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The Cohomology of the Space of Magnetic Monopoles

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Abstract: Denote by X_q the reduced space of SU_2 monopoles of charge q in $\mathbb{R}^3$. In this paper the cohomology of X_q, the cohomology with compact supports of X_q, and the image of the latter in the former are all calculated as representations of $\mathbb{Z}/q\mathbb{Z}$ which acts on X_2. This provides a non-trivial "lower bound" for the L^2 cohomology of X_q which is compatible with some conjectures of Sen. It is also shown that, granted some assumptions about the metric on X_q, its L^2 cohomology does not exceed this bound in the situation referred to in the paper as the "coprime case".

1. Introduction

The moduli space $\mathscr{M}_q$ of SU_2-monopoles of magnetic charge q in $\mathbb{R}^3$ is a Riemannian manifold of dimension $4q$. It has remarkable geometric properties, of which a comprehensive account can be found in [A-H]. Recently, to test hypotheses concerning electric-magnetic duality in non-abelian gauge theories [Sen], there has been interest in determining the square-summable harmonic forms on $\mathscr{M}_q$ – or, more precisely, on a $(4q-4)$-dimensional "reduced" moduli space X_q contained in it. To define the reduced space we first get rid of the free action of the group $\mathbb{R}^3$ of translations by restricting to monopoles whose centre of mass is at the origin in $\mathbb{R}^3$. There is still a free action of the circle group $\mathbb{T}$ which rotates the "phase" of a monopole. We cannot normalize the phase away completely, but we can fix it up to a q^{th} root of unity. This gives us a simply connected manifold X_q, on which the cyclic group μ_q of q^{th} roots of unity still acts freely by rotating the phase.

Let $\mathscr{H}_q^i$ denote the space of square-summable harmonic i-forms on X_q. We can decompose $\mathscr{H}_q^i$ according to the induced action of μ_q

$$\mathscr{H}_q^i = \bigoplus \mathscr{H}_{q,p}^i ,$$

where $\mathscr{H}_{q,p}^i$ is the part where the elements $\zeta \in \mu_q$ act by multiplication by ζ^p. Sen

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[Sen] has conjectured that

(i) If p and q are relatively prime then $\mathscr{H}^i_{q,p} = 0$ except in the middle dimension $i = 2q-2$, in which $\mathscr{H}^{2q-2}_{q,p} \cong \mathbb{C}$, and

(ii) if q and p have a common factor then $\mathscr{H}^i_{q,p} = 0$ for all i.

In fact it is conjectured that there is a natural action of the group $SL_2\mathbb{Z}$ on the L^2 harmonic forms on all of the X_q together, which transforms the (p,q)-bigrading according to the natural action of $SL_2\mathbb{Z}$ on $\mathbb{Z}^2$.

In this paper we shall provide some evidence in support of Sen's conjecture by calculating the cohomology of the space X_q together with the action of μ_q on it. In particular we shall show that part (i) of the conjecture follows from some mild and very plausible assumptions about the nature of the metric on X_q.

We begin by stating our results about the cohomology. We write $H^*(X_q)$ for the cohomology with complex coefficients, and $H^*(X_q)_p$ for the part where the group μ_q acts by $\zeta \mapsto \zeta^p$.

Theorem 1.1.

$$\begin{aligned} H^i(X_q)_p &\cong \mathbb{C} \quad \textit{if } i = 2q-2(p,q)\,, \\ &= 0 \quad \textit{otherwise}\,. \end{aligned}$$

Here (p,q) denotes the greatest common divisor of p and q.

This result is compatible with the existence of an $SL_2\mathbb{Z}$-action on $\bigoplus_{p,q} H^*(X_q)_p$ which does not preserve the dimension of the cohomology classes, but *does* preserve the dimension relative to the middle dimension $2q-2$.

Because X_q is an orientable open manifold of dimension $4q-4$ we can deduce its cohomology with compact supports from Theorem 1.1 by Poincaré duality.

Corollary 1.2.

$$\begin{aligned} H^i_{cpt}(X_q)_p &\cong \mathbb{C} \quad \textit{if } i = 2q+2(p,q)-4\,, \\ &= 0 \quad \textit{otherwise}\,. \end{aligned}$$

The map $H^i_{cpt} \to H^i$ is trivially zero except perhaps in the middle dimension $i = 2q-2$. We shall prove

Theorem 1.3.

$$H^{2q-2}_{cpt}(X_q) \xrightarrow{\cong} H^{2q-2}(X_q)\,.$$

In view of these topological results the following arguments show that the spaces $\mathscr{H}^i_{q,p}$ of L^2 harmonic forms are at least as large as Sen conjectures.

