

# Zeta Function Determinant of the Laplace Operator on the $D$ -Dimensional Ball

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**Abstract:** We present a direct approach for the calculation of functional determinants of the Laplace operator on balls. Dirichlet and Robin boundary conditions are considered. Using this approach, formulas for any value of the dimension,  $D$ , of the ball, can be obtained quite easily. Explicit results are presented here for dimensions  $D = 2, 3, 4, 5$  and  $6$ .

## 1. Introduction

Motivated by the need to give answers to some fundamental questions in quantum field theory, during the last years there has been (and continues to be) a lot of interest in the problem of calculating the determinant of a differential operator,  $L$  (see for example [1, 2]). Often one has to deal in these situations with positive elliptic differential operators acting on sections of a vector bundle over a compact manifold. In such cases  $L$  has a discrete spectrum  $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ . The determinant,  $\det L = \prod_i \lambda_i$ , is generally divergent and one needs to make sense out of it by means of some kind of analytic continuation. A most appropriate way of doing that is by using the zeta function regularization prescription introduced by Ray and Singer [3] (see also [4, 5]). In this procedure  $\ln \det L$  is defined by analytically continuing the function  $\sum_i \lambda_i^{-s} \ln \lambda_i$  in the exponent  $s$ , from the domain of the complex plane where the real part of  $s$  is large to the point  $s = 0$ . Introducing the zeta function associated with the spectrum  $\lambda_i$  of  $L$ ,

$$\zeta(s) = \sum_i \lambda_i^{-s},$$

this is equivalent to defining

$$\ln \det L = -\zeta'(0).$$

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Only a few general methods for the exact evaluation of  $\ln \det L$  are available. Thus, for example, given that the manifold has a boundary, in [6] (see also [7–10]) the determinants of differential and difference operators have been related to the boundary values of solutions of the operators. When  $L$  is a conformally covariant differential operator, exact results may sometimes be obtained by transforming to a “more simple” operator  $\tilde{L}$  for which  $\ln \det \tilde{L}$  is known. Then, the knowledge of the associated heat-kernel coefficients – nowadays available [11, 12] – gives sometimes the exact value of  $\ln \det L$  [13–15]. This approach has been used by Dowker to find the functional determinants for a variety of sectors of Euclidean space, spheres and flat balls for dimensions  $D \leq 4$  [16–18]. Similar techniques have proven to be very powerful in order to obtain estimates of different types [19–21].

As a rule, however, explicit knowledge of the eigenvalues  $\lambda_i$  is necessary in order to obtain exact results for  $\ln \det L$ . This explicit knowledge of the eigenvalues is in general only guaranteed for highly symmetric regions of space, such as the torus, sphere or regions bounded by parallel planes. For these manifolds, detailed calculations have been performed in the context of Casimir energies and effective potential considerations (for a summary of results see [22]).

In this paper we want to focus on a class of situations for which the eigenvalues of the operator are not known explicitly, but nevertheless the exact calculation of  $\ln \det L$  is possible. The method developed is applicable whenever an implicit equation satisfied by the eigenvalues is known and some properties (later specified) of this equation are known. We exemplify our approach by taking  $L = -\Delta$  on the  $D$ -dimensional ball  $B^D = \{x \in \mathbb{R}^D; |x| \leq R\}$ , together with Dirichlet – or general Robin – boundary conditions. In dimensions  $D \leq 5$  a part of these results may also be obtained using the conformal transformation techniques mentioned above. (Here the restriction  $D \leq 5$  results from the need of the knowledge of the heat-kernel coefficients  $a_{D/2}$ , which has been determined only recently for  $D = 5$  [12].) This particular method was used by Dowker and Aapps for  $D \leq 4$  (this was before  $a_{5/2}$  were known) [23]. Here we will apply a direct approach which has already been shown to be quite powerful for the calculation of the heat-kernel coefficients in the situations described above [24]. Interesting quantum field theoretical applications of the results obtained can be found in quantum cosmology [25–29]. For these applications, the consideration of dimensions  $D$  higher than four are of interest, because the technicalities involved in dealing with higher-spin fields reduce essentially to the ones for scalar fields in those dimensions. Further applications are in statistical mechanics, in connection with finite size effects [30], and in conformal field theory [31].

The paper is organized as follows. In Sect. 2 we describe in detail our approach for the calculation of functional determinants using as an example the Laplace operator of the three-dimensional ball with Dirichlet boundary conditions. For generality and because the analytic continuation procedure employed is slightly easier, we start with the massive Laplacian, performing the limit  $m \rightarrow 0$  at a suitable point of our calculation. The result we find here agrees completely with the one recently given by Dowker and Aapps [23]. After having explained in detail the main ideas, we apply our approach to an (in principle) arbitrary dimension  $D$  and also to general Robin boundary conditions. We start in Sect. 3 with Robin boundary conditions for the three-dimensional ball. The Neumann boundary conditions – a special case of the Robin boundary conditions – cannot be obtained as a limit of the parameter involved in Robin boundary conditions. An extra consideration, necessary in order to deal with this situation, is given at the end of Sect. 3. In Sect. 4 we describe

in detail how our scheme can be applied in dimensions  $D > 3$ . Explicit results are given for dimensions  $D = 4, 5, 6$  and all the different boundary conditions. The case  $D = 2$  is briefly considered in Sect. 5. This case is slightly different because the lowest angular momentum  $l = 0$  needs to be treated in a specific way. In the conclusions (Sect. 6) we summarize the main results of our investigation. As we will see in the course of our procedure, the functional determinant is naturally split up into two pieces. The contributions of each of these pieces and a detail of how they are obtained are given in three appendixes.

## 2. Zeta Function Determinant on the 3-Dimensional Ball with Dirichlet Boundary Conditions

In this chapter we want to concentrate on the 3-dimensional ball with Dirichlet boundary conditions in order to exemplify our direct approach, which is actually applicable in any dimension and for completely general Robin boundary condition. We are thus interested in obtaining the zeta function of the operator  $(-\Delta + m^2)$  on the ball  $B^3 = \{x \in R^3; |x| \leq R\}$  endowed with Dirichlet boundary conditions. The zeta function is formally defined as

$$\zeta(s) = \sum_k \lambda_k^{-s}, \tag{2.1}$$

with the eigenvalues  $\lambda_k$  being determined through

$$(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \tag{2.2}$$

( $k$  is in general a multi-index here), together with the boundary condition. It is convenient to introduce a spherical coordinate basis, with  $r = |x|$  and angles  $\Omega = (\theta, \varphi)$ . In these coordinates, a complete set of solutions of Eq. (2.2) can be given in the form

$$\phi_{l,m,n}(r, \Omega) = r^{-\frac{1}{2}} J_{l+\frac{1}{2}}(w_{l,n} r) Y_{l+\frac{3}{2}}(\Omega), \tag{2.3}$$

the  $J_{l+1/2}$  being Bessel functions and the  $Y_{l+3/2}$  hyperspherical harmonics [32]. The  $w_{l,n} (> 0)$  are determined through the boundary condition by

$$J_{l+\frac{1}{2}}(w_{l,n} R) = 0. \tag{2.4}$$

In these notations, using  $\lambda_{l,n} = w_{l,n}^2 + m^2$ , the zeta function can be given in the form

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (2l+1)(w_{l,n}^2 + m^2)^{-s}. \tag{2.5}$$

Here the sum over  $n$  is extended over all possible solutions of Eq. (2.4) on the positive real axis, and  $(2l+1)$  is the number of independent harmonic polynomials in 3 dimensions.

