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Quantum Affine Algebras and Deformations of the Virasoro and $\mathcal{W}$-Algebras

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Abstract: Using the Wakimoto realization of quantum affine algebras we define new Poisson algebras, which are q-deformations of the classical $\mathcal{W}$-algebras. We also define their free field realizations, i.e. homomorphisms into some Heisenberg–Poisson algebras. The formulas for these homomorphisms coincide with formulas for spectra of transfer-matrices in the corresponding quantum integrable models derived by the Bethe–Ansatz method.

1. Introduction

1.1. In this paper we generalize some results concerning affine Kac–Moody algebras at the critical level to the corresponding quantized universal enveloping algebras. Here is a short description of these results for the affine algebras.

(i) Let $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ be a completion of the universal enveloping algebra of an affine algebra $\widehat{\mathfrak{g}}$ at the critical level $-h^\vee$ (the precise definition is given in Sect. 2). This algebra possesses a large center $Z(\widehat{\mathfrak{g}})$, which has a natural Poisson structure. B. Feigin and E. Frenkel have shown that $Z(\widehat{\mathfrak{g}})$ is isomorphic to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g}^L)$ associated to the simple Lie algebra $\mathfrak{g}^L$, which is Langlands dual to $\mathfrak{g}$ [1].

(ii) The $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g}^L)$ consists of functionals on a certain Poisson manifold $\mathscr{C}(\mathfrak{g}^L)$ obtained by the Drinfeld–Sokolov hamiltonian reduction [2] from a hyperplane in the dual space to the affine algebra $\widehat{\mathfrak{g}^L}$. Elements of $\mathscr{C}(\mathfrak{g}^L)$, called $\mathfrak{g}^L$-opers in [3], can be considered as connections on a certain G^L-bundle over the circle with some extra structure. To a $\mathfrak{g}^L$-oper one can attach a $\widehat{\mathfrak{g}}$-module, on which the center acts according to the corresponding character. These $\widehat{\mathfrak{g}}$-modules can be considered as analogues of admissible representations of a simple group over a local non-archimedean field. They can be used in carrying out the geometric Langlands correspondence proposed by A. Beilinson and V. Drinfeld [3].

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(iii) The Wakimoto realization of $\widehat{\mathfrak{g}}$ [4, 5, 1, 6] provides a map from $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ to the tensor product of a certain Heisenberg algebra and a commutative algebra $\mathscr{H}(\mathfrak{g})$. The restriction of this map to $Z(\widehat{\mathfrak{g}})$ gives us a homomorphism $Z(\widehat{\mathfrak{g}}) \to \mathscr{H}(\mathfrak{g})$, which is an analogue of the Harish–Chandra homomorphism. The corresponding map $\mathscr{W}(\mathfrak{g}^L) \to \mathscr{H}(\mathfrak{g})$ is nothing but the Miura transformation, which has been defined for an arbitrary $\mathfrak{g}$ by V. Drinfeld and V. Sokolov [2].

(iv) The algebra $\mathscr{H}(\mathfrak{g})$ consists of functionals on a hyperplane $\mathscr{F}$ in the dual space to the homogeneous Heisenberg subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$. The algebra $\mathscr{H}(\mathfrak{g})$ is the classical limit of a completion of $U(\widehat{\mathfrak{h}})$, and hence it is a Heisenberg–Poisson algebra. The Miura transformation $\mathscr{W}(\mathfrak{g}^L) \to \mathscr{H}(\mathfrak{g})$ is a homomorphism of Poisson algebras.

For example, the center of $\widetilde{U}(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ is generated by the Sugawara operators, and $\mathscr{C}(\mathfrak{sl}_2)$ is isomorphic to a hyperplane $\mathscr{L}$ in the dual space to the Virasoro algebra. Thus, the center of $\widetilde{U}(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ is isomorphic to the Poisson algebra $\mathscr{W}(\mathfrak{sl}_2)$ of functionals on $\mathscr{L}$. The Poisson structure on $\mathscr{L}$ is often called the second Gelfand–Dickey structure. We call $\mathscr{W}(\mathfrak{sl}_2)$ the classical Virasoro algebra.

Elements of $\mathscr{L}$ can be considered as projective connections on the circle, i.e. differential operators of the form $\partial_t^2 - q(t)$; these are the $\mathfrak{sl}_2$-opers. On the other hand, elements of $\mathscr{F}$ can be considered as connections on a rank one bundle over the circle, i.e. differential operators of the form $\partial_t - \frac{1}{2}\chi(t)$. The Miura transformation sends a connection $\partial_t - \frac{1}{2}\chi(t)$ to the projective connection

$$\partial_t^2 - q(t) = \left(\partial_t - \frac{1}{2}\chi(t)\right)\left(\partial_t + \frac{1}{2}\chi(t)\right) . \tag{1.1}$$

This gives us a homomorphism of Poisson algebras $\mathscr{W}(\mathfrak{sl}_2) \to \mathscr{H}(\mathfrak{sl}_2)$.

In [7, 8] these results were used to give a new interpretation of the Bethe ansatz in the Gaudin models of statistical mechanics. This allowed us to gain new insights into completeness of Bethe ansatz, and to relate Bethe ansatz to the geometric Langlands correspondence.

1.2. There are many indications that these results can be generalized to the center of a completion $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ at the critical level. An explicit construction of central elements of a quantum affine algebra at the critical level has been given by N. Reshetikhin and M. Semenov–Tian–Shansky [9]. Later, J. Ding and P. Etingof [10] showed that those elements generate all singular vectors of imaginary weight in Verma modules over $U_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$. This makes us believe that the center of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ is generated, in an appropriate sense, by the elements constructed in [9].

The center $Z_h(\widehat{\mathfrak{g}})$ of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ possesses a natural Poisson structure, which is a q-deformation of the Poisson structure on $Z(\widehat{\mathfrak{g}})$. A natural question is to describe $Z_h(\widehat{\mathfrak{g}})$ and its spectrum.

In this paper we do this explicitly for $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ by using its Wakimoto realization. Our results for $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ can be summarized as follows.

(i) The center $Z_h(\widehat{\mathfrak{sl}}_2)$ of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ contains the Fourier coefficients of a power series $\ell(z)$ given in [9] in terms of the Reshetikhin–Semenov–Tian–Shansky (RS) realization of $U_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$. We rewrite $\ell(z)$ in terms of the Drinfeld realization

[11], using the explicit isomorphism between the two realizations established by J. Ding and I. Frenkel [12]. This gives us a q-analogue of the Sugawara formula.

(ii) Wakimoto realizations of $U_q(\widehat{\mathfrak{sl}}_2)$ have been given in [13] in terms of the Drinfeld realization; we use the presentation due to H. Awata, S. Odake, and J. Shiraishi [14]. It gives us a homomorphism from $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ to the tensor product of a certain Heisenberg algebra and a Heisenberg–Poisson algebra $\mathcal{H}_h(\mathfrak{sl}_2)$. Its restriction to $Z_h(\widehat{\mathfrak{sl}}_2)$ is a homomorphism of Poisson algebras $Z_h(\widehat{\mathfrak{sl}}_2) \rightarrow \mathcal{H}_h(\mathfrak{sl}_2)$, which is a q-deformation of the Miura transformation.

(iii) We find the image of $\ell(z)$ in $\mathcal{H}_h(\mathfrak{sl}_2)$:

$$\ell(z) \rightarrow s(z) = \Lambda(zq) + \Lambda(zq^{-1})^{-1}\,, \tag{1.2}$$

where $\Lambda(z)$ is a generating function of elements of $\mathcal{H}_h(\mathfrak{sl}_2)$. Using the Poisson structure on $\mathcal{H}_h(\mathfrak{sl}_2)$ we compute the Poisson structure on $Z_h(\widehat{\mathfrak{sl}}_2)$:

$$\begin{aligned}\{\ell(z), \ell(w)\} = 2h(q-q^{-1}) \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}\,\ell(z)\ell(w) \\ + 2h \sum_{m\in\mathbb{Z}} \left(\frac{w}{zq^2}\right)^m - 2h \sum_{m\in\mathbb{Z}} \left(\frac{wq^2}{z}\right)^m .\end{aligned} \tag{1.3}$$

This gives us a q-deformation of the classical Virasoro algebra.

Formula (1.2) shows that if we attach to $\Lambda(z)$ a first order q-difference operator $\mathcal{D}_q - \Lambda(zq)$, where $[\mathcal{D}_q \cdot f](z) = f(zq^2)$, then to $s(z)$ we can attach in a natural way a second order q-difference operator of the form $\mathcal{D}_q + \mathcal{D}_q^{-1} - s(z)$. Indeed, let $Q(z)$ be a solution of the difference equation $Q(zq) = \Lambda(z)Q(zq^{-1})$. Then from formula (1.2) we obtain

$$(\mathcal{D}_q + \mathcal{D}_q^{-1} - s(z))Q(z) = 0\,.$$

The latter equation written as

$$s(z) = \frac{Q(zq^2)}{Q(z)} + \frac{Q(zq^{-2})}{Q(z)} \tag{1.4}$$

was used by R. Baxter [15] in his study of the eight vertex model. Similar formulas appeared in [16] as the result of computation of the spectrum of the transfer-matrix of the six vertex model, an integrable model of statistical mechanics associated to $U_q(\widehat{\mathfrak{sl}}_2)$. In this context, the function $Q(z)$ is a product of a "vacuum value" and a polynomial, whose zeroes are solutions of Bethe ansatz equations.

1.3. Thus, we have interpreted formulas (1.2) and (1.4) as a hamiltonian map, which can be considered as a q-analogue of the Miura transformation. In fact, the Miura transformation plays the same role as Baxter's formula (1.4) in the Gaudin models, cf. [17, 7, 8].

The Miura transformation (1.1) is the classical limit of the free field realization of the Virasoro algebra. Free field realizations play an important role in conformal field theory, cf. [6]. It is quite remarkable that a q-analogue of free field realization appears in the context of Bethe ansatz in statistical mechanics.

Analogues of formula (1.2) for transfer-matrices of integrable models associated to other quantum affine algebras are known, cf. e.g. [18, 21] for $U_q(\widehat{\mathfrak{sl}}_N)$, [19, 20, 22] for other $U_q(\widehat{\mathfrak{g}})$. Motivated by our computation in the case of $U_q(\widehat{\mathfrak{sl}}_2)$ we expect that the formulas for the q-deformation of the Miura transformation of the center of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ coincide with the formulas for the transfer-matrices corresponding to $U_q(\widehat{\mathfrak{g}})$.

In particular, for $\mathfrak{g} = \mathfrak{sl}_N$ we obtain the following picture:

(i) In [9] the generating functions of central elements $\ell_1(z), \ldots, \ell_{N-1}(z)$ of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ corresponding to the fundamental representations have been constructed. The Fourier coefficients of $\ell_i(z)$'s generate a central subalgebra $Z_h(\widehat{\mathfrak{sl}}_N)$ of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$, which is closed with respect to the Poisson structure.

(ii) The Wakimoto realization of $U_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ [14] gives rise to a homomorphism of Poisson algebras $Z_h(\widehat{\mathfrak{sl}}_N) \to \mathscr{H}_h(\mathfrak{sl}_N)$, where $\mathscr{H}_h(\mathfrak{sl}_N)$ is a Heisenberg–Poisson algebra. This is a q-deformation of the Miura transformation. We find a formula for the image $s_i(z)$ of each generating function $\ell_i(z)$ in $\mathscr{H}_h(\mathfrak{sl}_N)$. These formulas match formulas for the spectra of the corresponding transfer-matrices in integrable models associated to $U_q(\widehat{\mathfrak{sl}}_N)$ [18, 21].

(iii) We explicitly compute the Poisson brackets between $s_i(z)$'s in $\mathscr{H}_h(\mathfrak{sl}_N)$ generalizing formula (1.3). Thus, we obtain an interesting Poisson subalgebra $\mathscr{W}_h(\mathfrak{sl}_N)$ of the Heisenberg–Poisson algebra $\mathscr{H}_h(\mathfrak{sl}_N)$, which is a q-deformation of the classical $\mathscr{W}$-algebra $\mathscr{W}(\mathfrak{sl}_N)$.

