_{j=1}^{n} K_{ij} m_j$. With such a standpoint, the integers m_k are nothing but the integration constants entering the parametrization $s_i(z) = c_i \exp(m_i z)$, $\bar{s}_i(\bar{z}) = \bar{c}_i \exp(m_i \bar{z})$ of the arbitrary screening functions $s_i(z)$ and $\bar{s}_i(\bar{z})$, $1 \leqq i \leqq n$, which determine the general solutions (2.3) of the Toda system (2.2).

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Note added in proof Our conjecture concerning the global Plücker formulae for arbitrary Lie algebras was proven by F. E. Burstall a year after our article appeared in preprint form.

A. Appendix: Group Properties for D_n

A.1 The bosonic representations Let us determine $\mathscr{G}_{\|}^{[p]}$ and $\mathscr{G}_{\perp}^{[p]}$ at once for $1 \leqq p \leqq n-2$, where, according to (3.7), $|\lambda_p\rangle = b_p^+ b_{p-1}^+ \cdots b_1^+ |0\rangle$. Let us call $\mathscr{A}_+$ the nilpotent Lie algebra generated by the step operators with positive roots. It is easy to see that $\mathscr{G}_{\|}^{[p]}$ is given by

$$\mathscr{G}_{\|}^{[p]} = \mathscr{A}_+, \{h_i, i \neq p\}, \{E_{-\vec{e}_\gamma + \vec{e}_\beta}, \alpha, \beta \leqq p\},$$

$$\{E_{-\vec{e}_k - \vec{e}_l}, l > k > p\}, \{E_{-\vec{e}_k + \vec{e}_l}, l > k > p\} \tag{A.1}$$

These generators may be reorganized as follows

$$\begin{aligned}\mathscr{G}_{\|}^{[p]} = &\{E_{\vec{e}_\gamma + \vec{e}_\beta}, \alpha, \beta \leqq p\}, \{E_{\vec{e}_\gamma + \vec{e}_k}, \alpha \leqq p, k > p\}, \{E_{\vec{e}_\gamma - \vec{e}_k}, \alpha \leqq p, k > p\},\\ &\{h_\gamma, \alpha \leqq p-1\}; \{E_{-\vec{e}_\gamma + \vec{e}_\beta}, \alpha < \beta \leqq p\}, \{E_{\vec{e}_\gamma - \vec{e}_\beta}, \alpha < \beta \leqq p\},\\ &\{h_k; k > p\}, \{E_{-\vec{e}_k - \vec{e}_l}, l > k > p\}, \{E_{-\vec{e}_k + \vec{e}_l}; l > k > p\},\\ &\{E_{\vec{e}_k - \vec{e}_l}, l > k > p\}, \{E_{\vec{e}_k + \vec{e}_l}, l > k > p\}\end{aligned} \tag{A.2}$$

The first line generates a nilpotent algebra denoted $\mathscr{A}_+^{[p]}$ of dimension $2p(n-p) + p(p-1)/2 = 2np - p(3p+1)/2$. The next line clearly generates A_{p-1}, which has dimension (p^2-1). The remaining lines generate D_{n-p}, of dimension $(n-p)(2n-2p-1)$. The dimension of $\mathscr{G}_{\|}^{[p]}$ is therefore $2np - p(3p+1)/2 + p^2 - 1 + (n-p)(2n-2p-1)$, that is, $2n^2 - n - 2np + 3p^2/2 + p/2 - 1$.

Next, this coset is parametrized by exponentiating

$$\mathscr{G}_{\perp}^{[p]} = \{h_p\}, \{E_{-\vec{e}_\gamma + \vec{e}_k}, \alpha \leqq p, k > p\}, \{E_{-\vec{e}_\gamma - \vec{e}_k}, \alpha \leqq p, k > p\},$$

$$\{E_{-\vec{e}_\gamma - \vec{e}_\beta}, \alpha \leqq \beta \leqq p\} \tag{A.3}$$

The dimension is $2np - p(3p+1)/2 + 1$ [17] Adding the dimensions of $\mathscr{G}_{\|}^{[p]}$ and $\mathscr{G}_{\perp}^{[p]}$ correctly gives $n(2n-1)$ which is the dimension of D_n

