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Growth and Oscillations of Solutions of Nonlinear Schrödinger Equation

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To the memory of Natasha and Sergei Kozlov

Abstract: We study the nonlinear Schrödinger equation in an n-cube, $n = 1,2,3$, under Dirichlet boundary conditions, treating it as a dynamical system in a function space formed by sufficiently smooth functions of x. We show that this space contains a distinguished small subset $\mathfrak{A}$ which is a recursion subset for the dynamical system and describe the dynamics of the equation in terms of the trajectory's recurrence to $\mathfrak{A}$. We use this description to estimate from below the space- and time-space oscillations of solutions in terms of a quantity, similar to the Reynolds number of classical hydrodynamics.

Introduction

We consider the nonlinear Schrödinger equation

$$-i\dot{u} = \delta(-\Delta u + V(x)u) + |u|^2 u, \quad u = u(t,x), \ \delta > 0\,, \tag{1}$$

with the space-variable x in the n-cube $K^n = \{0 \leqq x_j \leqq \pi\}$, $n = 1,2,3$, under Dirichlet boundary conditions

$$u|_{\partial K^n} = 0\,. \tag{2}$$

We study the problem (1), (2) as a dynamical system in a function space Z formed by sufficiently smooth complex functions $u(x)$,

$$Z \subset C^m(K^n; \mathbb{C}), \quad m \geqq 3\,, \tag{3}$$

which vanish at ∂K^n. That is, given $u_0 \in Z$ we interpret the solution $u(t,x)$ of (1), (2) with $u(0,x) = u_0(x)$ as a curve $u(t) \in Z$ and study the trajectories $u(t)$ as well as the flow-maps $S^t : Z \to Z$, $u_0 \mapsto u(t)$.

The problem (1), (2) is well-known to be Hamiltonian with the Hamiltonian $\mathscr{H}$,

$$\mathscr{H}(u(x)) = \int \left(\frac{\delta}{2} |\nabla u(x)|^2 + \frac{\delta}{2} V(x)\, |u(x)|^2 + \frac{1}{4} |u(x)|^4 \right) dx \Big/ (2\pi)^n\,, \tag{4}$$

which is an integral of motion for the dynamical system:

$$\text{const} = \mathscr{H}(u(t)) =: E^2 , \tag{5}$$

(see [7] and below).

In the space Z we define the subset $\mathfrak{A}^\delta$,

$$\mathfrak{A}^\delta = \{u(x) \in Z \mid |u|_\infty < K\delta^\mu \|u\|^{1-2\mu}\} ,$$

where $|\cdot|_\infty$ is the L^∞-norm, $\|\cdot\|$ is the norm in Z, K is an absolute constant and

$$\mu = \mu(m) = \frac{m(m-1)}{2m^2+3m-3}$$

(so $\mu(3) = \frac{1}{4}$ and $\mu \nearrow \frac{1}{2}$ as m grows).

For reasons we explain later, we call $\mathfrak{A}^\delta$ "the essential part of the phase-space Z." The set $\mathfrak{A}^\delta$ becomes small with δ and becomes relatively small as $\|u\|$ grows. Indeed, when $\delta \searrow 0$ the sets $\mathfrak{A}^\delta$ form a sequence of embedded domains with zero intersection, besides for δ fixed and for $S_r = \{\|u\| = r\}$ the intersection $\mathfrak{A}^\delta \cap S_r$ in the L^∞-norm has diameter $\leqq 2K\delta^\mu r^{1-2\mu}$, thus forming a relatively small part of the sphere S_r which has the L^∞-diameter $\geqq C^{-1}r$.

For m large we have $\mu \approx 1/2$, $1 - 2\mu \approx 0$ and in a finite part of the space Z the set $\mathfrak{A}^\delta$ looks like a tube of the L_∞-diameter $\sim K\sqrt{\delta}$.

In parts 2, 3 we show that – in a sense – dynamics of (1), (2) outside $\mathfrak{A}^\delta$ is simple and the whole dynamics of (1), (2) is determined by its rather complicated behavior in $\mathfrak{A}^\delta$. More exactly, we prove

Theorem. *If $u(t,x)$ is a solution of* (1), (2) *with $u(0,x) = u_0 \notin \mathfrak{A}^\delta$, $\|u_0\| = r$, then there exist $t', t'' \leqq Cr^{-a}\delta^{-b}$ such that $\|u(t')\| = 2r$ and $u(t'') \in \mathfrak{A}^\delta$. The positive numbers a, b, C are δ, r-independent.*

Due to the theorem, a trajectory of (1), (2) either

i) moves outside $\mathfrak{A}^\delta$ toward this set, finally entering $\mathfrak{A}^\delta$ and increasing its Z-norm (at least doubling it),

or

ii) moves inside $\mathfrak{A}^\delta$.

Since the hamiltonian $\mathscr{H}$ is an integral of motion, then the growth of the "smooth" Z-norm at the stage i) means that low Fourier modes (in x) of the solution decrease while high modes increase – energy of this solution goes to high frequencies (the phenomenon also known as the *direct cascade of energy*, see more on the subject in [11, 8, 9]). Very likely during the stage ii) the Z-norm decreases (the energy goes to low modes – the *inverse cascade of energy*) and finally the solution leaves $\mathfrak{A}^\delta$, if E in (5) is sufficiently large (more precisely, if large is the ratio E/δ). This is certainly the case for solutions of the Zakharov–Shabat equation (Eq. (1) with $V \equiv 0$, $n = 1$) which start outside $\mathfrak{A}^\delta$ since this equation is integrable and all its solutions are almost-periodic in time; they must decrease the Z-norm somewhere, so – inside $\mathfrak{A}^\delta$.

In [4] Ju.S. Ilyashenko obtained a description of dynamics of the Kuramoto–Sivashinsky equation, similar to i), ii). Since that equation is parabolic, its solutions tend to a bounded finite-dimensional attractor in the corresponding function phase-space. On its way to the attractor each trajectory changes from i) to ii) a finite number of times and ends at the stage ii) in the vicinity of the attractor (see [4], especially Part 2.2).

Since $\|u\| \geqq K^{1/(2\mu)}\sqrt{\delta}$ for each $u \notin \mathfrak{A}^{\delta}$, then by the time

$$T_{\text{pull}} = C_1 \delta^{-b-a/2},^{1}$$

the flow $\{S^t\}$ will pull the whole space Z through a narrow slit formed by the set $\mathfrak{A}^{\delta}$. Phenomenons of this kind are typical for Hamiltonian partial differential equations considered in smooth function spaces [8]. For deep symplectic reasons they are impossible in some distinguished function phase-spaces of low smoothness (see [8, 9] and references therein).