(iv) Recall that elements of the spectrum of $\mathscr{W}(\mathfrak{sl}_N)$ can be considered as N^{th} order differential operators. We show that elements of the spectrum of $\mathscr{W}_h(\mathfrak{sl}_N)$ can be considered as N^{th} order q-difference operators of the form

$$\mathscr{D}_q^N - s_{N-1}(z)\mathscr{D}_q^{N-1} + s_{N-2}(z)\mathscr{D}_q^{N-2} - \cdots - (-1)^N s_1(z)\mathscr{D}_q + (-1)^N .$$

We generalize (i) and (ii) to all quantum affine algebras of classical types. The computation of Poisson brackets is straightforward. We plan to study them in our next paper [23] along with results regarding quantum affine algebras of exceptional types.

Using our results in the same way as in [7] we can give a new interpretation of the Bethe ansatz in integrable models associated to quantum affine algebras. This and other applications will be discussed in [23].

The paper is organized as follows. In Sect. 2 we recall results concerning the center of $\widetilde{U}(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ and Miura transformation. In Sects. 3–5 we consider the Drinfeld and the RS realizations of $U_h(\widehat{\mathfrak{sl}}_2)_k$ and the isomorphism between them. In Sect. 6 we rewrite the RS formula for the generating function $\ell(z)$ of central elements in $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ in terms of the Drinfeld realization. In Sects. 7–9 we recall the Wakimoto realization of $U_h(\widehat{\mathfrak{sl}}_2)_k$, and use it to find an explicit formula for the image of $\ell(z)$ in $\mathscr{H}_h(\mathfrak{sl}_2)$ and to compute the Poisson bracket on $Z_h(\widehat{\mathfrak{sl}}_2)$. In Sects. 10 and 11 we generalize these results to $U_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ and other quantum affine algebras of classical types.

2. The Center of $\widetilde{U}(\widehat{\mathfrak{sl}}_2)$ at the Critical Level

2.1. The structure of the center. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. The affine algebra $\widehat{\mathfrak{g}}$ is the extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by a one-dimensional center

$\mathbb{C}K$. For $A \in \mathfrak{g}$, $n \in \mathbb{Z}$, denote $A[n] = A \otimes t^n$ and put

$$A(z) = \sum_{n \in \mathbb{Z}} A[n] z^{-n-1} .$$

Introduce a completion $\widetilde{U}(\widehat{\mathfrak{g}})$ of $U(\widehat{\mathfrak{g}})$, the universal enveloping algebra of $\widehat{\mathfrak{g}}$:

$$\widetilde{U}(\widehat{\mathfrak{g}}) = \varprojlim U(\widehat{\mathfrak{g}})/U(\widehat{\mathfrak{g}})(\mathfrak{g} \otimes t^n \mathbb{C}[t]), \quad n > 0 .$$

This is an associative algebra. It acts on $\widehat{\mathfrak{g}}$-modules M which satisfy the following property: for any $x \in M$ there exists $N \in \mathbb{Z}_+$ such that $A[n] \cdot x = 0$ for any $A \in \mathfrak{g}$ if $n > N$. For $k \in \mathbb{C}$ put $\widetilde{U}(\widehat{\mathfrak{g}})_k = \widetilde{U}(\widehat{\mathfrak{g}})/(K - k)$, and let $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ be $\widetilde{U}(\widehat{\mathfrak{g}})_{-h^\vee}$.

The algebra $\widetilde{U}(\widehat{\mathfrak{g}})_k$ contains the local completion $U(\widehat{\mathfrak{g}})_{k,}$ introduced in [1]. The center of $U(\widehat{\mathfrak{g}})_{k,}$ has been described in [1]. It consists of the constants when $k \neq -h^\vee$, but becomes "large" when $k = -h^\vee$. Let us recall its description in the case $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$; in this case $h^\vee = 2$.

Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}_2$. Introduce the generating function of the Sugawara operators S_n by formula

$$S(z) = \sum_{n \in \mathbb{Z}} S_n z^{-n-2} = \frac{1}{4} : h(z)^2 : + \frac{1}{2} : e(z) f(z) : + \frac{1}{2} : f(z) e(z) : .$$

It is well-known that

$$[S_n, A[m]] = -(k+2) m A[n+m]$$

for any $A \in \mathfrak{g}$, and

$$[S_n, S_m] = (k+2) \left[(n-m) S_{n+m} + \frac{k}{4}(n^3 - n)\delta_{n,-m} \right] . \tag{2.1}$$

Therefore, if $k \neq -2$, the operators $L_n = S_n/(k+2)$ generate the Virasoro algebra. If $k = -2$, the operators S_n, $n \in \mathbb{Z}$, are central elements of $U(\widehat{\mathfrak{sl}}_2)_{-h^\vee, \mathrm{loc}}$.

There is a natural Poisson structure on the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{cr}}$: for any $A, B \in Z(\widehat{\mathfrak{g}})$, let A', B' be their liftings to $\widetilde{U}(\widehat{\mathfrak{sl}}_2)$. Then we have $[A', B'] = (K + h^\vee)C' + (K + h^\vee)^2(\cdots)$. Let C be the projection of $C' \in \widetilde{U}(\widehat{\mathfrak{sl}}_2)$ to $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{cr}}$. Then the formula $\{A, B\} = C$ defines a Poisson bracket on $Z(\widehat{\mathfrak{g}})$, which does not depend on the liftings.

We obtain from formula (2.1):

$$\{S_n, S_m\} = (n-m) S_{n+m} - \frac{1}{2}(n^3 - n)\delta_{n,-m} . \tag{2.2}$$

Consider the hyperplane $\mathscr{L}$ in the dual space to the Virasoro algebra, which consists of those linear functionals on the Virasoro algebra which take value -6 on the central element (this corresponds to the factor $-1/2 = -6/12$ in the second term of formula (2.2)). This hyperplane is isomorphic to the space of projective connections on the formal punctured disc $\partial_z^2 - q(z)$, where $q(z) \in \mathbb{C}((z))$, in the sense that the natural action of vector fields on it coincides with the coadjoint action of the Virasoro algebra on $\mathscr{L}$. Let $\mathscr{W}(\mathfrak{sl}_2)$ be the Poisson algebra of local functionals on $\mathscr{L}$. It is the classical limit of the local completion of the universal enveloping algebra of the Virasoro algebra. Therefore we call $\mathscr{W}(\mathfrak{sl}_2)$ the classical Virasoro algebra.

In the case $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ the result of [1] is that the center $Z(\widehat{\mathfrak{sl}}_2)$ of $U(\widehat{\mathfrak{sl}}_2)_{-2,\mathrm{loc}}$ is isomorphic to $\mathscr{W}(\mathfrak{sl}_2)$. This isomorphism sends S_n to the local functional $\partial_z^2 - q(z) \to \int q(z) z^{n+1} dz$. According to formula (2.2), this is an isomorphism of Poisson algebras.

2.2. Wakimoto modules and Miura transformation. Consider the Heisenberg algebra Γ with generators a_n, a_n^*, $n \in \mathbb{Z}$, and relations

$$[a_n, a_m] = [a_n^*, a_m^*] = 0, \qquad [a_n, a_m^*] = \delta_{n,-m} ,$$

and the Heisenberg algebra $\mathscr{H}'(\mathfrak{sl}_2)$ with generators χ_n, $n \in \mathbb{Z}$, and $\mathbf{1}$ and relations

$$[\chi_n, \chi_m] = 2n\delta_{n,-m}\mathbf{1}, \qquad [\chi_n, \mathbf{1}] = 0 .$$

Introduce the generating functions

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}, \qquad \chi(z) = \sum_{n\in\mathbb{Z}} \chi_n z^{-n-1} .$$

We define an embedding ϕ of $\widehat{\mathfrak{sl}}_2$ into a completion of $\Gamma \otimes \mathscr{H}'(\mathfrak{sl}_2)$ by the formulas:

$$\begin{aligned}
\phi[e(z)] &= a(z) , \\
\phi[h(z)] &= -2 : a(z)a^*(z) : + \chi(z) , \\
\phi[f(z)] &= - : a(z)a^*(z)a^*(z) : + (\mathbf{1} - 2)\partial_z a^*(z) + \chi(z)a^*(z) , \\
\phi(K) &= \mathbf{1} - 2 .
\end{aligned}$$

The algebra Γ has a standard Fock representation M generated by a vector v, such that

$$a_n v = 0, \quad n \geqq 0; \qquad a_n^* v = 0, \quad n > 0 .$$

The algebra $\mathscr{H}'(\mathfrak{sl}_2)$ has a family of Fock representations $\pi_{\mu,\kappa}$, $\mu \in \mathbb{C}$, $\kheight \in \mathbb{C}$, $\kappa \neq 0$, generated by a vector $v_{\mu,\kappa}$, such that

$$\chi_n v_{\mu,\kappa} = \mu\delta_{n,0} v_{\mu,\kappa}, \quad n \geqq 0; \qquad \mathbf{1} v_{\mu,\kappa} = \kappa v_{\mu,\kappa} .$$

It also has an infinite-dimensional family of one-dimensional representations $\mathbb{C}_{x(z)}$, where $x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1} \in \mathbb{C}((z))$, on which χ_n acts by multiplication by x_n, and $\mathbf{1}$ acts by 0. Using the homomorphism ϕ we obtain representations of $\widehat{\mathfrak{sl}}_2$ of non-critical level in $M \otimes \pi_{\mu,\kappa}$, and representations of critical level in $M \otimes \mathbb{C}_{x(z)}$. These are called Wakimoto modules.

The algebra $\mathscr{H}'(\mathfrak{sl}_2)$ can be considered as a deformation of the commutative algebra $\mathscr{H}(\mathfrak{sl}_2) = \mathscr{H}'(\mathfrak{sl}_2)/(\mathbf{1})$. It induces a Poisson structure on $\mathscr{H}(\mathfrak{sl}_2)$, which is called the Heisenberg–Poisson structure. It is determined by the following Poisson brackets of the generators:

$$\{\chi_n, \chi_m\} = 2n\delta_{n,-m} . \tag{2.3}$$

The homomorphism ϕ defines a homomorphism ϕ_{-2} from $\widehat{\mathfrak{sl}}_2$ to a completion of $\Gamma \otimes \mathscr{H}(\mathfrak{sl}_2)$, which maps K to -2. One can check that under ϕ_{-2}, the Sugawara series $S(z)$ is mapped to

$$\frac{1}{4}\chi(z)^2 - \frac{1}{2}\partial_z \chi(z) . \tag{2.4}$$

Therefore ϕ_{-2} defines an embedding of $Z(\widehat{\mathfrak{sl}}_2)$ into a completion of $\mathscr{H}(\mathfrak{sl}_2)$, which we call the Miura transformation.

The Poisson structure between central elements $A, B \in Z(\widehat{\mathfrak{sl}}_2)$ has been defined via the commutator of their liftings to $\widetilde{U}(\widehat{\mathfrak{sl}}_2)$. The result does not depend on the choice of a lifting. Moreover, we will obtain the same result if we take the commutator between the liftings of the images A', B' of A, B in the completion of $\Gamma \otimes \mathscr{H}(\mathfrak{sl}_2)$. Since the image of $Z(\widehat{\mathfrak{sl}}_2)$ actually lies in the completion of $\mathscr{H}(\mathfrak{sl}_2)$, we can take liftings lying in the completion of $\mathscr{H}'(\mathfrak{sl}_2)$. But then the image of the Poisson bracket between $A, B \in Z(\widehat{\mathfrak{sl}}_2)$ in $\mathscr{H}(\mathfrak{sl}_2)$ will coincide with the Poisson bracket between $A', B' \in \mathscr{H}(\mathfrak{sl}_2)$. Hence the Miura transformation $Z(\widehat{\mathfrak{sl}}_2) \to \mathscr{H}(\mathfrak{sl}_2)$ is a homomorphism of Poisson algebras.

Therefore we can compute the Poisson structure on $Z(\widehat{\mathfrak{sl}}_2)$ using formulas (2.4) and (2.3). This gives us the Poisson bracket between the Sugawara operators, which coincides with formula (2.2).

