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Stable Singularities in String Theory

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Abstract: We study a topological obstruction of a very stringy nature concerned with deforming the target space of an $N = 2$ non-linear σ-model. This target space has a singularity which may be smoothed away according to the conventional rules of geometry, but when one studies the associated conformal field theory one sees that such a deformation is not possible without a discontinuous change in some of the correlation functions. This obstruction appears to come from torsion in the homology of the target space (which is seen by deforming the theory by an irrelevant operator). We discuss the link between this phenomenon and orbifolds with discrete torsion as studied by Vafa and Witten.

1. Introduction

A very interesting aspect of string theory is the way in which space-time is described. In physics, thanks to the success of general relativity, we are accustomed to picturing space-time as being a manifold equipped with a metric. The physics of space-time is then described in terms of this metric. Such a picture has some potential shortcomings. In particular we may wish to consider some space-time which is not smooth and thus may not admit a metric in the conventional sense. One way to treat such a space may be as a limit of a sequence of smooth manifolds which converges to the desired space. Thus the "metric" on the singular space is approximated by this sequence of smooth metrics.

While such a picture appears natural from a viewpoint of general relativity it may be that it is not so natural from a string theory point of view. In this paper we illustrate precisely this point by considering a singular space which classically appears as the limit of a sequence of smooth manifolds and then showing that a string theory on the singular space cannot be deformed into a string theory on any of the smooth manifolds which approximate it.

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The framework in which we will work is one of the most successful for studying stringy aspects of geometry. That is, we look at $N = (2,2)$ superconformal field theories and their associated Calabi–Yau target spaces. We also restrict ourselves in this paper to the rich class of complex dimension three target spaces. The usefulness of $N = (2,2)$ theories is that one can understand the geometry of the target Calabi–Yau manifold without any explicit reference to the target space metric (see, for example, [2] for a review). One may also study many singular spaces, such as orbifolds [18], without any inherent difficulties.

The deformations of a Calabi–Yau manifold can be understood in terms of marginal operators in the associated conformal field theory. If the target space is singular, rather than being a manifold, the marginal operators presumably still tell one how to deform the singular space. Certain of these deformations may remove some, or perhaps all, of the singularities. This process is well-understood in many cases of orbifolds (see, for example, [3]) where twisted marginal operators in the conformal field theory can be matched to the "blow-ups" of the orbifold, i.e., deformations which resolve (at least partially) the quotient singularities of the orbifold.

In [28] some examples of more troublesome orbifolds were studied. It was found that certain of the deformations of the classical orbifold appeared to be "missing" in the conformal field theory language. That is, these geometric deformations could not be seen by the string theory. The purpose of this paper is to shed some light on the geometrical explanation for such a phenomenon. We will see that there is a truly stringy explanation for such obstructions. These obstructions are due to world-sheet instantons wrapping themselves around particular elements of the second homology group of the target space.

The construction of [28] rests upon the study of "discrete torsion." There is some potential for confusion on the subject of discrete torsion and we will clearly set out our definitions here. Given a conformal field theory for a Calabi–Yau manifold V with a discrete symmetry group G, one may build the theory for the quotient V/G in a systematic way. There is an ambiguity in this construction however. Phases may be introduced when building the partition function for the characters without disturbing modular invariance. It was shown in [26] that these phases must be elements of $H^2(G, U(1))$. If G is finite then this group is isomorphic to $H_2(G)$. (The coefficient group $\mathbb{Z}$ is assumed for homology and cohomology if omitted.) Thus each element $\varepsilon \in H^2(G, U(1))$ gives rise to a possible conformal field theory for the orbifold V/G. We call ε the "2-cocycle" for the theory. In [28] some examples of orbifolds with a nontrivial 2-cocycle were studied and it was shown that each of the marginal operators could be associated to a deformation of the orbifold space itself, but that some deformations appeared to be "missing," i.e., corresponded to no marginal operator.

The singular cohomology groups $H^*(X)$ of a manifold X need not be free abelian groups. Of particular interest to us in this paper will be the torsion part of $H_2(X)$ (or equivalently $H^3(X)$), where X is a Calabi–Yau manifold. We impose the condition $h^{2,0}(X) = 0$. In this case the torsion group is isomorphic to the "Brauer group" of X. We will use this terminology here for convenience although the reader is not required to know the full definition of the Brauer group.[1] It was suggested in [26] that there should be some connection between the Brauer group and the 2-cocycles in an orbifold. An example studied in [7] had trivial Brauer group but admitted nontrivial 2-cocycles. Thus these two concepts are not equivalent. As we

[1] Further information about the Brauer group is provided in the appendix.

shall see in this paper however there is some intimate connection between them. Because the term "discrete torsion" has been used at times to refer to either the Brauer group or the group of 2-cocycles, we will try to avoid using it in this paper to save any confusion.

Although motivated by the orbifold construction of [28] we shall see that stable singularities are probably not confined to such examples. The topological obstruction to deforming away the singularities may be thought of as "hiding away" in the singularity itself. To understand this geometrically we will blow up the singularity to expose its contents. This blow-up will not be a marginal perturbation as one is accustomed to in orbifold theory but rather will be an irrelevant perturbation.

In Sect. 2 we discuss how the Brauer group affects the correlation functions of an $N=(2,2)$ superconformal field theory. In particular we only need concern ourselves with that part of the conformal field theory which is present in the A-model (which is one of the topological field theories obtained by twisting the original $N=2$ model). We will review how the Brauer group adds a degree of freedom to the A-model that cannot be expressed in terms of the Kähler form or the B-field.

In Sect. 3 we discuss blow-ups as irrelevant operators. This generalizes the usual notion of blow-ups in the context of orbifold theories which correspond to truly marginal operators. We will need such a generalization to deal with the singularities discussed in this paper. This allows us to study the examples of stable singularities in Sect. 4.

Finally we present a discussion in Sect. 5.

2. The A-Model

In this section we will study the form of the correlation functions of the A-model with target space X, where X may have a non-trivial Brauer group, that is, when $H^3(X)$ contains a torsion subgroup. From the universal coefficient theorem (see, for example, [11]) the torsion part of $H^3(X)$ is isomorphic to the torsion part of $H_2(X)$.

The A-model is a topological field theory [30] in which the correlation functions depend upon non-trivial instanton effects. The instantons are holomorphic maps from the world-sheet, Σ, to the target space X. Further, the action of this instanton is assumed to depend only upon the homology class of the image of this map in X. For an instanton I with homology class $[I] \in H_2(X)$, let us denote e^{-S_I} by $\mu([I])$, where S_I is the action of the instanton. In order for string interactions to behave correctly [26] we further demand that the action depend linearly upon the homology class of I, i.e.,

$$\mu \in \mathrm{Hom}(H_2(X), \mathbb{C}^*)\,, \tag{1}$$

where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers.

Recall that $H_2(X)$ is an abelian group and thus may be decomposed into free and torsion subgroups:

$$H_2(X) \cong \mathbb{Z}^{h^{1,1}(X)} \times \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2} \times \cdots\,, \tag{2}$$

where t_i are finite positive integers labeling the torsion part of $H_2(X)$.

A simple application of the universal coefficient theorem yields $\mathrm{Hom}(H_2(X), \mathbb{C}^*) \cong H^2(X, \mathbb{C}^*)$. Given the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0\,, \tag{3}$$

we obtain the long exact sequence

$$0 \to \mathrm{Free}(H^2(X)) \to H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C}^*) \to \mathrm{Tors}(H^3(X)) \to 0\,, \qquad (4)$$

where Free(G) and Tors(G) denote the free and torsion parts of the abelian group G respectively.