By the theorem a solution $u(t)$ of (1), (2) either is in $\mathfrak{A}^{\delta}$, or it moves fast toward $\mathfrak{A}^{\delta}$. In Part 4 we use this description of the dynamics of (1), (2) to estimate oscillations of its solutions $u(t,x)$. As a measure of the oscillations we propose the function

$$\omega_m(t) = \|u(t)\| / |u(t)|_\infty .$$

(Due to (3) this is something like the C^m-norm of the solution divided by its L^∞-*norm*). *The theorem implies that* $\omega_m(t)$ *becomes large at some point* t_* *of each* time-interval of the length l,

$$l = C/\sqrt{E\delta}$$

(E was defined in (5); the factor $\delta^{-1/2}$ corresponds to a natural time-scaling, see Part 4). Here "large" means that

$$\omega_m(t_*) \geqq C^{-1}(E/\delta)^{\kappa} \tag{6}$$

with some $\kappa > 1/5$.

We also consider a quantity which takes into account time-oscillations of solutions. We define the function $\Omega_m(t)$ as

$$\Omega_m = \frac{\|u(t)\| + C\,\|\dot{u}(t)\|/\sqrt{E\delta}}{|u(t)|_\infty}$$

and prove that averaging of Ω_m along each time-interval of length $\geqq 3l$ is at least one-sixth of the r.h.s. of (6). Roughly,

$$\langle \Omega_m \rangle_{\text{loc}} \geqq C^{-1}(E/\delta)^{\kappa}$$

(here $\langle \cdot \rangle_{\text{loc}}$ stands for local averaging in t).

Thus, solutions of (1), (2) oscillate at least as (E/δ) in a positive degree. We suppose that for (1), (2) (and other Hamiltonian PDE's) the quantity E/δ plays a role similar to the role of the Reynolds number for the equations of hydrodynamics [MY, Chapter 1].

The theorem describes behavior of solutions of the problem (1), (2) for $0 \leqq T_{\text{pull}}$ (since $t', t'' \leqq T_{\text{pull}}$) and gives no information on long-time behavior of individual solutions. Based on the fast[2] growth of the norms $\|u(t)\|$ of solutions outside $\mathfrak{A}^{\delta}$ one could conjecture that the solutions grow fast (at least – grow indefinitely) as t grows. This guess fails since the problem (1), (2) with $n = 1$, $V \equiv 0$ is integrable – all its solutions are almost-periodic in time (and so are bounded). One could try

[1] $=C_1\delta^{-1}$ since $a + b/2 = 1$ – see in Part 3.

[2] In fact – super exponentially fast.

to save the conjecture imposing the additional restriction that the Eq. (1) must be "typical" (and so non integrable), but this also does not help since for $n = 1,2$ and for typical $V(x)$ the problem (1), (2) has abundance of time-quasiperiodic solutions (see [7] for $n = 1$ and [3] for $n = 2$). These solutions jointly are dense near zero and their Z-norms jointly are unbounded (cf. Proposition in the introduction of [8]). It is an interesting open problem if the problem (1), (2) with $n \geqq 2$ has unbounded (in Z) solutions.

We remark that even in the integrable case ($V = 0,\ n = 1$) the Eq. (1) with small δ is rather complicated. Limiting (as δ goes to zero) behaviour of its solutions is studied in [5]; detailed analysis of solutions with fixed $0 < \delta \ll 1$ was not done yet.[3]

A few words on the notations we use: by C, C_1 etc. we denote different positive constants, independent of the main parameters (like δ and E), by $\|\cdot\|_s$ denote the C^s-norms, by $|\cdot|_p$ – the L^p-norms. We write a function $u(t,x)$ as $u(t)$ when we treat it as a curve in a function space of functions of x.

1. The Equation and its Phase-Space

We study the nonlinear Schrödinger equation in the n-cube $K^n = \{x \in \mathbb{R}^n \mid 0 \leqq x_j \leqq \pi\}$, $n = 1,2,3$, under Dirichlet boundary conditions:

$$-i\dot{u} = \delta(-\Delta u + V(x)u) + |u|^2 u, \quad u = u(t,x),\ x \in K^n\,, \tag{1.1}$$

$$u|_{\partial K^n} = 0\,, \tag{1.2}$$

where $0 < \delta \leqq 1$ and V is a smooth real potential. It is convenient to extend $V(x)$ to an even 2π-periodic function $V(x)$, $x \in T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$,

$$V(x_1,\ldots,x_n) = V(x_1,\ldots,-x_j,\ldots,x_n), \quad j = 1,\ldots,n\,,$$

and extend solution $u(t,x)$ to an odd 2π-periodic function $u(t,x)$, $x \in T^n$,

$$u(x_1,\ldots,x_n) = -u(x_1,\ldots,-x_j,\ldots,x_n), \quad j = 1,\ldots,n\,, \tag{1.3}$$

thus recasting (1.2) as odd periodic boundary conditions (1.3).[4]

We suppose that the potential $V(x)$ as a function on the torus T^n is even and smooth,

$$V \in C^\infty(T^n, \mathbb{R})\,,$$

and denote by A the linear operator

$$A(u(x)) = -\Delta u + V(x)u\,.$$

Let us take any Banach space H of complex functions $u(x)$, $x \in T^n$, with the norm $\|\cdot\|$, which is a Banach algebra with involution (i.e., $\|\bar{u}\| = \|u\|$ and $\|uv\| \leqq C\|u\|\,\|v\|$). Suppose that H is embedded to some space $C^m = C^m(T^n, \mathbb{C})$, $m \geqq 3$:

$$H \subset C^m, \qquad \|u\|_m \leqq \|u\|\,. \tag{1.4}$$

[3] Investigation of finite-gap solutions of (1) ($V = 0,\ n = 1$) with small δ is interesting since there are good hopes that using the KAM-theorems [7] one can study solutions of (1) with nonzero potential V and large values of E/δ as perturbations of the finite-gap trajectories.

[4] Clearly $u(x)$ ($x \in T^n$) as in (1.3) meets (1.2) being restricted to K^n.

Suppose also that H is invariant for the flow of (1.1) (solution of (1.1) with initial data from H stays in H for all t). Then the subspace $Z \subset H$ formed by odd functions,

$$Z = \{u(x) \in H \mid u \text{ meets } (1.3)\},$$

also is invariant. We equip Z with the norm $\|\cdot\|$ and take it for the phase space of the problem (1.1), (1.3).