As it stands, the zeta function in Eq. (2.5) is defined for  $\Re s > 3/2$ . The way to construct the analytical continuation of Eq. (2.5) to the left has been explained in detail in Ref. [24] for the calculation of the heat-kernel coefficients associated with the operator (2.2). For this reason our description here will be brief. The starting point of the consideration is the representation of  $\zeta(s)$  by the contour integral

$$\zeta(s) = \sum_{l=0}^{\infty} (2l+1) \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln J_{l+\frac{1}{2}}(kR), \tag{2.6}$$

where the contour  $\gamma$  runs counterclockwise and must enclose all solutions of Eq. (2.4) on the positive real axis (for a similar treatment as a contour integral see [25, 26, 33]). Subtracting and adding the leading asymptotic terms of  $I_\nu(zv)$  for  $v \rightarrow \infty, \nu = l + 1/2$ , the following representation of  $\zeta(s)$  valid in the strip  $-1/2 < \Re s < 1$  is found to be valid (for details see [24]),

$$\zeta(s) = 2 \sum_{l=0}^{\infty} \nu Z_D^\nu(s) + \sum_{i=-1}^2 A_i^D(s), \tag{2.7}$$

with the definitions

$$Z_D^\nu(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[ \left( \frac{zv}{R} \right)^2 - m^2 \right]^{-s} \\ \times \frac{\partial}{\partial z} \left\{ \ln[I_\nu(zv)] - \ln \left[ \frac{1}{\sqrt{2\pi\nu}} \frac{e^{v\eta}}{(1+z^2)^{\frac{1}{4}}} \right] - \sum_{n=1}^2 \frac{D_n(t)}{\nu^n} \right\}, \tag{2.8}$$

$$A_{-1}^D(s) = \frac{R^{2s}}{2\sqrt{\pi}\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \frac{\Gamma(j+s-\frac{1}{2})}{s+j} \zeta_H(2j+2s-2; 1/2), \tag{2.9}$$

$$A_0^D(s) = -\frac{R^{2s}}{2\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \Gamma(s+j) \zeta_H(2j+2s-1; 1/2), \tag{2.10}$$

and, for  $i = 1, 2$ ,

$$A_i^D(s) = -\frac{R^{2s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \zeta_H(-1+i+2j+2s; 1/2) \\ \times \sum_{a=0}^i x_{i,a} \frac{(i+2a)\Gamma(s+a+j+\frac{i}{2})}{\Gamma(1+a+\frac{i}{2})}. \tag{2.11}$$

Here

$$D_1(t) = \sum_{a=0}^1 x_{1,a} t^{1+2a} = \frac{1}{8}t - \frac{5}{24}t^3, \\ D_2(t) = \sum_{a=0}^2 x_{2,a} t^{2+2a} = \frac{1}{16}t^2 - \frac{3}{8}t^4 + \frac{5}{16}t^6,$$

and  $t = 1/\sqrt{1+z^2}, \eta = \sqrt{1+z^2} + \ln(z/1 + \sqrt{1+z^2})$ . Equation (2.7) is a very suitable starting point for the calculation of the zeta function determinant  $\zeta'(0)$ ; here we consider, for definiteness,  $m = 0$ . In the limit  $m \rightarrow 0$  only the  $j = 0$  term survives in the terms  $A_n^D(s), n = -1, 0, 1, 2$ , and one immediately finds

$$\left. \frac{d}{ds} \sum_{i=-1}^2 A_i^D(s) \right|_{s=0} = -\frac{3}{32} + \frac{\ln 2}{24} - \frac{\ln R}{24} + \frac{3}{2} \zeta'_R(-2) + \frac{1}{2} \zeta'_R(-1). \tag{2.12}$$

For the part  $Z_D^v(s)$  some additional calculation is needed. First of all using the analyticity of  $Z_D^v(s)$  around  $s = 0$ , the derivative  $Z_D^v(0)$  is found to be

$$Z_D^v(0) = - \left[ \ln I_v(mz) - v\eta + \ln(\sqrt{2\pi v}(1+z^2)^{1/4}) - \frac{D_1(t)}{v} - \frac{D_2(t)}{v^2} \right] \Big|_{z=(mR)/v}, \tag{2.13}$$

and, in the limit  $m \rightarrow 0$ , this reduces to

$$Z_D^v(0) = \ln \Gamma(v + 1) + v - v \ln v - \frac{1}{2} \ln(2\pi v) - \frac{1}{12v}. \tag{2.14}$$

To perform afterwards the sum over  $v$ , it is very convenient to use the integral representation of  $\ln \Gamma(v + 1)$  [34], to find

$$Z_D^v(0) = \int_0^\infty dt \left( -\frac{t}{12} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tv}}{t}. \tag{2.15}$$

Performing the sum, this yields

$$Z_D'(0) = 2 \int_0^\infty dt \left( -\frac{1}{2t^2} + \frac{2}{t^3} + \frac{d}{dt} \frac{e^{-t}}{t(1 - e^{-t})} \right) \frac{e^{-\frac{t}{2}}}{1 - e^{-t}}. \tag{2.16}$$

As is seen by studying the asymptotics for  $t \rightarrow 0$ , the integral is well defined. For the explicit calculation of  $Z_D'(0)$ , Eq. (2.16), it is suitable to introduce a regularisation parameter and to define

$$Z_D'(0, z) = 2 \int_0^\infty dt t^z \left( -\frac{1}{2t^2} + \frac{2}{t^3} + \frac{d}{dt} \frac{e^{-t}}{t(1 - e^{-t})} \right) \frac{e^{-\frac{t}{2}}}{1 - e^{-t}}, \tag{2.17}$$

with

$$Z_D'(0, 0) = Z_D'(0). \tag{2.18}$$

The individual pieces of the integral Eq. (2.17) may then be calculated by means of [34]

$$\int_0^\infty dx \frac{x^{v-1} e^{-\mu x}}{1 - e^{-\beta x}} = \frac{\Gamma(v)}{\beta^v} \zeta_H \left( v; \frac{\mu}{\beta} \right), \tag{2.19}$$

and differentiating with respect to  $\beta$ ,

$$\int_0^\infty dx \frac{x^v e^{-(\mu+\beta)x}}{(1 - e^{-\beta x})^2} = \frac{\Gamma(v+1)}{\beta^{v+1}} \zeta_H \left( v; \frac{\mu}{\beta} \right) - \frac{\Gamma(v+1)\mu}{\beta^{v+2}} \zeta_H \left( v+1; \frac{\mu}{\beta} \right). \tag{2.20}$$

As a result we have

$$Z_D'(0, z) = \zeta_H \left( z - 2; \frac{1}{2} \right) \Gamma(z - 2)[6 + z - z^2] + \frac{1}{4} \zeta_H \left( z; \frac{1}{2} \right) \Gamma(z),$$

and thus

$$Z_D'(0) = -\frac{9}{4} \zeta_R'(-2) - \frac{1}{8} \ln 2. \tag{2.21}$$

Adding up the contributions from Eqs. (2.12) and (2.21) we end up with

$$\zeta'(0) = -\frac{3}{32} - \frac{1}{12} \ln 2 - \frac{3}{4} \zeta'_R(-2) + \frac{1}{2} \zeta'_R(-1) - \frac{1}{24} \ln R \tag{2.22}$$

in agreement with Dowker and Apps [23].