For the time being let us assume that $\mathrm{Tors}(H^3(X)) \cong 0$. Then we see from (4) that

$$H^2(X,\mathbb{C}^*) \cong \frac{H^2(X,\mathbb{C})}{\mathrm{Free}(H^2(X))}\,. \qquad (5)$$

We usually think of the A-model correlation functions as depending upon the "complexified Kähler form." This complexified Kähler form is written $B + iJ$, where J is the usual Kähler form and B is a real 2-form of X defined modulo elements of de Rham cohomology which are elements of integral cohomology. It is easy to see that this agrees with (5).

Let us form a basis of Free($H^2(X)$) with elements $e_j, j = 1 \cdots h^{1,1}(X)$. We may then expand

$$B + iJ = \sum_j (B + iJ)_j e_j\,, \qquad (6)$$

where $B_j \cong B_j + 1$ for all j. We now introduce the usual q-variables:

$$q_j = \exp\{2\pi i(B + iJ)_j\}\,. \qquad (7)$$

We then see that

$$\mu([I]) = \prod_j q_j^{n_j([I])}\,, \qquad (8)$$

where n_j are non-negative integers labeling the homology class of the instanton.

When we calculate a correlation function in the A-model we use the usual methods of intersection theory in topological field theory [29]. That is to say, for each instanton background, the contribution to the correlation function is given by the intersection number of some cycles representing the observables in the moduli space of the instanton. This intersection number is an integer.[2] Such a contribution to the correlation function is then weighted by $\mu([I])$. We thus see that the correlation functions in this A-model take the form of power series in the variables q_j with *integer* coefficients. All of the many examples studied so far confirm this (see, for example [13, 12]).

Now let us consider the case when the Brauer group of X is not trivial. We take the simplest case where $\mathrm{Tors}(H^3(X)) \cong \mathbb{Z}_m$ for some integer m. This implies that $\mathrm{Tors}(H_2(X)) \cong \mathbb{Z}_m$. Let t be a 2-cycle so that $[t]$ generates this torsion class. That is, t is a cycle such that

$$p[t] \cong 0 \;\Leftrightarrow\; m \text{ divides } p\,. \qquad (9)$$

[2] It is *a priori* possible that the instanton moduli spaces in some examples could have orbifold singularities, leading to rational numbers rather than integers. No examples of this phenomenon are known, and perhaps it does not occur. In any case, although we have assumed here that the intersection numbers are integers, the discussions in this paper are unchanged if rational numbers are used in place of integers.

This implies that $(\mu([t]))^m = 1$, i.e.,

$$\mu([t]) = \exp\left(\frac{2\pi i}{m}\alpha\right), \tag{10}$$

for $\alpha = 0, \ldots, m-1$.

The choice of α is required, in addition to $B + iJ$, to determine the correlation functions. That is, the position of the specific A-model in the moduli space of theories is not entirely determined by the complexified Kähler form but also requires a specification of the discrete parameter α.

We may consider the shape of the moduli space of A-models as follows. The "large radius limit" of an A-model is the limit point where the action of all non-trivial instantons becomes infinite. That is, $\mu([I]) \to 0$ for $[I] \not\cong 0$ (as always we have $\mu(0) = 1$). At this limit point therefore the choice of α does not matter. Thus different "sheets" of the moduli space, each parametrized by $B + iJ$ but having a different value of α, are joined at the large radius limit. It is important to remember that when describing the moduli space in terms of A-models we assume that we are in the neighbourhood of the large radius limit and that each correlation function is completely determined by a power series centered at the limit point. Thus, we will not discuss further aspects of the global geometry of the moduli space which take us outside this region. This region of the moduli space is shown in Fig. 1.

Let us illustrate the new form of the correlation function by an example in which $m = 3$ and we are considering observables corresponding to divisors in X, where X is a Calabi–Yau threefold. For simplicity let us also assume that $h^{1,1}(X) = 1$ so that there is only one such observable. Thus we have $H_2(X) \cong \mathbb{Z} \times \mathbb{Z}_3$. Now we may perform the usual expansion of the A-model correlation functions in terms of the rational curves on X as was done in [13]. In this case we expect

$$\langle \mathcal{O}_D \mathcal{O}_D \mathcal{O}_D \rangle = \#(D \cap D \cap D) + (n_1 + \omega^\alpha n_2 + \omega^{2\alpha} n_3) q + O(q^2), \tag{11}$$

where ω is a nontrivial cube root of unity. It is not possible for any algebraic curve to lie in a torsion class of $H_2(X)$. This is because the area of an algebraic curve is given by the integral of the Kähler form over the curve. This area may be recast

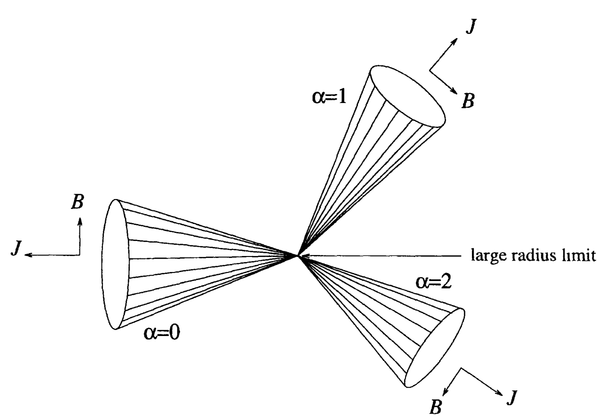

Fig. 1. The A-model moduli space for target space with $\mathbb{Z}_3 \subset H_2(X)$

as the intersection of a 4-cycle representing the dual of the Kähler form with the curve. Since any cycle in a torsion class must have zero intersection number with any other cycle, any curve in a torsion class would have zero area, which is not possible. It may happen however that the difference of two curves is a torsion cycle. The lines (i.e., rational curves intersecting D once) on X can therefore lie in one of three homology classes and these are counted by n_1, n_2, n_3. The number of lines is thus $n_1 + n_2 + n_3$ which is counted by (11) in the usual way when $\alpha = 0$. When α is 1 or 2 however we distinguish between these lines.

Note that for $\alpha = 1$ or 2 the series (11) is *not* a power series with integer (or rational) coefficients. Thus the fact that all examples studied so far did lead to a series with integer coefficients implies that the examples had a trivial Brauer group (or at least, only elements of order 2). It would be interesting to study an example of a smooth Calabi–Yau manifold with non-trivial Brauer group and so generate a solid example of the series of the form (11). Unfortunately at this point in time we are not aware of any such examples.

3. Away from Criticality

As is well-known (see, for example, [20]) the β-function for the metric of a non-linear σ-model is given by the Ricci-tensor to first order

$$\beta(g_{ij}) = -\tfrac{1}{2\pi} R_{ij} + \cdots . \tag{12}$$

If the target space is at large radius, the later terms in the series are negligible. Consider the flow towards the infra-red (low-energy) limit. If the target space has positive curvature then the sign of the β-function shows that the space will shrink under this flow. If the space has negative curvature it will expand, and if it is Ricci-flat then it will be stable (to leading order). Calabi–Yau manifolds fall into the last category and thus may provide conformally invariant σ-models.

A complex projective space is a space of positive curvature. The non-linear σ-model on such a space is a massive field theory [17] and thus naïvely appears to flow to something trivial in the infra-red limit. This may be viewed geometrically as a process in which the target space shrinks down to a point in the limit. Actually one needs to be a little careful about this statement. Although one might think that a non-linear σ-model with a point target space is, by its very definition, a trivial theory, one may reach different conclusions by treating the point as a limit point of theories on a complex projective space [15]. In the latter case the limit of the infra-red flow is better thought of as a target space whose size is $-\infty$. This has become a recurring theme in recent works [31] and may be viewed as part of the inherent difficulty in clearly defining the concept of sizes below the Planck scale.