3. The Drinfeld Realization

Let $U_q(\widehat{\mathfrak{sl}}_2)$ be the associative algebra over the ring $\mathbb{C}[q, q^{-1}]$, with generators e_i, f_i, and K_i, K_i^{-1}, $i = 0, 1$, which satisfy the following relations [24, 25]:

$$K_i K_j = K_j K_i \,,$$
$$K_i e_j = q^{A_{ij}} e_j K_i, \qquad K_i f_j = q^{-A_{ij}} f_j K_i \,,$$
$$[e_i, f_j] = \delta_{ij}(q - q^{-1})(K_i - K_i^{-1}) \,,$$
$$e_i^3 e_j - (q + 1 + q^{-1})(e_i^2 e_j e_i - e_i e_j e_i^2) - e_j e_i^3 = 0 \,,$$
$$f_i^3 f_j - (q + 1 + q^{-1})(f_i^2 f_j f_i - f_i f_j f_i^2) - f_j f_i^3 = 0 \,,$$

where A_{ij}, $i, j = 0, 1$, are the entries of the Cartan matrix of $\widehat{\mathfrak{sl}}_2$: $A_{00} = A_{11} = -A_{01} = -A_{10} = 2$.

For any $h, k \in \mathbb{C}$ consider the quotient $U_h(\widehat{\mathfrak{sl}}_2)_k$ of $U_q(\widehat{\mathfrak{sl}}_2)$ by the relations $q = e^h$, $K_0 K_1 = e^{hk}$. This is the Drinfeld–Jimbo realization of $U_h(\widehat{\mathfrak{sl}}_2)_k$.

There exist two other realizations of $U_h(\widehat{\mathfrak{sl}}_2)_k$: Drinfeld's realization [11], and Reshetikhin–Semenov–Tian–Shansky (RS) realization. The equivalence between the Drinfeld–Jimbo and the Drinfeld realizations has been established by V. Drinfeld [11], cf. also [26–28]. The equivalence between the Drinfeld and the RS realizations has been established by J. Ding and I. Frenkel [12]. The equivalence between the Drinfeld–Jimbo and the RS realization follows from these two equivalences, but it can also be established directly along the lines of [12].

First we consider the Drinfeld realization. It is important for us because the Wakimoto realization is defined in terms of this realization.

Introduce formal power series in x

$$f(x) = \frac{(x; q^4)(xq^4; q^4)}{(xq^2; q^4)^2} \,, \tag{3.1}$$

where

$$(a; b) = \prod_{n=0}^{\infty} (1 - ab^n) \,.$$

Remark 1. Each coefficient of $f(x)$ is itself a series in q, which converges for $|q| < 1$ and can be analytically continued to the whole complex plane except for some roots of unity. Thus, we can extend $f(x)$ as formal power series in x to all q except for the roots of unity. In what follows we will exclude roots of unity from consideration. $\square$

Let $h\in\mathbb{C}\backslash\{2\pi i\mathbb{Q}\}$, $k\in\mathbb{C}$; put $q = e^h$. We define an associative algebra $U_h(\widehat{\mathfrak{gl}}_2)_k$ over $\mathbb{C}$ with generators $E[n], F[n]$, $n \in \mathbb{Z}$, and $k_i^{\pm}[n]$, $i = 1,2$; $n \in \mp\mathbb{Z}_+$. Introduce the generating functions

$$E(z) = \sum_{n\in\mathbb{Z}} E[n]z^{-n}, \qquad F(z) = \sum_{n\in\mathbb{Z}} F[n]z^{-n}, \qquad k_i^{\pm}(z) = \sum_{n=0}^{\infty} k_i^{\pm}[\mp n]z^{\pm n}.$$

The defining relations in $U_h(\widehat{\mathfrak{gl}}_2)_k$ are

$$k_i^+[0]k_i^-[0] = k_i^-[0]k_i^+[0] = 1\,,$$

$$k_i^{\pm}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\pm}(z)\,,$$

$$k_i^-(z)k_i^+(w) = \frac{f(\frac{w}{z}q^{-k})}{f(\frac{w}{z}q^{k})}k_i^+(w)k_i^-(z)\,,$$

$$k_1^-(z)k_2^+(w) = \frac{f(\frac{w}{z}q^{k+2})}{f(\frac{w}{z}q^{-k+2})}k_2^+(w)k_1^-(z)\,,$$

$$k_2^-(z)k_1^+(w) = \frac{f(\frac{w}{z}q^{k-2})}{f(\frac{w}{z}q^{-k-2})}k_1^+(w)k_2^-(z)\,,$$

$$k_1^{\pm}(z)E(w) = \frac{zq^{\mp\frac{k}{2}-1} - wq}{zq^{\mp\frac{k}{2}} - w}E(w)k_1^{\pm}(z)\,,$$

$$k_1^{\pm}(z)F(w) = \frac{zq^{\pm\frac{k}{2}} - w}{zq^{\pm\frac{k}{2}-1} - wq}F(w)k_1^{\pm}(z)\,,$$

$$k_2^{\pm}(z)E(w) = \frac{zq^{\mp\frac{k}{2}+1} - wq^{-1}}{zq^{\mp\frac{k}{2}} - w}E(w)k_2^{\pm}(z)\,,$$

$$k_2^{\pm}(z)F(w) = \frac{zq^{\pm\frac{k}{2}} - w}{zq^{\pm\frac{k}{2}+1} - wq^{-1}}F(w)k_2^{\pm}(z)\,,$$

$$E(z)E(w) = \frac{zq^2 - w}{z - wq^2}E(w)E(z)\,,$$

$$F(z)F(w) = \frac{z - wq^2}{zq^2 - w}F(w)F(z)\,,$$

$$[E(z), F(w)] = (q - q^{-1})\left(\delta\left(\frac{w}{z}q^k\right) k_2^-(wq^{\frac{k}{2}})k_1^-(wq^{\frac{k}{2}})^{-1}\right.$$
$$\left. -\delta\left(\frac{w}{z}q^{-k}\right) k_2^+(wq^{-\frac{k}{2}})k_1^+(wq^{-\frac{k}{2}})^{-1}\right)\,,$$

where

$$\delta(x) = \sum_{m\in\mathbb{Z}} x^m .$$

These relations are understood as relations between formal powers series (cf. Remark 1).

Lemma 1. *The Fourier coefficients of the power series $k_1^{\pm}(z)k_2^{\pm}(zq^{-2}) - 1$ are central elements of $U_h(\widehat{\mathfrak{gl}}_2)_k$.*

Consider the quotient of $U_h(\widehat{\mathfrak{gl}}_2)_k$ by the ideal generated by these elements. It has $E[n], F[n], k_1^{\pm}[n]$, $n \in \mathbb{Z}$, as generators. There is a one-to-one correspondence between them and Drinfeld's generators which preserves relations: Drinfeld's $\psi(z)$ ($\psi_+(z)$ of [14]) is $k_1^-(zq^2)^{-1}k_1^-(z)^{-1}$, $\phi(z)$ ($\psi_-(z)$ of [14]) is $k_1^+(zq^2)^{-1}k_1^+(z)^{-1}$, $\xi^+(z)$ is $E(z)$ and $\xi^-(z)$ is $h(q-q^{-1})F(z)$. The following proposition then follows from [11], cf. also [26–28].

Proposition 1. *The quotient of $U_h(\widehat{\mathfrak{gl}}_2)_k$ by the ideal generated by the Fourier coefficients of the power series $k_1^{\pm}(z)k_2^{\pm}(zq^{-2}) - 1$ is isomorphic to $U_h(\widehat{\mathfrak{sl}}_2)_k$.*

4. The RS Realization

Now we turn to the RS realization [9]. This realization originated from the Quantum Inverse Scattering Method, cf. [29, 30]. It is important for us because in this realization we can write explicit formulas for central elements [9].

Introduce the R-matrix

$$R(x) = f(x)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1-x}{q-xq^{-1}} & \dfrac{x(q-q^{-1})}{q-xq^{-1}} & 0 \\ 0 & \dfrac{q-q^{-1}}{q-xq^{-1}} & \dfrac{1-x}{q-xq^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where $f(x)$ is given by formula (3.1).

The matrix (4.1) is the result of computation of the universal R-matrix of $U_q(\widehat{\mathfrak{sl}}_2)$ on the tensor product of two two-dimensional evaluation representations, cf. e.g. [31]. It satisfies the crossing-symmetry property:

$$(((R(x)^{-1})^{t_1})^{-1})^{t_1} = \left(\begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes I_2\right) R(xq^4) \left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes I_2\right),$$

which follows from the existence of an isomorphism between the two-dimensional evaluation module and its double dual [31]. This is related to the fact that $f(x)$ satisfies the q-difference equation

$$f(xq^4) = \frac{(1-xq^2)^2}{(1-x)(1-xq^4)} f(x) .$$

Remark 2. Our R-matrix differs from that of [12] by the factor $f(x)$ and by replacement of q by q^{-1}. It also differs from the R-matrix used in [32] by the factor which is a product of theta-functions. □

Let again $h \in \mathbb{C}\backslash\{2\pi i\mathbb{Q}\}$, $k \in \mathbb{C}$, $q = e^h$. We define an associative algebra $U_h'(\widehat{\mathfrak{gl}}_2)_k$ over $\mathbb{C}$ with generators $l_{ij}^{\pm}[n]$, where $i,j = 1,2$, and $n \in \mp\mathbb{Z}_+\backslash 0$, and $l_{ij}^{+}[0]$, $l_{ji}^{-}[0]$, $1 \leqq j \leqq i \leqq 2$. Introduce the generating functions

$$l_{ij}^{\pm}(z) = \sum_{n=0}^{\infty} l_{ij}^{\pm}[\mp n] z^{\pm n} ,$$

where we put $l_{ij}^{+}[0] = l_{ji}^{-}[0] = 0$ for $1 \leqq i < j \leqq 2$. Let $L^{\pm}(z)$ be the 2×2 matrix $(l_{ij}^{\pm}(z))_{i,j=1,2}$.

The defining relations in $U_h'(\widehat{\mathfrak{gl}}_2)_k$ are:

$$l_{ii}^{+}[0] l_{ii}^{-}[0] = l_{ii}^{-}[0] l_{ii}^{+}[0] = 1, \quad i = 1,2 ,$$

$$R\left(\frac{z}{w}\right) L_1^{\pm}(z) L_2^{\pm}(w) = L_2^{\pm}(w) L_1^{\pm}(z) R\left(\frac{z}{w}\right) , \tag{4.2}$$

$$R\left(\frac{z}{w} q^{-k}\right) L_1^{+}(z) L_2^{-}(w) = L_2^{-}(w) L_1^{+}(z) R\left(\frac{z}{w} q^{k}\right) . \tag{4.3}$$

Here $L_1^{\pm}(z)$ and $L_2^{\pm}(w)$ are elements of $\mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^2) \otimes U_h'(\widehat{\mathfrak{gl}}_2)_k$, i.e. 4×4 matrices with entries from $U_h'(\widehat{\mathfrak{gl}}_2)_k$, such that $L_1^{\pm}(w) = L^{\pm}(w) \otimes I_2$, $L_2^{\pm}(z) = I_2 \otimes (L^{\pm}(z))$.

The relations (4.2) and (4.3) are understood as relations between formal power series in $\frac{z}{w}$, cf. Remark 1.

5. The Isomorphism of Two Realizations

Following [12] one can construct an explicit isomorphism between the algebras $U_h(\widehat{\mathfrak{gl}}_2)_k$ and $U_h'(\widehat{\mathfrak{gl}}_2)_k$.

Consider the following decomposition:

$$\begin{aligned} L^{\pm}(z) &= \begin{pmatrix} 1 & 0 \\ e^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z) & 0 \\ 0 & k_2^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & f^{\pm}(z) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z) f^{\pm}(z) \\ e^{\pm}(z) k_1^{\pm}(z) & k_2^{\pm}(z) + e^{\pm}(z) k_1^{\pm}(z) f^{\pm}(z) \end{pmatrix} . \end{aligned} \tag{5.1}$$

In particular, we see that $l_{21}^{+}[0]$ is the constant term of $e^{+}(z) = \sum_{m \geqq 0} e[-m] z^m$ and $l_{12}^{-}[0]$ is the constant term of $f^{-}(z) = \sum_{m \leqq 0} f[-m] z^m$, while $e^{-}(z) = \sum_{m<0} e[-m] z^m$ and $f^{+}(z) = \sum_{m>0} f[-m] z^m$ have no constant terms.