A 2 Explicit parametrization of $\mathscr{C}^{[p]}$ for bosonic representations We treat the generic case $1 \leqq p \leqq n-2$, where $|\lambda_p\rangle = b_p^+ \quad b_1^+|0\rangle$ According to (A 3), $\mathscr{C}^{[p]}$ is parametrized by[18] $e^{\kappa_p h_p} e^{\Omega_p} b_p^+ \quad b_1^+|0\rangle$, with

$$\Omega_p = \sum_{l>p,\, i \leqq p} \sum (x_l^{[p],i} E_{-\vec{e}_i + \vec{e}_l} + x_{-l}^{[p],i} E_{-\vec{e}_i - \vec{e}_l}) + \sum_{1 \leqq i < \delta \leqq p} u_{i,\delta}^{[p]} E_{-\vec{e}_i - \vec{e}_\delta} \tag{A 4}$$

The coset parameters are $x_l^{[p],i}$, $x_{-l}^{[p],i}$ and $u_{i,\delta}^{[p]}$ Turning the same crank as for the first fundamental representation, one computes (for $\alpha < p$)

$$e^{\Omega_p} b_\alpha^+ e^{-\Omega_p} = b_\alpha^+ + \sum_{k>p,\, \varepsilon=\pm 1} x_{\varepsilon k}^{[p]\gamma} b_{\varepsilon k}^+ + \left(\sum_{p \geqq \beta > \gamma} u_{\alpha,\beta}^{[p]} - \sum_{p \geqq \gamma > \beta} u_{\beta,\alpha}^{[p]} \right) b_{-\beta}^+$$

$$- \frac{1}{2} \sum_{k>p,\, \varepsilon=\pm 1} \sum_{i \leqq p} x_{\varepsilon k}^{[p]\gamma} x_{-\varepsilon k}^{[p],i} b_{-i}^+ \tag{A 5}$$

Thus we obtain

$$e^{\kappa_p h_p} e^{\Omega_p} |\lambda_p\rangle = \sum_{A_1, \ldots, A_p} X^{[p]\, p, A_p} X^{[p]\, p-1, A_{p-1}} \quad X^{[p]\, 1, A_1} b_{A_p}^+ \quad b_{A_1}^+ |0\rangle \,, \tag{A 6}$$

where for $1 \leqq \alpha \leqq p$, $1 \leqq \beta \leqq p$, and $k > p$,

$$X^{[p]\,\alpha,\beta} = \delta_{\alpha,\beta}, \quad \text{for} \quad \alpha, \beta < p \,,$$

$$X^{[p]\, p, \beta} = X^{[p]\,\beta, p} = e^{\kappa_p} \delta_{p\,\beta} \,,$$

$$X^{[p]\,\alpha,-\beta} = \begin{cases} -\frac{1}{2} \sum_{k>p,\, \varepsilon=\pm} (x_{\varepsilon k}^{[p]\gamma} x_{\varepsilon k}^{[p]\beta}) + u_{\alpha,\beta}^{[p]}, & \text{if } p > \beta > \alpha \\ -e^{-\kappa_p} \frac{1}{2} \sum_{k>p,\, \varepsilon=\pm} (x_{\varepsilon k}^{[p]\gamma} x_{\varepsilon k}^{[p]p}) + e^{-\kappa_p} u_{\alpha,p}^{[p]}, & \text{if } \beta = p,\, p > \alpha \\ -\frac{1}{2} \sum_{k>p,\, \varepsilon=\pm} (x_{\varepsilon k}^{[p]\gamma} x_{\varepsilon k}^{[p]\beta}) - u_{\beta,\alpha}^{[p]}, & \text{if } \alpha > \beta \\ -\frac{1}{2} \sum_{k>p,\, \varepsilon=\pm} (x_{\varepsilon k}^{[p]\gamma} x_{\varepsilon k}^{[p]\beta}), & \text{if } \alpha = \beta \neq p \\ -e^{-\kappa_p} \frac{1}{2} \sum_{k>p,\, \varepsilon=\pm} (x_{\varepsilon k}^{[p]p} x_{\varepsilon k}^{[p]p}), & \text{if } \alpha = \beta = p \end{cases}$$

$$X^{[p]\,\alpha,\pm(p+1)} = e^{\mp\kappa_p} x_{\pm(p+1)}^{[p]\alpha}, \qquad X^{[p]\,\gamma,\pm k} = x_{\pm k}^{[p]\alpha}, \; k > p+1 \tag{A 7}$$