The best method of giving geometrical interpretations to spaces away from the large radius limit is probably that of the linear σ-model of [31]. The behaviour of the complex projective space, $\mathbb{P}^n$, as a target space was studied in the latter and gave the following picture which is the most complete version of what happens in the infra-red limit. The linear σ-model contains a real parameter, r, which, in the case $r \gg 0$ gives the size of the complex projective target space (or, to be more precise, the area of complex lines in the target space). The renormalization group acts on this parameter to drive it towards $-\infty$ in the I.R. limit. In this limit however the geometrical interpretation changes. When $r \ll 0$ the target space becomes that of $(n+1)$ disjoint points. The conformal field theory associated with such a target space has $c = 0$ but consists of $(n+1)$ $Sl(2,\mathbb{C})$-invariant vacua. That is, we have a reducible but trivial representation of the Virasoro algebra. This is the sense in which the $\mathbb{P}^n$-model flows to a trivial theory in the I.R. limit. Note that this picture preserves the Witten index, $\mathrm{Tr}(-1)^F$, or Euler characteristic, of the theory during the flow. The Euler characteristic of both $\mathbb{P}^n$ and $(n+1)$ disjoint points is $n+1$.

We wish now to consider something intermediate between a projective space and a Calabi–Yau manifold. That is, we want a theory which is not conformally invariant but flows to a non-trivial conformal field theory in the infrared limit. Such an example may be provided by blowing-up a smooth point on a Calabi–Yau manifold. Blowing up points is familiar in string theory for resolving orbifold singularities (see [3] for a review). Blowing up singularities may result in a smooth Calabi–Yau manifold. If one blows up a point on a Calabi–Yau manifold, X, that is already smooth however one obtains a manifold, $\tilde{X}$, which does not admit a Ricci-flat metric (although it is still complex and Kähler).

In the case of a Calabi–Yau threefold, blowing up a smooth point replaces that point by a divisor isomorphic to the projective space $\mathbb{P}^2$. The normal bundle of this divisor is $\mathcal{O}(-1)$ (i.e., the inverse of the Hopf bundle). Consider a curve C which is a projective space $\mathbb{P}^1$ lying within this $\mathbb{P}^2$. It is a simple matter to show that the normal bundle of the curve is $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. Given that its tangent bundle is $\mathcal{O}(2)$ we obtain

$$\int_C c_1(\tilde{X}) = 2 + 1 - 1 = 2\,. \tag{13}$$

Thus $c_1 \neq 0$. In particular since $\int_C c_1(\tilde{X}) > 0$, the curve C will shrink during the flow to the infra-red limit. Since all such curves shrink, the "exceptional divisor" $\mathbb{P}^2$ will also shrink. Curves away from this blowup will satisfy $\int_C c_1(\tilde{X}) = 0$ and should be stable under this flow. So long as a neighbourhood of such a curve is stable under the flow, the process must lead to a birational transformation and so the complex structure remains fixed. Thus, the net result would appear to be that the limit of this flow is to turn $\tilde{X}$ back into X. That is, we take a manifold X corresponding to a conformal field theory. We then perturb it to obtain a field theory that is not conformally invariant, but flows back to the original under flow to the infra-red limit. In other words, *blowing up a smooth point is equivalent to perturbation by an irrelevant operator.*

It is worth describing an example of this picture in terms of the linear σ-model, as we now do for completeness. The reader who is already convinced of our assertions concerning the effects of the renormalization group may skip this section. Let us consider the case of the quintic hypersurface in $\mathbb{P}^4$. This is a smooth Calabi–Yau manifold. A generic line (i.e., linearly embedded $\mathbb{P}^1$) in the ambient $\mathbb{P}^4$ will intersect this Calabi–Yau manifold at five distinct points. Thus by blowing up such a line we blow up the Calabi–Yau manifold at five points. The toric picture of this blown-up ambient space leads to the following gauged linear σ-model. Consider seven chiral superfields with lowest components $x_1,\ldots,x_5,p,f$ in a theory with gauge group

$U(1)^2$. The charges are as follows:

	$Q_i^{(1)}$	$Q_i^{(2)}$
x_1	1	1
x_2	1	1
x_3	1	1
x_4	1	0
x_5	1	0
p	-5	0
f	0	-1

(14)

Part of the classical potential comes from the D-terms of this theory and the vanishing of this requires that two parameters of the theory r_1 and r_2 be set as follows:

$$\begin{aligned} r_1 &= |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 - 5|p|^2, \\ r_2 &= |x_1|^2 + |x_2|^2 + |x_3|^2 - |f|^2. \end{aligned} \tag{15}$$

We also consider the invariant superpotential

$$W = p\left(f^5(x_1^5 + x_2^5 + x_3^5) + x_4^5 + x_5^5\right), \tag{16}$$

as in [31]. The vanishing of the classical potential requires all of the derivatives of (16) to be zero.

Now consider the phase where $r_1 - r_2 > 0$ and $r_2 > 0$. With this choice, the classical vacuum requires that at least one of $\{x_1, x_2, x_3\}$ does not vanish and that at least one of $\{x_4, x_5, f\}$ does not vanish either. Suppose first that $f \neq 0$. We may fix the phase of f (we then normalize $f = 1$) using one of the $U(1)$ groups. The derivatives of the superpotential then require $p = 0$ and that

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\,. \tag{17}$$

Let the other $U(1)$ action be used to form $[x_1, x_2, x_3, x_4, x_5]$ as the homogeneous coordinates of $\mathbb{P}^4$ (expressing $\mathbb{P}^4$ in the familiar form of a symplectic reduction $S^9/U(1)$). Thus the classical vacuum appears to be the quintic Calabi–Yau hypersurface in $\mathbb{P}^4$. Note however that we are missing the line $[0,0,0,x_4,x_5]$ and hence 5 points of this Calabi–Yau manifold. Now let $f = 0$. This forces either x_4 or x_5 to be nonzero and thus $p = 0$. We also have the constraint $x_4^5 + x_5^5 = 0$ from one of the derivatives of the superpotential. Use one of the $U(1)$'s to fix the phases of x_4 and x_5. The other $U(1)$ may be used to form $\mathbb{P}^2$ with homogeneous coordinates $[x_1, x_2, x_3]$. The result is that each of the 5 points $[0,0,0,x_4,x_5]$ in the quintic hypersurface have been replaced by $\mathbb{P}^2$. That is, we have blown-up 5 smooth points as promised.

Next consider the phase where $r_1 > 0$ and $r_2 < 0$. Now f must be nonzero and we fix it using one of the $U(1)$ groups. Also one of $\{x_1, x_2, x_3, x_4, x_5\}$ are nonzero and we use the other $U(1)$ to form $\mathbb{P}^4$. It follows that $p = 0$ and we lie on the quintic hypersurface. At first sight therefore, this phase appears to be simply the quintic Calabi–Yau manifold in $\mathbb{P}^4$. This is not the full story however. Thus far we have neglected some of the fields in the theory – namely the lowest components of the twisted chiral superfields coming from the field strength of the two $U(1)$ gauge fields. Call these fields σ_1 and σ_2 consistent with the notation of [31]. The classical potential of these fields is given by [24]:

$$U_\sigma = 2\sum_{a,b} \bar{\sigma}_a\sigma_b \sum_i Q_i^{(a)} Q_i^{(b)} |\phi_i|^2$$
$$= 2|\sigma_1|^2(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_4|^2 + 25|p|^2)$$
$$+ 2(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2)(|x_1|^2 + |x_2|^2 + |x_3|^2) + 2|\sigma_2|^2(|x_1|^2 + |x_2|^2 + |x_3|^2 + |f|^2)\,. \quad (18)$$

It would appear that for $r_1 \gg 0$ and $r_2 \ll 0$ the fields σ_a are very massive and so should be set equal to zero. It turns out however that there are large quantum corrections to the potential when $x_1 = x_2 = x_3 = f = x_4^5 + x_5^5 = 0$ and σ_2 appears to be massless. One may show [31] that there are $\sum_i Q_i^{(2)} = 2$ extra solutions for σ_2 when $r_2 \ll 0$.