First, because X_q is a *complete* Riemannian manifold, we know ([deR] Sect. 35 Theorem 26) that any L^2 harmonic form on it is closed and coclosed, and so there is a map

$$\mathscr{H}^i_{q,p} \to H^i(X_q)_p\,.$$

We also know ([deR] Sect. 32 Theorem 24) that the space $\Omega^{\cdot}_{(L^2)}$ of all L^2 forms on X_q has an orthogonal decomposition

$$\Omega^{\cdot}_{(L^2)} = \overline{d(\Omega^{\cdot}_{cpt})} \oplus \mathscr{H}^i_q \oplus \overline{\delta(\Omega^{\cdot}_{cpt})}\,,$$

where $\Omega^{\cdot}_{cpt}$ denotes the smooth forms with compact support, and the bars denote the closure in L^2. So there is a map

$$H^i_{cpt}(X_q) \to \mathscr{H}^i_q$$

defined by orthogonal projection of smooth closed forms with compact support. We have

Lemma 1.4. *The composition*

$$H^{\cdot}_{cpt}(X_q) \to \mathscr{H}^{\cdot}_q \to H^{\cdot}(X_q)$$

is the obvious map.

Proof. The map takes a closed form $\alpha \in \Omega^{\cdot}_{cpt}$ to $\alpha - \beta$, where $\beta = \lim d\gamma_i$ in L^2, with $\gamma_i \in \Omega^{\cdot}_{cpt}$. But β represents zero in $H^{\cdot}(X_q)$ because of Poincaré duality, for

$$\int \beta\theta = \lim \int d(\gamma_i\theta) = 0$$

for any closed $\theta \in \Omega^{\cdot}_{cpt}$.

Thus Theorem 1.3 tells us that $\dim(\mathscr{H}^{2q-2}_{q,p}) \geqq 1$ when p and q are coprime. We have not used any property of the metric of X_q except completeness. To prove that $\dim(\mathscr{H}^{2q-2}_{q,p}) \leqq 1$, in the coprime case, we need some more precise properties of the metric. This is explained in Sect. 3.

2. Calculation of the Cohomology

The proofs of Theorems 1.1 and 1.3 depend on two elementary observations. First, the functor $X \mapsto H^*(X)_p$ is a cohomology theory on the category of spaces with μ_q-action, in the sense that after taking the p-component we still have a long exact sequence associated to a space and a subspace, and we have a Mayer–Vietoris sequence for the cohomology of a union. Secondly, if g is a generator of μ_q, and g^k is homotopic to the identity on X, then $H^*(X)_p = 0$ unless the order of the root of unity $e^{2\pi i p/q}$ divides k, i.e. unless kp is divisible by q.

We shall use different descriptions of X_q for the proofs of the two theorems. For Theorem 1 we use Donaldson's identification of the moduli space $\mathscr{M}_q$ with the space of rational functions of the form

$$\frac{\varphi}{\psi} = \frac{a_{q-1}z^{q-1} + \cdots + a_1 z + a_0}{z^q + b_{q-1}z^{q-1} + \cdots + b_1 z + b_0},$$

where φ and ψ are polynomials with complex coefficients and no common root. The last condition can be expressed

$$\mathscr{R}(\varphi, \psi) \neq 0,$$

where the *resultant* $\mathscr{R}(\varphi, \psi)$ is the polynomial in the a_i and b_i defined by

$$\mathscr{R}(\varphi, \psi) = \prod \varphi(\beta_i),$$

where $\beta_1, \ldots, \beta_q$ are the roots of ψ. We notice that $\mathscr{R}(\varphi, \psi)$ is homogeneous of degree q in $a_0, \ldots, a_{q-1}$. The group $\mathbb{T}$ acts on $\mathscr{M}_q$ simply by multiplication. The reduced moduli space X_q is the subspace of $\mathscr{M}_q$ defined by the two conditions

$b_{q-1} = 0$ and $\mathscr{R}(\varphi, \psi) = 1$. It is a nonsingular algebraic hypersurface in $\mathbb{C}^{2q-1}$. (The hypersurfaces defined by $\mathscr{R}(\varphi, \psi) = \lambda$ must be nonsingular for almost all λ; but by the homogeneity of $\mathscr{R}(\varphi, \psi)$ they are all isomorphic when $\lambda \neq 0$.)