Proposition 2 ([12]). *The map $\psi' : U_h'(\widehat{\mathfrak{gl}}_2)_k \to U_h(\widehat{\mathfrak{gl}}_2)_k$ defined on generators by*

$$\psi'[k_1^{\pm}(z)] = k_1^{\pm}(z), \qquad \psi'[k_2^{\pm}(z)] = k_2^{\pm}(z) ,$$

$$\psi'[e^{+}(zq^{\frac{k}{2}}) - e^{-}(zq^{-\frac{k}{2}})] = E(z) ,$$

$$\psi'[f^{+}(zq^{-\frac{k}{2}}) - f^{-}(zq^{\frac{k}{2}})] = F(z) ,$$

is an isomorphism.

Lemma 2. *The Fourier coefficients of the power series*

$$l_{11}^{\pm}(zq^2)(l_{22}^{\pm}(z) - l_{21}^{\pm}(z)l_{11}^{\pm}(z)^{-1}l_{12}^{\pm}(z)) - 1 \tag{5.2}$$

are central elements of $U_h'(\widehat{\mathfrak{gl}}_2)_k$.

The image of the power series (5.2) under the map ψ is the series $k_1^{\pm}(zq^2)k_2^{\pm}(z) - 1$, whose coefficients are central elements of $U_h(\widehat{\mathfrak{gl}}_2)_k$. According to Proposition 1, the quotient of $U_h(\widehat{\mathfrak{gl}}_2)_k$ by the ideal generated by these central elements is isomorphic to $U_h(\widehat{\mathfrak{sl}}_2)_k$. Hence we obtain

Corollary 1. *ψ' induces an isomorphism ψ between the quotient of $U_h'(\widehat{\mathfrak{gl}}_2)_k$ by the Fourier coefficients of the power series $l_{11}^{\pm}(zq^2)(l_{22}^{\pm}(z) - l_{21}^{\pm}(z)l_{11}^{\pm}(z)^{-1}l_{12}^{\pm}(z)) - 1$ and $U_h(\widehat{\mathfrak{sl}}_2)_k$.*

6. The q-Analogues of the Sugawara Operators

We define a completion $\widetilde{U}_h(\widehat{\mathfrak{g}})_k$ of $U_h(\widehat{\mathfrak{g}})_k$ as follows:

$$\widetilde{U}_h(\widehat{\mathfrak{g}})_k = \lim_{\longleftarrow} U_h(\widehat{\mathfrak{g}})_k/J_n, \quad n > 0,$$

where J_n is the left ideal of $U_h(\widehat{\mathfrak{g}})_k$ generated by $l_{ij}^-[m]$, $m \geqq n$.

Let

$$L(z) = L^+(zq^{-\frac{k}{2}})L^-(zq^{\frac{k}{2}})^{-1}.$$

It is easy to see that all Fourier coefficients of the power series

$$\ell(z) = q^{-1}L_{11}(z) + qL_{22}(z), \tag{6.1}$$

lie in $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_k$. It follows from [9] that when $k = -2$ the coefficients of $\ell(z)$ are central elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$.

We will now express $\ell(z)$ in terms of the Drinfeld realization using the isomorphism ψ. Let us put $k = -2$. Using formula (5.1) we obtain:

$$\begin{aligned} L_{11}(z) = {} & k_1^+(zq)k_1^-(zq^{-1})^{-1} - k_1^+(zq) \\ & \times (f^+(zq) - f^-(zq^{-1}))k_2^-(zq^{-1})^{-1}e^-(zq^{-1}), \end{aligned} \tag{6.2}$$

$$\begin{aligned} L_{22}(z) = {} & k_2^+(zq)k_2^-(zq^{-1})^{-1} + e^+(zq)k_1^+(zq) \\ & \times (f^+(zq) - f^-(zq^{-1}))k_2^-(zq^{-1})^{-1}. \end{aligned} \tag{6.3}$$

Applying formula (4.45) from [12] (in which the sign of the second summand in the right-hand side has to be reversed) we obtain

$$k_2^-(zq^{-1})^{-1}e^-(zq^{-1}) = qe^-(zq)k_2^-(zq^{-1})^{-1}, \tag{6.4}$$

and applying formula (4.25) from [12] we obtain

$$e^+(zq)k_1^+(zq) = q^{-1}k_1^+(zq)e^+(zq^{-1}). \tag{6.5}$$

Substituting (6.4) into (6.2) we obtain

$$L_{11}(z) = k_1^+(zq)k_1^-(zq^{-1})^{-1} - qk_1^+(zq)(f^+(zq) - f^-(zq^{-1}))e^-(zq)k_2^-(zq^{-1})^{-1}\,, \tag{6.6}$$

and substituting (6.5) into (6.3) we obtain

$$L_{22}(z) = k_2^+(zq)k_2^-(zq^{-1})^{-1} + q^{-1}k_1^+(zq)e^+(zq^{-1}) \times (f^+(zq) - f^-(zq^{-1}))k_2^-(zq^{-1})^{-1}\,. \tag{6.7}$$

Combining formula (6.1) with (6.6) and (6.7) we obtain

$$\ell(z) = q^{-1}k_1^+(zq)k_1^-(zq^{-1})^{-1} + qk_2^+(zq)k_2^-(zq^{-1})^{-1}\,, \tag{6.8}$$

$$+ k_1^+(zq) : E(z)F(z) : k_2^-(zq^{-1})^{-1}\,, \tag{6.9}$$

where

$$: E(z)F(z) := e^+(zq^{-1})(f^+(zq) - f^-(zq^{-1})) - (f^+(zq) - f^-(zq^{-1}))e^-(zq)\,.$$

Now we apply the isomorphism ψ. We see that $\psi(: E(z)F(z) :)$ is a normally ordered product of the power series $E(z)$ and $F(z)$:

$$: E(z)F(z) := E_-(z)F(z) + F(z)E_+(z)\,,$$

where

$$E_-(z) = \sum_{n \leqq 0} E[n]z^{-n}, \qquad E_+(z) = \sum_{n>0} E[n]z^{-n}\,.$$

Thus, we obtain

Proposition 3. *The Fourier coefficients of the power series*

$$\ell(z) = q^{-1}k_1^+(zq)k_1^-(zq^{-1})^{-1} + qk_1^+(zq^3)^{-1}k_1^-(zq) \tag{6.10}$$

$$+ k_1^+(zq) : E(z)F(z) : k_1^-(zq) \tag{6.11}$$

are central elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$.

These are the q-analogues of the Sugawara operators.

7. The Wakimoto Realization of $U_h(\widehat{\mathfrak{sl}}_2)_k$

Now we will describe a homomorphism $\phi_{h,k}$ from $U_h(\widehat{\mathfrak{sl}}_2)_k$ to a completion of a quantum Heisenberg algebra. The map $\phi_{h,k}$ is a q-analogue of the map ϕ defined in Sect. 2. Such homomorphisms have been constructed in [13]. We will use the Awata–Odake–Shiraishi construction [14] with some modifications.

Introduce the quantum Heisenberg algebra $\mathscr{A}_{h,k}(\mathfrak{sl}_2)$. The generators are λ_n, b_n, c_n, $n \in \mathbb{Z}$, $n \neq 0$, $\exp(\pm\lambda_0/2)$, $\exp(\pm(q-q^{-1})b_0/2)$, $\exp(\pm(q-q^{-1})c_0)$, and p_b, p_c.

The relations are

$$
\begin{aligned}
[\lambda_n, \lambda_m] &= \frac{1}{n}\frac{[(k+2)n]_q [n]_q^2}{[2n]_q}(q-q^{-1})^2 \delta_{n,-m} , \\
[b_n, b_m] &= -\frac{1}{n}[n]_q^2 \delta_{n,-m}, \qquad [b_0, p_b] = -\frac{q-q^{-1}}{2h} , \\
[c_n, c_m] &= \frac{1}{n}[n]_q^2 \delta_{n,-m}, \qquad [c_0, p_c] = \frac{q-q^{-1}}{2h} ,
\end{aligned} \tag{7.1}
$$

where $q = e^h$ and $[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$. The generators b_n and c_n coincide with the corresponding generators of [14]. The generators λ_n are related to the generators a_n of [14] by the formula

$$
\lambda_n = \frac{q-q^{-1}}{q^n + q^{-n}} a_n
$$

(recall that we have assumed that q is not a root of unity).

We form the generating functions:

$$
\begin{aligned}
\Lambda_\pm(z) &= \exp\left(\pm\frac{\lambda_0}{2} \pm \sum_{n=1}^{\infty} \lambda_{\pm n} z^{\mp n}\right) , \\
b_\pm(z) &= \pm(q-q^{-1})\left(\frac{b_0}{2} + \sum_{n=1}^{\infty} b_{\pm n} z^{\mp n}\right) , \\
b(z) &= -\sum_{n \neq 0} \frac{b_n}{[n]_q} z^{-n} + \frac{q-q^{-1}}{2h} b_0 \log z + p_b , \\
c_\pm(z) &= \pm(q-q^{-1})\left(\frac{c_0}{2} + \sum_{n=0}^{\infty} c_{\pm n} z^{\mp n}\right) , \\
c(z) &= -\sum_{n \neq 0} \frac{c_n}{[n]_q} z^{-n} + \frac{q-q^{-1}}{2h} c_0 \log z + p_c .
\end{aligned}
$$

The series $\Lambda_\pm(z)$ is related to the series $a_\pm(z)$ from [14] by the formula

$$
\Lambda_\pm(z)\Lambda_\pm(zq^{\pm 2}) = e^{a_\pm(zq^{\pm 1})} .
$$

The other series are the same as in [14].

The relations between these series, in the sense of a formal power series (cf. Remark 1), are the following (cf. [14]):

$$
\begin{aligned}
\Lambda_+(z)\Lambda_-(w) &= \frac{f(\frac{w}{z}q^{-k-2})}{f(\frac{w}{z}q^{k+2})}\Lambda_-(w)\Lambda_+(z) , \\
e^{b_+(z)} : e^{b(w)} : &= \frac{z-wq}{zq-w} : e^{b(w)} : e^{b_+(z)} ,
\end{aligned} \tag{7.2}
$$

$$: e^{b(z)} : e^{b_-(w)} = \frac{z - wq}{zq - w} e^{b_-(w)} : e^{b(z)} := q \frac{z - wq}{zq - w} : e^{b_-(w) + b(z)} : . \tag{7.3}$$

$$e^{b_+(z)} e^{b_-(w)} = \frac{(z-w)^2}{(z - wq^2)(z - wq^{-2})} e^{b_-(w)} e^{b_+(z)} , \tag{7.4}$$

$$e^{c_+(z)} e^{c_-(w)} = \frac{(z - wq^2)(z - wq^{-2})}{(z-w)^2} e^{c_-(w)} e^{c_+(z)} .$$

We define the completion $\widetilde{\mathscr{A}}_{h,k}(\mathfrak{sl}_2)$ of $\mathscr{A}_{h,k}(\mathfrak{sl}_2)$ as follows:

$$\widetilde{\mathscr{A}}_{h,k}(\mathfrak{sl}_2) = \varprojlim \mathscr{A}_{h,k}/I_n, \quad n > 0,$$

where I_n is the left ideal of $\mathscr{A}_{h,k}(\mathfrak{sl}_2)$ generated by all polynomials in λ_m, b_m, c_m, $m > 0$, of degrees greater than or equal to n (we put $\deg \lambda_m = \deg b_m = \deg c_m = m$).

The next proposition follows from [14].