Our target space for this latter phase is thus as follows. We have a smooth hypersurface in $\mathbb{P}^4$ where the σ fields are zero and we have a completely disjoint set of 10 points given by the 5 points on the quintic with $x_1 = x_2 = x_3 = 0$ each with two possible nonzero values for σ_2, and σ_1 is still zero. Note that the Euler characteristic of this set is equal to $-200 + 10 = -190$ which is precisely that of the quintic blown-up at 5 points.

The effect of the I.R. flow is to force $r_2 \to -\infty$. Thus if we begin with a target space of the quintic threefold with 5 points blown-up and go to the I.R. limit, we end up with the quintic threefold with 10 disjoint points. This is shown in Fig. 2. Note that the resulting conformal field theory consists of that of the quintic together with 10 trivial representations of the Virasoro group. Thus it is only when we focus on the nontrivial irreducible part of the conformal field theory that the blow-up is, strictly speaking, an irrelevant operator. Since our main concern in this paper is the behaviour of the correlation functions only in this part of the theory, this meaning of an irrelevant operator is good enough for our purposes. Note also that the complex structure on the target space is unaffected by this process as expected.

Let us briefly note that the "monomial-divisor mirror map" of [6, 5] may also be extended to this picture. This map maps a toric divisor in the target space of the A-model to a monomial which may be used to deform the complex structure of the mirror B-model. This is done by identifying both the sets of divisors and the set of monomials with a set of points lying in a hyperplane intersecting a lattice based on the ideas of [8]. The blow-up of a smooth point can be represented by a point outside this hyperplane, this in turn maps to a monomial with the wrong weight to be considered as part of the original quasi-homogeneous defining equation. In fact, the monomial's weight is too high and thus is of no importance in the infrared limit as was argued in [27]. Thus we see again that the blow-up is an irrelevant operator. Presumably given a good definition of the B-model away from criticality, we could understand the meaning of the trivial representations that appear in the I.R. limit.

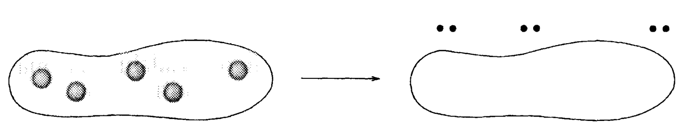

Fig. 2. The infra-red limit of the quintic threefold with 5 points blown up

Thus far we have gained little. We already knew how to handle a non-linear σ-model on a smooth Calabi–Yau. Where blow-ups prove useful is where they resolve singularities. If the target space X is singular then one may study the σ-model by blowing X up. Depending on the first Chern class of the resulting target space this may be a relevant, marginal or irrelevant perturbation of the original theory. In the case of orbifolds, the cases considered are usually marginal (see, for example, [3]). One may also have a case of a singularity being resolved by an irrelevant operator as we now show.

Consider $\mathbb{C}^4$ with coordinates (w,x,y,z) and the hypersurface defined by the equation

$$xy = wz . \tag{19}$$

This hypersurface has an isolated singularity, or "node", at the origin. There are many ways to remove such a singularity. Consider a compact variety X which contains such a node. The ways in which X can be smoothed depend upon the global geometry of X. That is, there may be global obstructions to processes which removed the singularity locally. Whether or not the resulting smooth space is Kähler is also a global question.

If X is a projective algebraic variety then there is always at least one way of smoothing X to form a Kähler manifold, $\tilde{X}$, as follows. Blow up the origin of $\mathbb{C}^4$ in which the node (19) is embedded. Thus, the origin is replaced by $\mathbb{P}^3$. The intersection of the hypersurface (19) with this $\mathbb{P}^3$ is obtained by treating the coordinates (w,x,y,z) as homogeneous coordinates. Thus the effect of the blow-up is to replace the node by a quadric hypersurface in $\mathbb{P}^3$. It is a well-known result in algebraic geometry that such a complex 2-fold is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Now consider the first Chern class of this blow-up. Let C by any one of the two families of rational curves in the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. Clearly because of the product structure, the normal bundle of this curve within the exceptional divisor is $\mathcal{O}(0)$. The other normal direction of the curve is that of the way $\mathbb{P}^3$ embeds in $\mathbb{C}^4$. Thus the total normal bundle of the curve is $\mathcal{O}(0) \oplus \mathcal{O}(-1)$. Adding this to the tangent bundle we obtain

$$\int_C c_1(\tilde{X}) = 2 + 0 - 1 = 1 . \tag{20}$$

This is thus similar to the case of a blow-up of a smooth point – the blow-up is an irrelevant operator. In contrast to the latter case however, the irrelevant perturbation has been of some use – we have smoothed the target space.

4. Examples

We are now in a position to study some examples for the target space which exhibit stable singularities thanks to the analysis of the preceding sections.

4.1. A Double Cover of $\mathbb{P}^3$. Let K' be a smooth hypersurface in $\mathbb{P}^3$ defined by an equation of degree 8 in the homogeneous coordinates. Let X' be a double cover of this $\mathbb{P}^3$ branched over K'. The space X' is a smooth Calabi–Yau manifold and was studied in the physics literature long ago [25]. The resulting space has $h^{1,1} = 1$ given by the original $\mathbb{P}^3$ and $h^{2,1}$=149, where the 149 corresponding deformations of complex structure of X' can be provided by the 149 inequivalent deformations of the octic defining equation for K'. The Brauer group of X' is trivial.

By deforming the octic equation to special values we may make the double cover singular. Let X be such a singular degeneration of X' branched over the singular octic surface K. We define K as follows. Let W be a generic polynomial in the homogeneous coordinates $[x_0,\dots,x_3],[p_0,\dots,p_3]$ of degree (2,2) with respect to the x's and the p's. The x's form the homogeneous coordinates of our $\mathbb{P}^3$. K is defined by

$$\left|\frac{\partial^2 W}{\partial p_i \partial p_j}\right| = 0\,. \tag{21}$$

The number of nodes may be calculated using the methods of [22].[3] This surface has 80 nodes and as a result X has 80 isolated nodes of the form (19). K, and thus, X have 69 deformations. The fact that $69+80=149$ shows that each of the nodes appears to have "eaten up" one of the original deformations of X'.

Given X how may we remove the singularities? Obviously one way is to deform it back into X' which is a smooth Calabi–Yau manifold. Another way one might be tempted to try is to use "small resolutions." This amounts to replacing each node by a $\mathbb{P}^1$. This process was used in the string context in [14] to continuously change the topology of the target space. It is also reviewed in [2] together with its relation to the "flop". Anyway, in this case the small resolutions do not work – the resulting smooth space is not Kähler.

Consider $\tilde{X}$ as the blow-up of X replacing each of the 80 nodes by exceptional divisors in the form of $\mathbb{P}^1 \times \mathbb{P}^1$ as described in the previous section. See Fig. 3 (where only 2 of the 80 nodes are shown). The space $\tilde{X}$ is smooth and Kähler but not Calabi–Yau. As shown in the appendix however, this space is very interesting for our purposes because $H_2(\tilde{X})$ contains a $\mathbb{Z}_2$ subgroup. That is, we have an example with a nontrivial Brauer group.