[17] For $p = 1$ this is actually $2n-1$ which coincides with the dimension of D_n B_{n-1}

[18] Here again we leave aside the bar components which are similar

It is easy to see that the coordinates $X^{[p]\alpha,A}$ satisfy the polynomial conditions,

$$\sum_A X^{[p]\alpha,A} X^{[p]\beta,-A} = 0, \quad 1 \leqq \alpha \leqq p, \ 1 \leqq \beta \leqq p \tag{A 8}$$

This explicit computation gives a parametrization such that for $1 \leqq A \leqq p$, and $1 \leqq \bar{A} \leqq p$,

$$X^{[p]\alpha,A} = \delta_{\alpha,A} \tag{A 9}$$

With this parametrization, we may easily solve the constraints (3 37), and write (for $0 \leqq \alpha, \beta \leqq p$),

$$X^{[p]\alpha,-\beta} + X^{[p]\beta,-\alpha} = -\sum_{A=p+1}^{n} X^{[p]\alpha,A} X^{[p]\beta,-A}, \tag{A 10}$$

so that the independent components are $X^{[p]\alpha,\pm A}$, $\bar{X}^{[p]\alpha,\bar{A}}$, $A,\bar{A} \geqq p+1$, and $X^{[p]\alpha,-\beta} - X^{[p]\beta,-\alpha}$, $\bar{X}^{[p]\alpha,-\beta} - \bar{X}^{[p]\beta,-\alpha}$

A 3 The three additional bosonic representations The previous description of the cosets extends to the representations generated by $|\lambda_{n-1}+\lambda_n\rangle$, $|2\lambda_n\rangle$, and $|2\lambda_{n-1}\rangle$ without any problem The first two cases are direct extensions of the formulae given for λ_p with $p \leqq n-1$ The last one is obtained from the calculation for $2\lambda_{n-1}$, by exchanging everywhere $\flat_n$ with $\flat_{-n}$ Using similar notations, we introduce

$$\begin{aligned}
\Omega_{\lambda_{n-1}+\lambda_n} &= \sum_{i \leqq n-1,\alpha} (x_n^{[\lambda_{n-1}+\lambda_n],i} E_{-\vec{e}_i+\vec{e}_n} + x_{-n}^{[\lambda_{n-1}+\lambda_n],i} E_{-\vec{e}_i-\vec{e}_n}) \\
&\quad + \sum_{1 \leqq i<\delta \leqq n-1} u_{i,\delta}^{[\lambda_{n-1}+\lambda_n]} E_{-\vec{e}_i-\vec{e}_\delta}, \\
\Omega_{2\lambda_n} &= \sum_{1 \leqq i<\delta \leqq n} u_{i,\delta}^{[2\lambda_n]} E_{-\vec{e}_i-\vec{e}_\delta}, \\
\Omega_{2\lambda_{n-1}} &= \sum_{i \leqq n-1,\alpha} x_n^{[2\lambda_{n-1}],i} E_{-\vec{e}_i+\vec{e}_n} + \sum_{1 \leqq i<\delta \leqq n-1} u_{i,\delta}^{[2\lambda_{n-1}]} E_{-\vec{e}-\vec{e}_\delta}, \\
X^{[\lambda]\mu} &= e^{\kappa_\lambda h_\lambda} e^{\Omega_\lambda} \flat_\mu^+ e^{-\Omega_\lambda} e^{-\kappa_\lambda h_\lambda}
\end{aligned} \tag{A 11}$$

A 4 The fermionic representations The corresponding coset manifolds are studied in the same way as above They are parametrized by $e^{\kappa_p h_p^{(1\,2)}} e^{\Omega_p^{(1\,2)}} |\lambda_p\rangle$, where

$$\begin{aligned}
\Omega_{n-1}^{(1\,2)} &= \sum_{i,\alpha} x_n^{[n-1],i} E^{(1\,2)}_{-\vec{e}_i+\vec{e}_n} + \sum_{1 \leqq i<\delta \leqq n-1} u_{i,\delta}^{[n-1]} E^{(1\,2)}_{-\vec{e}_i-\vec{e}_\delta}, \\
\Omega_n^{(1\,2)} &= \sum_{1 \leqq i<\delta} u_{i,\delta}^{[n]} E^{(1\,2)}_{-\vec{e}_i-\vec{e}_\delta}
\end{aligned} \tag{A 12}$$

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