### 3. Zeta Function Determinant on the 3-Dimensional Ball with Robin Boundary Conditions

In order to treat Robin boundary conditions, only very little changes are necessary. Writing the boundary condition in the form

$$\frac{u}{R} J_{l+1/2}(\omega_{l,n}R) + w_{l,n} J'_{l+1/2}(w_{l,n}r)|_{r=R} = 0, \tag{3.1}$$

the starting point of the calculation, analogous to Eqs. (2.8)–(2.11), is

$$\begin{aligned} Z_R^v(s) = & \frac{\sin(\pi s)}{\pi} \int_{mR/v}^{\infty} dz \left[ \left( \frac{zv}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left\{ \ln \left[ \frac{u}{R} I_v(zv) + \frac{zv}{R} I'_v(vz) \right] \right. \\ & \left. - \ln \left[ \frac{v}{R\sqrt{2\pi v}} e^{\eta} (1+z^2)^{\frac{1}{4}} \right] - \sum_{n=1}^2 \frac{M_n(t)}{v^n} \right\}, \end{aligned} \tag{3.2}$$

$$A_{-1}^R(s) = A_{-1}^D(s), \tag{3.3}$$

$$A_0^R(s) = -A_0^D(s), \tag{3.4}$$

and for  $i = 1, 2$ ,

$$\begin{aligned} A_i^R(s) = & -\frac{R^{2s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \zeta_H(-1+i+2j+2s; 1/2) \\ & \times \sum_{a=0}^{2i} z_{i,a} \frac{(i+a)\Gamma(s+j+\frac{a+i}{2})}{\Gamma(1+\frac{a+i}{2})}. \end{aligned} \tag{3.5}$$

Here we need the polynomials

$$M_1(t, u) = \sum_{a=0}^2 z_{1,a} t^{1+a} = \left( -\frac{3}{8} + u \right) t + \frac{7}{24} t^3, \tag{3.6}$$

and

$$M_2(t, u) = \sum_{a=0}^4 z_{2,a} t^{2+a} = \left( -\frac{3}{16} + \frac{u}{2} - \frac{u^2}{2} \right) t^2 + \left( \frac{5}{8} - \frac{u}{2} \right) t^4 - \frac{7}{16} t^6. \tag{3.7}$$

The contribution from the asymptotic terms is

$$\begin{aligned} \frac{d}{ds} \sum_{i=-1}^2 A_i^R(s) \Big|_{s=0} = & \frac{1}{32} - \frac{\ln 2}{24} + \frac{\ln R}{24} + \frac{3}{2} \zeta'_R(-2) - \frac{1}{2} \zeta'_R(-1) \\ & + \frac{u}{2} + \gamma u^2 + 2u^2 \ln 2 + u^2 \ln R. \end{aligned} \tag{3.8}$$

For the rest, things are very similar to the Dirichlet case. We find

$$Z_R''(0) = \int_0^\infty dt \left( -\frac{t}{12} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tv}}{t} + \ln \left( \frac{v}{u+v} \right) + \frac{u}{v} - \frac{u^2}{2v^2}, \tag{3.9}$$

and thus, using Eq. (2.15), we have

$$Z_R'(0) = -\frac{9}{4}\zeta_R'(-2) - \frac{1}{8} \ln 2 + 2 \sum_{l=0}^\infty \left\{ -v \ln \left( 1 + \frac{u}{v} \right) + u - \frac{u^2}{2v} \right\}. \tag{3.10}$$

The remaining sum may be done by differentiating with respect to  $u$  and integrating back. As a result

$$\begin{aligned} & \sum_{l=0}^\infty \left\{ -v \ln \left( 1 + \frac{u}{v} \right) + u - \frac{u^2}{2v} \right\} \\ &= -\frac{1}{2}u^2(\gamma + 2 \ln 2) - u \ln \Gamma \left( \frac{1}{2} + u \right) + \int_0^u dx \ln \Gamma \left( \frac{1}{2} + x \right). \end{aligned} \tag{3.11}$$

Let us mention that the last equation can also be given in terms of the Barnes  $G$ -function [34].

Putting things together, we arrive at our final result,

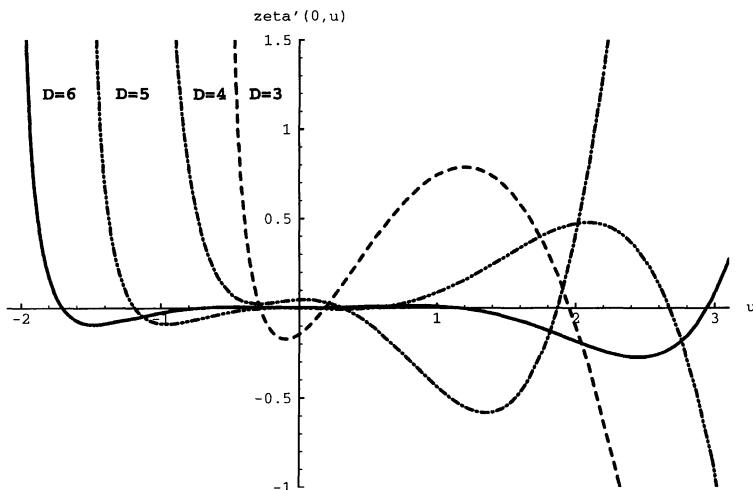
$$\begin{aligned} \zeta'(0, u) &= \frac{1}{32} - \frac{1}{6} \ln 2 - \frac{3}{4}\zeta_R'(-2) - \frac{1}{2}\zeta_R'(-1) + \frac{1}{24} \ln R \\ &+ \frac{u}{2} - 2u \ln \Gamma \left( \frac{1}{2} + u \right) + u^2 \ln R + 2 \int_0^u dx \ln \Gamma \left( \frac{1}{2} + x \right), \end{aligned} \tag{3.12}$$

which concludes the consideration of the three dimensional case with Robin boundary conditions. The dependence of  $\zeta'(0, u)$  on the parameter  $u$  is shown in Fig. 1 for  $R = 1$ .