Now let us consider the A-model on X. Since X is singular we need to think carefully about how to calculate correlation functions in the model. The $N=2$ σ-model on $\tilde{X}$ flows to the superconformal field theory on X as explained in Sect. 3. Thus if the exceptional divisors in $\tilde{X}$ are very small then we expect to have a theory with correlation functions very close to that of X. The exception to this will be correlation functions involving fields from the part of the theory that became trivial in the infra-red limit. From Sect. 3 we expect that the fields associated with the homology of the exceptional divisors themselves are such fields. The A-model on X can be considered as the limit of this infra-red flow, twisted to form a topological field theory. The A-model on X therefore would appear to be given by the A-model on $\tilde{X}$ with all the "massive" bits ignored since these disappear in the infra-red limit. That is, we ignore the contributions to the homology appearing from the exceptional divisors themselves. What we do not ignore however is the torsion in $H_2(\tilde{X})$ because this may be observed away from the exceptional divisors. In particular there are rational curves $C_1, C_2 \in \tilde{X}$ such that $[C_1]-[C_2]$ is a nontrivial element of $\mathrm{Tors}(H_2(\tilde{X}))$ and neither C_1 nor C_2 is contained in any of the exceptional divisors.

We propose therefore that one may define an A-model on X in terms of the homology classes on $\tilde{X}$ excluding classes lying exclusively within the exceptional divisors. This means that our $\mathbb{Z}_2$ group in the Brauer group allows us to introduce

[3] The matrix $\partial^2 W/\partial p_i \partial p_j$ represents a symmetric map $f: E \longrightarrow E^*$, where E is a vector bundle of rank 4 over $\mathbb{P}^3$. The fact that this matrix has entries which are quadratic in the homogeneous coordinates of this $\mathbb{P}^3$ shows that $E^* \cong \mathcal{O}(1)^{\oplus 4}$. Theorem 1 of [22] can then be used to calculate the number of nodes since nodes appear as the locus of corank $\geqq 2$ maps.

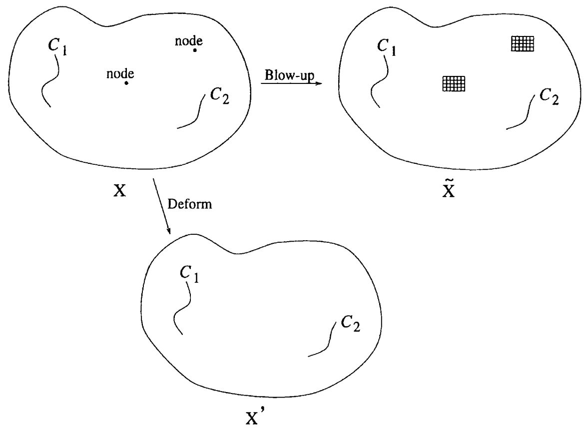

Fig. 3. The relationship between X, X' and $\tilde{X}$

a parameter α which may be 0 or 1 as in Sect. 2. In particular, if $\alpha = 0$, the curves C_1 and C_2 will contribute identically to correlation functions and if $\alpha = 1$ they will contribute differently.

Consider this A-model as we deform X slightly into X'. All the rational curves away from the nodes are deformed slightly but now the Brauer group is trivial and so C_1 and C_2 lie in the same homology class. In the underlying conformal field theory one would expect the correlation functions to change slightly. This is all well and good if $\alpha = 0$ but if $\alpha = 1$ then we are in trouble. The coefficients in the A-model correlation functions appear to jump as the homology classes of C_1 and C_2 change.

Thus, an A-model on X for which the parameter α is 0 may possibly be deformed into an A-model on X', but for A-models with $\alpha = 1$ this deformation appears to be obstructed at the level of correlation functions. Since this deformation is the only way of smoothing X into a Calabi–Yau manifold, if $\alpha = 1$ then we are unable to follow the A-model from the singular space to the smooth one. Thus, even though X may be classically deformed into the Calabi–Yau manifold X', this deformation is not compatible with string theory! The only deformations of complex structure of X allowed in the case $\alpha = 1$ are the 69 which preserve the 80 nodes.

This state of affairs is, of course, similar to that suggested in [28] to which we now turn our attention.

4.2. A Double Cover of $(\mathbb{P}^1)^3$. To discuss our next example, we need to introduce a whole plethora of spaces all of which may be deformed into each each other continuously. These are as follows:

$X^\sharp$: Let T be the torus of one complex dimension described as a quotient of the complex plane $\mathbb{C}$, parametrized by z, with identifications $z \cong z + 1$ and $z \cong z + i$. Take three copies of this torus parametrized by z_1, z_2, z_3. $X^\sharp$ is defined as the orbifold obtained by dividing this space T^3 by the group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$ and $(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3)$.

Y: This orbifold may be blown up in the usual way to form a Calabi–Yau manifold Y. Y has $h^{1,1} = 51$ and $h^{2,1} = 3$. Actually this blow-up is not unique and there are many topologies possible for Y. Which one we choose is not important for this paper.

X': Consider the space $(\mathbb{P}^1)^3$ and a smooth hypersurface K' within this space defined by an equation of weight $(4,4,4)$ (i.e., quartic in each of three sets of homogeneous coordinates). X' is the double cover of $(\mathbb{P}^1)^3$ branched over K'. It is a smooth Calabi–Yau manifold with torsion-free cohomology with $h^{1,1}=3$ and $h^{2,1}=115$. As explained in [28], $X^\sharp$ may be written as a double cover of $(\mathbb{P}^1)^3$ and may deformed into X'.

X: Let $f_{a_1a_2a_3}$ represent a generic polynomial of weight (a_1,a_2,a_3) in the homogeneous coordinates of $(\mathbb{P}^1)^3$. Let W be a symmetric matrix of the form

$$W=\begin{pmatrix} f_{400} & f_{220} & f_{202} \\ f_{220} & f_{040} & f_{022} \\ f_{202} & f_{022} & f_{004} \end{pmatrix}. \tag{22}$$

K is the hypersurface defined by $\det W=0$. X is the double cover of $(\mathbb{P}^1)^3$ branched over K. K, and therefore X, have 64 nodes.

$\tilde{X}$: The space X may be blown-up to a smooth manifold $\tilde{X}$ by replacing each of the 64 nodes by an exceptional divisor $\mathbb{P}^1\times\mathbb{P}^1$. As before $\tilde{X}$ is not a Calabi–Yau manifold.

These spaces are thus related as follows

$$\begin{array}{ccccc} X^\sharp & \xrightarrow{\text{def}} & X & \xrightarrow{\text{def}} & X' \\ \big\downarrow{\scriptstyle\text{blow-up}} & & \big\downarrow{\scriptstyle\text{blow-up}} & & \\ Y & & \tilde{X} & & \end{array} \tag{23}$$

where "def" refers to a deformation of complex structure.

String theory on $X^\sharp$ is understood from orbifold theory. Since $H_2(G)\cong\mathbb{Z}_2$ there are two possible theories depending on one's choice of the "discrete torsion" 2-cocycle [26]. With a trivial 2-cocycle one recovers the usual blow-up picture as expected [3]. That is, the chiral ring corresponds to the cohomology of Y. Thus Y may be taken to be the geometrical interpretation of a conformal field marginally perturbed from that of the orbifold $X^\sharp$ with trivial 2-cocycle.

