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Bethe Subalgebras in Twisted Yangians

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Abstract: We study analogues of the Yangian of the Lie algebra $\mathfrak{gl}_N$ for the other classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. We call them twisted Yangians. They are coideal subalgebras in the Yangian of $\mathfrak{gl}_N$ and admit homomorphisms onto the universal enveloping algebras $\mathrm{U}(\mathfrak{so}_N)$ and $\mathrm{U}(\mathfrak{sp}_N)$ respectively. In every twisted Yangian we construct a family of maximal commutative subalgebras parametrized by the regular semisimple elements of the corresponding classical Lie algebra. The images in $\mathrm{U}(\mathfrak{so}_N)$ and $\mathrm{U}(\mathfrak{sp}_N)$ of these subalgebras are also maximal commutative.

Introduction

In this article we study the Yangian of the Lie algebra $\mathfrak{gl}_N$ and its analogues for the other classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N[t])$ in the class of Hopf algebras [D1]. Moreover, it contains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ as a subalgebra and admits a homomorphism $\pi : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ identical on $\mathrm{U}(\mathfrak{gl}_N)$.

Let $\mathfrak{a}_N$ be one of the Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. In [D1] the Yangian $\mathrm{Y}(\mathfrak{a}_N)$ was defined as a deformation of the Hopf algebra $\mathrm{U}(\mathfrak{a}_N[t])$. It contains $\mathrm{U}(\mathfrak{a}_N)$ as a subalgebra but does not admit a homomorphism $\mathrm{Y}(\mathfrak{a}_N) \to \mathrm{U}(\mathfrak{a}_N)$ identical on $\mathrm{U}(\mathfrak{a}_N)$. In the present article we consider another analogue of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ for the classical Lie algebra $\mathfrak{a}_N$. It has been introduced in [O2] and called the twisted Yangian; see also [MNO]. The definition in [O2] was motivated by [O1] and [C2, S]. Algebras closely related to this analogue of $\mathrm{Y}(\mathfrak{gl}_N)$ were recently studied in [NS].

Consider $\mathfrak{a}_N$ as a fixed point subalgebra in the Lie algebra $\mathfrak{gl}_N$ with respect to an involutive automorphism σ. The twisted Yangian $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is a subalgebra in $\mathrm{Y}(\mathfrak{gl}_N)$. Moreover, it is a left coideal in the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$. It also contains $\mathrm{U}(\mathfrak{a}_N)$ as a subalgebra and does admit a homomorphism $\rho : \mathrm{Y}(\mathfrak{gl}_N, \sigma) \to \mathrm{U}(\mathfrak{a}_N)$ identical on $\mathrm{U}(\mathfrak{a}_N)$; see Sect. 3. The algebra $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is a deformation of the universal enveloping algebra for the twisted current Lie algebra

$$\{F(t) \in \mathfrak{gl}_N[t] \mid \sigma(F(t)) = F(-t)\} .$$

There is a remarkable family of maximal commutative subalgebras in $\mathrm{Y}(\mathfrak{gl}_N)$. They are parametrized by the regular semisimple elements of $\mathfrak{gl}_N$. As well as the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ itself, these subalgebras were studied in the works by mathematical physicists from St. Petersburg on the Bethe Ansatz; see for instance [KBI] and [KR, KS]. These subalgebras were also studied in [C1]. We will call them Bethe subalgebras. In Sect. 1 of the present article we recall their definition. Their images in $\mathrm{U}(\mathfrak{gl}_N)$ with respect to the homomorphism π are also maximal commutative; see Sect. 2.

The main aim of this article is to construct analogues of the Bethe subalgebras in $\mathrm{Y}(\mathfrak{gl}_N)$ for the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$. In Sect. 3 for any element $Z \in \mathfrak{a}_N$ we construct a certain commutative subalgebra $\mathrm{B}(\mathfrak{gl}_N, \sigma, Z)$ in $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$. This construction is a generalization of one result from [S]. If the element $Z \in \mathfrak{a}_N$ is regular semisimple then the corresponding commutative subalgebra in $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is maximal. Moreover, the image of this subalgebra in $\mathrm{U}(\mathfrak{a}_N)$ with respect to the homomorphism ρ is also maximal commutative; see Sect. 4. This image in $\mathrm{U}(\mathfrak{a}_N)$ is a quantization of a maximal involutive subalgebra in the Poisson algebra $\mathrm{S}(\mathfrak{a}_N)$ obtained by the so-called shift of argument method; see [K2] and [MF]. For further details on the involutive subalgebras in $\mathrm{S}(\mathfrak{a}_N)$ obtained by this method see for instance [RS]. Some results on the quantization of these subalgebras can be found in [V].

We are indebted to M. Raïs who explained to us that the methods of [K1] can be applied to the current Lie algebras; see [RT]. Together with [M] and [MNO] the present article is a part of a project on representation theory of Yangians initiated by [O1, O2]. It is our joint project with A. Molev, and we are grateful to him for collaboration.

1. Bethe Subalgebras in Yangians

We will start this section with recalling several known facts from [D1] and [KR, KS] about the *Yangian* of the Lie algebra $\mathfrak{gl}_N$; see also [MNO, Sects. 1–2]. This is a complex associative unital algebra $\mathrm{Y}(\mathfrak{gl}_N)$ with the countable set of generators $T_{ij}^{(r)}$, where $r = 1, 2, \ldots$ and $i, j = 1, \ldots, N$. The defining relations in the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ are

$$[\, T_{ij}^{(p+1)}, T_{kl}^{(q)} \,] - [\, T_{ij}^{(p)}, T_{kl}^{(q+1)} \,] = T_{kj}^{(p)}\, T_{il}^{(q)} - T_{kj}^{(q)}\, T_{il}^{(p)}; \quad p, q = 0, 1, 2, \ldots, \tag{1.1}$$

where $T_{ij}^{(0)} = \delta_{ij} \cdot 1$. The collection (1.1) is equivalent to the collection of relations

$$[\, T_{ij}^{(p)}, T_{kl}^{(q)} \,] = \sum_{r=1}^{\min(p,q)} (T_{kj}^{(r-1)}\, T_{il}^{(p+q-r)} - T_{kj}^{(p+q-r)}\, T_{il}^{(r-1)}); \quad p, q = 1, 2, \ldots . \tag{1.2}$$

Let $E_{ij} \in \mathrm{End}\,(\mathbb{C}^N)$ be the standard matrix units. We will also use the following matrix form of the relations (1.1). Introduce a formal variable u and consider the *Yang R-matrix*

$$R(u) = u \cdot \mathrm{id} - \sum_{i,j} E_{ij} \otimes E_{ji} \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}[u]\,,$$

the indices i and j run through the set $\{1, \ldots, N\}$. We will employ the equality

$$R(u)\, R(-u) = (\, 1 - u^2\,) \cdot \mathrm{id}\ . \tag{1.3}$$

Introduce the formal power series in u^{-1},

$$T_{ij}(u) = T_{ij}^{(0)} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \cdots ,$$

and combine all these series into the single element

$$T(u) = \sum_{i,j} E_{ij} \otimes T_{ij}(u) \in \mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)\,[[u^{-1}]] .$$

We will also regard E_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. The algebra $\mathrm{Y}(\mathfrak{gl}_N)$ contains $\mathrm{U}(\mathfrak{gl}_N)$ as a subalgebra: due to (1.1) the assignment $E_{ij} \mapsto T_{ij}^{(1)}$ defines the embedding. Moreover, there is a homomorphism

$$\pi : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N) : T_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1} . \tag{1.4}$$

The homomorphism π is by definition identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_N)$. It is called the *evaluation homomorphism* for the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. There is a natural Hopf algebra structure on $\mathrm{Y}(\mathfrak{gl}_N)$. The comultiplication $\Delta : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_N)^{\otimes 2}$ is defined by the assignment

$$T_{ij}(u) \mapsto \sum_k T_{ik}(u) \otimes T_{kj}(u) . \tag{1.5}$$

Here the tensor product is taken over the subalgebra $\mathbb{C}[[u^{-1}]]$ in $\mathrm{Y}(\mathfrak{gl}_N)\,[[u^{-1}]]$.

Throughout this article we will denote by ι_s the embedding of the algebra $\mathrm{End}\,(\mathbb{C}^N)$ into a finite tensor product $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$ as the sth tensor factor:

$$\iota_s(X) = 1^{\otimes (s-1)} \otimes X \otimes 1^{\otimes (n-s)}; \quad s = 1, \ldots, n .$$

For any series $Y(u) \in \mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ we set

$$Y_s(u) = \iota_s \otimes \mathrm{id}\ (Y(u)) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] .$$

Let v be another formal variable. In the above notation the defining relations (1.1) can be rewritten as the single relation in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_N)\,((u^{-1}, v^{-1}))$,

$$R(u-v) \otimes 1 \cdot T_1(u)\, T_2(v) = T_2(v)\, T_1(u) \cdot R(u-v) \otimes 1 . \tag{1.6}$$

Let tr_n be the standard matrix trace on $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$. We will also use various embeddings of the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ into $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$ for any $m \leqq n$. When $1 \leqq s_1 < \cdots < s_m \leqq n$ and $X \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ we put

$$X_{s_1 \cdots s_m} = \iota_{s_1} \otimes \cdots \otimes \iota_{s_m}(X) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$$

For any series $Y(u) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ we will also put

$$Y_{s_1 \cdots s_m}(u) = \iota_{s_1} \otimes \cdots \otimes \iota_{s_m} \otimes \mathrm{id}\ (Y(u)) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] .$$

For each $k = 1, \ldots, N$ let $H_k \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ be the antisymmetrisation map normalized so that $H_k^2 = H_k$. We will make use of the decomposition into an ordered

product in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$,

$$k!\,(k-1)!\cdots 1!\,H_k = \overleftarrow{\prod_{1\leqq p<k}}\left(\overleftarrow{\prod_{p<q\leqq k}} R_{pq}(q-p)\right), \tag{1.7}$$

where the arrows indicate the order in which the factors are arranged when the indices p and q increase. Due to this decomposition the relation (1.2) implies that in the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ we have

$$H_k\otimes 1\cdot T_1(u-1)\cdots T_k(u-k) = T_k(u-k)\cdots T_1(u-1)\cdot H_k\otimes 1\,. \tag{1.8}$$

Here the series in $(u-1)^{-1},\dots,(u-k)^{-1}$ should be re-expanded in u^{-1}.

Denote by $\mathrm{F}_k(\mathbb{C}^N)$ the subalgebra in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ formed by all elements X such that $H_k X = H_k X H_k$. Then by (1.8) we have

$$T_1(u-1)\cdots T_k(u-k)\in \mathrm{F}_k(\mathbb{C}^N)\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]\,.$$

We will identify the algebra $\mathrm{End}(\Lambda^k\mathbb{C}^N)$ with the subalgebra in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ which consists of all the elements of the form $H_k X H_k$. Denote by φ_k the homomorphism

$$\mathrm{F}_k(\mathbb{C}^N)\to \mathrm{End}(\Lambda^k\mathbb{C}^N):\ X\mapsto H_k X\,.$$

Let an arbitrary element $Z\in\mathrm{End}\,(\mathbb{C}^N)$ be fixed. We will now define a remarkable commutative subalgebra $\mathrm{B}(\mathfrak{gl}_N,Z)$ in $\mathrm{Y}(\mathfrak{gl}_N)$. We will call it the *Bethe subalgebra.* Our definition is only a slight modification of that given in [KR, Sect. 2].