Proposition 4. *There is a homomorphism $\phi_{h,k}$ from $U_h(\widehat{\mathfrak{sl}}_2)_k$ to $\widetilde{\mathscr{A}}_{h,k}(\mathfrak{sl}_2)$, which is defined on generators as follows:*

$$\begin{aligned}
\phi_{h,k}[E(z)] &= - : e^{b_+(z) - (b+c)(zq)} : + : e^{b_-(z) - (b+c)(zq^{-1})} : , \\
\phi_{h,k}[F(z)] &= \Lambda_+(zq^{\frac{k}{2}}) \Lambda_+(zq^{\frac{k}{2}+2}) : e^{b_+(zq^{k+2}) + (b+c)(zq^{k+1})} : , \\
&\quad - \Lambda_-(zq^{-\frac{k}{2}}) \Lambda_-(zq^{-\frac{k}{2}-2}) : e^{b_-(zq^{-k-2}) + (b+c)(zq^{-k-1})} : , \\
\phi_{h,k}[k_1^+(z)] &= \Lambda_-(zq^{-2})^{-1} e^{-b_-(zq^{-\frac{k}{2}-2})} , \\
\phi_{h,k}[k_1^-(z)] &= \Lambda_+(z)^{-1} e^{-b_+(zq^{\frac{k}{2}})} .
\end{aligned}$$

Remark 3. Under the homomorphism ϕ defined in Sect. 2.2, the affine algebra $\widehat{\mathfrak{sl}}_2$ embeds into a completion of the Heisenberg algebra $\Gamma \otimes \mathscr{H}'(\mathfrak{sl}_2)$ generated by a_n, a_n^* and χ_n. The power series $a(z)$ and $a^*(z)$ form the so-called $\beta\gamma$-system while the power series $\chi(z)$ is called a free scalar field. The $\beta\gamma$-system can be represented via exponentials of a pair of free scalar fields. The homomorphism ϕ then gives rise to a homomorphism ϕ' from $\widehat{\mathfrak{sl}}_2$ to a completion of the Heisenberg algebra generated by the Fourier coefficients of these two scalar fields and $\chi(z)$.

The power series $b(z)$ and $c(z)$ are q-analogues of the scalar bosonic fields representing the $\beta\gamma$ system when $q = 1$. Thus, the homomorphism $\phi_{h,k}$ is a q-deformation of ϕ' rather than ϕ. □

When $k \neq -2$, the homomorphism $\phi_{h,k}$ provides representations of $U_h(\widehat{\mathfrak{sl}}_2)_k$ in the Fock representation of the Heisenberg algebra $\mathscr{A}_{h,k}$, cf. [14]. These representations have one parameter – the action of λ_0 on the highest weight vector, cf. [14]. When $k = -2$, the generators λ_n commute among themselves and generate a commutative algebra $\mathscr{H}_h(\mathfrak{sl}_2)$. Therefore representations of $U_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ can be realized via $\phi_{h,k}$ in a smaller space: the tensor product of the Fock representation of the subalgebra of $\mathscr{A}_{h,-2}$ generated by b_n, c_n, $n \in \mathbb{Z}$, and a one-dimensional representation of $\mathscr{H}_h(\mathfrak{sl}_2)$. For the action of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ to be well-defined on this space, the action of $\Lambda(z)$ should be given by a Laurent power series $\lambda(z)$ (for example, this is the case if λ_n, $n > 0$, act by 0). The corresponding representations $W_{\lambda(z)}$ are the q-analogues of the Wakimoto modules over $U(\widehat{\mathfrak{sl}}_2)_{\rm cr}$ from Sect. 2.2.

8. Deformation of the Miura Transformation

In this section we will apply $\phi_h^{\rm cr} \equiv \phi_{h,-2}$ to the generating function of central elements $\ell(z)$ given by (6.10). For brevity, in what follows we will use the same notation for elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\rm cr}$ and their images in $\widetilde{\mathscr{A}}_{h,k}(\mathfrak{sl}_2)$.

The normally ordered product $: E(z)F(z) :$ can be written as

$$: E(z)F(z) := \int\limits_{C_R} \frac{E(w)F(z)}{w-z} dw - \int\limits_{C_r} \frac{F(z)E(w)}{w-z} \,, \tag{8.1}$$

where C_R and C_r are circles around the origin or radii $R > |w|$ and $r < |w|$, respectively.

Using the Wakimoto realization of $U_h(\widehat{\mathfrak{sl}}_2)_{\rm cr}$ and formulas (7.2)–(7.4), we find in the region $|w| > |z|$, $|w| > q^2|z|$, $|w| > q^{-2}|z|$:

$$\begin{aligned} E(w)F(z) = &- \frac{w-z}{wq-zq^{-1}} \Lambda_+(zq^{-1})\Lambda_+(zq) e^{(b+c)(zq^{-1})-(b+c)(wq)} e^{b_+(z)+b_+(w)} \\ &- \frac{w-z}{wq^{-1}-zq} \Lambda_-(zq^{-1})\Lambda_-(zq) e^{b_-(z)+b_-(w)} e^{(b+c)(zq)-(b+c)(wq^{-1})} \\ &+ q^{-1}\Lambda_-(zq^{-1})\Lambda_-(zq) e^{b_-(z)} e^{(b+c)(zq)-(b+c)(wq)} e^{b_+(w)} \\ &+ q\Lambda_+(zq^{-1})\Lambda_+(zq) e^{b_-(w)} e^{(b+c)(zq^{-1})-(b+c)(wq^{-1})} e^{b_+(z)} \,. \end{aligned}$$

We obtain the same formula for $F(z)E(w)$ in the region $|w| < |z|$, $|w| < q^2|z|$, $|w| < q^{-2}|z|$. We can therefore rewrite (8.1) as the integral of this expression over the contour on the w plane surrounding the points z, zq^2, zq^{-2}.

Evaluating the residues, we find that

$$: E(z)F(z) := - q^{-1}\Lambda_+(zq^{-1})\Lambda_+(zq) e^{b_+(zq^{-2})+b_+(z)} \,, \tag{8.2}$$

$$- q\Lambda_-(zq^{-1})\Lambda_-(zq) e^{b_-(zq^2)+b_-(z)} \,, \tag{8.3}$$

$$+ q^{-1}\Lambda_-(zq^{-1})\Lambda_-(zq) e^{b_-(z)} e^{b_+(z)} \,, \tag{8.4}$$

$$+ q\Lambda_+(zq^{-1})\Lambda_+(zq) e^{b_-(z)} e^{b_+(z)} \,. \tag{8.5}$$

Using this formula we obtain

$$\begin{aligned} k_1^+(zq) : E(z)F(z) : k_1^-(zq) = &- q^{-1}\Lambda_-(zq^{-1})^{-1}\Lambda_+(zq^{-1}) e^{-b_-(z)} e^{b_+(zq^{-2})} \\ &- q\Lambda_-(zq)\Lambda_+(zq)^{-1} e^{b_-(zq^2)} e^{-b_+(z)} \\ &+ q^{-1}\Lambda_-(zq)\Lambda_+(zq)^{-1} \\ &+ q\Lambda_-(zq^{-1})^{-1}\Lambda_+(zq^{-1}) \,. \end{aligned}$$

On the other hand, we have

$$k_1^+(zq)k_1^-(zq^{-1})^{-1} = \Lambda_-(zq^{-1})^{-1}\Lambda_+(zq^{-1})e^{-b_-(z)}e^{b_+(zq^{-2})}$$

and

$$k_1^+(zq^3)^{-1}k_1^-(zq) = \Lambda_-(zq)\Lambda_+(zq)^{-1}e^{b_-(zq^2)}e^{-b_+(z)}.$$

Substituting these formulas into (6.10) we obtain a formula expressing the image of $\ell(z)$ in $\mathscr{H}_h(\mathfrak{sl}_2)$ in terms of $\Lambda_\pm(z)$.

Theorem 1. *Under the homomorphism* ϕ_h^{cr},

$$\ell(z) \to s(z) = \Lambda(zq) + \Lambda(zq^{-1})^{-1}\,, \tag{8.6}$$

where

$$\Lambda(z) = q^{-1}\Lambda_-(z)\Lambda_+(z)^{-1}\,. \tag{8.7}$$

This is a q-deformation of the Miura transformation (2.4).

Remark 4. There is a simpler way to obtain formula (8.6). Consider the action of $s(z) = \phi_h^{\mathrm{cr}}[\ell(z)]$ on the module $W_{\lambda(z)}$ introduced at the end of Sect. 7. In the limit $q \to 1$ this module becomes a Wakimoto module over $\widehat{\mathfrak{sl}}_2$. Wakimoto modules are irreducible for generic values of parameters. Therefore the same is true for the modules $W_{\lambda(z)}$. Hence any central element of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ acts on $W_{\lambda(z)}$ by a constant. In particular, $\ell(z)$ acts by multiplication by a Laurent power series $\widetilde{s}(z)$. We can compute $\widetilde{s}(z)$ by taking the matrix element of $\ell(z)$ between the generating vector of $W_{\lambda(z)}$ and its dual using formulas (6.1), (5.1) and the maps $\psi, \phi_h^{\mathrm{cr}}$. Explicit computation shows that this matrix element is equal to $\lambda(zq) + \lambda(zq^{-1})^{-1}$. This implies that $s(z)$ lies in $\mathscr{H}_h(\mathfrak{sl}_2)$ and gives us formula (8.6) for $s(z)$. $\square$

Remark 5. In [9] a generating function of central elements of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ has been associated to an arbitrary finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$. Thus, in the case of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ we have a generating function $\ell^{(n)}(z)$ of central elements associated to the representation of $U_q(\widehat{\mathfrak{sl}}_2)$ of dimension $n+1$ for each positive integer n. In particular, $\ell(z) = \ell^{(1)}(z)$. These generating functions satisfy the following relation:

$$\ell^{(1)}(zq^{2n})\ell^{(n)}(z) = \ell^{(n+1)}(z) + \ell^{(n-1)}(z), \quad n > 0\,.$$

Using this relation and formula (8.6) it is easy to find recursively:

$$\begin{aligned}
\ell^{(n)}(z) &= \Lambda(zq)\Lambda(zq^3)\Lambda(zq^5)\cdots\Lambda(zq^{2n-1})\\
&+ \Lambda(zq^{-1})^{-1}\Lambda(zq^3)\Lambda(zq^5)\cdots\Lambda(zq^{2n-1})\\
&+ \Lambda(zq^{-1})^{-1}\Lambda(zq)^{-1}\Lambda(zq^5)\cdots\Lambda(zq^{2n-1})\\
&+ \Lambda(zq^{-1})^{-1}\Lambda(zq)^{-1}\Lambda(zq^3)^{-1}\cdots\Lambda(zq^{2n-1})\\
&+ \cdots\\
&+ \Lambda(zq^{-1})^{-1}\Lambda(zq)^{-1}\Lambda(zq^3)^{-1}\cdots\Lambda(zq^{2n-3})^{-1}
\end{aligned}$$

(compare with [33]). $\square$

9. Poisson Bracket

We can now compute the Poisson bracket between Fourier coefficients of $s(z)$. They generate a central subalgebra of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$. Let $Z_h(\widehat{\mathfrak{sl}}_2)$ be its completion in $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$.

Consider the algebra $\mathcal{H}_{h,k}(\mathfrak{sl}_2)$ with generators λ_n, $n \in \mathbb{Z}$, $\exp(\pm\lambda_0)$ and relations (7.1). Let $\widetilde{\mathcal{H}_{h,k}}(\mathfrak{sl}_2)$ be its completion defined as follows:

$$\widetilde{\mathcal{H}_{h,k}}(\mathfrak{sl}_2) = \varprojlim \mathcal{H}_{h,k}(\mathfrak{sl}_2)/I_n, \quad n > 0\,,$$

where I_n is the left ideal of $\mathcal{H}_{h,k}(\mathfrak{sl}_2)$ generated by all polynomials in λ_m, $m > 0$, of degrees greater than or equal to n.

The family $\widetilde{\mathcal{H}_{h,k}}(\mathfrak{sl}_2)$ induces a Poisson structure on $\mathcal{H}_h(\mathfrak{sl}_2) \equiv \widetilde{\mathcal{H}_{k,-2}}(\mathfrak{sl}_2)$, such that

$$\{\lambda_n, \lambda_m\} = 2h(q - q^{-1})\frac{[n]_q^2}{[2n]_q}\delta_{n,-m} \tag{9.1}$$

(recall that $q = e^h$ and that $h \notin 2\pi i\mathbb{Q}$).