When the nontrivial 2-cocycle is chosen, one obtains a chiral ring mirror to that with a trivial 2-cocycle. That is, $h^{1,1}=3$ and $h^{2,1}=51$. These numbers precisely agree with the degrees of freedom of X. X has 3 deformations of its "Kähler form," that is, the sizes of the three $\mathbb{P}^1$'s may be varied. (X itself is singular so it does not really have a Kähler form as such.) Varying W gives K 30 deformations of complex structure but this does not actually account for all the deformations of K which preserve the 64 nodes. Since X' has 115 deformations there will be $115-64=51$ deformations of K maintaining 64 nodes and thus 51 deformations of $\tilde{X}$. Thus there are 51 deformations of complex structure for X for our purposes.

As the reader may have guessed by now, the group $H_2(\tilde{X})$ contains a $\mathbb{Z}_2$ torsion part. Thus by analogous reasoning to the previous example, if we set $\alpha=1$ for the A-model on X, we obstruct the deformations taking X into X'. This then appears to give the correct geometrical picture for allowing X to be regarded as the geometrical interpretation of a conformal field theory marginally perturbed from that of the orbifold $X^\sharp$ with nontrivial 2-cocycle. Note however that in addition to just knowing the classical geometry of X, we also need to put $\alpha=1$ to stop X' from providing the geometrical interpretation.

5. Discussion

We have observed that conformal field theory, or topological field theory, on a target space with nodes may have degrees of freedom which are "hidden away" in the nodes. By perturbing by an irrelevant operator we have been able to probe the secrets of these nodes to discover the Brauer group at work.

It is important to realize in the above description that the hidden degrees of freedom cannot be expressed as some local property of each of the nodes. The appearance of torsion in $H_2(\tilde{X})$ is a global property – many nodes in just the right place are required to produce the element of the Brauer group on blowing up. This demonstrates further some of the peculiar properties of the stringy description of space.

A question we have not addressed is that of the existence of a conformal field theory associated to some singular target space X. Given a smooth Calabi–Yau manifold near the large radius limit we may assume the existence of a conformal field theory approximated by the non-linear σ-model with Ricci-flat target space. In the case of a singular target space however some of the correlation functions of the supposed conformal field theory may contain divergences.

Consider the conformal field theory on the manifold X' in the example in either Sect. 4.1 or 4.2 and consider the process of deforming the target space continuously to X. Such a degeneration of complex structure leads to infinities in the chiral ring. That is, the B-model on X' appears bad in the limit $X' \to X$. This appears to rule out a good conformal field theory corresponding to the A-model with $\alpha = 0$. For the case $\alpha = 1$ we have removed precisely the offending fields from the B-model causing the divergences. Thus the case $\alpha = 1$ contains no infinities and may describe a good conformal field theory.

This agrees with the analysis of [28] where there are only two choices of orbifold theories on $X^\sharp$. One consists of the theory which may be blown up to Y. The other is the theory which is deformed to X with $\alpha = 1$. There is no third possibility of a theory which may be deformed to X with $\alpha = 0$ since such a theory would be a limit of X' and, as such, contain divergences. Thus although we have introduced the Brauer group as an extra parameter in the space of A-models, it would appear that to obtain a finite conformal field theory on a singular space such as X, one is forced to rule out the choice $\alpha = 0$. Presumably it is only on smooth manifolds that one is really free to choose α.

The example in Sect. 4.2 appears to show a link between elements of the Brauer group and nontrivial 2-cocycles for orbifolds. The imposition of a (non)trivial 2-cocycle for the orbifold $X^\sharp$ appears to match the (non)trivial choice for α for the theory on X. Can we therefore claim to have a complete geometrical understanding of these 2-cocycles? Unfortunately the picture is not complete. It is not possible to blow-up $X^\sharp$ to obtain some manifold with nontrivial Brauer group. We must deform $X^\sharp$ into X before blowing up for our construction to work.

The desired theorem for a general case might appear along the lines as follows. *Given an orbifold $X^\sharp = V/G$, for finite group G, there exists some X obtained by a deformation of complex structure of $X^\sharp$ such that the blow-up, $\tilde{X}$, of X satisfies* $\mathrm{Tors}(H_2(\tilde{X})) \cong H_2(G)$. In light of the example of [7] we must also exclude the trivial case where $X^\sharp$ is a manifold. It is not at all clear that this conjecture is true and it is certainly worthy of further study.

It was observed in [28] that X and Y from Sect. 4.2 are a mirror pair. This is a fact that we have not used yet. Since Y may be written as a complete intersection in a toric variety one should be able to use the method of [10, 4] to construct its mirror. This example is very similar to that studied in Sect. 3.4 of [4]. The result is that the mirror of Y is a hybrid model which is a "trivial" (i.e., quadratic) Landau–Ginzburg theory in $\mathbb{C}^6/\mathbb{Z}_2$ fibred over $(\mathbb{P}^1)^3$. The superpotential of this Landau–Ginzburg theory "degenerates" (i.e., some directions become massless) over a subspace of $(\mathbb{P}^1)^3$. This subspace appears in the form of K in Sect. 4.2. That is to say, the description of the mirror of Y in the language of [4] is precisely X except that "double cover" is replaced by "Landau–Ginzburg fibration" and "branched over" is replaced by "with superpotential degenerating over." This is a curious point which should be pursued further.

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Appendix: Some Calculations of Brauer Groups

In this appendix, all cohomology groups will be defined using the étale topology unless otherwise noted. See [23] for a basic reference for the étale topology and étale cohomology. By the *Brauer group* of an algebraic variety X, we mean the cohomological Brauer group, $\mathrm{Br}'(X) = H^2(X, \mathbb{G}_m)$, where $\mathbb{G}_m$ is the sheaf of units in $\mathcal{O}_X$. See [21] for an introduction to Brauer groups of varieties. We will assume X is defined over $\mathbb{C}$, or any algebraically closed field of characteristic zero.

There is an exact sequence

$$0 \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H^2(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}'(X) \to 0\,.$$

(See [21], II Thm 3.1.) If X is a non-singular variety over the complex numbers, then $H^2(X, \mathbb{Q}/\mathbb{Z})$ coincides with the singular cohomology group $H^2_{\mathrm{sing}}(X, \mathbb{Q}/\mathbb{Z})$ in the usual topology. If furthermore, $\mathrm{Pic}(X) \cong H^2_{\mathrm{sing}}(X, \mathbb{Z})$, as is the case if $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$, then this exact sequence along with the universal coefficient theorem shows that $\mathrm{Br}'(X) \cong H^3_{\mathrm{sing}}(X, \mathbb{Z})_{\mathrm{tors}}$.

We use the cohomological Brauer group rather than the description "torsion in H^3" because, for the second example below, we will need some technical machinery which has already been set up using étale cohomology in [19]. Furthermore, in the first example, we will use the following interpretation for elements in the Brauer group.

If X is a variety, a Brauer–Severi variety over X is a variety $\mathbf{P}$ along with a map $f : \mathbf{P} \to X$ which is a $\mathbb{P}^n$-bundle. (See [21], I, Sect. 8 for details.) Not all such $\mathbb{P}^n$-bundles are projectivizations of vector bundles on X, and the Brauer group gives obstructions for a Brauer–Severi variety to come from a vector bundle. From the exact sequence

$$0 \to \mathbb{G}_m \to GL_{n+1} \to PGL_{n+1} \to 0$$

we get, using suitably defined cohomology groups, an exact sequence

$$H^1(X, GL_{n+1}) \longrightarrow H^1(X, PGL_{n+1}) \xrightarrow{\delta_{n+1}} H^2(X, \mathbb{G}_m)\,.$$

Giving a $\mathbb{P}^n$-bundle over X is equivalent to giving a class $\xi \in H^1(X, PGL_{n+1})$. If $\delta_{n+1}(\xi) \neq 0$, then ξ does not come from a rank $n+1$ vector bundle. Furthermore, $\operatorname{im} \delta_{n+1}$ is annihilated by multiplication by $n+1$ (see [21], I, 1.4). Thus, in particular, if $f : \mathbf{P} \to X$ is a $\mathbb{P}^1$-bundle which is not the projectivization of a rank 2 vector bundle, then it gives rise to a non-trivial 2-torsion element in $\mathrm{Br}'(X)$.