Consider the formal power series $T_1(u),\dots,T_N(u)$ with the coefficients in the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes N}\otimes \mathrm{Y}(\mathfrak{gl}_N)$. The subalgebra $\mathrm{B}(\mathfrak{gl}_N,Z)$ in $\mathrm{Y}(\mathfrak{gl}_N)$ is generated by the coefficients of all the series

$$B_k(u) = \mathrm{tr}_N\otimes \mathrm{id}\,(H_N\otimes 1\cdot T_1(u-1)\cdots T_k(u-k)\cdot Z_{k+1}\cdots Z_N\otimes 1\,)\,, \tag{1.9}$$

where $k=1,\dots,N$. Let $z_{ij}\in\mathbb{C}$ be the matrix elements of Z;

$$Z=\sum_{i,j} z_{ij}E_{ij}\,.$$

Then in a more conventional notation $B_k(u)$ equals the sum

$$\sum_{g,h} T_{g(1)h(1)}(u-1)\cdots T_{g(k)h(k)}(u-k)\cdot z_{g(k+1)h(k+1)}\cdots z_{g(N)h(N)}\cdot \varepsilon/N!\,, \tag{1.10}$$

where g and h run through the set of all permutations of $1,2,\dots,N$ while ε stands for $\mathrm{sgn}\,g\cdot\mathrm{sgn}\,h$. Note that the projector $H_N\in\mathrm{End}\,(\mathbb{C}^N)^{\otimes N}$ is one-dimensional. So by using (1.8) when $k=N$ we obtain for the series $B_N(u)$ another expression,

$$B_N(u)=\sum_g T_{g(1),1}(u-1)\cdots T_{g(N),N}(u-N)\cdot \mathrm{sgn}\,g\,, \tag{1.11}$$

where g runs through the set of all permutations of $1,2,\dots,N$. The series $B_N(u)$ is called the *quantum determinant* for the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. The following proposition is well known; its detailed proof can be found in [MNO, Sect. 2].

Proposition 1.1. *The coefficients at u^{-1}, $u^{-2},\dots$ of the series $B_N(u)$ are free generators for the centre of the algebra* $\mathrm{Y}(\mathfrak{gl}_N)$.

The proof of the next proposition is also known. Nevertheless, we will give it here.

Proposition 1.2. *All the coefficients of the series $B_1(u),\dots,B_N(u)$ commute.*

Proof. By making use of the equality (1.8) when $k=N$ we obtain that

$$H_N \otimes 1 \cdot T_1(u-1)\cdots T_N(u-N) = H_N \otimes B_N(u)\,. \tag{1.12}$$

The element $T(u)$ belongs to

$$\mathrm{id}\otimes 1 + \mathrm{End}\,(\mathbb{C}^N)\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]\cdot u^{-1}$$

and is therefore invertible in the algebra $\mathrm{End}\,(\mathbb{C}^N)\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$. Let $\widehat{T}(u)$ be the inverse series. By (1.12) the series $B_k(u)$ then equals

$$B_N(u)\cdot \mathrm{tr}_N\otimes \mathrm{id}\,(H_N\otimes 1\cdot \widehat{T}_N(u-N)\cdots \widehat{T}_{k+1}(u-k-1)\cdot Z_{k+1}\cdots Z_N\otimes 1)\,.$$

Put

$$\widehat{B}_k(u) = \mathrm{tr}_k\otimes \mathrm{id}\,(H_k\otimes 1\cdot \widehat{T}_k(u-k)\cdots \widehat{T}_1(u-1)\cdot Z_1\cdots Z_k\otimes 1)\,, \tag{1.13}$$

where $\widehat{T}_1(u),\dots,\widehat{T}_k(u)$ are regarded as elements of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$. Then

$$B_k(u) = B_N(u)\widehat{B}_{N-k}(u-k)\cdot \binom{N}{k}\,. \tag{1.14}$$

By Proposition 1.1 all the coefficients of the series $B_N(u)$ are central in $\mathrm{Y}(\mathfrak{gl}_N)$. Hence it suffices to prove that $[\widehat{B}_k(u),\widehat{B}_l(v)]=0$ for all the indices $k,l=1,\dots,N$.

Let us consider the ordered product

$$P(u) = \overrightarrow{\prod_{1\leqq p\leqq k}}\left(\overleftarrow{\prod_{k<q\leqq k+l}} R_{pq}(u-p+q)\right)\in \mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}[u]\,. \tag{1.15}$$

The product $P(u)$ has an inverse in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}(u)$ and commutes with the element $Z^{\otimes(k+l)}$. Moreover, due to the *Yang–Baxter equation* in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 3}[u,v]$

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,, \tag{1.16}$$

and to the decomposition (1.7) we have

$$H_k\otimes 1\cdot P(u) = \overleftarrow{\prod_{1\leqq p\leqq k}}\left(\overleftarrow{\prod_{k<q\leqq k+l}} R_{pq}(u-p+q)\right)\cdot H_k\otimes 1\,,$$

$$1\otimes H_l\cdot P(u) = \overrightarrow{\prod_{1\leqq p\leqq k}}\left(\overrightarrow{\prod_{k<q\leqq k+l}} R_{pq}(u-p+q)\right)\cdot 1\otimes H_l\,.$$

In particular, we have

$$P(u)\in \mathrm{F}_k(\mathbb{C}^N)\otimes \mathrm{F}_l(\mathbb{C}^N)\,[u]\,.$$

We will denote

$$\varphi_k\otimes\varphi_l\,(P(u)) = \bar{P}(u)\,. \tag{1.17}$$

The element $\bar{P}(u)$ has an inverse in $\mathrm{End}(\Lambda^k\mathbb{C}^N)\otimes \mathrm{End}(\Lambda^l\mathbb{C}^N)(u)$.

By (1.6) we also have the equality in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)} \otimes \mathrm{Y}(\mathfrak{gl}_N)((u^{-1},v^{-1}))$,

$$P(u-v)\otimes 1 \cdot T_1(u-1)\cdots T_k(u-k)\cdot T_{k+1}(v-1)\cdots T_{k+l}(v-l) =$$

$$T_{k+1}(v-1)\cdots T_{k+l}(v-l)\cdot T_1(u-1)\cdots T_k(u-k)\cdot P(u-v)\otimes 1\,. \quad (1.18)$$

Introduce the elements of the algebra $\mathrm{End}\,(\Lambda^k\mathbb{C}^N)\otimes \mathrm{End}\,(\Lambda^l\mathbb{C}^N)\otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$,

$$K(u) = \varphi_k\otimes\varphi_l\otimes \mathrm{id}\,(\widehat{T}_k(u-k)\cdots \widehat{T}_1(u-1))\,,$$

$$L(u) = \varphi_k\otimes\varphi_l\otimes \mathrm{id}\,(\widehat{T}_{k+l}(u-l)\cdots \widehat{T}_{k+1}(u-1))\,,$$

and $W = \Lambda^k Z\otimes \Lambda^l Z\otimes 1$. Then we have

$$\widehat{B}_k(u)\widehat{B}_l(v) = \mathrm{tr}\,\otimes \mathrm{id}\,(K(u)L(v)\cdot W)\,,$$

where tr stands for the restriction of tr_{k+l} onto $\mathrm{End}(\Lambda^k\mathbb{C}^N)\otimes \mathrm{End}(\Lambda^l\mathbb{C}^N)$. But by applying the homomorphism $\varphi_k\otimes\varphi_l\otimes \mathrm{id}$ to (1.18) we get the equalities

$$K(u)L(v)\cdot W\cdot \bar{P}(u-v)\otimes 1 = K(u)L(v)\cdot \bar{P}(u-v)\otimes 1\cdot W$$

$$= \bar{P}(u-v)\otimes 1\cdot L(v)K(u)\cdot W\,.$$

Therefore we obtain that

$$\mathrm{tr}\,\otimes \mathrm{id}\,(K(u)L(v)\cdot W) = \mathrm{tr}\,\otimes \mathrm{id}\,(L(v)K(u)\cdot W) = \widehat{B}_l(v)\widehat{B}_k(u)\,. \qquad \square$$

Theorem 1.3. *Suppose that the element $Z\in \mathrm{End}\,(\mathbb{C}^N)$ has a simple spectrum. Then the subalgebra $\mathrm{B}(\mathfrak{gl}_N,Z)$ in $\mathrm{Y}(\mathfrak{gl}_N)$ is maximal commutative. The coefficients at $u^{-1}, u^{-2},\ldots$ of the series $B_1(u),\ldots,B_N(u)$ are free generators for $\mathrm{B}(\mathfrak{gl}_N,Z)$.*

The proof of this theorem will be given in Sect. 2. We will end up this section with comparing our definition of the Bethe subalgebra in $\mathrm{Y}(\mathfrak{gl}_N)$ with that given in [KR, Sect. 2]. The defining relations (1.6) show that the assignment $T(u)\mapsto \widehat{T}(u)$ defines an antiautomorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Denote by η this antiautomorphism. It then follows from (1.12) that

$$\eta(B_N(u)) = B_N(u)^{-1}\,.$$

Furthermore, we have

$$\eta^2(T(u)) = T(u+N)\cdot B_N(u)/B_N(u+1)\,;$$

the proof of the latter statement can be found for instance in [MNO, Sect. 5].

The commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$ considered in [KR, Sect. 2] is generated by the coefficients of all the series

$$\eta^{-1}(\widehat{B}_k(u)) = \mathrm{tr}_k\otimes \mathrm{id}\,(H_k\otimes 1\cdot T_1(u-1)\cdots T_k(u-k)\cdot Z_1\cdots Z_k\otimes 1)\,, \quad (1.19)$$

where $k=1,\ldots,N$. This commutative subalgebra is not maximal if the element $Z\in \mathrm{End}\,(\mathbb{C}^N)$ is not invertible. In that case we have $\widehat{B}_N(u)=0$. However, due to (1.14) the commutative subalgebra in $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the coefficients of all the series (1.19) along with $B_N(u)$ coincides with $\eta^{-1}(\mathrm{B}(\mathfrak{gl}_N,Z))$. The latter subalgebra is maximal for any element Z with a simple spectrum by Theorem 1.3.

2. Proof of Theorem 1.3

We will reduce the proof to several lemmas. Some of them are rather general and will be used again in Sect. 4. We will employ methods from [C3] and [K1].

Let us equip the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ with an ascending filtration by setting degrees of its generators as $\deg T_{ij}^{(r)} = r$. The linear subspace in $\mathrm{Y}(\mathfrak{gl}_N)$ consisting of all the elements with degrees not greater than r will be denoted by $\mathrm{Y}_r(\mathfrak{gl}_N)$. Consider the graded algebra

$$\mathrm{X}(\mathfrak{gl}_N) = \bigoplus_{r \geqq 0} \mathrm{Y}_r(\mathfrak{gl}_N)/\mathrm{Y}_{r-1}(\mathfrak{gl}_N)$$

corresponding to the filtered algebra $\mathrm{Y}(\mathfrak{gl}_N)$. The defining relations (1.2) show that the algebra $\mathrm{X}(\mathfrak{gl}_N)$ is commutative.

There is a natural Poisson algebra structure on $\mathrm{X}(\mathfrak{gl}_N)$. For any two elements X, Y in $\mathrm{Y}(\mathfrak{gl}_N)$ of degrees p, q respectively the Poisson bracket of their images x, y in $\mathrm{X}(\mathfrak{gl}_N)$ is

$$\{x, y\} = [X, Y] \bmod \mathrm{Y}_{p+q-2}(\mathfrak{gl}_N)\,.$$

Let $\mathrm{J}(\mathfrak{gl}_N, Z)$ be the image in $\mathrm{X}(\mathfrak{gl}_N)$ of the Bethe subalgebra $\mathrm{B}(\mathfrak{gl}_N, Z)$ in the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. To prove the first statement of Theorem 1.3 it suffices to show that the subalgebra $\mathrm{J}(\mathfrak{gl}_N, Z)$ in $\mathrm{X}(\mathfrak{gl}_N)$ is maximal involutive.