Example 1 ($n = 1$). Since the L_2-norm and the Hamiltonian $\mathscr{H}$ (see below) are integrals of motion for Eq. (1.1), then the Sobolev H^1-norm (in x) of a solution $u(t,x)$ can be estimated via the H^1-norm of the initial data $u(0,x)$ uniformly in t. Therefore, Eq. (1.1) is well-posed in the space $H^1(T^1;\mathbb{C})$. Simple induction shows that it is also well-posed in the spaces $H^l(T^1;\mathbb{C})$, $l \geqq 1$. Since $H^l \subset C^m$ if $l > m + \frac{1}{2}$, then we can take

$$H = H^l(T^1;\mathbb{C}), \quad l > m + \frac{1}{2}.$$

Example 2 ($n = 2, 3$, $V \equiv 0$). Now the equation also is well-posed in the Sobolev spaces H^l, $l \geqq 1$ – this is a rather nontrivial result of J. Bourgain [2] – and we can take $H = H^l(\mathbb{T}^n;\mathbb{C})$, where

$$l > \begin{cases} m+1, & n=2, \\ m+\frac{3}{2}, & n=3, \end{cases}$$

since these spaces are embedded to $C^m(T^n;\mathbb{C})$. For V nonzero see Part 5.4 below.

We supply the linear space Z with the skew-symmetric two-form, defining the skew-product of functions $u(x)$, $v(x) \in Z$ as

$$-\mathrm{Im} \int u\bar{v}\, dx \,/\, (2\pi)^n.$$

This two-form defines a constant-coefficient symplectic structure in the phase-space Z and (1.1) becomes the Hamiltonian equation with the Hamiltonian $\mathscr{H}$,

$$\mathscr{H}(u) = \int_{T^n} \left(\frac{\delta}{2} |\nabla u(x)|^2 + \frac{\delta}{2} V(x) |u(x)|^2 + \frac{1}{4} |u(x)|^4 \right) dx \Big/ (2\pi)^n.$$

In particular, $\mathscr{H}(u(t,\cdot)) = \text{const}$ for any solution u of (1.1) in Z (see e.g. in [7]).

Multiplying (1.1) by $\bar{u}(t,x)$, integrating over T^n and taking the imaginary part of the equality we get that

$$|u(t,\cdot)|_2 = \text{const}$$

for any solution u.

We also observe that since $H \subset C^m$, then by the Gagliardo–Nirenberg inequality

$$\|u\|_k \leqq C_k |u|_\infty^{\frac{m-k}{m}} \|u\|^{\frac{k}{m}}, \quad 0 \leqq k \leqq m \tag{1.5}$$

(see [1]; the special case of the inequality we use now goes back to Hadamard–Landau–Kolmogorov).

2. The Main Estimate for the Flow

In this part, we study solutions $u(t) \in Z$ of (1.1) such that $u(0) = u_0$, where

$$\|u_0\| = 1, \qquad |u_0|_\infty = \varrho \tag{2.1}$$

and

$$\varrho \geqq K\delta^{\mu} \quad \text{with} \quad \mu = \mu(m) = \frac{m(m-1)}{2m^2+3m-3}$$

(clearly $\varrho \leqq 1$). The function μ increases with $m \geqq 3$ and

$$\mu(3) = \frac{1}{4}, \qquad \mu(m) \nearrow \frac{1}{2} \quad \text{as } m \longrightarrow \infty\,. \tag{2.2}$$

We study the solutions for $0 \leqq t \leqq T$, where

$$T = K_1 \varrho^{-\frac{2m+1}{m}} \leqq K_1 K^{-\frac{2m+1}{m}} \delta^{-\nu}, \quad \nu = \frac{2m+1}{m}\mu\,. \tag{2.3}$$

The m-independent positive constants K, K_1 are such that

$$K \geqq 2\sqrt{C_+ K_1}, \quad K \geqq 2\,, \tag{2.4}$$

where C_+ is an absolute constant from the inequality (2.8) below. The values of K, K_1 will be specified later.

We shall show that each solution of (1.1), (2.1) doubles its Z-norm somewhere on the time-segment $[0, T]$. To prove the doubling we suppose that on the contrary

$$\|u(t)\| \leqq 2 \quad \text{for all } 0 \leqq t \leqq T\,, \tag{2.5}$$

and shall extract a contradiction from this assumption.

Below by C, C_1, etc., we denote different positive constants independent of δ, ϱ and K, K_1 (which can depend on m).

Elementary calculations show that for T as above we have

$$T \leqq K_2 \varrho^{2/m} \delta^{-1}, \qquad K_2 = K_1 K^{-\frac{2m+3}{m}} < \frac{1}{4C_+}\,. \tag{2.6}$$

Suppose that for some $T_1 \leqq T$ we have

$$|u(t)|_\infty \leqq 2\varrho \quad \text{if } 0 \leqq t \leqq T_1\,. \tag{2.7}$$

Then by (2.5) and (1.5),

$$|Au(t)|_\infty \leqq C\,\|u(t)\|_2 \leqq C_+\,|u(t)|_\infty^{\frac{m-2}{m}} \leqq 2C_+ \varrho^{\frac{m-2}{m}}\,. \tag{2.8}$$

Take any $x \in T^n$. Multiplying (1.1) by $\bar{u}(t,x)$ and taking the imaginary part we get

$$\left| \frac{1}{2}\frac{d}{dt}\,|u(t,x)|^2 \right| \leqq \delta |Au(t,x)|\;|u(t,x)| \leqq 2\delta C_+ \varrho^{2-\frac{2}{m}}\,. \tag{2.9}$$

In particular, $|u(t,x)|^2 \leqq \varrho^2 + 4t\delta C_+ \varrho^{2-\frac{2}{m}}$ and using (2.6) we see that

$$|u(t,x)|^2 \leqq \varrho^2(1 + 4K_2 C_+) < 2\varrho^2\,.$$

It means that for each solution which satisfies (2.1), (2.5) the estimate (2.7) holds with $T_1 = T$:

$$|u(t)|_\infty \leqq 2\varrho \quad \text{if } 0 \leqq t \leqq T\,. \tag{2.10}$$

Now let us denote

$$w(t,x) = |u(t,x)|^2 - |u_0(x)|^2\,.$$

Then $w(0,x) \equiv 0$ and by (2.9),

$$\left|\frac{d}{dt}\, w(t,x)\right| \leqq C\delta\varrho^{2-\frac{2}{m}} .$$

Therefore

$$|w(t)|_\infty \leqq C\, t\delta\varrho^{2-\frac{2}{m}} . \tag{2.11}$$

Since H is a Banach algebra, then by (2.5), $\|w(t)\| \leqq C$ for all $0 \leqq t \leqq T$. Now by (1.5) and (2.11),

$$\|w(t)\|_k \leqq C(t\delta\varrho^{2-\frac{2}{m}})^{\frac{m-k}{m}} . \tag{2.12}$$

We rewrite (1.1) as

$$\dot{u} - i(|u_0|^2 + w)u = i\delta Au .$$

Treating this equation as a nonautonomous linear ordinary differential equation for the complex function $t \mapsto u(t,x)$ with the right-hand side $i\delta Au$, we write its solution u as

$$u(t,x) = i\delta \int_0^t \exp\left(i(t-\tau)\,|u_0|^2 + i\int_\tau^t w(\theta,x)\,d\theta \right) Au(\tau,x)\,d\tau$$
$$+ \exp(it|u_0|^2)\exp\left(i\int_0^t w(\tau,x)\,d\tau \right) u_0(x) =: u_1 + u_2$$

(u_1 denotes the first summand and u_2 – the second). We wish to estimate $\|u(t)\|_1$ from below. To do so we shall estimate $\|u_1(t)\|_1$ from above and $\|u_2(t)\|_1$ from below.