Equation (3.12) can be given in an alternative form using the following expression, first obtained in [35] (see also [22]):

$$\begin{aligned} \int_0^u dx \ln \Gamma \left( \frac{1}{2} + x \right) &= \zeta_H(-1, u + 1/2) + \zeta_H'(-1, u + 1/2) \\ &+ \frac{u}{2} \ln(2\pi) + \frac{\ln 2 - 1}{24} + \frac{1}{2}\zeta_R'(-1). \end{aligned} \tag{3.13}$$

As is seen, the limit  $u \rightarrow -1/2$ , corresponding to Neumann boundary conditions, is not smooth, since a logarithmic divergence appears due to the  $\ln \Gamma(1/2 + u)$  term. This logarithmic divergence might be traced back to the  $v = 1/2$  term in Eq. (3.9). Thus, in order to find the functional determinant corresponding to Neumann boundary condition, the term  $v = 1/2$  has to be treated separately. In fact, looking in detail at the derivation of Eq. (3.9) starting from Eq. (3.2), it is seen that the case  $u = -1/2$  has to be treated specifically, because the behavior of  $(u/R)I_\nu(zv) + (zv/R)I_\nu'(zv)$  for  $z \rightarrow 0$  is different for  $u = -1/2$ .



**Fig. 1.** Plot of the dependence of  $\zeta'(0, u)$  on the parameter  $u$  for  $R = 1$  and for dimensions  $D = 3, 4, 5, 6$ . Notice the divergence that appears for  $u = 1 - D/2$  in each dimension, corresponding to the case of Neumann boundary conditions.

Probably the easiest way to find the results for the Neumann boundary conditions is to write

$$\zeta(s, -1/2) = \zeta^{l=0}(s, -1/2) + \lim_{u \rightarrow -1/2} (\zeta(s, u) - \zeta^{l=0}(s, u)), \tag{3.14}$$

because then we can use all the results for the Robin boundary conditions that we have derived before. Here  $\zeta^{l=0}(s, u)$  is the contribution from the angular momentum component  $l = 0$  to  $\zeta(s, u)$ ,

$$\zeta^{l=0}(s, u) = \frac{\sin(\pi s)}{\pi} \int_m^\infty dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln \left[ \frac{u}{R} I_{1/2}(kR) + k I'_{1/2}(kR) \right]. \tag{3.15}$$

Proceeding with the calculations as for the Robin boundary conditions, one easily finds

$$\frac{d}{ds} (\zeta^{l=0}(s, -1/2) - \zeta^{l=0}(s, u)) \Big|_{s=0} = -2 \ln R + \ln(3/2) + \ln(u + 1/2). \tag{3.16}$$

In the limit  $u \rightarrow -1/2$ , the logarithmic divergence in Eq. (3.16) cancels the divergence in  $\zeta(s, u)$ , Eq. (3.12). Thus the limit  $u \rightarrow -1/2$  is well defined and the functional determinant for Neumann boundary conditions reads

$$\begin{aligned} \zeta'(0, -1/2) = & -\frac{7}{32} - \frac{7 \ln 2}{6} + \ln 3 - \frac{41 \ln R}{24} - \frac{3 \zeta'_R(-2)}{4} - \frac{\zeta'_R(-1)}{2} \\ & - 2 \int_0^{1/2} dx \ln \Gamma(x). \end{aligned} \tag{3.17}$$

This concludes the consideration of the 3-dimensional case: all ordinary boundary conditions have been dealt with explicitly. As for the Robin boundary condition, an



alternative form may be presented using

$$\int_0^{1/2} dx \ln \Gamma(x) = \frac{1}{8} + \frac{5}{24} \ln 2 + \frac{1}{4} \ln \pi - \frac{3}{2} \zeta'_R(-1).$$

**4. Zeta Function Determinants on  $D > 3$  Dimensional ball**

For higher dimensional balls, exactly the same procedure may be employed in order to calculate the zeta function determinant. The starting point is here (for details see [24])

$$\zeta_D(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln J_{l+\frac{D-2}{2}}(kR), \tag{4.1}$$

with the number  $d_l(D)$  of independent harmonic polynomials, explicitly

$$d_l(D) = (2l + D - 2) \frac{(l + D - 3)!}{l!(D - 2)!}. \tag{4.2}$$

It is suitable to introduce the coefficients  $e_{\alpha}(D)$  by

$$d_l(D) = \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \left( l + \frac{D-2}{2} \right)^{\alpha}. \tag{4.3}$$

A representation which shows the analytic structure around  $s = 0$  may then be given in analogy with Eq. (2.7) (now one defines  $\nu = l + (D - 2)/2$ )

$$\zeta_D(s) = \sum_{l=0}^{\infty} d_l(D) Z_D^{\nu}(s) + \sum_{i=-1}^{D-1} A_i^D(s). \tag{4.4}$$

Here we have used the following definitions. First

$$\begin{aligned} Z_D^{\nu}(s) &= \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[ \left( \frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \\ &\times \frac{\partial}{\partial z} \left[ \ln I_{\nu}(z\nu) - \nu\eta + \ln(\sqrt{2\pi\nu}(1+z^2)^{1/4}) - \sum_{n=1}^{D-1} \frac{D_n(t)}{\nu^n} \right]. \end{aligned} \tag{4.5}$$

The polynomials  $D_n(t)$  arise from the asymptotic expansion of  $I_{\nu}(z\nu)$  [36]. In detail one has

$$I_{\nu}(z\nu) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right], \tag{4.6}$$

with the recursion relation

$$u_{k+1}(t) = \frac{1}{2} t^2 (1 - t^2) u'_k(t) + \frac{1}{8} \int_0^t d\tau (1 - 5\tau^2) u_k(\tau) \tag{4.7}$$

for the polynomials  $u_k(t)$ . The coefficient functions  $D_n(t)$  are then defined through the cumulant expansion

$$\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{v^k} \right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{v^n} \tag{4.8}$$

and are easily found with the help of a simple computer program.

The  $A_i^D(s)$  have already been determined in [24] and we give only their final form for completeness. Introducing

$$D_i(t) = \sum_{a=0}^i x_{i,a} t^{i+2a}, \tag{4.9}$$

they read

$$\begin{aligned} A_{-1}^D(s) &= \frac{R^{2s}}{4\sqrt{\pi}\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \frac{\Gamma(j+s-\frac{1}{2})}{s+j} \\ &\times \left[ \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \zeta_H(2j+2s-1-\alpha; (D-2)/2) \right], \end{aligned} \tag{4.10}$$

$$\begin{aligned} A_0^D(s) &= -\frac{R^{2s}}{4\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \Gamma(s+j) \\ &\times \left[ \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \zeta_H(2j+2s-\alpha; (D-2)/2) \right], \end{aligned} \tag{4.11}$$

$$\begin{aligned} A_i^D(s) &= -\frac{R^{2s}}{2\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \\ &\times \left[ \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \zeta_H(-\alpha+i+2j+2s; (D-2)/2) \right] \\ &\times \sum_{a=0}^i x_{i,a} \frac{(i+2a)\Gamma(s+a+j+\frac{i}{2})}{\Gamma(1+a+\frac{i}{2})}. \end{aligned} \tag{4.12}$$

Their contributions to the functional determinant are easily determined and are listed in Appendix A, for the dimensions  $D = 4, 5, 6$ .