Proof of Theorem 1.1. We first consider the part of $H^*(X_q)$ fixed under the μ_q-action, i.e. the case $p = 0$. Then $H^*(X_q)_0 \cong H^*(X_q/\mu_q) = H^*(\mathscr{M}_q/\mathbb{C}^\times)$. The theorem asserts that $\mathscr{M}_q/\mathbb{C}^\times$ has the rational homotopy type of a point. Now $\pi_1(\mathscr{M}_q) = \mathbb{Z}$, and the map $\mathscr{M}_q \to \mathbb{C}^\times$ defined by $\varphi/\psi \mapsto \mathscr{R}(\varphi, \psi)$ induces an isomorphism of rational homology. (This is a simple consequence of the results of [S], where it is proved that $H_*(\mathscr{M}_q)$ – with arbitrary coefficients – is a summand in $H_*(\Omega^2 S^3)$; for, rationally, we have $\Omega^2 S^3 \simeq S^1$.) The multiplication action of $\mathbb{C}^\times$ on $\mathscr{M}_q$ gives us maps $\mathbb{C}^\times \to \mathscr{M}_q$, and the composites $\mathbb{C}^\times \to \mathscr{M}_q \to \mathbb{C}^\times$ are $u \mapsto cu^q$. Homotopically, therefore, $\mathscr{M}_q/\mathbb{C}^\times$ is a bundle over the classifying space $B\mathbb{C}^\times = \mathbb{C}P^\infty$ with fibre $\mathscr{M}_q$, and so it is rationally a circle bundle over $\mathbb{C}P^\infty$ with Chern class $q \in H^2(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}$. The Gysin sequence for this circle bundle shows that $\mathscr{M}_q/\mathbb{C}^\times$ is rationally a point.

To treat the other cases of Theorem 1.1 we stratify X_q according to the multiplicities of the roots of ψ: if $\lambda = (\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_r)$ is a partition of q we write X_λ for the part of X_q where the denominator ψ is of the form

$$\psi = \prod_{i=1}^{r} (z - \beta_i)^{\lambda_i}$$

with $\beta_1, \ldots, \beta_r$ distinct.

If p is a non-zero residue class modulo q, let $l > 1$ divide the order of the root of unity $e^{2\pi i p/q}$. Let Y be the part of X_q consisting of rational functions φ/ψ such that $\psi = \chi^l$ for some polynomial χ of degree $r = q/l$. Equivalently, Y is the union of the strata X_λ for all partitions λ, all of whose parts λ_i are divisible by l. We shall prove

Proposition 2.1. *Y is a closed μ_q-invariant submanifold of X_q, algebraically isomorphic to the disjoint union of l copies of $X_r \times \mathbb{C}^{q-r}$. In terms of the μ_q-action*

$$Y \cong \mu_q \times_{\mu_r} (X_r \times \mathbb{C}^{q-r}),$$

where μ_r acts on X_r and on $\mathbb{C}^{q-r}$ by multiplication.

Proposition 2.2.

$$H^*(X_q - Y)_p = 0 .$$

Before proving Propositions 2.1 and 2.2 let us show that they imply Theorem 1.1. The submanifold Y of X_q has complex codimension $(2q-2) - (2r-2) - (q - r) = q - r$. Let U be a μ_q-invariant tubular neighborhood of Y in X_q. We have

$$\begin{aligned} H^i(X_q)_p &\cong H^i(X_q, X_q - Y)_p \quad \text{from Proposition 2.2} \\ &\cong H^i(U, U - Y)_p \quad \text{by excision} \\ &\cong H^{i-2q+2r}(Y)_p \quad \text{by the Thom isomorphism theorem.} \end{aligned}$$

Proposition 2.1 implies that $H^*(Y)$ is the representation of μ_q induced from the representation $H^*(X_r)$ of μ_r. It follows that $H^i(X_q)_p \cong H^{i-2q+2r}(X_r)_p$. This completes the proof of Theorem 1.1, for we can assume by induction that the theorem is true when q is replaced by r.