Since both the functions $|u_0(x)|^2$ and $w(\theta,x)$ are real, then the norm of the exponential factor under the integral in u_1 equals one. Any first-order x-derivative of u_1 contains two terms (since we differentiate either Au or the exponent). By (1.5) and (2.10),

$$\|u(t)\|_k \leqq C\varrho^{\frac{m-k}{m}} \quad \text{for } 0 \leqq k \leqq m . \tag{2.13}$$

As $\|Au\|_k \leqq C_k\|u\|_{k+2}$, then from (2.13) with $k = 2, 3$ we have

$$\|u_1(t)\|_1 \leqq C\delta \int_0^t \left((t-\tau) \left(\|u_0\bar{u}_0\|_1 + \sup_{\theta \leqq t} \|w(\theta)\|_1 \right) \varrho^{\frac{m-2}{m}} \right) d\tau + C\delta\varrho^{\frac{m-3}{m}}\, t .$$

By (2.13) with $k = 1$ and (2.12), (2.5), $\|u_0\bar{u}_0\|_1 + \|w(\theta)\|_1 \leqq C\varrho^{2\frac{m-1}{m}}$. Therefore

$$\|u_1(t)\|_1 \leqq C\delta\varrho^{\frac{3m-4}{m}}\, t^2 + C\delta\varrho^{\frac{m-3}{m}}\, t . \tag{2.14}$$

Now we pass to the function $u_2(t,x)$ and write it as $u_2 = u_2^1 u_2^2 u_0$, where

$$u_2^1 = \exp(it|u_0|^2), \qquad u_2^2 = \exp i\int_0^t w(\tau,x)\,d\tau .$$

Clearly

$$\|u_2\|_1 \geqq \sup_x |(\nabla_x u_2^1)u_2^2 u_0| - \|u_2^2\|_1 |u_2^1|_\infty |u_0|_\infty - \|u_0\|_1\, |u_2^2|_\infty\, |u_2^1|_\infty . \tag{2.15}$$

We shall estimate the three terms in the r.h.s. As $|u_2^1| \equiv 1$, $|u_2^2| \equiv 1$, then we already know that

$$|u_2^1|_\infty = 1, \qquad |u_0|_\infty = \varrho, \qquad |u_2^2|_\infty = 1, \qquad \|u_0\|_1 \leqq C\varrho^{\frac{m-1}{m}} . \tag{2.16}$$

Since $|u_2^2| \equiv 1$, then

$$\sup_x |(\nabla_x u_2^1) u_2^2 u_0| = t \sup_x |u_0(x)\, \nabla_x |u_0|^2| = \frac{2}{3} t \sup_x |\nabla_x |u_0|^3| \geqq t C^{-1} \varrho^3 , \qquad (2.17)$$

where the last estimate follows from (2.1) since u_0 vanishes at ∂K^n.

It remains to estimate $\|u_2^2\|_1$. *Using (2.12) we get that*

$$\|u_2^2\|_1 \leqq t \sup_{0 \leqq \tau \leqq t} \|w(\tau)\|_1 \leqq Ct(t\delta\varrho^{2-\frac{2}{m}})^{\frac{m-1}{m}} . \qquad (2.18)$$

Now the estimates (2.14)–(2.18) jointly imply the estimate for $u(t,x)$ from below:

$$\|u(t)\|_1 \geqq C_1^{-1} t\varrho^3 - C_2 t^2 \delta \varrho^{\frac{3m-4}{m}} - C_3 t \delta \varrho^{\frac{m-3}{m}}$$

$$- C_4 t^{\frac{2m-1}{m}} \varrho^{1+2(\frac{m-1}{m})^2} \delta^{\frac{m-1}{m}} - C_5 \varrho^{\frac{m-1}{m}} . \qquad (2.19)$$

By (2.13) we have

$$\|u(t)\|_1 \leqq C_* \varrho^{\frac{m-1}{m}} \quad \text{for } 0 \leqq t \leqq T . \qquad (2.20)$$

On the other hand, by (2.19),

$$\|u(t_*)\|_1 \geqq 2C_* \varrho^{\frac{m-1}{m}} ,$$

if t_* meets the following system of inequalities:

$$\begin{cases} C_1^{-1} t\varrho^3 \geqq (5C_* + C_5)\varrho^{\frac{m-1}{m}} , \\ C_2 t^2 \delta \varrho^{\frac{3m-4}{m}} < C_* \varrho^{\frac{m-1}{m}} , \\ C_3 t \delta \varrho^{\frac{m-3}{m}} < C_* \varrho^{\frac{m-1}{m}} , \\ C_4 t^{\frac{2m-1}{m}} \varrho^{1+2(\frac{m-1}{m})^2} \delta^{\frac{m-1}{m}} < C_* \varrho^{\frac{m-1}{m}} , \end{cases}$$

or

$$\begin{cases} t \geqq C_1' \varrho^{-2-\frac{1}{m}} , \\ t < C_2' \delta^{-\frac{1}{2}} \varrho^{-\frac{2m-3}{2m}} , \\ t < C_3' \delta^{-1} \varrho^{\frac{2}{m}} , \\ t < C_4' \delta^{-\frac{m-1}{2m-1}} \varrho^{-\frac{2m^2-3m+2}{m(2m-1)}} . \end{cases} \qquad (2.21)$$

The first two inequalities in (2.21) are consistent if

$$\varrho > C_1'' \, \delta^{\frac{m}{2m+5}} ;$$

the first and the third are if

$$\varrho > C_2'' \, \delta^{\frac{m}{2m+3}} ;$$

and the first and the fourth are if

$$\varrho > C_3'' \, \delta^{\mu}, \qquad \mu = \mu(m) \quad \text{as in (2.2)} .$$

Let us denote $\tilde{C} = \max\{C_1'', C_2'', C_3''\}$. Then the system (2.21) is consistent if

$$\varrho > \tilde{C} \delta^{\mu} .$$

We choose in (2.3) $K_1 = C_1'$ and take for K any number such that

$$K > \tilde{C}, \qquad K \geqq 2, \qquad K \geqq 2\sqrt{C_+ K_1} .$$

With this choice of K, K_1 the assumptions (2.4) are met and $t = T := K_1 \varrho^{-2-1/m}$ satisfies (2.21). Thus, $\|u(T)\|_1 \geqq 2C_* \varrho^{(m-1)/m}$ in a contradiction with (2.20).