For the calculation of  $Z_D^v(s)$  we go on as for the three dimensional case. We have found

$$\begin{aligned} Z_D^v(0) &= \ln \Gamma(v+1) + v - v \ln v - \frac{1}{2} \ln(2\pi v) + \sum_{n=1}^{D-1} \frac{D_n(1)}{v^n} \\ &= \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tv}}{t} + \sum_{n=1}^{D-1} \frac{D_n(1)}{v^n} \\ &= \int_0^{\infty} \left( \sum_{n=1}^{D-1} \frac{D_n(1)}{(n-1)!} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tv}}{t}. \end{aligned} \tag{4.13}$$

This is the suitable starting point for the summation of the angular momentum. Introducing the regularized version of  $Z_D(s)$  as in Eq. (2.17) one obtains

$$\begin{aligned}
 Z'_D(0, z) = & \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \int_0^{\infty} dt t^z \frac{e^{-t \frac{D-2}{2}}}{1 - e^{-t}} \frac{d^{\alpha}}{dt^{\alpha}} \left( \frac{1}{t} \frac{1}{e^t - 1} \right) \\
 & + \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \left\{ \sum_{n \geq \alpha+1}^{D-1} \frac{D_n(1)}{\Gamma(n - \alpha)} \zeta_H \left( z + n - \alpha; \frac{D-2}{2} \right) \Gamma(z + n - \alpha) \right. \\
 & + \frac{1}{2} (-1)^{\alpha} \alpha! \zeta_H \left( z - \alpha; \frac{D-2}{2} \right) \Gamma(z - \alpha) \\
 & \left. - (-1)^{\alpha} (\alpha + 1)! \zeta_H \left( z - \alpha - 1; \frac{D-2}{2} \right) \Gamma(z - \alpha - 1) \right\}, \tag{4.14}
 \end{aligned}$$

which we need at  $z = 0$ . The remaining task is the calculation of the integral term in Eq. (4.14). It may be given by repeated differentiation of Eq. (2.19). Defining

$$f(1, \beta, \gamma, \nu) = \int_0^{\infty} \frac{x^{\nu-1} e^{-\gamma x}}{1 - e^{-\beta x}} dx = \frac{\Gamma(\nu)}{\beta^{\nu}} \zeta \left( \nu, \frac{\gamma}{\beta} \right) \tag{4.15}$$

and

$$f(k, \beta, \gamma, \nu) = \int_0^{\infty} \frac{x^{\nu+k-2} e^{-[\gamma+(k-1)\beta]x}}{(1 - e^{-\beta x})^k} dx, \quad k = 1, 2, \dots, \tag{4.16}$$

we get the recurrence

$$\begin{aligned}
 f(k, \beta, \gamma, \nu) = & \frac{-1}{k-1} \frac{\partial}{\partial \beta} f(k-1, \beta, \gamma, \nu) \\
 & - \frac{k-2}{k-1} f(k-1, \beta, \gamma, \nu+1). \tag{4.17}
 \end{aligned}$$

The first few cases,  $k = 2, 3, 4$ , and a general formula for  $f(k, \beta, \gamma, \nu)$  is given in Appendix B. All terms appearing in Eq. (4.14) and resulting from the integration may be given in terms of these functions  $f(k, \beta, \gamma, \nu)$ . In detail this is seen as follows. First of all one may show that

$$\frac{d^{\alpha}}{dt^{\alpha}} \left( \frac{1}{t} \frac{1}{e^t - 1} \right) = \sum_{k=0}^{\alpha} B_k^{(\alpha)} \sum_{i=1}^{k+1} A_i^{(k)} \frac{t^{k-1-\alpha}}{(e^t - 1)^i},$$

with the recursion relations

$$B_0^{(\alpha)} = (-1)^{\alpha} \alpha!,$$

$$B_{\alpha}^{(\alpha)} = 1,$$

$$B_k^{(\alpha)} = (k - \alpha) B_k^{(\alpha-1)} + B_{k-1}^{(\alpha-1)}, \quad \text{for } k = 1, \dots, \alpha - 1,$$

for  $B_k^{(\alpha)}$ , and similar ones for  $A_i^{(k)}$  [43],

$$\begin{aligned} A_1^{(k)} &= (-1)^k, \\ A_{k+1}^{(k)} &= (-1)^k k!, \\ A_i^{(k)} &= -iA_i^{(k-1)} - (i-1)A_{i-1}^{(k-1)}, \quad \text{for } 2 \leq i \leq k, \end{aligned}$$

coming from

$$\frac{d^k}{dt^k} \frac{1}{e^t - 1} = \sum_{i=1}^{k+1} \frac{A_i^{(k)}}{(e^t - 1)^i}.$$

As a result,

$$\begin{aligned} &\sum_{\alpha=1}^{D-2} e_\alpha(D) \int_0^\infty dt t^z \frac{e^{-t \frac{D-2}{2}}}{1 - e^{-t}} \frac{d^\alpha}{dt^\alpha} \left( \frac{1}{t} \frac{1}{e^t - 1} \right) \\ &= \sum_{\alpha=1}^{D-2} e_\alpha(D) \sum_{k=0}^\alpha B_k^{(\alpha)} \sum_{i=1}^{k+1} A_i^{(k)} f \left( i + 1, 1, \frac{D-2}{2}, z + k - \alpha - i \right). \end{aligned}$$

Thus we have derived all necessary equations, showing that  $Z'_D(0, z)$  may be given solely in terms of  $\Gamma$ -functions and Riemann zeta functions and the limit  $z \rightarrow 0$  may be taken. The results are listed in Appendix C for  $D = 4, 5, 6$ .

Adding up the contributions from the asymptotic terms and  $Z'_D(0)$ , we have found the following final results for the zeta function determinants of the Laplace operator with Dirichlet boundary conditions in  $D = 4, 5$  and 6 dimensions,

$$\begin{aligned} \zeta'_4(0) &= \frac{173}{30240} + \frac{1}{90} \ln 2 - \frac{1}{90} \ln R + \frac{1}{3} \zeta'_R(-3) \\ &\quad - \frac{1}{2} \zeta'_R(-2) + \frac{1}{6} \zeta'_R(-1), \\ \zeta'_5(0) &= \frac{47}{9216} + \frac{17}{2880} \ln 2 + \frac{17}{5760} \ln R - \frac{5}{64} \zeta'_R(-4) \\ &\quad + \frac{7}{48} \zeta'_R(-3) - \frac{1}{32} \zeta'_R(-2) - \frac{1}{48} \zeta'_R(-1), \\ \zeta'_6(0) &= -\frac{4027}{6486480} - \frac{1}{756} \ln 2 + \frac{1}{756} \ln R + \frac{1}{60} \zeta'_R(-5) - \frac{1}{24} \zeta'_R(-4) \\ &\quad + \frac{1}{24} \zeta'_R(-2) - \frac{1}{60} \zeta'_R(-1). \end{aligned}$$

The result for  $D = 4$  agrees with Dowker and Apps [23]. As is clear from our presentation, *any* higher dimension  $D$  can be treated in exactly the same way without additional problems (they just get a bit arithmetically cumbersome).