Proof of Proposition 2.1. If $\varphi/\psi = \varphi/\chi^l$ belongs to Y, then $\mathscr{R}(\varphi,\psi) = \mathscr{R}(\varphi,\chi)^l$, so $\mathscr{R}(\varphi,\chi)$ is an l^{th} root of unity, and Y breaks up into l components, Y_u, according to the value of $u = \mathscr{R}(\varphi,\chi)$. We can write

$$\varphi = \varphi_0\chi + \varphi_1$$

uniquely, where φ_1 has degree less than r, and φ_0 has degree less than $q-r$. Then $\mathscr{R}(\varphi,\chi) = \mathscr{R}(\varphi_1,\chi)$, so the correspondence

$$\varphi/\chi^l \leftrightarrow (\varphi_1/\chi, \varphi_0)$$

is an isomorphism between Y_1 and $X_r \times \mathbb{C}^{q-r}$. Finally, multiplication by $\zeta \in \mu_q$ is an isomorphism $Y_1 \to Y_{\zeta^r}$.

Proof of Proposition 2.2. We begin with

Lemma 2.3. *Suppose that $q = q_1 + q_2$, and that U is the open μ_q-invariant subset of X_q consisting of all rational functions φ/ψ with $\psi \in W$, where W is an open subset of the space of monic polynomials such that all $\psi \in W$ can be factorized $\psi = \psi_1\psi_2$ into ordered coprime factors ψ_1, ψ_2 depending continuously on $\psi \in W$, with $\deg(\psi_i) = q_i$. Suppose also that q_1 and q_2 are not divisible by the order of $e^{2\pi i p/q}$. Then $H^*(U)_p = 0$.*

Proof of the Lemma. If $\varphi/\psi \in U$ we have a unique decomposition

$$\frac{\varphi}{\psi} = \frac{\varphi_1}{\psi_1} + \frac{\varphi_2}{\psi_2}\,.$$

We can define an action of the group

$$G = \{(\zeta_1,\zeta_2) \in \mathbb{T}\times\mathbb{T} : \zeta_1^{q_1}\zeta_2^{q_2} = 1\} \tag{2.4}$$

on U by

$$(\zeta_1,\zeta_2)\cdot(\varphi/\psi) = \zeta_1(\varphi_1/\psi_1) + \zeta_2(\varphi_2/\psi_2)\,.$$

For

$$\begin{aligned}\mathscr{R}(\varphi,\psi) &= \mathscr{R}(\varphi_1\psi_2 + \varphi_2\psi_1, \psi_1\psi_2)\\ &= (-1)^{q_1q_2}\mathscr{R}(\varphi_1,\psi_1)\mathscr{R}(\varphi_2,\psi_2)\mathscr{R}(\psi_1,\psi_2)^2\,,\end{aligned}$$

and so the action of $(\zeta_1,\zeta_2) \in G$ multiplies $\mathscr{R}(\varphi,\psi)$ by $\zeta_1^{q_1}\zeta_2^{q_2} = 1$. The action of G extends the action of the cyclic group μ_q, for μ_q can be identified with the diagonal subgroup

$$\{(\zeta_1,\zeta_2) \in G : \zeta_1 = \zeta_2\}\,.$$

Now G is isomorphic to $\mathbb{T}\times\mu_d$, where d is the greatest common divisor of q_1 and q_2. This means that if g is a generator of μ_q then the action of g^d on U is homotopic to the identity, and hence that $H^*(U)_p = 0$ unless the order of $e^{2\pi i p/q}$ divides d.

Returning to the proof of Proposition 2.2, we now see, by using the Mayer–Vietoris sequence and induction on n, that $H^*(U)_p = 0$ if U is the union of n open subsets $U_1,\dots,U_n$ of X_q which each satisfy the hypotheses of Lemma 2.3. (Notice that $U_n \cap (U_1 \cup \dots \cup U_{n-1})$ is a union of $n-1$ sets which satisfy the hypotheses.) This essentially completes the proof of Proposition 2.2, as the open subset $U = X_q - Y$ of X_q is a union of such sets U_i. For if ψ is a monic polynomial with a root β of multiplicity q_1 not divisible by l then any polynomial near ψ can be factorized canonically as $\psi_1\psi_2$, where ψ_1 has q_1 roots near β, and ψ_2 has no roots near β. This argument leads to a covering of U by a countable number of subsets $\{U_i\}_{i\in S}$ of the desired type. At first sight the Mayer–Vietoris method works only for a finite covering. But there is a well-known device for dealing with this situation. We choose a partition of unity $\{\lambda_i\}$ subordinate to the covering $\{U_i\}$ of U. We can assume that at most $N+1$ of the numbers $\lambda_i(x)$ are non-zero at any point x of U, where $N = \dim(U)$. Then for each finite subset σ of S we define

$$U'_\sigma = \left\{ x \in U : \inf_{i\in\sigma} \lambda_i(x) > \sup_{i\notin\sigma} \lambda_i(x) \right\} .$$