We now construct 2-torsion elements in the Brauer groups for the threefolds mentioned in the main text.

Construction 1. Let $W \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ be a generic hypersurface of bidegree (2,2), and let $K \subseteq \mathbb{P}^3$ be the discriminant locus of the fibration $p_1 : W \to \mathbb{P}^3$, where p_1 is the projection onto the first factor. It is easy to see that K is an octic surface with 80 ordinary nodes. Let $d : X \to \mathbb{P}^3$ be the double cover of $\mathbb{P}^3$ branched over K, and let $\pi : \tilde{X} \to X$ be the blowing-up of the 80 nodes of X, so that $\tilde{X}$ is non-singular.

Theorem 1. *There is a non-trivial 2-torsion element in* $\mathrm{Br}'(\tilde{X})$.

Proof. Let U be the non-singular locus of X, so that $U = \tilde{X} - \bigcup_{i=1}^{80} E_i$, where the E_i are the exceptional divisors obtained from blowing up the singular points of X. Each E_i is a non-singular quadric surface. By [21], III 6.2, there is an exact sequence

$$0 \to \mathrm{Br}'(\tilde{X}) \to \mathrm{Br}'(U) \to \bigoplus_{i=1}^{80} H^1(E_i, \mathbb{Q}/\mathbb{Z}) = 0,$$

so $\mathrm{Br}'(\tilde{X}) = \mathrm{Br}'(U)$. Now each point $x \in U$ corresponds to a choice of a ruling of the non-singular quadric or quadric cone $p_1^{-1}(d(x))$. Let $\mathbf{P} \subseteq \mathrm{Gr}(2,4) \times U$ be the variety such that $\mathbf{P}_x$ parametrizes the lines in the corresponding ruling of $p_1^{-1}(d(x))$, so that $f : \mathbf{P} \to U$ is a $\mathbb{P}^1$-bundle. Let $l_x \subseteq W$ be the line corresponding to a point $x \in \mathbf{P}$.

Claim: f does not have a rational section, i.e. a rational map $\sigma : U \to \mathbf{P}$ with $f \circ \sigma$ the identity wherever σ is defined.

Proof. Suppose that f has a rational section $\sigma : U \to \mathbf{P}$. Let $D \subseteq W$ be defined to be the Zariski closure of the set

$$\{ l_{\sigma(x_1)} \cap l_{\sigma(x_2)} \mid x_1, x_2 \in U \text{ are any distinct points on which } \sigma \text{ is defined such that } d(x_1) = d(x_2) \}\,.$$

If $d(x_1) = d(x_2)$ then $l_{\sigma(x_1)}$ and $l_{\sigma(x_2)}$ are lines in distinct rulings of $p_1^{-1}(d(x_1))$, so the intersection consists of one point. Thus the projection $D \to \mathbb{P}^3$ is generically one to one, and so the cup product of the cohomology class $[D]$ of D in $H^4_{\mathrm{sing}}(W, \mathbb{Z})$ with the cohomology class of a fibre of p_1 is one. But since W is ample in $\mathbb{P}^3 \times \mathbb{P}^3$, by the Lefschetz hyperplane theorem, $H^4_{\mathrm{sing}}(W, \mathbb{Z}) \cong H^4_{\mathrm{sing}}(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{Z})$ and so the intersection of every cohomology class in $H^4_{\mathrm{sing}}(W, \mathbb{Z})$ with a fibre of p_1 is always even. This is a contradiction, proving the claim. □

Now if f were the projectivization of a rank 2 vector bundle $\mathscr{E}$ on U, a section of $\mathscr{E}$ would yield a rational section of f. Thus $f : \mathbf{P} \to U$ gives rise to a non-trivial 2-torsion element in $\mathrm{Br}'(U) \cong \mathrm{Br}'(\tilde{X})$. □

Construction 2. Let $P = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with trihomogeneous coordinates $([p_0, p_1], [p_2, p_3], [p_4, p_5])$. We denote by $\mathscr{O}_P(a,b,c)$ the line bundle of tridegree (a,b,c). Let

M be a symmetric matrix

$$M = \begin{pmatrix} f_{400} & f_{220} & f_{202} \\ f_{220} & f_{040} & f_{022} \\ f_{202} & f_{022} & f_{004} \end{pmatrix}$$

with the tridegrees of the forms indicated by the subscripts. For general choice of M, M is rank $\leqq 2$ on a surface $K \subseteq P$ of tridegree $(4,4,4)$ whose singular locus is the locus where M is rank 1: from this, it is an easy Chern class calculation using [22] to see that K has 64 nodes.

Now consider the map $s : P \to P$ defined by $s([p_0, p_1],[p_2, p_3],[p_4, p_5]) = ([p_0^2, p_1^2],[p_2, p_3],[p_4, p_5])$, so that s is a double cover. Consider a general matrix

$$M_s = \begin{pmatrix} f_{200} & f_{120} & f_{102} \\ f_{120} & f_{040} & f_{022} \\ f_{102} & f_{022} & f_{004} \end{pmatrix}.$$

$\det M_s$ vanishes on a surface K_s of tridegree $(2,4,4)$, which for general M_s has 32 singular points. Now $s^{-1}(K_s)$ is a surface K of the type described above, but the matrix $M = s^* M_s$ determining it may not be general. Nevertheless, if K_s is general, K will have 64 nodes. If X and X_s are the double covers of P branched over K and K_s respectively, $\tilde{X}$ and $\tilde{X}_s$ the blow-ups of the nodes, then it is clear that $\tilde{X}$ can be deformed smoothly to the blow-up of the double cover branched over a surface determined by a general matrix M. The Brauer group is a topological invariant, and so showing $\mathrm{Br}'(\tilde{X})$ contains a 2-torsion element for this special M will show it contains a 2-torsion element for general M.

Consider the map $f_s : X_s \to \mathbb{P}^1 \times \mathbb{P}^1$ which is the composition of the maps $X_s \to P$ and $P \to \mathbb{P}^1 \times \mathbb{P}^1$ given by projection onto the second and third $\mathbb{P}^1$'s. f_s is a conic bundle. We also define $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ similarly, so that f is an elliptic fibration. The following lemma summarizes the geometric results about X and X_s we will need.

Lemma 2.

(1) *The discriminant locus Δ of f_s consists of two curves Δ_1 and Δ_2, each of type $(4,4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, meeting transversally at 32 points, and each fibre of f_s over Δ is a union of two $\mathbb{P}^1$'s.*

(2) *The Cartier divisor class group and the Weil divisor class group of X coincide, and $\mathrm{Pic}\,X \cong \mathbb{Z}^{\oplus 3}$, generated by $p_i^* \mathcal{O}_{\mathbb{P}^1}(1)$, $1 \leqq i \leqq 3$, where $p_i : X \to \mathbb{P}^1$ is the projection onto the i^{th} component of P.*

(3) *There is a non-singular threefold V birationally equivalent to X, and a map $g : V \to B$ birationally equivalent to $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$, where B is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points of $\Delta_1 \cap \Delta_2$. Furthermore*

(a) *g is flat.*

(b) *If we also denote by Δ_1 and Δ_2 the proper transforms of these two curves on B, $g^{-1}(\Delta_1)$ and $g^{-1}(\Delta_2)$ are irreducible divisors, and a fibre of g over a general point of Δ_1 or Δ_2 consists of a union of $\mathbb{P}^1$'s meeting at two points.*

Proof. (1) It is easy to see that for a general choice of M_s, the projection $p : K_s \to \mathbb{P}^1 \times \mathbb{P}^1$ is finite. p is a double cover, and the branch locus of p is the discriminant locus Δ of f_s. Furthermore, the singular points of Δ are precisely the images of the singular points of K_s. Since K_s has 32 nodes, Δ has 32 nodes.