Let us now describe briefly the calculation for Robin boundary condition. There the starting point is

$$\zeta(s, u) = \sum_{l=0}^\infty d_l(D) \int_\gamma \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln \left[ \frac{u}{R} J_\nu(kR) + k J'_\nu(kR) \right]. \quad (4.18)$$

The relevant asymptotic expansions are (4.6) together with [36]

$$I'_v(vz) \sim \frac{1}{\sqrt{2\pi v}} \frac{e^{v\eta}(1+z^2)^{1/4}}{z} \left[ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{v^k} \right]. \quad (4.19)$$

The relevant polynomials this time are determined by

$$\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{v^k} + \frac{u}{v} t \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{v^k} \right) \right] \sim \sum_{n=1}^{\infty} \frac{M_n(t)}{v^n}, \quad (4.20)$$

where, in analogy to Eq. (4.9), we define

$$M_n(t, u) = \sum_{a=0}^{2i} z_{i,a} t^{i+a}. \quad (4.21)$$

Continuing as for the 3-dimensional case, and in analogy to Eqs. (4.4), (4.5), we define

$$\begin{aligned} Z_R^v(s) &= \frac{\sin(\pi s)}{\pi} \int_{mR/v}^{\infty} dz \left[ \left( \frac{zv}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left\{ \ln \left[ \frac{u}{R} I_v(zv) + \frac{zv}{R} I'_v(vz) \right] \right. \\ &\quad \left. - \ln \left[ \frac{v}{R\sqrt{2\pi v}} e^{v\eta}(1+z^2)^{1/4} \right] - \sum_{n=1}^{D-1} \frac{M_n(t, u)}{v^n} \right\}. \end{aligned} \quad (4.22)$$

In the limit  $m \rightarrow 0$ , we obtain

$$Z_R^v(s) = \int_0^{\infty} dt \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tv}}{t} + \ln \frac{v}{u+v} + \sum_{n=1}^{D-1} \frac{M_n(1, u)}{v^n}, \quad (4.23)$$

and, realizing that

$$M_n(1, 0) = D_n(1), \quad M_n(1, u) - M_n(1, 0) = (-1)^{n+1} \frac{u^n}{n}, \quad (4.24)$$

we find, after performing the sum over the angular momentum,

$$Z'_R(0) = Z'_D(0) + \sum_{l=0}^{\infty} \sum_{\alpha=1}^{D-2} e_{\alpha}(D) v^{\alpha} \left\{ -\ln \left( 1 + \frac{u}{v} \right) + \sum_{n=1}^{D-1} (-1)^{n+1} \frac{1}{n} \left( \frac{u}{v} \right)^n \right\}. \quad (4.25)$$

The remaining sum may once more be done by differentiating and integrating back, and we end up with

$$\begin{aligned} Z'_R(0) &= Z'_D(0) + \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \left\{ (-1)^{\alpha+1} \Psi \left( \frac{D-2}{2} \right) \frac{u^{\alpha+1}}{\alpha+1} \right. \\ &\quad + \sum_{n=\alpha+2}^{D-1} (-1)^{n+1} \zeta_H(n-\alpha; (D-2)/2) \frac{u^n}{n} \\ &\quad - (-1)^{\alpha} \alpha \int_0^u dx x^{\alpha-1} \ln \Gamma \left( \frac{D-2}{2} + x \right) \\ &\quad \left. + (-1)^{\alpha} u^{\alpha} \ln \Gamma \left( \frac{D-2}{2} + u \right) \right\}. \end{aligned} \quad (4.26)$$

The results for the special dimensions  $D = 4, 5$  and  $6$  are not listed, because they are essentially given in Eq. (4.26) together with Appendix C. Let us only note that

$$\begin{aligned}
 e_1(4) &= 0, & e_2(4) &= 1, \\
 e_1(5) &= -\frac{1}{12}, & e_2(5) &= 0, & e_3(5) &= \frac{1}{3}, \\
 e_1(6) &= 0, & e_2(6) &= -\frac{1}{12}, & e_3(6) &= 0, & e_4(6) &= \frac{1}{12}.
 \end{aligned}$$

The contributions of the asymptotic terms are calculated from

$$A_{-1}^R(s) = A_{-1}^D(s), \quad A_0^R(s) = -A_0^D(s), \tag{4.27}$$

and

$$\begin{aligned}
 A_i^R(s) &= -\frac{R^{2s}}{2\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \\
 &\quad \times \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \zeta_H(-1 + i + 2j + 2s; (D - 2)/2) \\
 &\quad \times \sum_{a=0}^{2i} z_{i,a} \frac{(i + a)\Gamma(s + j + \frac{a+i}{2})}{\Gamma(1 + \frac{a+i}{2})}.
 \end{aligned} \tag{4.28}$$

They are collected in Appendix A.

Summing up, we have found the following final results for the functional determinant with Robin boundary conditions:

$$\begin{aligned}
 \zeta'_4(0, u) &= \frac{11}{4320} + \frac{u}{30} - \frac{5u^2}{12} - \frac{u^3}{3} + \frac{\ln(2)}{90} + \frac{u^3 \ln(2)}{3} \\
 &\quad - \frac{\ln(R)}{90} - \frac{u^3 \ln(R)}{3} + \frac{\zeta'_R(-3)}{3} + \frac{\zeta'_R(-2)}{2} + \frac{\zeta'_R(-1)}{6} \\
 &\quad + u^2 \ln \Gamma(1 + u) - 2 \int_0^u dx x \ln \Gamma(1 + x), \\
 \zeta'_5(0, u) &= -\frac{61}{46080} - \frac{11u}{576} - \frac{u^2}{16} + \frac{11u^3}{72} + \frac{u^4}{24} \\
 &\quad + \frac{7\ln(2)}{720} - \frac{17\ln(R)}{5760} - \frac{u^2 \ln(R)}{24} + \frac{u^4 \ln(R)}{12} \\
 &\quad - \frac{5\zeta'_R(-4)}{64} - \frac{7\zeta'_R(-3)}{48} - \frac{\zeta'_R(-2)}{32} + \frac{\zeta'_R(-1)}{48} \\
 &\quad - \frac{1}{12} \int_0^u dx \ln \Gamma\left(\frac{3}{2} + x\right) + \int_0^u dx x^2 \ln \Gamma\left(\frac{3}{2} + x\right) \\
 &\quad + \frac{1}{12} u \ln \Gamma\left(\frac{3}{2} + u\right) - \frac{1}{3} u^3 \ln \Gamma\left(\frac{3}{2} + u\right),
 \end{aligned}$$

$$\begin{aligned} \zeta'_6(0, u) = & -\frac{9479}{32432400} - \frac{u}{315} + \frac{517u^2}{15120} + \frac{83u^3}{1512} - \frac{19u^4}{480} \\ & - \frac{u^5}{45} - \frac{\log(2)}{756} - \frac{u^3 \ln(2)}{36} + \frac{u^5 \ln(2)}{60} + \frac{\ln(R)}{756} + \frac{u^3 \ln(R)}{36} \\ & - \frac{u^5 \ln(R)}{60} + \frac{\zeta'_R(-5)}{60} + \frac{\zeta'_R(-4)}{24} - \frac{\zeta'_R(-2)}{24} - \frac{\zeta'_R(-1)}{60} \\ & + \frac{1}{6} \int_0^u dx x \ln \Gamma(2+x) - \frac{1}{3} \int_0^u dx x^3 \ln \Gamma(2+x) \\ & - \frac{1}{12} u^2 \ln \Gamma(2+u) + \frac{1}{12} u^4 \ln \Gamma(2+u). \end{aligned}$$

The detailed dependence of  $\zeta'(0, u)$  on the parameter  $u$  for dimensions  $D = 4, 5, 6$  is given in Fig. 1, for  $R = 1$ .