The sets U'_σ cover U. Finally, for each $k \in \{0, 1, \ldots, N\}$ we define V_k as the union of the U'_σ when σ runs through the subsets of S with $k+1$ elements. The sets $V_0, V_1, \ldots, V_N$ cover U, and each V_i satisfies the hypotheses of Lemma 2.3, as it is a *disjoint* union of subsets each contained in one of the U_i.

Proof of Theorem 1.3. The simplest proof uses the same argument once again, but applies it to the alternative, more "physical", description of $\mathscr{M}_q$ as a space of clusters of monopoles in $\mathbb{R}^3$. Unfortunately, this description does not seem to have been worked out with sufficient precision for our purpose, so the following proof must be regarded as heuristic rather than complete. We have therefore given a complete proof in the Appendix, using the description of $\mathscr{M}_q$ by rational functions. It avoids the questionable use of clusters, but is much more complicated.

The cluster description depends on the fact that a q-monopole $\Phi \in \mathscr{M}_q$ centered at $x \in \mathbb{R}^3$ has an energy distribution in $\mathbb{R}^3$, and can sometimes be interpreted as the superposition of q_i-monopoles Φ_i centered at $x_i \in \mathbb{R}^3$ for $i = 1, 2, \ldots, k$, where $\Sigma q_i = q$, $\Sigma q_i x_i = qx$, and the points $x_1, \ldots, x_k$ are far apart in $\mathbb{R}^3$. In that case the assignment

$$\Phi \mapsto (T_{-x_1}\Phi_1, \ldots, T_{-x_k}\Phi_k; x_1, x_2, \ldots, x_k) ,$$

where T_y is the operation of translating a monopole by $y \in \mathbb{R}^3$, extends to a local diffeomorphism

$$f : U \to \mathscr{M}^0_{q_1} \times \mathscr{M}^0_{q_2} \times \cdots \times \mathscr{M}^0_{q_k} \times \mathbb{R}^{3k}$$

defined in a neighborhood U of Φ in $\mathscr{M}_q$, where $\mathscr{M}^0_{q_i}$ denotes the space of *centered* q_i-monopoles. We can suppose that the image of U is invariant under the action of $\mathbb{T}^k$ on $\mathscr{M}^0_{q_1} \times \cdots \times \mathscr{M}^0_{q_k}$ which rotates the phases of the monopoles, and f is automatically $\mathbb{T}$-equivariant for the diagonal action of $\mathbb{T}$ which rotates the overall phase. This enables us to extend the $\mathbb{T}$-action on U to a $\mathbb{T}^k$-action.

In the new picture the space X_q obtained from $\mathscr{M}^0_q$ by normalizing the overall phase must not be thought of as a subspace of $\mathscr{M}^0_q$. It is properly defined as the simply-connected covering space of $\mathscr{M}^0_q/\mathbb{T}$, whose fundamental group is cyclic of order q: we can identify it with the rational functions φ/ψ with $\mathscr{R}(\varphi,\psi) = 1$ only after choosing a direction in $\mathbb{R}^3$. Nevertheless, a $\mathbb{T}$-invariant open subset U of $\mathscr{M}^0_q$ defines an open subset U' of X_q which is the restriction of the q-fold covering to $U/\mathbb{T}$. Taking $k = 2$ in the preceding discussion, the $\mathbb{T} \times \mathbb{T}$ action on U induces an action on U' of the group G which was introduced in 2.4, but which now appears as the group of transformations of U' which cover the action of $(\mathbb{T} \times \mathbb{T})/\mathbb{T}$ on $U/\mathbb{T}$. Just as in the proof of Lemma 2.3, we can conclude that $H^*(U')_p = 0$ if p is prime to q, which is the only relevant case.