We will denote by $t_{ij}^{(r)}$ the image in $\mathrm{X}(\mathfrak{gl}_N)$ of the generator $T_{ij}^{(r)}$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. All the elements $t_{ij}^{(r)}$ are free generators of the commutative algebra $\mathrm{X}(\mathfrak{gl}_N)$. The proof of this assertion can be found for instance in [MNO, Sect. 1]. We will identify $\mathrm{X}(\mathfrak{gl}_N)$ with the symmetric algebra $\mathrm{S}(\mathfrak{gl}_N[t])$ of the polynomial current Lie algebra $\mathfrak{gl}_N[t]$. The generator $t_{ij}^{(r)}$ will be identified with the element $E_{ij}\, t^{r-1}$. We will also set $t_{ij}^{(0)} = \delta_{ij} \cdot 1$. Then by (1.11),

$$\{t_{ij}^{(p)}, t_{kl}^{(q)}\} = \sum_{r=1}^{\min(p,q)} (t_{kj}^{(r-1)}\, t_{il}^{(p+q-r)} - t_{kj}^{(p+q-r)}\, t_{il}^{(r-1)}); \quad p, q = 1, 2, \ldots\,. \tag{2.1}$$

Consider the formal power series in u^{-1} with the coefficients in $\mathrm{X}(\mathfrak{gl}_N)$,

$$t_{ij}(u) = t_{ij}^{(0)} + t_{ij}^{(1)}\, u^{-1} + t_{ij}^{(2)}\, u^{-2} + \cdots\,.$$

Then the image in $\mathrm{X}(\mathfrak{gl}_N)[[u^{-1}]]$ of the series $T_{ij}(u-c) \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ equals $t_{ij}(u)$ for any $c \in \mathbb{C}$. Therefore the image of the series $B_k(u)$ in the algebra $\mathrm{X}(\mathfrak{gl}_N)[[u^{-1}]]$ equals

$$\sum_{g,h} t_{g(1)h(1)}(u) \cdots t_{g(k)h(k)}(u) \cdot z_{g(k+1)h(k+1)} \cdots z_{g(N)h(N)} \cdot \varepsilon / N!\,, \tag{2.2}$$

where g, h and ε are the same as in (1.10). Note that the coefficients of the latter series are homogeneous in $\mathrm{X}(\mathfrak{gl}_N)$.

For each $M = 1, 2 \ldots$, let us denote by $\mathrm{X}_M(\mathfrak{gl}_N)$ the ideal in the algebra $\mathrm{X}(\mathfrak{gl}_N)$ generated by all the elements $t_{ij}^{(r)}$, where $r \geqq M$. Note that by (2.1) we have

$$\{t_{ij}^{(p)}, t_{kl}^{(q)}\} \in \mathrm{X}_M(\mathfrak{gl}_N), \quad M = \max(p, q)\,.$$

Therefore each $X_M(\mathfrak{gl}_N)$ is a Poisson ideal in $X(\mathfrak{gl}_N)$. Due to the following general lemma it suffices to prove that for each M the image of $J(\mathfrak{gl}_N, Z)$ in the quotient Poisson algebra $X(\mathfrak{gl}_N)/X_M(\mathfrak{gl}_N)$ is maximal involutive.

Let X be any graded Poisson algebra and $X_1 \supset X_2 \supset \cdots$ be a descending chain of Poisson ideals in X such that for each $M = 1, 2, \ldots$ the ideal X_M contains only elements of degrees not less than M. Let $b_1, b_2, \ldots$ be a sequence of homogeneous elements of X in involution.

Lemma 2.1. *Supose that for each $M = 1, 2, \ldots$ the images of $b_1, b_2, \ldots$ in the Poisson algebra X/X_M generate a maximal involutive subalgebra. Then $b_1, b_2, \ldots$ generate a maximal involutive subalgebra in* X.

Proof. Fix a homogeneous element $x \in X$ in involution with each of $b_1, b_2, \ldots$ and set $M = 1 + \deg x$. By our assumption $x = f(b_1, b_2, \ldots) + y$ for a certain polynomial f and some $y \in X_M$. Then x is the sum of the terms in $f(b_1, b_2, \ldots)$ of degrees smaller than M. □

By definition, we have $X_1(\mathfrak{gl}_N) = S(\mathfrak{gl}_N)$. From now we shall keep $M \geqq 1$ fixed. For each $r = 1, \ldots, M$ we will denote by $x_{ij}^{(r)}$ the image of the generator $t_{ij}^{(r)}$ in the quotient Poisson algebra $X(\mathfrak{gl}_N)/X_{M+1}(\mathfrak{gl}_N)$. By (2.1) in the latter algebra we have

$$\{x_{ij}^{(p)}, x_{kl}^{(q)}\} = \sum_{r=\max(1,\, p+q-M)}^{\min(p,q)} (x_{kj}^{(r-1)} x_{il}^{(p+q-r)} - x_{kj}^{(p+q-r)} x_{il}^{(r-1)}) , \tag{2.3}$$

where $x_{ij}^{(0)} = \delta_{ij} \cdot 1$. Consider the vector subspace in the Lie algebra $\mathfrak{gl}_N[t]$

$$\mathfrak{g}_{M,N} = \mathfrak{gl}_N + \cdots + \mathfrak{gl}_N \cdot t^{M-1} . \tag{2.4}$$

As a commutative algebra the quotient $X(\mathfrak{gl}_N)/X_{M+1}(\mathfrak{gl}_N)$ can be identified with the symmetric algebra $S(\mathfrak{g}_{M,N})$. Further, we will identify $X(\mathfrak{gl}_N)/X_{M+1}(\mathfrak{gl}_N)$ with the algebra $P(\mathfrak{g}_{M,N})$ of the polynomial functions on $\mathfrak{g}_{M,N}$. The generator $x_{ij}^{(r)}$ will be regarded as the coordinate function corresponding to the vector $E_{ij} \cdot t^{r-1}$.

Let us now make use of the following observation. The relations (1.6) imply that for any invertible element $G \in \mathrm{End}(\mathbb{C}^N)$ the assignment

$$T(u) \mapsto G \otimes 1 \cdot T(u) \cdot G^{-1} \otimes 1$$

determines an automorphism of the algebra $Y(\mathfrak{gl}_N)$. The image of the subalgebra $B(\mathfrak{gl}_N, Z)$ with respect to this automorphism coincides with $B(\mathfrak{gl}_N, G^{-1} Z G)$. Hence it suffices to assume that $z_{ij} = z_i \cdot \delta_{ij}$, where all $z_i \in \mathbb{C}$ are pairwise distinct. From now on until the end of this section we shall keep to this assumption.

Let $\mathfrak{b}_N$ be the Borel subalgebra in $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqq j$. Fix any principal nilpotent element of the opposite Borel subalgebra of the form

$$E = \varepsilon_1 E_{21} + \cdots + \varepsilon_{N-1} E_{N,N-1} ,$$

where $\varepsilon_1, \ldots, \varepsilon_{N-1} \neq 0$. We will consider the Poisson bracket (2.3) on $P(\mathfrak{g}_{M,N})$ in a neighbourhood of the point

$$E^{(M)} = E \cdot t^{M-1} \in \mathfrak{g}_{M,N} .$$

Introduce the affine subspace in $\mathfrak{g}_{M,N}$

$$\mathfrak{t}_{M,N} = E^{(M)} + \mathfrak{g}_{M,N} \cap \mathfrak{b}_N[t] .$$

For each $k = 1,\dots,N$ denote by $b_k(u)$ the image of $B_k(u) \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ in

$$\mathrm{X}(\mathfrak{gl}_N)/\mathrm{X}_{M+1}(\mathfrak{gl}_N)[[u^{-1}]] = \mathrm{P}(\mathfrak{g}_{M,N})[[u^{-1}]] .$$

This image is in fact a polynomial in u^{-1} of the degree kM. We will write

$$b_k(u) = b_k^{(0)} + b_k^{(1)} u^{-1} + \cdots + b_k^{(kM)} u^{-kM} .$$

Here all the coefficients $b_k^{(r)}$ are in involution due to Proposition 1.2. Consider them as polynomial functions on the vector space $\mathfrak{g}_{M,N}$. Of the proof of Theorem 1.3 the next lemma is the main part; cf. [K1, Sect. 4] and [RT, Sect. 4]. Another approach to the proof of this theorem was described in [RS, Sect. 3].

Lemma 2.2. *The restrictions of the functions $b_k^{(r)}$ onto the affine subspace $\mathfrak{t}_{M,N}$ generate the whole algebra of polynomial functions on $\mathfrak{t}_{M,N}$.*

Proof. Introduce the polynomial in u of the degree $M-1$

$$x_{ij}(u) = x_{ij}^{(1)} u^{M-1} + x_{ij}^{(2)} u^{M-2} + \cdots + x_{ij}^{(M)} .$$

Then

$$u^{kM} b_k(u) = \sum_{g,h} (u^M \delta_{g(1)h(1)} + x_{g(1)h(1)}(u)) \cdots (u^M \delta_{g(k)h(k)} + x_{g(k)h(k)}(u)) \\ \times z_{g(k+1)h(k+1)} \cdots z_{g(N)h(N)} \cdot \varepsilon / N! ,$$

where g, h and ε are the same as in (1.10) and (2.2). Denote by $X(u)$ the square matrix of order N formed by all the polynomials $x_{ij}(u)$. Then we have the Laplace expansion

$$\det(u^M + X(u) + Z v) = v^N \det Z + \sum_{k=1}^{N} u^{kM} b_k(u) v^{N-k} \binom{N}{k}, \tag{2.5}$$

where Z is now regarded as a diagonal matrix of order N with pairwise distinct diagonal entries $z_1,\dots,z_N \in \mathbb{C}$. Denote by $F(u,v)$ the polynomial in u,v obtained by restricting the coefficients of (2.5) onto $\mathfrak{t}_{M,N}$.

On the subspace $\mathfrak{t}_{M,N}$ we have $x_{i+1,i}(u) = \varepsilon_i$ and $x_{ij}(u) = 0$ if $i - j > 1$. Take the functions $x_{i,i+d}^{(r)}$ with $d = 0,\dots, N-1$ and $r = 1,\dots,M$ as coordinates on $\mathfrak{t}_{M,N}$. Endow the set of the pairs (d,r) with the lexicographical order. We will prove consecutively for each of these pairs that $x_{i,i+d}^{(r)}$ are polynomials in the coefficients of $F(u,v)$.

Assume that $\deg v = M$ while $\deg u = 1$. Consider the terms of $F(u,v)$ with the total degree $M(N-d) - r$. Their sum has the form

$$f(u,v) + (-1)^d \cdot \sum_{i=1}^{N-d} x_{i,i+d}^{(r)} u^{M-r} \cdot g_i(u^M, v)\, \varepsilon_i \cdots \varepsilon_{i+d-1} ,$$

where

$$g_i(u,v) = \prod_{j \neq i,\dots,i+d} (u + z_j v).$$

Here the coefficients of the polynomial $f(u,v)$ depend only on $z_1,\dots,z_N$ and $x_{ij}^{(s)}$ with the pair $(j-i,s)$ preceding (d,r). It now remains to observe that all the $N-d$

polynomials $g_i(u,v)$ are linearly independent. Indeed, for each $i = 1,\dots,N-d$, we have

$$g_i(-z_i,1) = \prod_{j \neq i,\dots,i+d} (z_j - z_i) \neq 0\,,$$

while for $1 \leqq i < j \leqq N-d$ we have the equality $g_j(-z_i,1) = 0$. $\square$

Note that here $b_k^{(0)} \in \mathbb{C}$ for any $k = 1,\dots,N$. The total number of the remaining coefficients $b_k^{(r)}$ with $r = 1,\dots,kM$ equals

$$M + 2M + \cdots + NM = MN(N+1)/2 = \dim \mathfrak{t}_{M,N}\,.$$

Corollary 2.3. *All the coefficients $b_k^{(r)}$ with $k = 1,\dots,N$ and $r = 1,\dots,kM$ are algebraically independent.*

This corollary is valid for any $M \geqq 1$ and therefore already implies the second statement of Theorem 1.3. Namely, all the coefficients at u^{-1}, u^{-2}, ... of the series $B_1(u),\dots,B_N(u)$ are free generators for $\mathrm{B}(\mathfrak{gl}_N, Z)$. Denote

$$D = \dim \mathfrak{g}_{M,N} - \dim \mathfrak{t}_{M,N} = MN(N-1)/2\,.$$

Lemma 2.4. *There is an open neighbourhood of the point $E^{(M)}$ in $\mathfrak{g}_{M,N}$, where the rank of the Poisson bracket* (2.3) *equals* $2D$.