Since the assumption (2.5) led to a contradiction, then the solution $u(t)$ doubles its norm for some $t \leqq T = K_1 \varrho^{-(2m+1)/m}$ with K_1 as above and we get

Theorem 1. *There exist constants K, K_1 such that if $u(t)$ is a solution of* (1.1) *and $u(0) = u_0$ satisfies* (2.1) *with some*

$$\varrho \geqq K\, \delta^{\mu}, \qquad \mu = \mu(m) = \frac{m(m-1)}{2m^2 + 3m - 3},$$

then there exists t_1,

$$t_1 \leqq K_1 \varrho^{-2-1/m} \leqq C \delta^{-\nu}, \qquad \nu = \frac{(2m+1)(m-1)}{2m^2+3m-3} < 1 ,$$

such that $\|u(t_1)\| \geqq 2$.

Remark 1. The statement and its proof remain true if we replace $\mu(m)$ and $\nu(m)$ by any $\tilde{\mu}, \tilde{\nu}$ such that $0 < \tilde{\mu} \leqq \mu(m)$, $\tilde{\nu} = \frac{2m+1}{m}\, \tilde{\mu}$.

3. Rescaled Estimates and the Essential Part of the Phase-Space

We define a subset $\mathfrak{A} \subset Z$ which we call "essential part of the phase-space Z" as follows:

$$\mathfrak{A} = \{u(x) \in Z \,|\, |u|_\infty < K \delta^{\mu} \|u\|^{1-2\mu}\} ,$$

with μ and K as in Theorem 1. Since $|u|_\infty \leqq \|u\| = (\|u\|/\sqrt{\delta})^{2\mu}\, \delta^{\mu} \|u\|^{1-2\mu}$, then $\mathfrak{A}$ contains the ball

$$\{u \in Z \,|\, \|u\| < \sqrt{\delta}\, K^{1/(2\mu)}\} \tag{3.1}$$

(and $\mathfrak{A}$ does not contain some points of the norm $C\sqrt{\delta}$, where $C > K^{1/(2\mu)}$ is δ-independent).

Now let us consider a solution of (1.1) with any initial condition outside $\mathfrak{A}$:

$$u(0,x) = u_0(x) \notin \mathfrak{A} .$$

We denote $r = \|u_0\|$, $v = u/r$. Then

$$v(0,x) = v_0(x) = u_0/r, \qquad \|v_0\| = 1 , \tag{3.2}$$

and

$$-i\dot{v} = \delta A v + r^2 |v|^2 v .$$

So if we stretch the time t as

$$t = r^{-2} \tau ,$$

then we get for v the rescaled equation:

$$-i v_\tau' = \delta_1 A v + |v|^2 v, \quad \delta_1 = \delta / r^2 . \tag{3.3}$$

Since $u_0 \notin \mathfrak{A}$, then by (3.1) $r \geqq \sqrt{\delta} K^{1/(2\mu)} > \sqrt{\delta}$. So

$$\delta_1 \leqq 1 \quad \text{and} \quad \|v_0\| = 1, \qquad |v_0|_\infty = |u_0|_\infty / r \geqq K(\delta/r^2)^\mu = K\delta_1^\mu .$$

Now application of Theorem 1 with $\delta = \delta_1$ to the problem (3.2), (3.3) implies existence of $\tau_1 \leqq C\delta_1^{-\nu}$, corresponding to $t_1 = r^{-2}\tau_1 \leqq C r^{-2}(\delta/r^2)^{-\nu}$, such that

$$r_1 := \|u(t_1)\| \geqq 2r = 2\|u_0\| .$$

If still $u(t_1) \notin \mathfrak{A}$, then we can iterate the procedure to find $t_2 \leqq C r_1^{-2}(\delta/r_1^2)^{-\nu}$ such that

$$r_2 := \|u(t_1 + t_2)\| \geqq 2^2 \|u_0\| .$$

We can iterate further, provided that $u(t) \notin \mathfrak{A}$, to find $t_1, \dots, t_M$ such that

$$r_M := \|u(t_1 + \dots + t_M)\| \geqq 2^M \|u_0\| \tag{3.4}$$

and

$$t_M \leqq C r_{M-1}^{-2} (\delta / r_{M-1}^2)^{-\nu} .$$

Denote $u_M = u(t_1 + \dots + t_M)$. Since the flow of (1.1) preserves the Hamiltonian $\mathscr{H}$, then

$$|u_M|_4 \leqq 4\, \mathscr{H}(u_M) = 4\mathscr{H}(u_0) .$$

By Sobolev's theorem and the Gagliardo–Nirenberg inequality (see [1]),

$$|u_M|_\infty \leqq C\|u_M\|_{1,4} \leqq C_1 \|u_M\|^{\frac{1}{m}} |u_M|_4^{\frac{m-1}{m}} \leqq C r_M^{\frac{1}{m}} , \tag{3.5}$$

where $\|\cdot\|_{1,4}$ stands for the norm in the Sobolev space $W^{1,4}(T^n; \mathbb{C})$ (we recall that $n \leqq 3$). If $u_M \notin \mathfrak{A}$, then we have $|u_M|_\infty \geqq K\delta^\mu r_M^{1-2\mu}$. Using (3.5) we find that

$$C \geqq K \delta^\mu r_M^{1-2\mu-1/m} .$$

Suppose for a moment that $n = 2$ or 3. Then $1 - 2\mu > 1/m$ and we see that the norms r_M are bounded by an M-independent constant. Now (3.4) implies that $M \leqq M(u_0)$. It means that after a finite number of steps the solution $u(t)$ hits to $\mathfrak{A}$. The total time the solution $u(t)$ spends outside $\mathfrak{A}$ is estimated:

$$\begin{aligned} t_1 + \dots + t_M &\leqq C\delta^{-\nu}(r^{2\nu-2} + r_1^{2\nu-2} + \cdots) \\ &\leqq C\delta^{-\nu} r^{2\nu-2}(1 + 4^{\nu-1} + 4^{2(\nu-1)} + \cdots) . \end{aligned}$$

Since $\nu = 1 - \frac{5}{2m} + O(m^{-2}) < 1$, then

$$t_1 + \dots + t_M \leqq mC\delta^{-\nu} r^{2\nu-2} . \tag{3.6}$$

If $n = 1$, then $|u_M| \leqq C\|u_M\|_{2/3,2} \leqq C_1 \|u_M\|^{2/(3m)} |u_M|_2^{(3m-2)/(3m)} \leqq C r_M^{2/(3m)}$ and (3.6) also follows.