Now let V be the open subset of X_q consisting of all monopoles which can be regarded as superpositions of two widely separated configurations. A basic property of X_q is that the complement of V in X_q is compact. The argument of the proof of Proposition 2.2 shows that $H^*(V)_p = 0$ if $(p,q) = 1$, and hence that all elements of $H^*(X_q)_p$ come from $H^*(X_q, V)_p$, and so have compact support.

To conclude this section, let us explain more precisely the mild caution we expressed about the preceding argument. If in an open subset U of $\mathscr{M}_q$ the monopoles split into clusters of charge q_1 and q_2, with $q_1 + q_2 = q$, there seems no real problem in defining a $\mathbb{T}^2$-action on U which rotates the phases of the clusters independently. What needs to be checked carefully is that if U' is another region with a similar splitting $q = q_1' + q_2'$, then in $U \cap U'$, where the monopoles break up into, say, three clusters, the $\mathbb{T}^2$-action coming from U and U' form parts of a $\mathbb{T}^3$-action on $U \cap U'$ which rotates the phases of the three clusters. This was used, implicitly, in the Mayer–Vietoris argument: essentially it is equivalent to the assertion that superposition of widely separated clusters is an associative operation. (The corresponding assertion in the rational function description is obvious, for there superposition is simply addition of the rational functions.)

3. L^2 Harmonic Forms in the Coprime Case

If we are prepared to make some further, rather plausible, assumptions about the metric of the space X_q we can use the preceding methods to prove Sen's conjecture completely in the case when p and q are coprime.

Assumptions. The space $\mathscr{M}_q^0$ of centered monopoles can be covered by a finite number of $\mathbb{T}$-invariant open subsets $V_0, \ldots, V_m$ with the following properties:

(i) The closure of V_0 is compact.

(ii) If $i \neq 0$ then V_i consists of configurations which can be broken into two ordered widely-separated parts with charges a_i and b_i such that $a_i + b_i = q$.

(iii) The action of $\mathbb{T}$ on V_i extends to an action of the group $\mathbb{T}^2$ which rotates the phases of the parts independently. This action is by *near-isometries*, in the sense that if $f \in \mathbb{T}^2$ then

$$A^{-1} g \leqq f^* g \leqq A g$$

for some $A \geqq 1$, where g is the metric of V_i. Furthermore, the orbits of the action of $\mathbb{T}^2$ on V_i are of bounded size.

(iv) There is a smooth partition of unity $\{\lambda_i\}$ subordinate to the covering $\{V_i\}$ of $\mathscr{M}_q^0$ such that each $d\lambda_i$ is bounded.

The assumption (iv) follows from a stronger statement, with three parts.

(iv)(a) We can take V_0 to be the interior of a compact smooth manifold $\overline{V}_0$ with boundary Y, and $\mathscr{M}_q^0 - V_0$ is diffeomorphic to $Y \times [0, \infty)$.

(iv)(b) The projection $\pi : \mathscr{M}_q^0 - V_0 \to Y$ is length-decreasing for the Riemannian metric.

(iv)(c) The sets V_i for $i \neq 0$ are of the form $\pi^{-1}(W_i)$ for some $W_i \subset Y$.

We hope to return to the justification of these assumptions in another paper. Meanwhile, let us show that they imply Sen's conjecture when p and q are coprime.

It is enough to show that

$$H^*_{cpt}(X_q)_p \to \mathscr{H}_{q,p}$$

is surjective.

For any Riemannian manifold M, let $h^*(M)$ denote the cohomology of the subcomplex of the de Rham complex of M consisting of smooth forms α such that both α and $d\alpha$ belong to L^2. If μ_q acts on M by near-isometries, we can calculate the component $h^*(M)_p$ from the subcomplex of forms α such that $g^*_\zeta \alpha = \zeta^p \alpha$.

For the complete manifold X_q it follows from the discussion at the end of Sect. 1 that there is an injective map

$$\mathscr{H}_{q,p} \to h^*(X_q)_p ,$$

so to prove the conjecture it is enough to show that $h^*(V'_1 \cup \cdots \cup V'_m)_p = 0$, where V'_i is the q-fold covering space of $V_i/\mathbb{T}$ described in Sect. 2.