Proof. The elements $x_{ij}^{(1)}$ generate a Poisson subalgebra isomorphic to the algebra $\mathrm{P}(\mathfrak{gl}_N)$ of the polynomial functions on $\mathfrak{gl}_N$ with the standard Poisson bracket:

$$\{x_{ij}^{(1)}, x_{kl}^{(1)}\} = \delta_{kj}\, x_{il}^{(1)} - \delta_{il}\, x_{kj}^{(1)}\,. \tag{2.6}$$

By (2.3) for any $p,q \geqq 1$, the value of the function $\{x_{ij}^{(p)}, x_{kl}^{(q)}\}$ at the point $E^{(M)}$ equals

$$\delta_{p+q,M+1} \cdot (\delta_{kj}\, \delta_{i,l+1}\, \varepsilon_l - \delta_{il}\, \delta_{k,j+1}\, \varepsilon_j)\,.$$

Therefore the rank of the bracket (2.3) at this point is M times the rank of the standard Poisson bracket on $\mathrm{P}(\mathfrak{gl}_N)$ in the point E. The latter rank is $N^2 - N$. So there exists an open neighbourhood of the point $E^{(M)}$ in $\mathfrak{g}_{M,N}$, where the rank of the Poisson bracket (2.3) is not smaller than $M(N^2-N) = 2D$. Due to Proposition 1.1 and Corollary 2.3 there are MN algebraically independent elements $b_N^{(1)},\dots,b_N^{(MN)}$ in the centre of the Poisson algebra $\mathrm{P}(\mathfrak{g}_{M,N})$. So the rank of the Poisson bracket cannot exceed $M(N^2-N) = 2D$ at any point. $\square$

Let us now fix any $x \in \mathrm{P}(\mathfrak{g}_{M,N})$ in involution with all the elements $b_k^{(r)}$. To complete the proof of Theorem 1.3 we have to demonstrate that the element x is then a polynomial in $b_k^{(r)}$. For any collection

$$H = (h_k^{(r)} \in \mathbb{C} \mid k = 1,\dots,N;\ r = 1,\dots,kM)\,,$$

denote by $\mathscr{S}_H$ the subset in $\mathfrak{g}_{M,N}$, where the values of the functions $b_k^{(r)}$ are $h_k^{(r)}$ respectively. Due to Lemma 2.2 there is an open neighbourhood $\mathscr{E}$ of the point $E^{(M)} \in \mathfrak{g}_{M,N}$ such that every non-empty intersection $\mathscr{E} \cap \mathscr{S}_H$ is transversal to $\mathfrak{t}_{M,N}$. Again due to Lemma 2.2 to demonstrate that x is a polynomial in $b_k^{(r)}$ it now suffices to prove the following statement; cf. [C3].

Lemma 2.5. *One can choose the open neighbourhood $\mathscr{E}$ so that the function x is constant on every intersection $\mathscr{E} \cap \mathscr{S}_H$.*

Proof. For any polynomial b in $b_k^{(r)}$ consider the respective flow in $\mathfrak{g}_{M,N}$,

$$dx_{ij}^{(r)}/dt = \{b, x_{ij}^{(r)}\}\,; \quad i,j = 1,\dots,N;\ r = 1,\dots,M\,, \tag{2.7}$$

where t stands for the coordinate on the line $\mathbb{R}$. Since $\{b,x\} = 0$ the function x is constant along every trajectory of this flow. For any point $F \in \mathfrak{g}_{M,N}$ denote by $\mathscr{T}_F$ the collection of the trajectories of the flows (2.7) passing through F for all polynomials b in $b_k^{(r)}$. Due to Lemma 2.2 and to Lemma 2.4 by the Liouville theorem we can choose the open neighbourhood $\mathscr{E}$ of $E^{(M)}$ so that

$$F \in S_H \ \Rightarrow\ \mathscr{T}_F \cap \mathscr{E} = \mathscr{S}_H \cap \mathscr{E}\,. \qquad \square$$

Now we will consider the commutative subalgebra $\pi(\mathrm{B}(\mathfrak{gl}_N, Z))$ in $\mathrm{U}(\mathfrak{gl}_N)$. There is a canonical filtration on the algebra $\mathrm{U}(\mathfrak{gl}_N)$. Consider the involutive subalgebra in the graded algebra $\mathrm{S}(\mathfrak{gl}_N)$ corresponding to $\pi(\mathrm{B}(\mathfrak{gl}_N, Z))$. We will identify the Poisson algebras $\mathrm{S}(\mathfrak{gl}_N)$ and $\mathrm{P}(\mathfrak{gl}_N)$; the element $E_{ij} \in \mathfrak{gl}_N$ will be identified with the respective coordinate function. Let X be the square matrix of order N formed by these functions. The subalgebra in $\mathrm{P}(\mathfrak{gl}_N)$ corresponding to $\pi(\mathrm{B}(\mathfrak{gl}_N, Z))$ is generated by the coefficients of the polynomial in u, v,

$$\det(u + X + Z\,v)\,.$$

It is well known that the coefficients of this polynomial at $u^k\,v^l$ with $k + l < N$ are algebraically independent for any Z with a simple spectrum; see [H, Sect. 2] and [MF, Sect. 4]. Lemma 2.2 at $M = 1$ provides an elementary proof of this fact. Moreover, it shows that these coefficients then generate a maximal involutive subalgebra in $\mathrm{P}(\mathfrak{gl}_N)$. Thus we obtain the following statement.

Proposition 2.6. *The subalgebra $\pi(\mathrm{B}(\mathfrak{gl}_N, Z))$ in $\mathrm{U}(\mathfrak{gl}_N)$ is maximal commutative.*

The analogues of this proposition for the universal enveloping algebras $\mathrm{U}(\mathfrak{so}_N)$ and $\mathrm{U}(\mathfrak{sp}_N)$ of the other classical Lie algebras will be given in Sect. 4.

3. Bethe Subalgebras in Twisted Yangians

In the previous section we considered only the Yangian of the Lie algebra $\mathfrak{gl}_N$. We will start this section with describing analogues of this Yangian for the other classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$; in the latter case N has to be even. These analogues have been introduced in [O2]; see also [MNO, Sect. 3]. Then we will construct the respective analogues of the Bethe subalgebra $\mathrm{B}(\mathfrak{gl}_N, Z)$ in $\mathrm{Y}(\mathfrak{gl}_N)$.

Let $\mathfrak{a}_N$ be one of the classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. We will regard $\mathfrak{a}_N$ as an involutive subalgebra in $\mathfrak{gl}_N$. Let σ be the corresponding involutive automorphism of the Lie algebra $\mathfrak{gl}_N$. The superscript $'$ will denote transposition in $\mathrm{End}(\mathbb{C}^N)$ with respect to the symmetric or alternating bilinear form on $\mathbb{C}^N$ preserved by the subalgebra $\mathfrak{a}_N$ in $\mathfrak{gl}_N$. As well as above $E_{ij} \in \mathrm{End}(\mathbb{C}^N)$ will be the standard matrix units. But from now on we will let the indices i and j run through the set $\{-n,\dots,-1,1,\dots,n\}$ if $N = 2n$ and the set $\{-n,\dots,-1,0,1,\dots,n\}$ if $N = 2n+1$.

Put $\varepsilon_{ij} = \operatorname{sgn} i \cdot \operatorname{sgn} j$ if $\mathfrak{a}_N = \mathfrak{sp}_N$ and $\varepsilon_{ij} = 1$ if $\mathfrak{a}_N = \mathfrak{so}_N$. We will choose the symmetric or alternating bilinear form on $\mathbb{C}^N$ so that

$$E'_{ij} = \varepsilon_{ij} \cdot E_{-j,-i} \ .$$

If we regard E_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ then $\sigma(E_{ij}) = -E'_{ij}$ so that $F_{ij} = E_{ij} - E'_{ij}$ are generators of the algebra $\mathrm{U}(\mathfrak{a}_N)$.

Let us now introduce the element of the algebra $\operatorname{End}(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$,

$$\widetilde{T}(u) = \sum_{i,j} E'_{ij} \otimes T_{ij}(u) \, ,$$

and consider the series with the coefficients in $\operatorname{End}(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$,

$$S(u) = T(u)\,\widetilde{T}(-u) = \sum_{i,j} E_{ij} \otimes S_{ij}(u),$$

$$\tilde{S}(u) = \sum_{i,j} E'_{ij} \otimes S_{ij}(u) \, . \tag{3.1}$$

Then

$$S_{ij}(u) = \delta_{ij} \cdot 1 + S_{ij}^{(1)} u^{-1} + S_{ij}^{(2)} u^{-2} + \cdots$$

for certain elements $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots \in \mathrm{Y}(\mathfrak{gl}_N)$. By definition, the *twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is the subalgebra in $\mathrm{Y}(\mathfrak{gl}_N)$ generated by all the elements $S_{ij}^{(r)}$. This definition along with (1.5) implies that $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is a left coideal in the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$:

$$\Delta(\mathrm{Y}(\mathfrak{gl}_N, \sigma)) \subset \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N, \sigma) \, .$$

Introduce also the element of $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}[u]$,

$$\tilde{R}(u) = u \cdot \mathrm{id} \ - \sum_{i,j} E'_{ij} \otimes E_{ji} = u \cdot \mathrm{id} \ - \sum_{i,j} E_{ij} \otimes E'_{ji} \ .$$

Later on we will employ the equality

$$\tilde{R}(u)\tilde{R}(-u+N) = (Nu - u^2) \cdot \mathrm{id} \ . \tag{3.2}$$

The relation (1.6) implies that in the algebra $\operatorname{End}(\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_N)((u^{-1}, v^{-1}))$,

$$\widetilde{T}_1(u) \cdot \tilde{R}(u-v) \otimes 1 \cdot T_2(v) = T_2(v) \cdot \tilde{R}(u-v) \otimes 1 \cdot \widetilde{T}_1(u) \, . \tag{3.3}$$

By making use of the latter relation and by applying (1.6) again we obtain the relation in $\operatorname{End}(\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_N, \sigma)((u^{-1}, v^{-1}))$,

$$\begin{aligned} &R(u,v) \otimes 1 \cdot S_1(u) \cdot \tilde{R}(-u-v) \otimes 1 \cdot S_2(v) \\ &\quad = \ S_2(v) \cdot \tilde{R}(-u-v) \otimes 1 \cdot S_1(u) \cdot R(u,v) \otimes 1 \, . \end{aligned} \tag{3.4}$$

The proof of the following proposition is contained in [MNO, Sect. 3]. Whenever in the present article the double sign $\pm$ or $\mp$ occurs, the upper one will correspond to the case $\mathfrak{a}_N = \mathfrak{so}_N$ while the lower sign will correspond to the case $\mathfrak{a}_N = \mathfrak{sp}_N$.

Proposition 3.1. *In the algebra* $\operatorname{End}(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N, \sigma)[[u^{-1}]]$ *we have the relation*

$$S(u) - \tilde{S}(-u) = \mp (S(u) - S(-u))/2u \ . \tag{3.5}$$

The relations (3.4) *and* (3.5) *yield the defining relations for the generators* $S_{ij}^{(r)}$ *of the subalgebra* $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$ *in* $\mathrm{Y}(\mathfrak{gl}_N)$.