We have proved the following result:

Theorem 2. *If $u(t,x)$ is a solution of (1.1) with $u(0,x) = u_0(x) \notin \mathfrak{A}$, $\|u_0\| = r$, then*

1) *there exists $t' \leqq C_1 r^{2\nu-2}\delta^{-\nu}$ such that $\|u(t')\| \geqq 2r$,*
2) *there exists $t'' \leqq C_2 r^{2\nu-2}\delta^{-\nu}$ such that $u(t'') \in \mathfrak{A}$.*

Here $\nu = \nu(m)$ as in Theorem 1.

Remark 2. The arguments we used to prove Theorems 1, 2 are applicable as well to negative times $t < 0$. Thus, we can find negative $-t_1', -t_1''$ such that t_1', t_1'' meet the same estimates as t', t'' and $\|u(-t_1')\| \geqq 2r$, $u(-t_1'') \in \mathfrak{A}$.

In particular, $\mathfrak{A}$ is a recursion subset of the phase-space Z – each trajectory of (1.1) in Z visits $\mathfrak{A}$ at arbitrarily large values of times.

In addition, since by (3.1) $r \geqq \sqrt{\delta}K^{1/(2\mu)}$ for each $u_0 \notin \mathfrak{A}$, then $t'' \leqq C\delta^{-1}$ and by the time

$$T_{\text{pull}} = C\,\delta^{-1}$$

the flow $\{S^t\}$ will pull the whole space Z through its essential part $\mathfrak{A}$.

4. Oscillations of Solutions

In this part we discuss properties of solutions $u(t)$ with large values of the quantity R,

$$R = \frac{E}{\delta} \quad \text{where } \mathscr{H}(u(t)) \equiv E^2 .$$

(We denote the ratio by R to note its similarity with the Reynolds number of classical hydrodynamics [10].) To do so, it is convenient to introduce the "fast time" $\tau = \sqrt{\delta}\,t$, so

$$\frac{\partial}{\partial t} = \sqrt{\delta}\,\frac{\partial}{\partial \tau} ,$$

and to write (1.1) as

$$-i\sqrt{\delta}\,u_\tau' = -\delta\,\Delta u + |u|^2 u . \tag{4.1}$$

Our goal is to study the function

$$\omega(\tau) = \omega_m(\tau) = \frac{\|u(\tau)\|}{|u(\tau)|_\infty} ,$$

which we propose to use as a measure of the oscillating solutions of (4.1). (The subindex m recalls that $H \subset C^m$ – we treat $\|\cdot\|$ as an "almost C^m-norm.") We shall show that Theorem 2 implies estimates for averaged characteristics of ω_m in terms of R.

Consider any solution $u(\tau) \in Z$, $0 \leqq \tau \leqq T$, of (4.1) such that $\mathscr{H}(u(\tau)) \equiv E^2$, where E is bounded from below by a positive δ-independent constant, $E \geqq C^{-1}$. Let us denote $\|u(\tau)\| = r(\tau)$. Since

$$C^{-1} \leqq E^2 = \frac{\delta}{2}\,|\nabla u|_2^2 + \frac{1}{4}\,|u|_4^4 \leqq \frac{\delta n}{2}\,r^2 + \frac{1}{4}\,r^4 ,$$

then

$$r(\tau)^2 \geqq E/C_1 . \tag{4.2}$$

Let us denote

$$\tilde{\Delta}(\tau) = r(\tau)^{2(\nu-1)}\,\delta^{-\nu+1/2} .$$

Then by (4.2),

$$\tilde{\Delta} \leqq C_2\Delta, \qquad \Delta = E^{\nu-1}\delta^{-\nu+1/2} ,$$

and by Theorem 2 if $u(\tau_0) \notin \mathfrak{A}$, then

$$\|u(\tau_0 + \tau_1)\| = 2\,\|u(\tau_0)\| \quad \text{for some } \tau_1 \leqq C_1\tilde{\Delta} \leqq C'\Delta , \tag{4.3}$$

$$u(\tau_0 + \tau_2) \in \mathfrak{A} \quad \text{for some } \tau_2 \leqq C_2\tilde{\Delta} \leq C\Delta . \tag{4.4}$$

Since for $u(\tau) \in \mathfrak{A}$ we have

$$\omega_m(\tau) \geqq K^{-1}(r^2/\delta)^\mu \geqq C_1^{-1}R^\mu \tag{4.5}$$

(we use (4.2)), then by (4.4) we get

Proposition 1. *For each solution as above we have*

$$\sup_{\tau_1 \leqq \tau \leqq \tau_1 + C\Delta} \omega_m(\tau) \geqq C_1^{-1}R^{\mu(m)} \tag{4.6}$$

if $0 \leqq \tau_1 \leqq T - C\Delta$.

We recall that $\mu = \mu(m) \geqq 1/4$ tends to $1/2$ as $m \to \infty$.

Next we estimate an average of the function

$$\widetilde{\Omega}_m = \omega_m + C\Delta\ |\omega_m'|$$

and start with

Lemma. *If $u(\tau_0) \in \overline{\mathfrak{A}}$ is such that $u(\tau_0 + \varepsilon) \notin \mathfrak{A}$ for all sufficiently small $\varepsilon > 0$, then there exists $\tau_* > \tau_0$, $\tau_* \leqq \tau_0 + C\Delta$, such that $u(\tau_*) \in \mathfrak{A}$ and*

$$\frac{1}{\tau_* - \tau_0} \int_{\tau_0}^{t_*} \widetilde{\Omega}_m(\tau)\, d\tau \geqq \omega_m(\tau_0)\,. \tag{4.7}$$

Proof. Since the set $\mathfrak{A}$ is open, then $u(\tau_0) \notin \mathfrak{A}$ and we can find $\tau' = \tau_1$ as in (4.3). We can suppose that $\|u(\tau)\| < 2\|u(\tau_0)\|$ for $\tau_0 \leqq \tau < \tau'$. Then by the estimate (2.10) from the proof of Theorem 1, $|u(\tau')|_\infty \leqq 2|u(\tau_0)|_\infty$. Therefore $\omega(\tau') \geqq \omega(\tau_0)$. If $\tau' \in \mathfrak{A}$ then at the moment τ'' of the next doubling of the H-norm we also have $\omega(\tau'') \geqq \omega(\tau') \geqq \omega(\tau_0)$. Finally, due to the proof of the second statement of Theorem 2, we shall find a point $\tau_* = \tau^{(N)} \in \mathfrak{A}$ such that $\omega(\tau_*) \geqq \omega(\tau_0)$ and $\tau_* \in [\tau_0, \tau_0 + C\Delta]$.