We define a Poisson bracket on the center of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_2)_{\mathrm{cr}}$ in the same way as in Sect. 2.1, as the leading term in the commutator of liftings of central elements. Formula (8.6) shows that the images of the Fourier coefficients of $\ell(z)$ under the homomorphism ϕ_h^{cr} lie in $\mathcal{H}_h(\mathfrak{sl}_2)$. As was explained in Sect. 2.2, we can take liftings of the images of central elements inside the deformation $\widetilde{\mathcal{H}_{h,k}}(\mathfrak{sl}_2)$ of $\mathcal{H}_h(\mathfrak{sl}_2)$. This shows that the homomorphism from the center to $\mathcal{H}_h(\mathfrak{sl}_2)$ is a homomorphism of Poisson algebras. We can use this fact to compute the Poisson brackets between Fourier coefficients of $s(z)$.

From formulas (9.1) and (8.7) we obtain

$$\{\Lambda(z), \Lambda(w)\} = 2h(q - q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}\Lambda(z)\Lambda(w)\,. \tag{9.2}$$

According to formula (8.6), we have:

$$\begin{aligned}\{s(z), s(w)\} &= \{\Lambda(zq), \Lambda(wq)\} + \{\Lambda(zq), \Lambda(wq^{-1})^{-1}\} \\ &\quad + \{\Lambda(zq^{-1})^{-1}, \Lambda(wq)\} + \{\Lambda(zq^{-1})^{-1}, \Lambda(wq^{-1})^{-1}\}\,.\end{aligned}$$

Substituting (9.2) into this formula, we obtain:

$$\begin{aligned}\{s(z), s(w)\} &= 2h(q - q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}\Lambda(zq)\Lambda(wq) \\ &\quad - 2h(q - q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}q^{-2m}\Lambda(zq)\Lambda(wq^{-1})^{-1} \\ &\quad - 2h(q - q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}q^{2m}\Lambda(zq^{-1})^{-1}\Lambda(wq) \\ &\quad + 2h(q - q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m \frac{[m]_q^2}{[2m]_q}\Lambda(zq^{-1})^{-1}\Lambda(wq^{-1})^{-1}\end{aligned}$$

$$
\begin{aligned}
&= 2h(q-q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m\frac{[m]_q^2}{[2m]_q}s(z)s(w)\\
&\quad -2h\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m(1-q^{-2m})\Lambda(zq)\Lambda(wq^{-1})^{-1}\\
&\quad -2h\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m(q^{2m}-1)\Lambda(zq^{-1})^{-1}\Lambda(wq)\,.
\end{aligned}
$$

The last two terms give us

$$
\begin{aligned}
&-2h\delta\left(\frac{w}{z}\right)\Lambda(zq)\Lambda(wq^{-1})^{-1}+2h\delta\left(\frac{w}{zq^2}\right)\Lambda(zq)\Lambda(wq^{-1})^{-1}\\
&-2h\delta\left(\frac{wq^2}{z}\right)\Lambda(zq^{-1})^{-1}\Lambda(wq)+2h\delta\left(\frac{w}{z}\right)\Lambda(zq^{-1})^{-1}\Lambda(wq)\\
&=2h\delta\left(\frac{w}{zq^2}\right)-2h\delta\left(\frac{wq^2}{z}\right),
\end{aligned}
$$

where

$$\delta(x)=\sum_{m\in\mathbb{Z}}x^m\,.$$

Finally, we obtain

Theorem 2.

$$
\begin{aligned}
\{s(z),s(w)\} &= 2h(q-q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m\frac{[m]_q^2}{[2m]_q}\,s(z)s(w)\\
&\quad +2h\delta\left(\frac{w}{zq^2}\right)-2h\delta\left(\frac{wq^2}{z}\right). \qquad (9.3)
\end{aligned}
$$

This implies that the Poisson bracket between $\ell(z)$ and $\ell(w)$ is given by formula (1.3).

Formula (9.3) gives us the following formula for the Poisson bracket between the Fourier coefficients of $s(z)$:

$$\{s_n,s_m\}=2h\sum_{l\in\mathbb{Z}}\frac{q^l-q^{-l}}{q^l+q^{-l}}s_{n-l}s_{m+l}-2h(q^{2n}-q^{-2n})\delta_{n,-m}\,, \qquad (9.4)$$

where we put

$$s(z)=\sum_{n\in\mathbb{Z}}s_nz^{-n}\,.$$

The elements s_n generate a Poisson algebra $\mathscr{W}_h(\mathfrak{sl}_2)$, which is a q-deformation of the classical Virasoro algebra $\mathscr{W}(\mathfrak{sl}_2)$. This Poisson algebra is embedded into $\mathscr{H}_h(\mathfrak{sl}_2)$ via the q-deformation of the Miura transformation (8.6). The Poisson algebra $Z_h(\widehat{\mathfrak{sl}}_2)$ is isomorphic to $\mathscr{W}_h(\mathfrak{sl}_2)$.

Remark 6. In the limit $h\to 0$ we have:

$$\ell(z)=2+4h^2\left(z^2S(z)+\frac{1}{4}\right)+h^3(\cdots)$$

(cf. [10]). On the other hand

$$\Lambda(z)=1-h(z\chi(z)+1)+h^2(\cdots)\,.$$

Substituting these formulas into formula (8.6) and expanding in powers of h up to h^2, we obtain:

$$2 + 4h^2 z^2 S(z) + h^2 = 2 + h^2 (z\chi(z) + 1)^2 - 2h^2 z\partial_z (z\chi(z)) ,$$

which coincides with the Miura transformation (2.4).

Now let us consider formula (9.4). The leading term in the expansion of the left-hand side is $16h^4\{S_n, S_m\}$. Expanding the right-hand side of (9.4) in powers of h up to h^4, we obtain for the first term

$$8h^2 n\delta_{n,-m} + 16h^4(n-m)\left(S_{n+m} + \frac{1}{4}\delta_{n,-m}\right) - h^4\frac{8}{3}n^3\delta_{n,-m} ,$$

and for the second term

$$-8h^2 n\delta_{n,-m} - h^4\frac{16}{3}n^3\delta_{n,-m} .$$

Taking the sum, we see that the leading term in the right-hand side is $16h^4$ times the right-hand side of formula (2.2).

Note that we can obtain a different Poisson algebra by placing an arbitrary overall factor in the right-hand side of formula (9.3). In particular, if we put the overall factor $-c/6$ in the right-hand side of the formula, then in the limit $h \longrightarrow 0$ we will recover the classical Virasoro algebra with central charge c.

We can also replace the overall factor $2h$ in the right-hand side by $(q - q^{-1})$ without changing the asymptotics $h \longrightarrow 0$. After that we can consider $\mathcal{W}_h(\mathfrak{sl}_2)$ as a Poisson algebra over the ring of rational functions in q. $\square$

Remark 7. Formula (9.3) gives another asymptotics as $h \longrightarrow 0$, if we postulate that $s(z)$ does not depend on h. If we divide the right-hand side of the formula by $2h(q-q^{-1})$, we obtain for $h = 0$:

$$\{s(z), s(w)\} = \frac{1}{2}\delta'\left(\frac{w}{z}\right) s(z)s(w) - 2\delta'\left(\frac{w}{z}\right) .$$

The corresponding limiting Poisson algebra $\widetilde{\mathcal{W}}(\mathfrak{sl}_2)$ has a nice interpretation in terms of the algebra $\mathcal{A}_k(\widehat{SL}_2^*)$ of functions on the Poisson–Lie group $\widehat{SL}_2^*$ dual to $\widehat{SL}_2$ (this algebra is the classical limit of $U_h(\widehat{\mathfrak{sl}}_2)_k$, cf. [9]). Namely, the limits of the coefficients of $\ell(z)$ generate a central Poisson subalgebra of $\mathcal{A}_0(\widetilde{SL}_2^*)$ [9]. Deforming the level k we obtain a new Poisson structure on this subalgebra, which coincides with $\widetilde{\mathcal{W}}(\mathfrak{sl}_2)$. We will discuss this in more detail in [23]. $\square$

10. Generalization to $U_q(\widehat{\mathfrak{sl}}_N)$

10.1. A q-deformation of the Heisenberg–Poisson algebra. For a simple Lie algebra $\mathfrak{g}$ of rank l, denote by $B = (B_{ij})_{i,j=1,\dots,l}$ the symmetrized Cartan matrix of $\mathfrak{g}$; recall that $B_{ij} = (\alpha_i, \alpha_j)$. Let $\mathcal{H}_{h,k}(\mathfrak{g})$ be the Heisenberg algebra with generators $a_i[n]$, $i = 1, \dots, l$; $n \in \mathbb{Z}$, and relations

$$[a_i[n], a_i[m]] = \frac{1}{n}[(k + h^\vee)n]_q [B_{ij}n]_q \delta_{n,-m} .$$

The algebra $\mathscr{H}_{h,k}(\mathfrak{g})$ appears in [14] in the construction of the Wakimoto realization of $U_h(\widehat{\mathfrak{sl}}_N)_k$.

The family $\mathscr{H}_{h,k}(\mathfrak{g})$ induces a Poisson structure on the commutative algebra $\mathscr{H}_{h,-h^\vee}(\mathfrak{g})$. The Poisson brackets between the generators of $\mathscr{H}_{h,-h^\vee}(\mathfrak{g})$ are

$$\{a_i[n], a_j[m]\} = \frac{2h}{q-q^{-1}}[B_{ij}n]_q \delta_{n,-m} \,. \tag{10.1}$$

Let $\mathscr{H}_h(\mathfrak{g})$ be the completion of $\mathscr{H}_{h,-h^\vee}(\mathfrak{g})$ defined in the same way as $\mathscr{H}_h(\mathfrak{sl}_2)$.

Consider the case $\mathfrak{g} = \mathfrak{sl}_N$. Introduce new generators $\lambda_i[n]$, $i = 1,\dots,N$; $n \in \mathbb{Z}$, of $\mathscr{H}_h(\mathfrak{sl}_N)$, which are related to the generators $a_i[n]$ by the formulas

$$\lambda_i[n] - \lambda_{i+1}[n] = q^{ni}(q-q^{-1})a_i[n], \quad i = 1,\dots,N-1;\ n \in \mathbb{Z}\,, \tag{10.2}$$

and which satisfy the linear relation

$$\sum_{i=1}^{N} q^{2(1-i)n}\lambda_i[n] = 0\,. \tag{10.3}$$

We find from formulas (10.2) and (10.3) the inverse change of variables:

$$\lambda_1[n] = (q-q^{-1})\sum_{j=1}^{N-1}\frac{[(N-j)n]_q}{[Nn]_q}a_j[n]\,,$$

$$\lambda_2[n] = -q^{Nn}(q-q^{-1})\frac{[n]_q}{[Nn]_q}a_1[n] + (q-q^{-1})\sum_{j=2}^{N-1}\frac{[(N-j)n]_q}{[Nn]_q}a_j[n]\,,$$

$$\dots$$

$$\lambda_N[n] = -q^{Nn}(q-q^{-1})\sum_{j=1}^{N-1}\frac{[jn]_q}{[Nn]_q}a_j[n]\,.$$

From these formulas and the brackets (10.1) we find

$$\{\lambda_i[n], \lambda_i[m]\} = 2h(q-q^{-1})\frac{[(N-1)n]_q[n]_q}{[Nn]_q}\delta_{n,-m}\,, \tag{10.4}$$

$$\{\lambda_i[n], \lambda_j[m]\} = -2h(q-q^{-1})\frac{[n]_q^2}{[Nn]_q}q^{-Nn}\delta_{n,-m}, \quad i<j\,. \tag{10.5}$$

Introduce the generating functions

$$\Lambda_i(z) = q^{-N+2i-1}\exp\left(-\sum_{m\in\mathbb{Z}}\lambda_i[m]z^{-m}\right). \tag{10.6}$$

From (10.4) and (10.5) we find:

$$\{\Lambda_i(z), \Lambda_i(w)\} = 2h(q-q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m\frac{[(N-1)m]_q[m]_q}{[Nm]_q}\Lambda_i(z)\Lambda_i(w)\,, \tag{10.7}$$

$$\{\Lambda_i(z), \Lambda_j(w)\} = -2h(q-q^{-1})\sum_{m\in\mathbb{Z}}\left(\frac{w}{zq^N}\right)^m\frac{[m]_q^2}{[Nm]_q}\Lambda_i(z)\Lambda_j(z), \quad i<j\,. \tag{10.8}$$

10.2. A q-deformation of the $\mathcal{W}$-algebra. Let us define generating functions $s_i(z)$, $i = 0,\dots,N$, whose coefficients lie in $\mathscr{H}_{h(\mathfrak{sl}_N)}$: $s_0 = 1$, and

$$s_i(z) = \sum_{1 \leqq j_1 < \cdots < j_i \leqq N} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2)\cdots\Lambda_{j_{i-1}}(zq^{2(i-2)})\Lambda_{j_i}(zq^{2(i-1)}), \tag{10.9}$$

$i = 1,\dots,N$. In particular,

$$s_1(z) = \sum_{j=1}^{N} \Lambda_j(z),$$

$$s_2(z) = \sum_{1 \leqq j_1 < j_2 \leqq N} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2),$$

etc. Formula (10.3) implies that

$$s_N(z) = \Lambda_1(z)\Lambda_2(zq^2)\cdots\Lambda_N(zq^{2N-2}) = 1.$$

These formulas coincide with formulas for spectra of transfer-matrices in integrable models associated to $U_q(\widehat{\mathfrak{sl}}_N)$ [18, 21]. Note that for $\widehat{\mathfrak{sl}}_2$ we have $s_1(z) = s(zq^{-1})$, where $s(z)$ is given by formula (8.6).