In a more conventional notation (3.4) and (3.5) can be rewritten respectively as the collections of relations for all possible indices i, j, k, l,

$$\begin{aligned}(u^2-v^2)\cdot[S_{ij}(u),S_{kl}(v)] &= (u+v)\cdot(S_{kj}(u)S_{il}(v)-S_{kj}(v)S_{il}(u))\\ &-(u-v)\cdot(\varepsilon_{k,-j}\,S_{i,-k}(u)S_{-j,l}(v)-\varepsilon_{i,-l}\,S_{k,-i}(v)S_{-l,j}(u))\\ &+\varepsilon_{i,-j}\cdot(S_{k,-i}(u)S_{-j,l}(v)-S_{k,-i}(v)S_{-j,l}(u))\,,\end{aligned}\tag{3.6}$$

and for all i, j,

$$S_{ij}(u)-\varepsilon_{ij}\,S_{-j,-i}(-u)=\mp(S_{ij}(u)-S_{ij}(-u))/2u\ .\tag{3.7}$$

The algebra $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{a}_N)$ as a subalgebra; due to (3.6) the assignment $F_{ij}\mapsto S_{ij}^{(1)}$ defines the embedding. Moreover, due to (3.6) and (3.7) we obtain the following corollary to Proposition 3.1.

Corollary 3.2. *There is a homomorphism*

$$\rho:\ \mathrm{Y}(\mathfrak{gl}_N,\sigma)\to\mathrm{U}(\mathfrak{a}_N):\ S_{ij}(u)\mapsto\delta_{ij}+F_{ij}\,(u\pm 1/2)^{-1}\ .\tag{3.8}$$

The homomorphism ρ is by definition identical on the subalgebra $\mathrm{U}(\mathfrak{a}_N)$. We will regard (3.8) as an analogue of the evaluation homomorphism (1.4).

The Yang–Baxter equation (1.16) in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 3}[u,v]$ along with (1.3) implies

$$R_{12}(u)\tilde{R}_{13}(v)\tilde{R}_{23}(u+v)=\tilde{R}_{23}(u+v)\tilde{R}_{13}(v)R_{12}(u)\,,\tag{3.9}$$

$$R_{13}(u)\tilde{R}_{12}(v)\tilde{R}_{23}(u+v)=\tilde{R}_{23}(u+v)\tilde{R}_{12}(v)R_{13}(u)\,,\tag{3.10}$$

$$R_{23}(u)\tilde{R}_{12}(v)\tilde{R}_{13}(u+v)=\tilde{R}_{13}(u+v)\tilde{R}_{12}(v)R_{23}(u)\,.\tag{3.11}$$

For each $k=1,\dots,N$ consider the element of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)[[u^{-1}]]$,

$$S(u,k)=\omega(u)\cdot\prod_{1\leqq p\leqq k}^{\rightarrow}\left(S_p(u-p)\cdot\prod_{p<q\leqq k}^{\rightarrow}\tilde{R}_{pq}(p+q-2u)\otimes 1\right),$$

where

$$\omega(u)=\prod_{1\leqq p<q\leqq k}(p+q-2u)^{-1}\ .$$

Due to the decomposition (1.7) by applying the relation (3.4) repeatedly along with (3.9) to (3.11) we obtain the equality in the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)((u^{-1}))$,

$$H_k\otimes 1\cdot S(u,k)=\omega(u)\cdot\prod_{1\leqq p\leqq k}^{\leftarrow}\left(\prod_{p<q\leqq k}^{\leftarrow}\tilde{R}_{pq}(p+q-2u)\otimes 1\cdot S_p(u-p)\right)\cdot H_k\otimes 1\ ;$$

it is an analogue of the equality (1.8) in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}\otimes\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$. In particular,

$$S(u,k)\in\mathrm{F}_k(\mathbb{C}^N)\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)[[u^{-1}]]\,.\tag{3.12}$$

Let us keep fixed an arbitrary element $Z\in\mathrm{End}\,(\mathbb{C}^N)$ but now assume that $Z'=Z$ or $Z'=-Z$. We will now construct a certain commutative subalgebra in

$\mathrm{Y}(\mathfrak{gl}_N,\sigma)$. It will be regarded as an analogue of the Bethe subalgebra in the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ and denoted by $\mathrm{B}(\mathfrak{gl}_N,\sigma,Z)$.

For each $k=1,\dots,N$ introduce the element of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}[[u^{-1}]]$,

$$Z(u,k)=\omega(u)\cdot\overrightarrow{\prod_{1\leqq p\leqq k}}\left(Z_p\cdot\overrightarrow{\prod_{p<q\leqq k}}\tilde{R}_{pq}(p+q-2u)\right)\,. \tag{3.13}$$

Observe that due to the assumption that $Z'=Z$ or $Z'=-Z$ a relation similar to (3.4) holds in the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}[u,v]$. It can be verified directly that

$$R(u,v)Z_1\,\tilde{R}(-u-v)Z_2=Z_2\,\tilde{R}(-u-v)Z_1\,R(u,v)\,. \tag{3.14}$$

By applying the relation (3.14) repeatedly along with the equalities (3.9) to (3.11) we obtain that

$$H_k\cdot Z(u,k)=\omega(u)\cdot\overleftarrow{\prod_{1\leqq p\leqq k}}\left(\overleftarrow{\prod_{p<q\leqq k}}\tilde{R}_{pq}(p+q-2u)\cdot Z_p\right)\cdot H_k\,.$$

In particular, we obtain that $Z(u,k)\in\mathrm{F}_k(\mathbb{C}^N)(u)$. Introduce the formal power series in u^{-1} with the coefficients in $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$,

$$A_k(u)=\mathrm{tr}_N\otimes\mathrm{id}\,(H_N\otimes 1\cdot S_{1,\dots,k}(u,k)\cdot I(u)\otimes 1$$
$$Z_{k+1,\dots,N}(u+N/2-k,N-k)\otimes 1)\,,$$

where

$$I(u)=\overrightarrow{\prod_{1\leqq p\leqq k}}\left(\overrightarrow{\prod_{k<q\leqq N}}\tilde{R}_{pq}(p+q-2u)/(p+q-2u)\right).$$

By definition, the subalgebra $\mathrm{B}(\mathfrak{gl}_N,\sigma,Z)$ in the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$ is generated by all coefficients of the series $A_1(u),\dots,A_N(u)$. By using (3.12) when $k=N$ we get the analogue of (1.12),

$$H_N\otimes 1\cdot S(u,N)=H_N\otimes A_N(u)\,. \tag{3.15}$$

The proof of the next proposition is also contained in [MNO, Sect. 3]. Let us put $\theta(u)=1+N/(1-2u)$ if $\mathfrak{a}_N=\mathfrak{sp}_N$. If $\mathfrak{a}_N=\mathfrak{so}_N$ we put $\theta(u)=1$.

Proposition 3.3. *We have the equality in the algebra* $\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$,

$$A_N(u)\,\theta(u)=B_N(u)B_N(N-u+1)\,.$$

The coefficients at $u^{-2},u^{-4},\dots$ *of the series* $A_N(u)$ *are free generators for the centre of the algebra* $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$.

The series $A_N(u)\,\theta(u)$ is called the *quantum determinant* for the algebra $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$, or *Sklyanin determinant* [MNO]. There is an analogue of the expression (1.11) for this series, it has been proposed in [M]. Now we will prove the following theorem.

Theorem 3.4. *All the coefficients of the series* $A_1(u),\dots,A_N(u)$ *commute.*

Proof. We will employ arguments already used in the proof of Proposition 1.3. We shall keep to the notation introduced therein. We will also use a method from

[S, Sect. 3]. The element $S(u,k)$ belongs to

$$\mathrm{id} \otimes 1 + \mathrm{End}\,(\mathbb{C}^N)^{\otimes k} \otimes \mathrm{Y}(\mathfrak{gl}_N,\sigma)[[u^{-1}]] \cdot u^{-1}\,,$$

and is therefore invertible in the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k} \otimes \mathrm{Y}(\mathfrak{gl}_N,\sigma)[[u^{-1}]]$. Let $\widehat{S}(u,k)$ be the inverse series. Put

$$\widehat{A}_k(u) = \mathrm{tr}_k \otimes \mathrm{id}\,(H_k \otimes 1 \cdot \widehat{S}(u,k) \cdot Z(u+N/2,k) \otimes 1)\,. \tag{3.16}$$

Then by (3.15),

$$A_k(u) = A_N(u)\widehat{A}_{N-k}(u-k) \cdot \binom{N}{k}\,.$$

By Proposition 3.3 all the coefficients of the series $A_N(u)$ are central in $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$. Hence it suffices to prove that $[\,\widehat{A}_k(u),\widehat{A}_l(v)\,] = 0$ for all the indices $k,l = 1,\dots,N$.

Let $P(u)$ denote the same ordered product as in (1.15). Consider also the product

$$Q(u) = \prod_{1\leqq p\leqq k}^{\rightarrow} \left(\prod_{k<q\leqq k+l}^{\rightarrow} \tilde{R}_{pq}(u+p+q) \right) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}[u]\,. \tag{3.17}$$

The product $Q(u)$ has an inverse in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}(u)$. Due to the decomposition (1.7) and Eq. (3.9) to (3.11) we have

$$H_k \otimes 1 \cdot Q(u) = \prod_{1\leqq p\leqq k}^{\leftarrow} \left(\prod_{k<q\leqq k+l}^{\rightarrow} R_{pq}(u+p+q) \right) \cdot H_k \otimes 1\,,$$

$$1 \otimes H_l \cdot Q(u) = \prod_{1\leqq p\leqq k}^{\rightarrow} \left(\prod_{k<q\leqq k+l}^{\leftarrow} R_{pq}(u+p+q) \right) \cdot 1 \otimes H_l\,.$$

In particular,

$$Q(u) \in \mathrm{F}_k(\mathbb{C}^N) \otimes \mathrm{F}_l(\mathbb{C}^N)\,[u]\,.$$

We will denote

$$\varphi_k \otimes \varphi_l(Q(u)) = \bar{Q}(u)\,.$$

The element $\bar{Q}(u)$ has an inverse in $\mathrm{End}(\Lambda^k\mathbb{C}^N) \otimes \mathrm{End}(\Lambda^l\mathbb{C}^N)(u)$ as well as the element $\bar{P}(u)$ defined by (1.17).

Further, let κ and λ denote the simultaneous transpositions in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}$ with respect to the first k and the last l tensor factors. Due to the equalities (1.3) and (3.2) the definition (3.17) implies that

$$\kappa(Q(u)^{-1}) \cdot \kappa(Q(u-N)) \cdot \beta(u) = \mathrm{id}\,,$$

$$\lambda(Q(u)^{-1}) \cdot \lambda(Q(u-N)) \cdot \beta(u) = \mathrm{id}\,,$$

where $\beta(u)$ stands for the rational function

$$\prod_{1\leqq p\leqq k}\ \prod_{k<q\leqq k+l} \frac{(u+p+q)(u+p+q-N)}{(u+p+q-N)^2-1}\,.$$

Note that the elements $H_k \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ and $H_l \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes l}$ are invariant with respect to the simultaneous transpositions in all the tensor factors. Therefore in

$\mathrm{End}(\Lambda^k\mathbb{C}^N)\otimes\mathrm{End}(\Lambda^l\mathbb{C}^N)(u)$ we get

$$\kappa(\bar{Q}(u)^{-1})\cdot\kappa(\bar{Q}(u-N))\cdot\beta(u)=\mathrm{id}\ , \tag{3.18}$$

$$\lambda(\bar{Q}(u)^{-1})\cdot\lambda(\bar{Q}(u-N))\cdot\beta(u)=\mathrm{id}\ . \tag{3.19}$$

By (3.4) and (3.9) to (3.11) we obtain in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)((u^{-1},v^{-1}))$,

$$\begin{aligned}P(u-v)\otimes 1\cdot S_{1,\dots,k}(u,k)\cdot Q(-u-v)\otimes 1\cdot S_{k+1,\dots,k+l}(v,l)\\ =\ S_{k+1,\dots,k+l}(v,l)\cdot Q(-u-v)\otimes 1\cdot S_{1,\dots,k}(u,k)\cdot P(u-v)\otimes 1\ .\end{aligned} \tag{3.20}$$