Let us denote

$$\theta = \sup\{\tau \in [\tau_0, \tau_*] \,|\, \omega(\tau) \leqq \omega(\tau_0)\}\,.$$

Then $\omega(\theta) = \omega(\tau_0)$ and $\omega \geqq \omega(\tau_0)$ everywhere in $[\theta, \tau_*]$. Therefore denoting $f(\tau) = \omega(\tau) - \omega(\tau_0)$ we get:

$$\begin{aligned}\frac{1}{\theta - \tau_0} \int_{\tau_0}^{\theta} \widetilde{\Omega}_m(\tau)\, d\tau &= \omega(\tau_0) + \frac{1}{\theta - \tau_0} \int_{\tau_0}^{\theta} (f(\tau) + C\Delta\, |f'(\tau)|)\, d\tau \\ &\geqq \omega(\tau_0) + \int_{\tau_0}^{\theta} \left(-|f'(\tau)| + \frac{C\Delta}{\theta - \tau_0}\, |f'(\tau)|\right) d\tau \geqq \omega(\tau_0)\end{aligned}$$

(to prove the last inequality we use that $f(\tau_0) = f(\theta) = 0$). Since $\widetilde{\Omega}_m \geqq \omega_m \geqq \omega(\tau_0)$ in $[\theta, \tau_*]$ then (4.7) follows. □

Proposition 2. *For $\tau_2 > \tau_1 + 3C\Delta$ we have*

$$\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \widetilde{\Omega}_m(\tau)\, d\tau \geqq \frac{1}{3}\, C_1^{-1} R^\mu\,. \tag{4.8}$$

The constants C, C_1 are the same as in Proposition 1.

Proof. Let $a_1 \geqq \tau_1$ be the first moment when the trajectory enters $\overline{\mathfrak{A}}$ and $b_1 \geqq a_1$ be the moment of its first exit from $\overline{\mathfrak{A}}$. Let $a_2 = \tau_*(b_1) > b_1$, where τ_* is as in the lemma (with $\tau_0 = b_1$). Let $b_2 \geqq a_2$ be the moment of the next exit from $\overline{\mathfrak{A}}$, etc.

We consider only the worst situation when $a_1 > \tau_1$ and $b_r < \tau_2 < a_{r+1}$ for some r. Since at each segment $[a_j, b_j]$ we have $\widetilde{\Omega}_m \geqq \omega_m \geqq C_1^{-1} R^\mu$ (see (4.5)), then averaging of $\widetilde{\Omega}_m$ along $[a_j, b_j]$ is $\geqq C_1^{-1} R^\mu$. Averaging of $\widetilde{\Omega}_m$ along any segment $[b_j, a_{j+1}]$ by the lemma also is $\geqq C_1^{-1} R^\mu$. Therefore averaging of $\widetilde{\Omega}_m$ along $[a_1, b_r]$ is $\geqq C_1^{-1} R^\mu$.

By (4.4), $|a_1 - \tau_1| + |\tau_2 - b_r| \leqq 2C\Delta$. So $|\tau_2 - \tau_1| \leqq 3|b_r - a_1|$ and (4.8) follows. □

The number Δ contains the factor $\delta^{-\nu+1/2}$, where $\nu > 1/2$ and the factor is large for small δ. We can drop it, simultaneously changing the r.h.s. of (4.6), (4.8). Indeed, by Remark 1 we can replace μ and ν by $\widetilde{\mu} \leqq \mu$, $\widetilde{\nu} = \frac{2m+1}{m}\widetilde{\mu}$. Since $\mu \geqq 1/4$ (see (2.2)), then we can take $\mu = m/(4m+2) < 1/4$. Then $\widetilde{\nu} = 1/2$ and we get versions of the Propositions 1 and 2:

Proposition 1′. *For* $\tau_2 \geqq \tau_1 + C/\sqrt{E}$ *we have*

$$\sup_{\tau_1 \leqq \tau \leqq \tau_2} \omega_m(\tau) \geqq C^{-1} R^{m/(4m+2)} . \tag{4.6′}$$

Proposition 2′. *If* $\tau_2 \geqq \tau_1 + 3C/\sqrt{E}$, *then*

$$\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \left(\omega_m + \frac{C}{\sqrt{E}} |\omega_m'| \right) d\tau \geqq \frac{1}{3} C^{-1} R^{\frac{m}{4m+2}} . \tag{4.8′}$$

We note that since $\mu(m)$ tends to 1/2 as m grows, then for large m the exponents of R in the r.h.s.'s of (4.6′), (4.8′) are almost twice as small as in (4.6), (4.8).

If we treat (4.1) as a partial differential equation rather than a dynamical system in Z, then it is more natural to choose as a measure of space-time oscillations of a solution u not $\widetilde{\Omega}_m$ but the function Ω_m,

$$\Omega_m(\tau) = \frac{\|u(\tau)\| + C'\Delta\, \|u'(\tau)\|}{|u(\tau)|_\infty}$$

with C' as in (4.3). Observe that

$$\Omega_m(\tau) \geqq \frac{\|u(\tau)\| + C'\Delta\, |\frac{\partial}{\partial\tau} \|u(\tau)\|\, |}{|u(\tau)|_\infty} . \tag{4.9}$$

We think that the quantity Ω_m is a natural measure for solution's oscillations and name the related statement "theorem":

Theorem 3. *If* $\tau_2 \geqq \tau_1 + 3C'\Delta$, *then*

$$\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Omega_m(\tau)\, d\tau \geqq \frac{1}{6} C_1^{-1} R^\mu ; \tag{4.10}$$

if $\tau_2 \geqq \tau_1 + 3\, C/\sqrt{E}$, *then*

$$\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \frac{\|u(\tau)\| + C E^{-1/2}\, \|u'(\tau)\|}{|u(\tau)|_\infty}\, d\tau \geqq \frac{1}{6} C_2^{-1} R^{\frac{m}{4m+2}} . \tag{4.11}$$

Proof of (4.10) essentially follows the proof of Proposition 2. Let $b_1 \geqq a_1 \geqq \tau_1$ be the numbers from that proof and $b_1^2 > b_1$ be the first moment of doubling the solution's norm. If $u(b_1^2) \notin \mathfrak{A}$, let b_1^3 be the moment of the second doubling, etc. Finally, we shall find $b_1^j \in \mathfrak{A}$, $j \geqq 2$, and by (4.3),

$$|b_1^p - b_1^{p-1}| \leqq C'\Delta \quad \forall p \leqq j\,. \tag{4.12}$$

Denote $a_2 = b_1^j$, denote by b_2 the moment of the next exit from $\overline{\mathfrak{A}}$, construct $b_2, b_2^2, \ldots, a_3$ as above, etc.