Consider the curve $\Delta_1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined by $f_{040}f_{004} - f_{022}^2 = 0$. This is a curve of bidegree $(4,4)$. Over this curve, $\det M_s$ reduces to

$$-f_{120}^2 f_{004} + 2f_{120}f_{102}f_{022} - f_{102}^2 f_{040}\ .$$

If we consider this as a quadratic expression in the variables f_{120} and f_{102}, its discriminant is $-4(f_{040}f_{004} - f_{022}^2)$, which is zero over Δ_1. Thus K_s is branched over Δ_1, and $\Delta_1 \subseteq \Delta$. It is easy to see that Δ is of bidegree $(8,8)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and thus $\Delta = \Delta_1 \cup \Delta_2$ with Δ_2 of bidegree $(4,4)$. Since Δ has 32 nodes, this leaves no choice but for Δ_1 and Δ_2 to be non-singular curves meeting transversally.

(2) Let $\mathrm{Cl}(X)$ denote the Weil divisor class group of X. The *defect* of X is defined as $\mathrm{rk}(\mathrm{Cl}(X)/\operatorname{Pic}(X))$, and this can be computed via the methods of [16], Sect. 3 to be

$$\dim H^0(\mathscr{I}_{Z/P}(4,4,4)) - \dim H^0(\mathscr{O}_P(4,4,4)) + \#\text{ of nodes of } K\ ,$$

where Z is the singular locus of K and $\mathscr{I}_{Z/P}$ is the ideal sheaf of Z in P.

Z is defined by the 2×2 minors of the symmetric matrix M. There are six such distinct minors, and using them, one obtains a three step resolution of $\mathscr{I}_{Z/P}$ by direct sums of line bundles on P. (One can do this by hand or very quickly using Macaulay [9].) From this one computes that $\dim H^0(\mathscr{I}_{Z/P}(4,4,4)) = 61$. We omit the details. This then gives that the defect is zero.

Now $\operatorname{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$ since K is an ample divisor in P, and the local class group of a node is torsion free, so $\mathrm{Cl}(X)/\operatorname{Pic}(X)$ is torsion free and rank 0. We conclude that $\mathrm{Cl}(X) \cong \operatorname{Pic}(X)$.

(3) X is a double cover of X_s branched over a non-singular surface S which is contained in the non-singular part of X_s. If $p \in \Delta_1 \cap \Delta_2$, then $f^{-1}(p) = l_1 \cup l_2$ with l_1 and l_2 being $\mathbb{P}^1$'s intersecting at a node of X_s. Blow-up the node and then the proper transforms of l_1 and l_2. Doing this for all $p \in \Delta_1 \cap \Delta_2$, we obtain a non-singular threefold V_s with a flat morphism $V_s \to B$. (Equivalently, V_s is obtained by blowing up the singular locus of $X_s \times_{\mathbb{P}^1 \times \mathbb{P}^1} B$.) Let S' be the proper transform of S in V_s. Since S intersects l_1 and l_2 transversally, S' is non-singular. Let V be the double cover of V_s branched along S, and $g : V \to B$ the composition of $V \to V_s$ and $V_s \to B$. It is then clear from the construction that g is flat. For (b), observe that if $g^{-1}(\Delta_i)$ was not irreducible, then the Weil divisor class group of X would be larger than (2) permits. The last statement follows from the above description of V. □

Theorem 3. $\mathrm{Br}'(\tilde{X})$ *contains a non-trivial 2-torsion element.*

Proof. The Brauer group is a birational invariant ([21], III, Theorem 7.4), so it will be enough to show that $\mathrm{Br}'(V)$ contains 2-torsion.

We will follow the notation of [19]. Let η be the generic point of B, $i : \eta \to B$ the inclusion, $P_{V/B} = R^1 g_* \mathbb{G}_m$, and $\mathscr{E} = \ker(P_{V/B} \to i_* i^* P_{V/B})$. By Lemma 2, (3), $g : V \to B$ is what is called a good model in [19], Definition 1.1, so we can apply the results of [19], Sect. 1.

Let $D_i = g^{-1}(\Delta_i)$, $\tilde{D}_i$ be the normalization of D_i, and let $\tilde{D}_i \to \tilde{\Delta}_i \to \Delta_i$ be the Stein factorization. By Lemma 2, (1), $\tilde{\Delta}_i \to \Delta_i$ is an unramified double cover, as one obtains the same double covering using the map f_s. By [19], Proposition 1.13,

$$H^1(B,\mathscr{E}) = \bigoplus_{i=1}^{2} \ker(H^1(\Delta_i, \mathbb{Q}/\mathbb{Z}) \to H^1(\tilde{\Delta}_i, \mathbb{Q}/\mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}\,.$$

By the exact sequence

$$0 \to \mathscr{E} \to P_{V/B} \to i_* i^* P_{V/B} \to 0$$

(surjectivity on the right follows from [19], Proposition 1.10) we obtain a sequence

$$H^0(B, P_{V/B}) \xrightarrow{\alpha} H^0(B, i_* i^* P_{V/B}) \longrightarrow H^1(B,\mathscr{E}) \longrightarrow H^1(B, P_{V/B})\,.$$

We also have an exact sequence of sheaves on η:

$$0 \longrightarrow A \longrightarrow i^* P_{V/B} \xrightarrow{d} \mathbb{Z} \longrightarrow 0,$$

where d is the degree map and A is the Jacobian of $V_\eta = X_\eta$. First $H^0(A) = 0$: a non-trivial degree zero line bundle on X_η would extend to give a divisor on X not allowed by Lemma 2, (2), and so $H^0(\eta, i^* P_{V/B}) \subseteq \mathbb{Z}$. Thus coker$(\alpha)$ is a cyclic group, and since it injects into $H^1(B,\mathscr{E})$, we must have

$$\operatorname{coker}(\alpha) = 0 \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z},$$

and thus

$$\operatorname{im}(H^1(B,\mathscr{E}) \to H^1(B, P_{V/B})) = \mathbb{Z}/2\mathbb{Z} \quad \text{or} \quad (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}\,.$$

In fact, it is the first case which occurs, but since that does not matter to us, we do not prove this here. Finally, we have an exact sequence ([19], 1.5)

$$0 = \operatorname{Br}'(B) \to \operatorname{Br}'(V) \to H^1(B, P_{V/B}) \to H^3(B, \mathbb{G}_m) = 0\,,$$

with the left and right terms being zero since B is a rational surface, so $\operatorname{Br}'(V)$ contains a two-torsion element. □

Remark 4. The above computation obscures the actual source of the 2-torsion, which can be seen in the following manner: Using the results of [1], $H^4_{\mathrm{sing}}(V_s, \mathbb{Z})$ has 2-torsion, generated by $l_0 - l_1$, where l_0 and l_1 are the two components of a fibre of g over a point in Δ_1 or Δ_2. (V_s is as in the proof of Lemma 2, (3).) This cycle then lifts to a difference of two rational curves in V, generating the 2-torsion in $H^4_{\mathrm{sing}}(V, \mathbb{Z})$ we have produced above. However, this requires a more detailed analysis of the geometry of V. Furthermore, the method of proof of Theorem 4 is a more suitable approach for some other examples.

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