The coefficients of the series $s_i(z)$, $i = 1,\dots,N-1$, generate a Poisson subalgebra $\mathcal{W}_h(\mathfrak{sl}_N)$ of $\mathscr{H}_{h(\mathfrak{sl}_N)}$. The relations between them can be computed directly from formulas (10.7) and (10.8). Introduce the functions

$$C_{ij}(x) = \sum_{m \in \mathbb{Z}} C_{ij}^{(m)} x^m, \quad i,j = 1,\dots,N-1, \tag{10.10}$$

where

$$C_{ij}^{(m)} = \frac{[(N - \max\{i,j\})m]_q [\min\{i,j\}m]_q}{[Nm]_q}. \tag{10.11}$$

The relations are:

$$\begin{aligned} \{s_i(z), s_j(w)\} &= 2h(q - q^{-1})C_{ij}\left(\frac{wq^{j-i}}{z}\right) s_i(z)s_j(w) \\ &+ 2h\sum_{p=1}^{i} \delta\left(\frac{w}{zq^{2p}}\right) s_{i-p}(w)s_{j+p}(z) \\ &- 2h\sum_{p=1}^{i} \delta\left(\frac{wq^{2(j-i+p)}}{z}\right) s_{i-p}(z)s_{j+p}(w), \end{aligned}$$

if $i \leqq j$ and $i + j \leqq N$; and

$$\begin{aligned} \{s_i(z), s_j(w)\} &= 2h(q - q^{-1})C_{ij}\left(\frac{wq^{j-i}}{z}\right) s_i(z)s_j(w) \\ &+ 2h\sum_{p=1}^{N-j} \delta\left(\frac{w}{zq^{2p}}\right) s_{i-p}(w)s_{j+p}(z) \\ &- 2h\sum_{p=1}^{N-j} \delta\left(\frac{wq^{2(j-i+p)}}{z}\right) s_{i-p}(z)s_{j+p}(w), \end{aligned}$$

if $i \leqq j$ and $i + j > N$.

For example, in the case of $\mathfrak{sl}_3$ we have the following relations:

$$\{s_1(z), s_1(z)\} = 2h(q-q^{-1}) \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{[2m]_q[m]_q}{[3m]_q} s_1(z)s_1(w)$$
$$+ 2h\delta\left(\frac{w}{zq^2}\right) s_2(z) - 2h\delta\left(\frac{wq^2}{z}\right) s_2(w)\,,$$
$$\{s_1(z), s_2(w)\} = 2h(q-q^{-1}) \sum_{m\in\mathbb{Z}} \left(\frac{wq}{z}\right)^m \frac{[m]_q^2}{[3m]_q} s_1(z)s_2(w)$$
$$+ 2h\delta\left(\frac{w}{zq^2}\right) - 2h\delta\left(\frac{wq^4}{z}\right),$$
$$\{s_2(z), s_2(w)\} = 2h(q-q^{-1}) \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{[2m]_q[m]_q}{[3m]_q} s_2(z)s_2(w)$$
$$+ 2h\delta\left(\frac{w}{zq^2}\right) s_1(w) - 2h\delta\left(\frac{wq^2}{z}\right) s_1(z)\,.$$

The Poisson algebra $\mathcal{W}_h(\mathfrak{sl}_N)$ is a q-deformation of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_N)$. The asymptotic expansion of $s_i(z)$ around $h=0$ has the form

$$s_i(z) = \binom{N}{i} + h^2 C_i W_2(z) + \cdots\,,$$

where C_i is some coefficient, and $W_2(z)$ is the quadratic (Virasoro) generating series of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_N)$. For each $n=2,\ldots,N-1$, one can find a combination of $s_i(z)$'s having expansion of the form $M + h^n W_n(z) + \cdots$, where M is a constant and $W_n(z)$ is a multiple of an n^{th} order generating series of $\mathcal{W}(\mathfrak{sl}_N)$. The Poisson structure on $\mathcal{W}(\mathfrak{sl}_N)$ can be recovered from that on $\mathcal{W}_h(\mathfrak{sl}_N)$.

Remark 8. If we replace $2h$ by $(q-q^{-1})$ in the formulas above, we will obtain a Poisson algebra over the ring of rational functions in q. □

Remark 9. As in the case of $\mathfrak{sl}_2$ (cf. Remark 7), the Poisson algebra $\mathcal{W}_h(\mathfrak{sl}_N)$ has another limit as $h \to 0$, which can be interpreted in terms of the central subalgebra of the algebra of functions on the Poisson–Lie group dual to $\widehat{SL}_N$ [23]. □

10.3. The center of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)$ at the critical level. Following [9], cf. also [31, 10], for any finite-dimensional representation W of $U_q(\widehat{\mathfrak{g}})$, one can construct matrices $L_W^{\pm}(z) = (L_W^{\pm}(z))_{i,j=1,\ldots,\dim W}$ consisting of generating functions of elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\text{cr}}$. It is shown in [9] that the Fourier coefficients of the power series

$$\ell^W(z) = \operatorname{tr}_W q^{2\rho} L_W^+(z) L_W^-(zq^{-h^\vee})^{-1} \tag{10.12}$$

are central elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\text{cr}}$.

In particular, for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_N$, let $\ell_i(z) \equiv \ell^{W_{\omega_i}}(z)$ be the generating function of central elements of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\text{cr}}$ corresponding to the i^{th} fundamental representation W_{ω_i} of $U_q(\widehat{\mathfrak{sl}}_N)$. Note that for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ we have $\ell_1(z) = \ell(zq^{-1})$, where $\ell(z)$ is given by (6.1).

The next to leading term in the h-expansion of $\ell_i(z)$ is a multiple of the Sugawara series of $\widetilde{U}(\widehat{\mathfrak{sl}}_N)_{\text{cr}}$, cf. [10]. Higher Sugawara elements of $\widetilde{U}(\widehat{\mathfrak{sl}}_N)_{\text{cr}}$ can be obtained from higher order terms of the expansions of $\ell_i(z)$'s.

Let $Z_h(\widehat{\mathfrak{sl}}_N)$ be the completion of the central subalgebra of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ generated by the coefficients of the series $\ell_1(z),\ldots,\ell_{N-1}(z)$. Using the Wakimoto realization of $\widetilde{U}_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ [14], we obtain a homomorphism of Poisson algebras $Z_h(\widehat{\mathfrak{sl}}_N) \to \mathscr{H}_{h(\mathfrak{sl}_N)}$, which we call the q-deformation of the Miura transformation.

Using the method of Remark 4 we can find the image of $\ell_1(z)$ in $\mathscr{H}_{h(\mathfrak{sl}_N)}$ by computing the matrix element of $\ell_1(z)$ between the generating vector of a Wakimoto module over $U_h(\widehat{\mathfrak{sl}}_N)_{\mathrm{cr}}$ and its dual. But for that we only need the diagonal part of the "Gauss decomposition" of $L^{\pm}(z)$ [12] and a formula expressing the corresponding diagonal elements $k_i^{\pm}(z)$, $i=1,\ldots,N$, in terms of $\Lambda_i(z)$, $i=1,\ldots,N-1$. This formula can be obtained from [14] and (10.6). Explicit computation shows that the image of $\ell_1(z)$ in $\mathscr{H}_{h(\mathfrak{sl}_N)}$ is equal to $s_1(z)$.

The generating functions $\ell_i(z)$, $i=2,\ldots,N-1$, corresponding to other fundamental representations can be expressed in terms of $\ell_1(z)$ by the fusion procedure, cf. [34, 35] and references therein. Using this procedure, we can show that the image of $\ell_i(z)$ in $\mathscr{H}_{h(\mathfrak{sl}_N)}$ is equal to $s_i(z)$ given by formula (10.9) for all $i=1,\ldots,N-1$. Thus, we obtain that $Z_h(\widehat{\mathfrak{sl}}_N)$ is isomorphic to $\mathscr{W}_h(\mathfrak{sl}_N)$ as a Poisson algebra. We will discuss this isomorphism in more detail in [23].

In conclusion of this section, recall that elements of the spectrum of $\mathscr{W}(\mathfrak{sl}_N)$ can be considered as N^{th} order differential operators, cf. [2]. The classical Miura transformation corresponds to splitting of such an operator into a product of first order operators.

The spectrum of $\mathscr{W}_h(\widehat{\mathfrak{sl}}_N)$ and the q-deformation of the Miura transformation can be interpreted in a similar fashion. Namely, elements of the spectrum of $\mathscr{H}_{h(\mathfrak{sl}_N)}$ can be considered as first order q-difference operators, and elements of the spectrum of $Z_h(\widehat{\mathfrak{sl}}_N)$ can be considered as N^{th} order q-difference operators. The q-deformation of the Miura transformation corresponds to the splitting of the N^{th} order operator into a product of first order operators.

Indeed, we can write:

$$\begin{aligned}&\mathscr{D}_q^N - s_{N-1}(z)\mathscr{D}_q^{N-1} + s_{N-2}(z)\mathscr{D}_q^{N-2} - \cdots - (-1)^N s_1(z)\mathscr{D}_q + (-1)^N \\ &\quad = (\Lambda_1(z)\mathscr{D}_q - 1)(\Lambda_2(z)\mathscr{D}_q - 1)\cdots(\Lambda_N(z)\mathscr{D}_q - 1)\,,\end{aligned}$$

where $[\mathscr{D}_q f](z) = f(zq^2)$.

Let $Q_i(z)$, $i=1,\ldots,N-1$, be solutions of the q-difference equations

$$(\mathscr{D}_q - \Lambda_i(z))\frac{Q_i(z)}{Q_{i-1}(zq^{-2})} = 0\,, \quad i=1,\ldots,N\,,$$

where we put $Q_0(z) = Q_N(z) = 1$. Then $Q_{N-1}(z)$ satisfies the q-difference equation

$$(\mathscr{D}_q^N - s_{N-1}(z)\mathscr{D}_q^{N-1} - \cdots - (-1)^N s_1(z)\mathscr{D}_q + (-1)^N)Q_{N-1}(zq^{-2}) = 0.$$

Thus, elements of the spectrum of $\mathscr{W}_h(\mathfrak{sl}_N)$ can be considered as q-difference operators of the form

$$\mathscr{D}_q^N - s_{N-1}(z)\mathscr{D}_q^{N-1} - \cdots - (-1)^N s_1(z)\mathscr{D}_q + (-1)^N\,.$$

11. Generalization to Other Quantum Affine Algebras

Let us first adopt notation we used in the case of $\widehat{\mathfrak{sl}}_N$ to the general case. We pass to another set of generators of $\mathscr{H}_h(\mathfrak{sl}_N)$, $y_i[n]$, $i=1,\ldots,N-1$; $n \in \mathbb{Z}$, such that

$$\lambda_i[n] = q^{(i-1)n} y_i[n] - q^{in} y_{i-1}[n], \quad i = 1, \ldots, N ,$$

and $y_0[n] = y_N[n] = 0$. Using formulas (10.4) and (10.5) we find the following Poisson brackets:

$$\{y_i[n], y_j[m]\} = 2h(q-q^{-1})C_{ij}^{(m)}\delta_{n,-m} , \tag{11.1}$$

where $C_{ij}^{(m)}$ is given by (10.11),

$$\{y_i[n], a_j[m]\} = 2h[n]_q\delta_{n,-m}\delta_{ij} . \tag{11.2}$$

Thus, the generators $y_i[n]$ are "dual" to the generators $a_i[n]$. In fact, it is easy to see that

$$B^{(m)}C^{(m)} = [m]_q^2 I_{N-1} , \tag{11.3}$$

where $C^{(m)} = (C_{ij}^{(m)})_{i,j=1,\ldots,N-1}$ and $B^{(m)} = ([B_{ij}m]_q)_{i,j=1,\ldots,N-1}$, $(B_{ij})_{i,j=1,\ldots,N-1}$ being the Cartan matrix of $\mathfrak{sl}_N$.