Clearly, the average of Ω_m along each segment $[a_r, b_r]$ (or along $[a_r, \tau_2]$ if $a_r < \tau_2 < b_r$) is $\geqq C^{-1}R^\mu$. To study a segment $[b_r^p, b_r^{p+1}]$ let us take any sequence $b_r, b_r^2, \ldots, b_r^j = a_{r+1}$ and denote for short

$$x_1 = b_r, \qquad x_2 = b_r^2, \ldots, x_j = b_r^j = a_{r+1}\,.$$

As in the proof of the lemma,

$$\omega(x_1) \leqq \omega(x_2) \leqq \cdots \leqq \omega(x_j)\,. \tag{4.13}$$

Besides for each $\tau \in [x_p, x_{p+1}]$ we have $|u(\tau)|_\infty \leqq 2\,|u(x_p)|_\infty$. By this inequality and (4.9), (4.12) for each $p < j$ we have:

$$\begin{aligned}
&\frac{1}{x_{p+1} - x_p} \int_{x_p}^{x_{p+1}} \Omega_m(\tau)\,d\tau \\
&\geqq \frac{1}{2(x_{p+1} - x_p)\,|\,u(x_p)\,|_\infty} \int_{x_p}^{x_{p+1}} \left(\|u(\tau)\| + C'\Delta \left| \frac{\partial}{\partial\tau}\|u(\tau)\| \right| \right) d\tau \\
&\geqq \frac{\|u(x_p)\|}{2\,|u(x_p)|_\infty} + \frac{1}{2\,|u(x_p)|_\infty} \int_{x_p}^{x_{p+1}} \left(-\left|\frac{\partial}{\partial\tau}\|u\|\right| + \frac{C'\Delta}{x_{p+1} - x_p}\left|\frac{\partial}{\partial\tau}\|u\|\right| \right) d\tau \\
&\geqq \frac{\|u(x_p)\|}{2\,|u(x_p)|_\infty} = \frac{1}{2}\omega(x_p)\,,
\end{aligned}$$

where we have used the elementary inequality $A^{-1}\int_0^A f(t)\,dt \geqq f(0) - \int_0^A |f'(t)|dt$, valid for each C^1-smooth f.

Now (4.13) implies that

$$\frac{1}{x_{p+1} - x_p} \int_{x_p}^{x_{p+1}} \Omega_m(\tau)\,d\tau \geqq \frac{1}{2}\,C_1^{-1}R^\mu\,.$$

So we estimated the averages along all the segments with the possible exception of the segments $[\tau_1, a_1]$ (and $[b_r^j, \tau_2]$ if $\tau_2 \in [b_r^j, b_r^{j+1}[$). Since both the segments are shorter than $C'\Delta$, the result follows.

The estimate (4.11) results from (4.10) in the same way as Proposition 2′ from Proposition 2. □

5. Some Generalizations

5.1. Periodic Boundary Conditions. The Eq. (1.1) can be studied under periodic boundary conditions – i.e., in the whole space H of the periodic functions

$u(x), x \in T^n$ (see (1.4)). In this case for $c > 0$ we should consider the subsets $\mathscr{L}_c \subset H$,

$$\mathscr{L}_c = \{u(x) \in H \mid |u(x)| = c \ \forall x \in T^n\} .$$

The sets $\mathscr{L}_c$ are smooth Lagrangian submanifolds of the phase-space H which is given the symplectic structure defined in Part 1. Next we define the "frame" $\mathscr{L}$ as the union of all $\mathscr{L}_c$,

$$\mathscr{L} = \cup_c \mathscr{L}_c ,$$

and define essential part $\mathfrak{A}_p$ of the phase-space H as

$$\mathfrak{A}_p = \{u(x) \mid \operatorname{dist}_\infty(u, \mathscr{L}) \leqq K\delta^\mu \|u\|^{1-2\mu}\} .$$

Theorem 2 remains true for the periodic boundary value problem for Eq. (1.1) if we replace $\mathfrak{A}$ by $\mathfrak{A}_p$, cf. [8] where a weaker form of this result was obtained for nonlinear wave equations.

5.2. Polynomial Nonlinearities. Theorems 1 and 2 and propositions from Part 4 remain true (after correcting the constants K, K_1 and the exponents μ and ν) if we replace the nonlinearity $|u|^2 u$ by any $|u|^{2p}u$ (p is a positive integer) or by a polynomial

$$(a_1|u|^2 + a_2|u|^4 + \cdots + a_p|u|^{2p})u ,$$

where $a_j \geqq 0$ for all j and $a_p > 0$. The proofs go without any changes.

5.3. Focusing Equation. Consider Eq. (1.1) with $V = 0$ and with the changed sign in front of the Laplacian:

$$-i\dot{u} = \delta\Delta u + |u|^2 u . \tag{5.1}$$

If the phase-space Z is a Sobolev space as in Examples 1 and 2 from Part 1, then solutions of (5.1) in Z are well-defined locally in time (see e.g., [2]) but can blow up in finite time if $n > 1$ (see [6]). So (5.1) defines in Z a local flow. Still statements of Theorems 1 and 2 remain true with the natural refinement that the solutions of (5.1) exist until the correspondent times t_1 and t', t''. The only difference comes during proving the second statement of Theorem 2 since in the focusing case (5.1) we cannot use the Hamiltonian $\mathscr{H}$ to majorize the L^4-norm of the solution. Instead we can use in the estimate (3.5) the L^2-norm (which still is an integral of motion) under the additional restriction:

$$m \geqq 4 \quad \text{if } n = 3 .$$

5.4. Local Flow. Suppose that a phase space $Z \subset C^m$ for the problem (1.1), (1.2) is chosen in such a way that solutions $u(t) \in Z$ are defined only locally in time (or we cannot prove that they are defined globally, as in the case when $n = 2, 3$, $V(x) \not\equiv 0$ and Z is the Sobolev space as in Example 2). Then – as in the previous item – the statements of Theorems 1, 2 remain true with the same obvious refinement that the solutions do exist till the times t_1, t', t''.

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