Finally, we are left with the task of the calculation of the zeta function determinant for Neumann boundary conditions. As we have already seen, the  $l = 0$  term corresponding to  $\nu = (D - 2)/2$  has to be treated separately. Employing the same procedure as for 3 dimensions, one finds

$$\begin{aligned} & \left. \frac{d}{ds} (\zeta^{l=0}(s, 1 - D/2) - \zeta^{l=0}(s, u)) \right|_{s=0} \\ & = \sum_{\alpha=1}^{D-2} e_{\alpha}(D) \left( \frac{D-2}{2} \right)^{\alpha} [-2 \ln R + \ln(D/2) + \ln((D-2)/2 + u)], \end{aligned}$$

which results in the following final results for Neumann boundary conditions,

$$\begin{aligned} \zeta'_4(0, -1) = & -\frac{493}{4320} + \frac{61 \ln 2}{90} - \frac{151 \ln R}{90} \\ & + \frac{\zeta'_R(-3)}{3} + \frac{\zeta'_R(-2)}{2} + \frac{\zeta'_R(-1)}{6} \\ & + 2 \int_0^1 dx (x-1) \ln \Gamma(x), \\ \zeta'_5(0, -3/2) = & -\frac{19261}{46080} - \frac{713 \ln 2}{720} + \ln 5 - \frac{9647 \ln R}{5760} \\ & - \frac{5 \zeta'_R(-4)}{64} - \frac{7 \zeta'_R(-3)}{48} - \frac{\zeta'_R(-2)}{32} + \frac{\zeta'_R(-1)}{48} \\ & + \frac{1}{12} \int_0^{3/2} dx \ln \Gamma(x) - \int_0^{3/2} dx \left( x - \frac{3}{2} \right)^2 \ln \Gamma(x), \end{aligned}$$

$$\begin{aligned} \zeta'_6(0, -2) = & -\frac{7087979}{32432400} - \frac{1181 \ln 2}{3780} + \ln 3 - \frac{6379 \ln R}{3780} \\ & + \frac{\zeta'_R(-5)}{60} + \frac{\zeta'_R(-4)}{24} - \frac{\zeta'_R(-2)}{24} - \frac{\zeta'_R(-1)}{60} \\ & - \frac{1}{6} \int_0^2 dx (x-2) \ln \Gamma(x) + \frac{1}{3} \int_0^2 dx (x-2)^3 \ln \Gamma(x). \end{aligned}$$

This terminates the explicit calculation for all kind of boundary conditions and for dimensions  $D = 3, 4, 5, 6$  (that can be extended immediately to any dimension  $D > 6$ ). Once more the results may be given in an alternative form by using Eq. (3.13), but this makes little difference for numerical calculations.

Finally, we are left with the case  $D = 2$ .

### 5. Zeta Function Determinants on the 2-Dimensional Ball

For the 2-dimensional ball the procedure has to be changed slightly. Here the degeneracy of every  $l \geq 1$  is 2,  $l = 0$  has to be counted only once. Due to the presence of this term  $l = 0$ , the starting point Eqs. (2.7)–(2.11) is not valid any more and may be applied only to  $l \geq 1$ . The  $l = 0$  term may be treated as before for the Neumann boundary conditions. We shall not give any further details for this case but only write down the final results – which on the other hand have appeared in part in the literature [37]. We quote them for completeness.

For Dirichlet boundary conditions we have

$$\zeta'_D(0) = \frac{5}{12} + 2\zeta'_R(-1) + \frac{1}{2} \ln \pi + \frac{1}{6} \ln 2 + \frac{1}{3} \ln R.$$

The zeta function determinant for general Robin boundary conditions reads

$$\zeta'_R(0) = -\frac{7}{12} + \frac{1}{3} \ln R + 2\zeta'_R(-1) - \frac{5}{6} \ln 2 - \frac{1}{2} \ln \pi + 2u \ln(2/R) - \ln u + 2 \ln \Gamma(1 + u).$$

Finally, the results for Neumann boundary condition is

$$\zeta'_N(0) = -\frac{7}{12} - \frac{5}{3} \ln R + 2\zeta'_R(-1) + \frac{1}{6} \ln 2 - \frac{1}{2} \ln \pi.$$

And this concludes the list of examples of zeta function determinants on the ball that we had promised to consider.

### 6. Conclusions

In this paper we have developed a systematic approach for the calculation of functional determinants of (elliptic differential) operators, which is very useful in all cases when the basis of functions – constrained by the equations corresponding to the boundary conditions – is known. Using our approach we have calculated the zeta function determinant of the Laplacian on the ball for Dirichlet, Neumann and the general Robin boundary conditions. Explicit results for dimensions  $D \leq 6$  have been



given. All necessary formulas to iterate the calculation of  $\zeta'(0)$  and to obtain it in any dimension, by means of a simple computer program, have been given explicitly.

An extension of the present work to higher-dimensional bundles, such as those for spinors and vectors, is envisaged. This should allow to establish connections with recent work dealing with mixed boundary conditions [38–40].

*Comment.* Using a different approach, based on work by Moss [41] and Voros [42], Dowker has also considered the calculation of heat-kernel coefficients and functional determinants for the Laplace operator on the ball with Robin boundary conditions [43]. We are indebted to him for interesting and fruitful correspondence during the course of the present work.

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## A. Appendix: Contributions to the Zeta Function Determinant from the Asymptotic Terms

In this appendix we list the contributions of the asymptotic terms to the zeta function determinant for Dirichlet and Robin boundary conditions.

*A.1. Dirichlet Boundary Condition.* For Dirichlet boundary conditions we find

$$\begin{aligned} (\text{Asym})_D(D=4) &= -\frac{83}{6048} - \frac{\gamma}{360} + \frac{\ln 2}{90} - \frac{\ln R}{90} - \zeta'_R(-3) - \frac{\zeta'_R(-2)}{2}, \\ (\text{Asym})_D(D=5) &= \frac{47}{9216} + \frac{\pi^2}{8640} - \frac{11}{5760} \ln 2 + \frac{17}{5760} \ln R \\ &\quad + \frac{5}{16} \zeta'_R(-4) + \frac{7}{48} \zeta'_R(-3) - \frac{1}{16} \zeta'_R(-2) - \frac{1}{48} \zeta'_R(-1), \\ (\text{Asym})_D(D=6) &= \frac{40091}{25945920} + \frac{\gamma}{3360} - \frac{\ln 2}{756} + \frac{\ln R}{756} - \frac{\zeta_R(3)}{15120} - \frac{\zeta'_R(-5)}{12} \\ &\quad - \frac{\zeta'_R(-4)}{24} + \frac{\zeta'_R(-3)}{12} + \frac{\zeta'_R(-2)}{24}. \end{aligned}$$