Further, by using (3.14) along with (3.9) to (3.11) we obtain in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(k+l)}(u,v)$ the following analogue of (3.20):

$$\begin{aligned}P(u-v)\cdot Z_{1,\dots,k}(u,k)\cdot Q(-u-v)\cdot Z_{k+1,\dots,k+l}(v,l)\\ =\ Z_{k+1,\dots,k+l}(v,l)\cdot Q(-u-v)\cdot Z_{1,\dots,k}(u,k)\cdot P(u-v)\ .\end{aligned} \tag{3.21}$$

Introduce the elements of the algebra $\mathrm{End}(\Lambda^k\mathbb{C}^N)\otimes\mathrm{End}(\Lambda^l\mathbb{C}^N)\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)[[u^{-1}]]$,

$$K(u)=\varphi_k\otimes\varphi_l\otimes\mathrm{id}\,(\widehat{S}_{1,\dots,k}(u,k))\ ,$$

$$L(u)=\varphi_k\otimes\varphi_l\otimes\mathrm{id}\,(\widehat{S}_{k+1,\dots,k+l}(u,l))\ ,$$

and

$$U(u)=\varphi_k(Z(u+N/2\,,k))\otimes\mathrm{id}\ \otimes 1\ ,$$

$$V(u)=\mathrm{id}\ \otimes\varphi_l(Z(u+N/2\,,l))\otimes 1\ .$$

The equalities (3.20) and (3.21) then imply that

$$\begin{aligned}K(u)\cdot\bar{Q}(-u-v)^{-1}\otimes 1\cdot L(v)\cdot\bar{P}(u-v)\otimes 1\\ =\bar{P}(u-v)\otimes 1\cdot L(v)\cdot\bar{Q}(-u-v)^{-1}\otimes 1\cdot K(u)\,,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\bar{P}(u-v)\otimes 1\cdot U(u)\cdot\bar{Q}(-u-v-N)\otimes 1\cdot V(v)\\ =V(v)\cdot\bar{Q}(-u-v-N)\otimes 1\cdot U(u)\cdot\bar{P}(u-v)\otimes 1\,.\end{aligned} \tag{3.23}$$

By the definition (3.16) we have

$$\widehat{A}_k(u)\widehat{A}_l(v)=\mathrm{tr}\ \otimes\mathrm{id}\,(K(u)L(v)\cdot V(v)\,U(u)).$$

Let us call two elements of $\mathrm{End}(\Lambda^k\mathbb{C}^N)\otimes\mathrm{End}(\Lambda^l\mathbb{C}^N)\otimes\mathrm{Y}(\mathfrak{gl}_N,\sigma)((u^{-1},v^{-1}))$ *equivalent* and relate them by the symbol $\sim$ if the values of $\mathrm{tr}\otimes\mathrm{id}$ on these two elements are the same. Then

$$\begin{aligned}K(u)L(v)\cdot V(v)\,U(u)&\sim\lambda\otimes\mathrm{id}\,(K(u)L(v))\cdot\lambda\otimes\mathrm{id}\,(V(v)\,U(u))\\ &=K(u)\cdot\lambda\otimes\mathrm{id}\,(L(v))\cdot\lambda\otimes\mathrm{id}\,(V(v))\ \cdot\ U(u)\\ &=K(u)\cdot\lambda\otimes\mathrm{id}\,(L(v))\cdot\lambda(\bar{Q}(-u-v)^{-1})\otimes 1\\ &\quad\times\lambda(\bar{Q}(-u-v-N))\otimes 1\cdot\lambda\otimes\mathrm{id}(V(v))\cdot U(u)\cdot\beta(-u-v)\,,\end{aligned}$$

where we made use of (3.19). The product in the last two lines is equivalent to

$$\begin{aligned}
&\lambda \otimes \mathrm{id}\,(K(u) \cdot \lambda \otimes \mathrm{id}\,(L(v)) \cdot \lambda(\bar{Q}(-u-v)^{-1}) \otimes 1) \\
&\quad \times\ \lambda \otimes \mathrm{id}\,(\lambda(\bar{Q}(-u-v-N)) \otimes 1 \cdot \lambda \otimes \mathrm{id}\,(V(v)) \cdot U(u)) \cdot \beta(-u-v) \\
&\sim\ K(u) \cdot \bar{Q}(-u-v)^{-1} \otimes 1 \cdot L(v) \\
&\quad \times\ V(v) \cdot \bar{Q}(-u-v-N) \otimes 1 \cdot U(u) \cdot \beta(-u-v) \\
&=\ K(u) \cdot \bar{Q}(-u-v)^{-1} \otimes 1 \cdot L(v) \cdot \bar{P}(u-v) \otimes 1 \\
&\quad \times\ \bar{P}(u-v)^{-1} \otimes 1 \cdot V(v) \cdot \bar{Q}(-u-v-N) \otimes 1 \cdot U(u) \cdot \beta(-u-v) \\
&=\ \bar{P}(u-v) \otimes 1 \cdot L(v) \cdot \bar{Q}(-u-v)^{-1} \otimes 1 \cdot K(u) \\
&\quad \times\ U(u) \cdot \bar{Q}(-u-v-N) \otimes 1 \cdot V(v) \cdot \bar{P}(u-v)^{-1} \otimes 1 \cdot \beta(-u-v)\,,
\end{aligned}$$

where we used (3.2) and (3.23). The product in the last two lines is equivalent to

$$\begin{aligned}
&L(v) \cdot \bar{Q}(-u-v)^{-1} \otimes 1 \cdot K(u) \\
&\quad \times\ U(u) \cdot \bar{Q}(-u-v-N) \otimes 1 \cdot V(v) \cdot \beta(-u-v) \\
&\sim\ \kappa \otimes \mathrm{id}\,(L(v) \cdot \bar{Q}(-u-v)^{-1} \otimes 1 \cdot K(u)) \\
&\quad \times\ \kappa \otimes \mathrm{id}\,(U(u) \cdot \bar{Q}(-u-v-N) \otimes 1 \cdot V(v)) \cdot \beta(-u-v) \\
&=\ L(v) \cdot \kappa \otimes \mathrm{id}\,(K(u)) \cdot \kappa \otimes \mathrm{id}\,(U(u)) \cdot V(v) \sim L(v)K(u) \cdot U(u)V(v)\,,
\end{aligned}$$

where we made use of (3.18). It now remains to take into account that

$$\mathrm{tr} \otimes \mathrm{id}\,(L(v)K(u) \cdot U(u)V(v)) = \widehat{A}_l(v)\widehat{A}_k(u)\,. \qquad \square$$

Theorem 3.5. *Suppose that the element $Z \in \mathrm{End}\,\mathbb{C}^N$ has a simple spectrum and $Z' = -Z$. Then the subalgebra $\mathrm{B}(\mathfrak{gl}_N, \sigma, Z)$ in $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$ is maximal commutative. The coefficients at $u^{-2}, u^{-4}, \ldots$ of the series $A_N(u), A_{N-2}(u), \ldots$ and the coefficients at $u^{-1}, u^{-3}, \ldots$ of the series $A_{N-1}(u), A_{N-3}(u), \ldots$ are free generators for $\mathrm{B}(\mathfrak{gl}_N, \sigma, Z)$.*

The proof of this theorem will be given in Sect. 4. We will end up this section with making the following observation. Consider the series $\widehat{A}(u)$ defined by (3.16).

Proposition 3.6. *Suppose that $Z' = \pm Z$, where the upper sign corresponds to the case $\mathfrak{a}_N = \mathfrak{so}_N$ while the lower one corresponds to $\mathfrak{a}_N = \mathfrak{sp}_N$. Then*

$$\widehat{A}_k(u) = \mathrm{tr}_k \otimes \mathrm{id}\,(H_k \otimes 1 \cdot \widehat{S}(u,k) \cdot Z_1 \cdots Z_k \otimes 1)\,.$$

Proof. It can be verified directly that for $Z' = \pm Z$ we have in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}[u]$ the equality

$$Z_1\, \tilde{R}(u)\, Z_2\, H_2 = Z_1\, Z_2\, H_2\ .$$

By using repeatedly this equality we obtain from (3.13) and (3.16) the required statement. $\square$

4. Proof of Theorem 3.5

We will employ arguments already used in the proof of Theorem 1.3. Consider the ascending filtration on the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ introduced in Sect. 2. Then by (3.1) for the generators of the subalgebra $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$ in $\mathrm{Y}(\mathfrak{gl}_N)$ we have $\deg S_{ij}^{(r)} = r$. Denote by $\mathrm{X}(\mathfrak{gl}_N,\sigma)$ and $\mathrm{J}(\mathfrak{gl}_N,\sigma,Z)$ the images in the graded Poisson algebra $\mathrm{X}(\mathfrak{gl}_N)$ of the subalgebras $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$ and $\mathrm{B}(\mathfrak{gl}_N,\sigma,Z)$ in $\mathrm{Y}(\mathfrak{gl}_N)$ respectively. We shall prove that the subalgebra $\mathrm{J}(\mathfrak{gl}_N,\sigma,Z)$ in the Poisson algebra $\mathrm{X}(\mathfrak{gl}_N,\sigma)$ is maximal involutive.

We will denote by $s_{ij}^{(r)}$ the image in $\mathrm{X}(\mathfrak{gl}_N,\sigma)$ of the generator $S_{ij}^{(r)}$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N,\sigma)$. Due to the relation (3.5) we then have

$$s_{ij}^{(r)} = \varepsilon_{ij}\ s_{-j,-i}^{(r)} \cdot (-1)^r; \quad r = 1,2,\dots \tag{4.1}$$

in $\mathrm{X}(\mathfrak{gl}_N,\sigma)$. Moreover, the relations (4.1) are defining relations for generators $s_{ij}^{(r)}$ of the commutative algebra $\mathrm{X}(\mathfrak{gl}_N,\sigma)$. The proof of the latter statement is contained in [MNO, Sect. 3]. We will identify $\mathrm{X}(\mathfrak{gl}_N,\sigma)$ with the symmetric algebra over the *twisted* polynomial current Lie algebra

$$\{F(t) \in \mathfrak{gl}_N[t] \mid \sigma(F(t)) = F(-t)\}\,.$$

The generator $s_{ij}^{(r)}$ can be identified with the element

$$E_{ij} \cdot t^{r-1} - E'_{ij} \cdot (-t)^{r-1} \tag{4.2}$$

of this Lie algebra. We will set $s_{ij}^{(0)} = \delta_{ij} \cdot 1$.