Introduce generating functions

$$Y_i(z) = q^{-i(N-i)} \exp\left(-\sum_{m\in\mathbb{Z}} y_i[m]z^{-m}\right) .$$

We have:

$$\Lambda_i(z) = Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \quad i = 1, \ldots, N ,$$

where we put $Y_0(z) = Y_N(z) = 1$. Note that $Y_i(z)$ can be written as $Q_i(zq^{i+1})/Q_i(zq^{i-1})$ in terms of $Q_i(z)$ introduced at the end of last section.

From formula (11.1) we find the Poisson brackets between $Y_i(z)$ and $Y_j(w)$:

$$\{Y_i(z), Y_j(w)\} = 2h(q-q^{-1})C_{ij}\left(\frac{w}{z}\right) Y_i(z)Y_j(w) ,$$

where $C_{ij}(x)$ is given by formula (10.10).

We now define analogous generating functions $Y_i(z)$, $i = 1, \ldots, l$, for an arbitrary simple Lie algebra $\mathfrak{g}$. Namely, let $y_i[n]$, $i = 1, \ldots, l$; $n \in \mathbb{Z}$, be the elements of $\mathscr{H}_h(\mathfrak{g})$ uniquely defined by the Poisson bracket (11.2). We put

$$Y_i(z) = q^{-2(\rho,\omega_i)} \exp\left(-\sum_{m\in\mathbb{Z}} y_i[m]z^{-m}\right) ,$$

where ω_i is the i^{th} fundamental weight of $\mathfrak{g}$. Then we have:

$$\{Y_i(z), Y_j(w)\} = 2h(q-q^{-1})C_{ij}\left(\frac{w}{z}\right) Y_i(z)Y_j(w) , \tag{11.4}$$

where

$$C_{ij}(x) = \sum_{m\in\mathbb{Z}} C_{ij}^{(m)}x^m ,$$

and the matrix $(C_{ij}^{(m)})_{i,j=1,\ldots,l}$ is defined by formula (11.3) (with $N-1$ replaced by l) with respect to the symmetrized Cartan matrix B of $\mathfrak{g}$.

Remark 10. It is interesting that the functions $C_{ij}(x)$ appear in the Thermodynamic Bethe Ansatz equations [36, 21]. □

For each dominant integral highest weight λ of $\mathfrak{g}$ there exists an irreducible finite-dimensional representation W_λ of $U_q(\widehat{\mathfrak{g}})$ which satisfies the following property. Its restriction to the subalgebra $U_q(\mathfrak{g})$ is completely reducible, the irreducible representation of $U_q(\mathfrak{g})$ with highest weight λ has multiplicity one, and all other irreducible components of W_λ have highest weights $\mu < \lambda$.

Let $\ell_i(z)$, $i = 1, \ldots, l$, be the generating functions of central elements of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ corresponding to W_{ω_i}. Let $Z_h(\widehat{\mathfrak{g}})$ be the central subalgebra of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$ generated by the coefficients of $\ell_i(z)$, $i = 1, \ldots, l$. Recall from Sects. 1, 2 that the Miura transformation is the homomorphism of Poisson algebras from the center of $\widetilde{U}(\widehat{\mathfrak{g}})$ to the Heisenberg–Poisson algebra $\mathscr{H}(\mathfrak{g})$.

Conjecture 1. (a) *$Z_h(\widehat{\mathfrak{g}})$ is closed with respect to the Poisson structure on the center of $\widetilde{U}_h(\widehat{\mathfrak{g}})_{\mathrm{cr}}$.*

(b) *There exists a homomorphism of Poisson algebras $Z_h(\widehat{\mathfrak{g}}) \to \mathscr{H}_h(\mathfrak{g})$, which is a deformation of the Miura transformation.*

(c) *The formulas for the images $s_i(z)$ of the generating functions $\ell_i(z)$ from $Z_h(\widehat{\mathfrak{g}})$ in $\mathscr{H}_h(\mathfrak{g})$ coincide with the formulas for spectra of the corresponding transfer-matrices in integrable models associated to $U_q(\widehat{\mathfrak{g}})$.*

Formulas for spectra of transfer-matrices in integrable models associated to $U_q(\widehat{\mathfrak{g}})$ have been given in [19–22] (although in different normalizations). We will now describe $s_i(z)$'s for all quantum affine algebras of classical types via the series $Y_i(z)$, $i = 1, \ldots, l$ (we put $Y_0(z) = 1$). The Poisson brackets between $Y_i(z)$'s given by (11.4) uniquely determine the Poisson brackets between $s_i(z)$'s.

11.1. The series $\widehat{\mathfrak{g}} = A_n^{(1)}$. Introduce

$$\Lambda_i(z) = Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \quad i = 1, \ldots, n+1 .$$

Let

$$s_i(z) = \sum_{1 \leqq j_1 < \cdots < j_i \leqq n+1} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2) \cdots \Lambda_{j_{i-1}}(zq^{2(i-2)})\Lambda_{j_i}(zq^{2(i-1)}), \quad i = 1, \ldots, n .$$

11.2. The series $\widehat{\mathfrak{g}} = B_n^{(1)}$. Introduce

$$\Lambda_i(z) = Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \quad i = 1, \ldots, n-1 ,$$

$$\Lambda_n(z) = Y_n(zq^{-n+3/2})Y_n(zq^{-n+1/2})Y_{n-1}(zq^{-n})^{-1} ,$$

$$\Lambda_{n+1}(z) = Y_n(zq^{-n+3/2})Y_n(zq^{-n-1/2})^{-1} ,$$

$$\Lambda_{n+2}(z) = Y_{n-1}(zq^{-n+1})Y_n(zq^{-n+1/2})^{-1}Y_n(zq^{-n-1/2})^{-1} ,$$

$$\Lambda_{2n-i+2}(z) = Y_{i-1}(zq^{-2n+i+1})Y_i(zq^{-2n+i})^{-1} \quad i = 1, \ldots, n-1 .$$

Let

$$s_i(z) = \sum_{\{j_1,\ldots,j_i\} \in S} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2) \cdots \Lambda_{j_i}(zq^{2i-2}), \quad i = 1, \ldots, n-1 ,$$

where S is the set of $\{j_1, \ldots, j_i\}$, such that $j_\alpha < j_{\alpha+1}$ or $j_\alpha = j_{\alpha+1} = n+1$, $\alpha = 1, \ldots, i-1$.

The formula for $s_n(z)$, which corresponds to the spinor representation of $B_n^{(1)}$, is more complicated:

$$s_n(z) = \sum_{\sigma_1,\ldots,\sigma_n = \pm 1} b_{\sigma_1}(z|n)b_{\sigma_2}(zq^{1-\sigma_1}|n-1) \cdots b_{\sigma_n}(zq^{n-1-\sigma_1-\cdots-\sigma_{n-1}}|1) ,$$

where

$$
\begin{aligned}
b_1(z|1) &= Y_n(zq^{-n-1/2})^{-1} , \\
b_1(z|k) &= 1, \quad k = 2,\dots,n ; \\
b_{-1}(z|1) &= Y_{n-1}(zq^{-n})^{-1} Y_n(zq^{-n+1/2}) , \\
b_{-1}(z|k) &= Y_{n-k}(zq^{-n+k-1})^{-1} Y_{n+1-k}(zq^{-n+k}), \quad k = 2,\dots,n .
\end{aligned}
$$

11.3. The series $\widehat{\mathfrak{g}} = C_n^{(1)}$. Introduce

$$
\begin{aligned}
\Lambda_i(z) &= Y_i(zq^{-(i-1)/2}) Y_{i-1}(zq^{-i/2})^{-1}, \quad i = 1,\dots,n , \\
\Lambda_{2n-i+1}(z) &= Y_{i-1}(zq^{-(2n-i+2)/2}) Y_i(zq^{-(2n-i+3)/2})^{-1}, \quad i = 1,\dots,n .
\end{aligned}
$$

Let

$$
s_i(z) = \sum_{\{j_1,\dots,j_i\}\in S} \Lambda_{j_1}(z)\Lambda_{j_2}(zq)\cdots\Lambda_{j_i}(zq^{i-1}), \quad i = 1,\dots,n ,
$$

where S is the set of $\{j_1,\dots,j_i\}$, such that $1 \leqq j_1 < \cdots < j_i \leqq 2n$ and if $j_\alpha = l$, $j_\beta = 2n+1-l$ for some $l = 1,\dots,n$, then $l \leqq n+\alpha-\beta$.

11.4. The series $\widehat{\mathfrak{g}} = D_n^{(1)}$. Introduce

$$
\begin{aligned}
\Lambda_i(z) &= Y_i(zq^{-i+1}) Y_{i-1}(zq^{-i})^{-1}, \quad i = 1,\dots,n-2 , \\
\Lambda_{n-1}(z) &= Y_n(zq^{-n+2}) Y_{n-1}(zq^{-n+2}) Y_{n-2}(zq^{-n+1})^{-1} , \\
\Lambda_n(z) &= Y_{n-1}(zq^{-n+2}) Y_n(zq^{-n})^{-1} , \\
\Lambda_{n+1}(z) &= Y_n(zq^{-n+2}) Y_{n-1}(zq^{-n})^{-1} , \\
\Lambda_{n+2}(z) &= Y_{n-2}(zq^{-n+1}) Y_{n-1}(zq^{-n})^{-1} Y_n(zq^{-n})^{-1} , \\
\Lambda_{2n-i+1}(z) &= Y_{i-1}(zq^{-2n+i+2}) Y_i(zq^{-2n+i+1})^{-1} .
\end{aligned}
$$

Let

$$
s_i(z) = \sum_{\{j_1,\dots,j_i\}\in S} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2)\cdots\Lambda_{j_i}(zq^{2i-2}), \quad i = 1,\dots,n-2 ,
$$

where S is the set of $\{j_1,\dots,j_i\}$, such that $j_\alpha < j_{\alpha+1}$ or $j_\alpha = j_{\alpha+1}+1 = n+1$, $\alpha = 1,\dots,i-1$.

The formulas for $s_{n-1}(z)$ and $s_n(z)$, which correspond to the spinor representations of $D_n^{(1)}$, are more complicated. In these formulas the subscript ε means n, if $\varepsilon = +$, and $n-1$, if $\varepsilon = -$. Thus, $s_+(z) = s_n(z)$, $s_-(z) = s_{n-1}(z)$. Now let

$$
s_\varepsilon(z) = \sum_{\sigma_1,\dots,\sigma_{n-1}=\pm 1} b^{\varepsilon}_{\sigma_1}(z|n)\, b^{\varepsilon\sigma_1}_{\sigma_2}(zq^{1-\sigma_1}|n-1)\cdots b^{\varepsilon\sigma_1\cdots\sigma_{n-1}}_{\sigma_{n-1}}(zq^{n-2-\sigma_1-\cdots-\sigma_{n-2}}|2) ,
$$

where

$$b_1^{\varepsilon}(z|2) = Y_{\varepsilon}(zq^{-n})^{-1}\,,$$
$$b_1^{\varepsilon}(z|k) = 1, \quad k = 3,\ldots,n\,;$$
$$b_{-1}^{\varepsilon}(z|2) = Y_{n-2}(zq^{-n+1})^{-1}Y_{\varepsilon}(zq^{-n+2})\,,$$
$$b_{-1}^{\varepsilon}(z|k) = Y_{n-k}(zq^{-n+k-1})^{-1}Y_{n+1-k}(zq^{-n+k}), \quad k = 3,\ldots,n\,.$$

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