If $M = M_1 \cup M_2$, where M_1 and M_2 are open submanifolds of M, there is a Mayer–Vietoris sequence

$$\cdots \to h^*(M) \to h^*(M_1) \oplus h^*(M_2) \to h^*(M_1 \cap M_2) \to \cdots ,$$

providing there is a partition of unity $\lambda_1 + \lambda_2 = 1$ for $M = M_1 \cup M_2$ for which $d\lambda_1 = -d\lambda_2$ is bounded. There is also a generalised Mayer–Vietoris argument applying to coverings by more than two sets which implies that if there is a partition of unity, $\{\lambda_i\}$, for $M = M_1 \cup \cdots \cup M_m$, with each $d\lambda_i$ bounded and if $h^*(\bigcap_{i \in A} M_i) = 0$ for all nonempty $A \subseteq \{1, \ldots, m\}$, then $h^*(M) = 0$ (this follows by considering the double complex). Consequently we have that $h^*(V'_1 \cup \cdots \cup V'_m)_p = 0$, in view of the following variant of Lemma 2.3, where G_i denotes the group defined by (2.4), but with (q_1, q_2) replaced by (a_i, b_i).

Lemma 3.1. *If M is a G_i-stable open submanifold of V'_i, and p and q are coprime, then $h^*(M)_p = 0$.*

Proof. Let α be a smooth L^2 form representing an element of $h^*(M)_p$. We have $g^*_\zeta \alpha = \zeta^p \alpha$, where ζ is a primitive q^{th} root of unity. The element $\varphi = g^d_\zeta$, where $d = GCD(a_i, b_i)$, belongs to the identity component of G_i, so we have an explicit homotopy $\varphi \simeq$ (identity) generated by a *bounded* vector field. This shows that $\varphi^* \alpha - \alpha = d\beta$ for some L^2 form β. But $\varphi^* \alpha = \zeta^{pd} \alpha$, and $\zeta^{pd} \neq 1$. So $\alpha = d((\zeta^{pd} - 1)^{-1} \beta)$, as we want.

4. Appendix

Here we calculate the map $H^{2q-2}_{cpt}(X_q) \to H^{2q-2}(X_q)$. In what follows, p and q are coprime, $\zeta = e^{2\pi i/q}$, and g is the generator of μ_q corresponding to multiplication of rational functions by ζ. It will be convenient to work in homology – there are natural isomorphisms:

$$H_{2q-2}(X_q)_p = H^{2q-2}_{cpt}(X_q)_p = (H^{2q-2}(X_q)_{-p})^* = (H^c_{2q-2}(X_q)_{-p})^* ,$$

where the last of these spaces, H^c, the *closed chain* homology, is introduced only to define a normalisation for Theorem A.1. In this language there is an intersection pairing between $H_{2q-2}(X_q)$ and $H^c_{2q-2}(X_q)$, which is perfect by Poincaré duality, and another between $H_{2q-2}(X_q)$ and itself, which we should like to prove is perfect.

We observe that the argument of Proposition 2.1 with $l = q$ implies that

$$\tau_p = \frac{1}{q}\sum_{k<q} \zeta^{-kp} Y_{\zeta^k} = \frac{1}{q}\sum_{k<q} \zeta^{-kp} g^k Y_1$$

is a generator of $H^c_{2q-2}(X_q)_p$. (The submanifold Y_{ζ^k} being complex has a standard orientation.)

Theorem A.1. *If $\sigma_p \in H_{2q-2}(X_q)_p$ is the dual basis to τ_{-p} (i.e., $\sigma_p\tau_{p'} = \delta_{p,-p'}$), then $\sigma_p\sigma_{p'} = (-1)^{q-1}\delta_{p,-p'}$.*

Corollary. *The map $H^{2q-2}_{cpt}(X_q) \to H^{2q-2}(X_q)$ is an isomorphism.*

Remark. Transforming $\{\sigma_p\}$ to a real basis we can calculate the signature of the pairing, with the result that it is positive definite if q is odd and negative definite if q is even.

Proof of Theorem A.1. A typical rational function in X_q is denoted by $\varphi(z)/\psi(z)$, the roots of ψ being $\{z_1,\dots,z_q\}$. Recall that $\sum z_i = 0$ and $\prod \varphi(z_i) = 1$. Let ε be a fixed positive number, taken sufficiently small in what follows.