Lemma 4.1. *In the Poisson algebra* $\mathrm{X}(\mathfrak{gl}_N,\sigma)$ *for any* $p,q \geqq 1$ *we have*

$$\begin{aligned}\{s_{ij}^{(p)}, s_{kl}^{(q)}\} = &\sum_{r=1}^{\min(p,q)} (s_{kj}^{(r-1)} s_{il}^{(p+q-r)} - s_{kj}^{(p+q-r)} s_{il}^{(r-1)}) \\ &+ \sum_{r=1}^{\min(p,q)} (\varepsilon_{k,-j}\ s_{i,-k}^{(r-1)} s_{-j,l}^{(p+q-r)} - \varepsilon_{i,-l}\ s_{k,-i}^{(p+q-r)} s_{-l,j}^{(r-1)}) \cdot (-1)^{p+r-1}\,. \end{aligned} \tag{4.3}$$

Proof. By the relations (3.6) and by the definition of the Poisson algebra $\mathrm{X}(\mathfrak{gl}_N,\sigma)$

$$\begin{aligned}\{s_{ij}^{(p)}, s_{kl}^{(q)}\} = &\sum_{r=1}^{p} (s_{kj}^{(p-r)} s_{il}^{(q+r-1)} - s_{kj}^{(q+r-1)} s_{il}^{(p-r)}) \\ &+ \sum_{r=1}^{p} (\varepsilon_{k,-j}\ s_{i,-k}^{(p-r)} s_{-j,l}^{(q+r-1)} - \varepsilon_{i,-l}\ s_{k,-i}^{(q+r-1)} s_{-l,j}^{(p-r)}) \cdot (-1)^r\,. \end{aligned} \tag{4.4}$$

Here the first of the two sums coincides with the first sum in (4.3). The second sum in (4.4) coincides with the second sum in (4.3) by the relations (4.1). □

Consider the formal power series in u^{-1}

$$s_{ij}(u) = s_{ij}^{(0)} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots$$

with the coefficients in the algebra $\mathrm{X}(\mathfrak{gl}_N,\sigma)$. The image of the series $A_k(u)$ in the algebra $\mathrm{X}(\mathfrak{gl}_N,\sigma)[[u^{-1}]]$ equals

$$\sum_{g,h} s_{g(1)h(1)}(u) \cdots s_{g(k)h(k)}(u) \cdot z_{g(k+1)h(k+1)} \cdots z_{g(N)h(N)} \cdot \varepsilon/N!\,, \tag{4.5}$$

where g, h run through the set of all permutations of $-n, \dots, -1, 1, \dots, n$ if $N = 2n$ and the set of all permutations $-n, \dots, -1, 0, 1, \dots, n$ if $N = 2n+1$. Here $\varepsilon = \operatorname{sgn} g \cdot \operatorname{sgn} h$ as well as in (1.10) and (2.2).

For each $M = 1, 2 \dots$ denote by $\mathrm{X}_M(\mathfrak{gl}_N, \sigma)$ the ideal in the algebra $\mathrm{X}(\mathfrak{gl}_N, \sigma)$ generated by all the elements $s_{ij}^{(r)}$, where $r \geqq M$. Note that by Lemma 4.1 we have

$$\{s_{ij}^{(p)}, s_{kl}^{(q)}\} \in \mathrm{X}_M(\mathfrak{gl}_N, \sigma); \quad M = \max(p, q) .$$

Therefore each $\mathrm{X}_M(\mathfrak{gl}_N, \sigma)$ is a Poisson ideal in $\mathrm{X}(\mathfrak{gl}_N, \sigma)$. Now let the index M be fixed. For each $r = 1, \dots, M$ denote by $y_{ij}^{(r)}$ the image of the generator $s_{ij}^{(r)}$ in the quotient Poisson algebra $\mathrm{X}(\mathfrak{gl}_N, \sigma)/\mathrm{X}_{M+1}(\mathfrak{gl}_N, \sigma)$. In the latter algebra we have

$$\{y_{ij}^{(p)}, y_{kl}^{(q)}\} = \sum_{r=\max(1,\, p+q-M)}^{\min(p,q)} (y_{kj}^{(r-1)} y_{il}^{(p+q-r)} - y_{kj}^{(p+q-r)} y_{il}^{(r-1)})$$

$$+ \sum_{r=\max(1,\, p+q-M)}^{\min(p,q)} (\varepsilon_{k,-j}\; y_{i,-k}^{(r-1)} y_{-j,l}^{(p+q-r)} - \varepsilon_{i,-l}\; y_{k,-i}^{(p+q-r)} y_{-l,j}^{(r-1)}) \cdot (-1)^{p+r-1} , \quad (4.6)$$

where $y_{ij}^{(0)} = 1 \cdot \delta_{ij}$. Moreover, by (4.1) we have the equalities

$$y_{ij}^{(r)} = \varepsilon_{ij} y_{-j,-i}^{(r)} \cdot (-1)^r; \quad r = 1, \dots, M .$$

Let $\mathfrak{g}_{M,N}$ be the same vector space as in (2.4). Consider the subspace in $\mathfrak{g}_{M,N}$,

$$\mathfrak{f}_{M,N} = \{F(t) \in \mathfrak{g}_{M,N} \mid \sigma(F(t)) = F(-t)\} .$$

As a commutative algebra the quotient $\mathrm{X}(\mathfrak{gl}_N, \sigma)/\mathrm{X}_{M+1}(\mathfrak{gl}_N, \sigma)$ can be identified with the symmetric algebra $\mathrm{S}(\mathfrak{f}_{M,N})$. Further, we will identify this quotient with the algebra $\mathrm{P}(\mathfrak{f}_{M,N})$ of the polynomial functions on $\mathfrak{f}_{M,N}$. The generator $y_{ij}^{(r)}$ will be regarded as the coordinate function corresponding to the vector (4.2) in $\mathfrak{f}_{M,N}$.

Let us introduce the polynomials in u of the degree $M-1$,

$$y_{ij}(u) = y_{ij}^{(1)} u^{M-1} + y_{ij}^{(2)} u^{M-2} + \dots + y_{ij}^{(M)} ,$$

and denote by $Y(u)$ the square matrix of order N formed by these polynomials. For each $k = 1, \dots, N$ denote by $a_k(u)$ the image of $A_k(u) \in \mathrm{Y}(\mathfrak{gl}_N, \sigma)[[u^{-1}]]$ in

$$\mathrm{X}(\mathfrak{gl}_N, \sigma)/\mathrm{X}_{M+1}(\mathfrak{gl}_N, \sigma)[[u^{-1}]] = \mathrm{P}(\mathfrak{f}_{M,N})[[u^{-1}]] .$$

This image is in fact a polynomial in u^{-1} of the degree kM, see (4.5). We have the Laplace expansion

$$\det (u^M + Y(u) + Zv) = v^N \det Z + \sum_{k=1}^{N} u^{kM} a_k(u) v^{N-k} \binom{N}{k} , \quad (4.7)$$

where Z is now regarded as a square matrix of order N. Since $Z' = -Z$ we have

$$a_k(-u) = a_k(u) \cdot (-1)^{N-k}; \quad k = 1, \dots, N .$$

We will write

$$a_k(u) = a_k^{(0)} + a_k^{(1)} u^{-1} + \dots + a_k^{(kM)} u^{-kM} .$$

Here $a_k^{(0)} \in \mathbb{C}$ for any $k = 1, \ldots, N$. Furthermore $a_k^{(r)} = 0$ unless $N - k + r \in 2\mathbb{Z}$. All the remaining coefficients $a_k^{(r)}$ are in involution with respect to the Poisson bracket (4.6) by Proposition 3.3.

From now on we will assume that $M = 2m + 1$ if $\mathfrak{a}_N = \mathfrak{so}_{2n+1}$ or $\mathfrak{a}_N = \mathfrak{sp}_{2n}$ and that $M = 2m$ if $\mathfrak{a}_N = \mathfrak{so}_{2n}$. We shall prove that all the coefficients $a_k^{(r)}$ with

$$1 \leqq r \leqq kM; \quad N - k + r \in 2\mathbb{Z} \tag{4.8}$$

are algebraically independent and generate a maximal involutive subalgebra in the quotient $\mathrm{X}(\mathfrak{gl}_N, \sigma)/\mathrm{X}_{M+1}(\mathfrak{gl}_N, \sigma)$. Lemma 2.1 will then imply that the subalgebra $\mathrm{J}(\mathfrak{gl}_N, \sigma, Z)$ in the Poisson algebra $\mathrm{X}(\mathfrak{gl}_N, \sigma)$ is also maximal involutive. So we will then have Theorem 3.5 proved.

It can be verified directly that for any element $G \in \mathrm{End}(\mathbb{C}^N)$ the following equality holds in the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}[u]$:

$$G_1 \tilde{R}(u) G_2' = G_2' \tilde{R}(u) G_1 \,. \tag{4.9}$$

Now suppose that $G' = G^{-1}$ so that the element G belongs to the orthogonal or symplectic group on $\mathbb{C}^N$ corresponding to the subalgebra $\mathfrak{a}_N$ in $\mathfrak{gl}_N$. By the above two equalities the defining relations (3.4) and (3.5) then imply that the assignment

$$S(u) \mapsto G \otimes 1 \cdot S(u) \cdot G^{-1} \otimes 1$$

determines an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_N, \sigma)$. The image of the subalgebra $\mathrm{B}(\mathfrak{gl}_N, \sigma, Z)$ with respect to this automorphism is $\mathrm{B}(\mathfrak{gl}_N, \sigma, G^{-1} Z G)$. Indeed, by applying (4.9) repeatedly we obtain that the images of the series $A_1(u), \ldots, A_N(u)$ determined by the element Z coincide with the corresponding series determined by the element $G^{-1} Z G$.

Hence it suffices to prove Theorem 3.5 assuming that $z_{ij} = z_i \cdot \delta_{ij}$, where all $z_i \in \mathbb{C}$ are pairwise distinct and $z_{-i} = -z_i$ for any index i. We shall keep to this assumption from now on until the end of this section.

Let $\mathfrak{b}_N$ be the Borel subalgebra in $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqq j$. Fix a principal nilpotent element of the Borel subalgebra in $\mathfrak{gl}_N$ opposite to $\mathfrak{b}_N$

$$E = \begin{cases} E_{n,n-1} - E_{1-n,-n} + \cdots + E_{21} - E_{-1,-2} + E_{1,0} - E_{0,-1} & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n+1}\,; \\ E_{n,n-1} - E_{1-n,-n} + \cdots + E_{21} - E_{-1,-2} + E_{1,-1} & \text{if } \mathfrak{a}_N = \mathfrak{sp}_{2n}\,; \\ E_{n,n-1} + E_{1-n,-n} + \cdots + E_{21} + E_{-1,-2} + E_{1,-1} & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n}\,. \end{cases}$$

Let $\mathfrak{gl}_N = \mathfrak{a}_N \oplus \mathfrak{x}_N$ be the decomposition into the eigenspaces of the involutive automorphism σ. Then $E \in \mathfrak{a}_N$ for $\mathfrak{a}_N = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$, but $E \in \mathfrak{x}_N$ for $\mathfrak{a}_N = \mathfrak{so}_{2n}$. Therefore

$$E^{(M)} = E \cdot t^{M-1} \in \mathfrak{f}_{M,N}$$

by our assumption on the parity of M. We will consider the Poisson bracket (4.6) on $\mathrm{P}(\mathfrak{f}_{M,N})$ in a neighbourhood of the point $E^{(M)}$.

Regard the coefficients $a_k^{(r)}$ as polynomial functions on $\mathfrak{f}_{M,N}$. Consider the affine subspace in $\mathfrak{f}_{M,N}$,

$$\mathfrak{s}_{M,N} = E^{(M)} + (\mathfrak{f}_{M,N} \cap \mathfrak{b}_N[t])\,.$$

It will be taken as an analogue of the subspace $\mathfrak{t}_{M,N}$ in $\mathfrak{g}_{M,N}$. For the principal nilpotent element $E \in \mathfrak{gl}_N$ fixed above we have

$$\mathfrak{f}_{M,N} \cap \mathfrak{t}_{M,N} = \mathfrak{s}_{M,N} .$$

Therefore by comparing (2.5) and (4.7) we obtain the next statement directly from Lemma 2.2.

Lemma 4.2. *The restrictions of the functions $a_k^{(r)}$ onto the affine subspace $\mathfrak{s}_{M,N}$ generate the whole algebra of polynomial functions on $\mathfrak{s}_{M,N}$.*

Observe that the total number of the coeficients $a_k^{(r)}$ with the indices satisfying (4.8) coincides with

$$\dim \mathfrak{s}_{M,N} = \begin{cases} (2mn+m+n)(n+1) & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n+1} ; \\ (2mn+m+n+1)n & \text{if } \mathfrak{a}_N = \mathfrak{sp}_{2n} ; \\ (2n+1)mn & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n} . \end{cases}$$

Corollary 4.3. *All the coefficients $a_k^{(r)}$ with the indices satisfying* (4.8) *are algebraically independent.*

To complete the proof of Theorem 3.5 we have to demonstrate that the involutive subalgebra in $\mathrm{P}(\mathfrak{f}_{M,N})$ generated by the coefficients $a_k^{(r)}$ is maximal. As an argument similar to that given in the end of Sect. 2 shows, this assertion follows from Lemma 4.2 and from the next lemma. Set

$$D = \dim \mathfrak{f}_{M,N} - \dim \mathfrak{s}_{M,N} = \begin{cases} (2mn+m+n)n & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n+1} ; \\ (2mn-m+n)n & \text{if } \mathfrak{a}_N = \mathfrak{sp}_{2n} ; \\ (2n-1)mn & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n} . \end{cases}$$

Lemma 4.4. *There is an open neighbourhood of the point $E^{(M)}$ in $\mathfrak{f}_{M,N}$, where the rank of the Poisson bracket* (4.6) *equals* $2D$.