*A.2. Robin Boundary Condition.* For Robin boundary conditions the final results read

$$\begin{aligned} (\text{Asym})_R(D=4) &= -\frac{73}{4320} - \frac{1}{360} \gamma + \frac{1}{90} \ln 2 - \frac{1}{90} \ln R - \zeta'_R(-3) + \frac{1}{2} \zeta'_R(-2) \\ &\quad + \frac{u}{30} - \frac{5}{12} u^2 - \frac{1}{3} u^3 - \frac{1}{3} \gamma u^3 + \frac{1}{3} u^3 \ln 2 - \frac{1}{3} u^3 \ln R, \end{aligned}$$

$$\begin{aligned}
(\text{Asym})_R(D=5) &= -\frac{61}{46080} + \frac{\pi^2}{8640} + \frac{11}{5760} \ln 2 - \frac{17}{5760} \ln R \\
&+ \frac{5}{16} \zeta'_R(-4) - \frac{7}{48} \zeta'_R(-3) - \frac{1}{16} \zeta'_R(-2) + \frac{1}{48} \zeta'_R(-1) \\
&- \frac{11}{576} u + \frac{1}{48} u^2 - \frac{1}{24} \gamma u^2 + \frac{1}{24} u^3 + \frac{1}{72} \pi^2 u^3 + \frac{1}{24} u^4 \\
&+ \frac{1}{12} \gamma u^4 - \frac{1}{12} u^2 \ln 2 + \frac{1}{6} u^4 \ln 2 - \frac{1}{24} u^2 \ln R \\
&+ \frac{1}{12} u^4 \ln R - \frac{7}{48} u^4 \zeta_R(3),
\end{aligned}$$

$$\begin{aligned}
(\text{Asym})_R(D=6) &= \frac{243079}{129729600} + \frac{1}{3360} \gamma - \frac{1}{756} \ln 2 - \frac{1}{15120} \zeta_R(3) \\
&- \frac{1}{12} \zeta'_R(-5) + \frac{1}{24} \zeta'_R(-4) + \frac{1}{12} \zeta'_R(-3) - \frac{1}{24} \zeta'_R(-2) \\
&- \frac{1}{315} u + \frac{517}{15120} u^2 + \frac{41}{1512} u^3 + \frac{1}{36} \gamma u^3 - \frac{3}{160} u^4 \\
&- \frac{1}{288} \pi^2 u^4 - \frac{1}{45} u^5 - \frac{1}{60} \gamma u^5 - \frac{1}{36} u^3 \ln 2 + \frac{1}{60} u^5 \ln 2 \\
&+ \frac{1}{756} \ln R + \frac{1}{36} u^3 \ln R - \frac{1}{60} u^5 \ln R + \frac{1}{60} u^5 \zeta_R(3).
\end{aligned}$$

## B. Appendix: Detail of the Calculation of $Z'_D(0)$

Using the recurrence relation (4.17), the function  $f(k, \beta, \gamma, \nu)$  for  $k = 2, 3$  and 4 read,

$$\begin{aligned}
f(2, \beta, \gamma, \nu) &= \int_0^\infty \frac{x^\nu e^{-(\gamma+\beta)x}}{(1-e^{-\beta x})^2} dx \\
&= \frac{\Gamma(\nu+1)}{\beta^{\nu+1}} \left[ \zeta\left(\nu, \frac{\gamma}{\beta}\right) - \frac{\gamma}{\beta} \zeta\left(\nu+1, \frac{\gamma}{\beta}\right) \right], \quad (2.1)
\end{aligned}$$

$$\begin{aligned}
f(3, \beta, \gamma, \nu) &= \int_0^\infty \frac{x^{\nu+1} e^{-(\gamma+2\beta)x}}{(1-e^{-\beta x})^3} dx \\
&= \frac{\Gamma(\nu+2)}{2\beta^{\nu+2}} \left[ \zeta\left(\nu, \frac{\gamma}{\beta}\right) - \left(1+2\frac{\gamma}{\beta}\right) \zeta\left(\nu+1, \frac{\gamma}{\beta}\right) \right. \\
&\quad \left. + \frac{\gamma}{\beta} \left(1+\frac{\gamma}{\beta}\right) \zeta\left(\nu+2, \frac{\gamma}{\beta}\right) \right], \quad (2.2)
\end{aligned}$$

$$\begin{aligned}
 f(4, \beta, \gamma, \nu) &= \int_0^\infty \frac{x^{\nu+2} e^{-(\gamma+3\beta)x}}{(1 - e^{-\beta x})^4} dx \\
 &= \frac{\Gamma(\nu+3)}{6\beta^{\nu+3}} \left[ \zeta\left(\nu, \frac{\gamma}{\beta}\right) - 3\left(1 + \frac{\gamma}{\beta}\right) \zeta\left(\nu+1, \frac{\gamma}{\beta}\right) \right. \\
 &\quad + \left[ 2 + 6\frac{\gamma}{\beta} + 3\left(\frac{\gamma}{\beta}\right)^2 \right] \zeta\left(\nu+2, \frac{\gamma}{\beta}\right) \\
 &\quad \left. - \frac{\gamma}{\beta} \left[ 2 + 3\frac{\gamma}{\beta} + \left(\frac{\gamma}{\beta}\right)^2 \right] \zeta\left(\nu+3, \frac{\gamma}{\beta}\right) \right]. \tag{2.3}
 \end{aligned}$$

For the general formula one has

$$f(k, \beta, \gamma, \nu) = \frac{\Gamma(\nu+k-1)}{(k-1)! \beta^{\nu+k-1}} \sum_{j=0}^{k-1} (-1)^j c_{k-1,j} \left(\frac{\gamma}{\beta}\right) \zeta\left(\nu+k, \frac{\gamma}{\beta}\right), \tag{2.4}$$

where

$$\begin{aligned}
 c_{k-1,0} \left(\frac{\gamma}{\beta}\right) &= 1, \quad c_{k-1,1} \left(\frac{\gamma}{\beta}\right) = \frac{\gamma}{\beta} + \left(\frac{\gamma}{\beta} + 1\right) + \dots + \left(\frac{\gamma}{\beta} + k - 2\right), \dots, \\
 c_{k-1,k-1} \left(\frac{\gamma}{\beta}\right) &= \frac{\Gamma\left(\frac{\gamma}{\beta} + k - 1\right)}{\Gamma\left(\frac{\gamma}{\beta}\right)}.
 \end{aligned}$$

### C. Appendix: Contribution of $Z_D(s)$ to the Zeta Function Determinant

In this appendix we list the contributions of  $Z_D(s)$  to the zeta function determinant for dimensions  $D = 4, 5$  and  $6$ . They are

$$\begin{aligned}
 Z_D^{4'}(0) &= \frac{7}{360} + \frac{1}{360}\gamma + \frac{4}{3}\zeta_R'(-3) + \frac{1}{6}\zeta_R'(-1), \\
 Z_D^{5'}(0) &= -\frac{\pi^2}{8640} + \frac{1}{128} \ln 2 - \frac{25}{64}\zeta_R'(-4) + \frac{1}{32}\zeta_R'(-2), \\
 Z_D^{6'}(0) &= -\frac{131}{60480} - \frac{\gamma}{3360} + \frac{1}{15120}\zeta_R(3) + \frac{1}{10}\zeta_R'(-5) - \frac{1}{12}\zeta_R'(-3) - \frac{1}{60}\zeta_R'(-1).
 \end{aligned}$$

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