Proof. Consider the alternating bilinear form $\alpha : \mathfrak{gl}_N \times \mathfrak{gl}_N \to \mathbb{C}$, where $\alpha(E_{ij}, E_{kl})$ is the value of (2.6) at the point E. The rank of this bilinear form is $N^2 - N$.

Any summand at the right-hand side of (4.6) vanishes at the point $E^{(M)}$ unless $p+q = M+1$ and $r = 1$. We will assume that $p+q = M+1$ and that $p, q \geqq 1$. Then the value of the bracket $\{y_{ij}^{(p)}, y_{kl}^{(q)}\}$ at the point $E^{(M)}$ is

$$\alpha(E_{ij} - \sigma(E_{ij}) \cdot (-1)^p, E_{kl} - \sigma(E_{kl}) \cdot (-1)^q) .$$

Suppose that $\mathfrak{a}_N = \mathfrak{so}_{2n}$ so that $M = 2m$. Then $E \in \mathfrak{x}_N$ and the restriction of the bilinear form α onto either of the subspaces $\mathfrak{a}_N \times \mathfrak{x}_N$ and $\mathfrak{x}_N \times \mathfrak{a}_N$ has the rank

$$(N^2 - N)/2 = n(2n-1) .$$

Hence there exists an open neighbourhood of the point $E^{(M)}$ in $\mathfrak{f}_{M,N}$, where the rank of the Poisson bracket (4.6) is not smaller than $2m \cdot n(2n-1) = 2D$.

Now suppose that $\mathfrak{a}_N = \mathfrak{so}_{2n+1}$ or $\mathfrak{a}_N = \mathfrak{sp}_{2n}$ so that $M = 2m+1$. Then $E \in \mathfrak{a}_N$. Moreover, E is a principal nilpotent element of the Borel subalgebra in $\mathfrak{a}_N$ opposite to $\mathfrak{a}_N \cap \mathfrak{b}_N$. Therefore the restriction of the bilinear form α onto $\mathfrak{a}_N \times \mathfrak{a}_N$ has the rank $2n^2$. The restriction of α onto $\mathfrak{x}_N \times \mathfrak{x}_N$ then has the rank

$$(N^2 - N) - 2n^2 = 2n(n \pm 1) .$$

So there exists an open neighbourhood of the point $E^{(M)}$ in $\mathfrak{f}_{M,N}$, where the rank of the Poisson bracket (4.6) is not smaller than

$$(m+1)\cdot 2n^2 + m\cdot 2n(n\pm 1) = 2D\,.$$

But for every $\mathfrak{a}_N$ due to Proposition 3.3 and to Corollary 4.3 in the centre of the Poisson algebra $\mathrm{P}(\mathfrak{f}_{M,N})$ there are algebraically independent elements $b_N^{(r)}$ with $1 \leqq r \leqq MN$, $r \in 2\mathbb{Z}$. So the rank of the Poisson bracket (4.6) at any point cannot exceed

$$2D = \dim \mathfrak{f}_{M,N} - \begin{cases} 2mn+n+m & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n+1}\,; \\ 2mn+n & \text{if } \mathfrak{a}_N = \mathfrak{sp}_{2n}\,; \\ 2mn & \text{if } \mathfrak{a}_N = \mathfrak{so}_{2n}\,. \end{cases}$$

The proof of Lemma 4.4 is now complete. □

Let us now consider the commutative subalgebra $\rho(\mathrm{B}(\mathfrak{gl}_N,\sigma,Z))$ in $\mathrm{U}(\mathfrak{a}_N)$. First consider the involutive subalgebra in the graded algebra $\mathrm{S}(\mathfrak{a}_N)$ corresponding to $\rho(\mathrm{B}(\mathfrak{gl}_N,\sigma,Z))$. We identify the Poisson algebras $\mathrm{S}(\mathfrak{a}_N)$ and $\mathrm{P}(\mathfrak{a}_N)$; the element $F_{ij} \in \mathfrak{a}_N$ is identified with the respective coordinate function. Denote by y_{ij} this function. Let Y be the square matrix of the order N formed by all these functions. The subalgebra in $\mathrm{P}(\mathfrak{a}_N)$ corresponding to $\pi(\mathrm{B}(\mathfrak{gl}_N,\sigma,Z))$ is generated by the coefficients of the polynomial in u, v,

$$\det(u + Y + Zv)\,. \tag{4.10}$$

This subalgebra in $\mathrm{P}(\mathfrak{a}_N)$ has been considered in [MF, Sect. 4]. It has been proved there that the coefficients of (4.10) form a *complete set* of elements of $\mathrm{P}(\mathfrak{a}_N)$ in involution: their gradients at a generic point of $\mathfrak{a}_N$ span a space of the maximal possible dimension

$$(\dim \mathfrak{a}_N + \operatorname{rank} \mathfrak{a}_N)/2\,.$$

Lemma 4.2 provides another proof of this fact for $\mathfrak{a}_N = \mathfrak{so}_{2n+1}$ and $\mathfrak{a}_N = \mathfrak{sp}_{2n}$. In these two cases it also shows that the coefficients of (4.10) generate a maximal involutive subalgebra in $\mathrm{P}(\mathfrak{a}_N)$. Indeed, when $\mathfrak{a}_N = \mathfrak{so}_{2n+1}$ or $\mathfrak{a}_N = \mathfrak{sp}_{2n}$ we can set $M = 1$. But when $\mathfrak{a}_N = \mathfrak{so}_{2n}$ by our assumption M has to be even. So we have to consider the latter case separately.

Suppose that $N = 2n$ and $\mathfrak{a}_N = \mathfrak{so}_{2n}$. Fix in $\mathfrak{so}_{2n}$ the Borel subalgebra $\mathfrak{b}_{2n} \cap \mathfrak{so}_{2n}$ and choose the principal nilpotent element of the opposite Borel subalgebra,

$$E = E_{n,n-1} - E_{1-n,-n} + \cdots + E_{21} - E_{-1,-2} + E_{2,-1} - E_{1,-2}\,.$$

Consider the affine subspace $\mathfrak{s}_{2n} = E + \mathfrak{b}_{2n} \cap \mathfrak{so}_{2n}$ in $\mathfrak{so}_{2n}$. To prove that the coefficients of (4.10) again generate a maximal involutive subalgebra in $\mathrm{P}(\mathfrak{so}_{2n})$ it suffices to establish the following lemma.

Lemma 4.5. *The restrictions of the coefficients of the polynomial* (4.10) *onto the affine subspace* $\mathfrak{s}_{2n}$ *generate the whole algebra of polynomial functions on* $\mathfrak{s}_{2n}$.

Proof. We will employ arguments already used in the proof of Lemma 2.2. Denote the polynomial in u, v obtained by restricting the coefficients of (4.10) onto $\mathfrak{s}_{2n}$ by $F(u,v)$. Let the indices i, j run through the set $\{1,\dots,n\}$. Take the functions y_{ij} with $i \leqq j$ and $y_{-i,j}$ as coordinates on $\mathfrak{s}_{2n}$. Put $\bar{y}_{ij} = y_{ij} - \delta_{1i} y_{-1,j}$. We will prove

consecutively for $d = 0, 1, \dots, 2n-2$ that here all $\bar{y}_{ij}$ with $j - i = d$ and all $y_{-i,j}$ with $i + j = d + 2$ are polynomials in the coefficients of $F(u, v)$.

Consider the terms of $F(u, v)$ with the total degree $2n - d - 1$ in u, v. If $d = 0, \dots, n-1$ their sum has the form

$$f(u,v) + (-1)^{n+d} \cdot \sum_{i=1}^{n-d} \bar{y}_{i,i+d}\, g_i(u,v) - (-1)^{n+d} \cdot \sum_{i=1}^{d+1} y_{-i,d-i+2}\, h_i(u,v)\,,$$

where

$$h_i(u,v) = 2u \cdot \prod_{j=i+1}^{n} (-u + z_j v) \cdot \prod_{j=d-i+2}^{n} (u + z_j v)\,, \tag{4.11}$$

while $g_i(u, v)$ equals

$$\prod_{j \neq i,\dots,i+d} (-u + z_j v) \cdot \prod_{j=1}^{n} (u + z_j v) + \prod_{j \neq i,\dots,i+d} (u + z_j v) \cdot \prod_{j=1}^{n} (-u + z_j v)\,.$$

Here the coefficients of the polynomial $f(u, v)$ depend only on $\bar{y}_{ij}$ with $j - i < d$ and on $y_{-i,j}$ with $i + j < d + 2$ along with $z_1, \dots, z_n$. But all the $n + 1$ polynomials $g_i(u, v)$ and $h_i(u, v)$ are linearly independent. To prove this consider their values at

$$(u, v) = (z_1, 1), \dots, (z_n, 1), (0, 1)\,.$$

Since all the numbers $z_1, -z_1, \dots, z_n, -z_n$ are pairwise distinct we have

$$h_i(z_i, 1) \neq 0, \quad 1 \leqq i \leqq d+1\,; \qquad h_j(z_i, 1) = 0, \quad 1 \leqq j < i \leqq d+1\,.$$

Moreover, we have

$$h_j(0, 1) = 0, \quad 1 \leqq j \leqq d+1\,; \qquad h_j(z_i, 1) = 0, \quad 1 \leqq j \leqq d+1 < i \leqq n\,.$$

Furthermore, we have $g_1(0, 1) \neq 0$ and

$$g_i(z_{d+i}, 1) \neq 0, \quad 2 \leqq i \leqq n-d\,; \qquad g_j(z_{d+i}, 1) = 0, \quad 1 \leqq j < i \leqq n-d\,.$$

If $d = n, \dots, 2n-2$ then the sum of the terms of $F(u, v)$ with the total degree $2n - d - 1$ has the form

$$f(u,v) - (-1)^{n+d} \cdot \sum_{i=d-n+2}^{n} y_{-i,d-i+2}\, h_i(u,v)\,,$$

where $h_i(u, v)$ is determined by (4.11) while the coefficients of the polynomial $f(u, v)$ depend only on $\bar{y}_{-i,j}$ with $i + j < d + 2$ and on $\bar{y}_{ij}$ along with $z_1, \dots, z_n$. Here all the $2n - d - 1$ polynomials $h_i(u, v)$ are again linearly independent. Indeed,

$$h_i(z_i, 1) \neq 0, \quad d - n + 2 \leqq i \leqq n\,;$$

$$h_j(z_i, 1) = 0, \quad d - n + 2 \leqq j < i \leqq n\,. \qquad \square$$

Thus for $\mathfrak{a}_N = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$ we have proved that the involutive subalgebra in $\mathrm{P}(\mathfrak{a}_N)$ corresponding to the commutative subalgebra $\rho(\mathrm{B}(\mathfrak{gl}_N, \sigma, Z))$ of $\mathrm{U}(\mathfrak{a}_N)$, is maximal. So for each of the above classical Lie algebras we get the next theorem.

Theorem 4.6. *The subalgebra* $\rho(\mathrm{B}(\mathfrak{gl}_N, \sigma, Z))$ *in* $\mathrm{U}(\mathfrak{a}_N)$ *is maximal commutative.*

This theorem is the analogue of Proposition 2.6 for the above classical Lie algebras.

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