Define the (closed) subset $F \subseteq \mathbb{C}^q \times \mathbb{T}^q$ by

$$F = \{(z_1,\dots,z_q,w_1,\dots,w_q) : \textstyle\sum z_i = 0, \prod w_i = 1, w_i \neq w_j \Rightarrow |z_i - z_j| \geqq \varepsilon\}\,.$$

Define the map $\beta : F \to X_q$ by prescribing $\varphi(z_i) = w_i$ and $\psi(z_i) = 0$ if z_i are distinct (i.e., the map of Lagrange interpolation). This extends smoothly to the whole of F because of the $w_i \neq w_j \Rightarrow |z_i - z_j| \geqq \varepsilon$ condition.

The objective will be to describe explicitly the generating cycles of $H_{2q-2}(X_q)$ and then calculate their intersections by using a convenient diffeomorphism of X_q to perturb σ_p. The idea is that the proof of Theorem 1.1 implies that $\beta_* : H_{2q-2}(F) \to H_{2q-2}(X_q)$ is surjective, so that we may work mainly in F. In fact, rather than describing the complete cycles, it will prove simpler to use relative cycles (see below).

Recall that the proof of Lemma 2.3 made use of homotopies which were only defined when the roots of ψ were separated into two sets. This means that it is necessary to split X_q into subsets in which the z_i are separated in a controlled manner. To do this we introduce the *permutahedron*, P_q in $\mathbb{R}^q$.

Choose q distinct real numbers $r_1 < r_2 < \cdots < r_q$ such that $\sum r_i = 0$. (The r_i will have to be chosen sufficiently small to make a later part of this argument work.) Define $P(a_1,\dots,a_n)$ to be the convex hull in $\mathbb{R}^n$ of the $n!$ points $(a_{\pi_1},\dots,a_{\pi_n})$, where $\pi \in S_n$ is a permutation. Let $P_q = P(r_1,\dots,r_q)$, a polyhedron of dimension $q-1$. If $A = \{a_1,\dots,a_k\} \subseteq \{1,2,\dots,q\}$ with $a_1 < \cdots < a_k$, then define $\pi_A : \mathbb{R}^q \to \mathbb{R}^k$ by $\pi_A(x_1,\dots,x_q) = (x_{a_1},\dots,x_{a_k})$. If $A_1,\dots,A_r$ is an ordered partition of $\{1,2,\dots,q\}$ into non-empty subsets then let $s_i = \sum_{j<i}|A_j|$ and define

$$(A_1|\cdots|A_r) = \{x \in \mathbb{R}^q : \pi_{A_i}(x) \in P(r_{s_i+1},\dots,r_{s_{i+1}})\}\,.$$

It is a fact that $(A_1|\cdots|A_r)$ is a face of P_q and that all faces arise in this way. So faces of P_q correspond to ordered partitions and are isomorphic to products of lower dimensional permutahedra.

The property of these faces we shall need is that if $(x_1,\dots,x_q) \in (A_1|\cdots|A_r)$ and $i \neq j$ then $|x_a - x_b| \geqq \varepsilon$ whenever $a \in A_i$ and $b \in A_j$. This is ensured by choosing ε smaller than $r_2 - r_1$, $r_3 - r_2 \dots, r_q - r_{q-1}$.

We wish to think of the faces of P_q as singular chains, so we need to give them an orientation. This will be done by giving a section of the top exterior power of the tangent bundle. If $A \subseteq \{1,\dots,q\}$ then define

$$O_A = \left(\frac{\partial}{\partial x_{a_2}} - \frac{\partial}{\partial x_{a_1}}\right) \wedge \cdots \wedge \left(\frac{\partial}{\partial x_{a_k}} - \frac{\partial}{\partial x_{a_1}}\right),$$

where $A = \{a_1,\dots,a_k\}$ and $a_1 < \cdots < a_k$. It is convenient to define the orientation of $(A_1|\cdots|A_r)$ to be

$$\operatorname{sgn}(A_1A_2\cdots A_r)\prod_{i=1}^{r}[(-1)^{(r-i)|A_i|}O_{A_i}],$$

where $\operatorname{sgn}(A_1A_2\cdots A_r)$ denotes the sign of the permutation in S_q obtained by listing the elements of A_1 in ascending order followed by those of A_2, and so on. This strange choice makes the formula for the boundary simple:

$$\partial(A_1|\cdots|A_r) = \sum_{i=1}^{r}(-1)^i \sum_{\emptyset \neq B \subsetneq A_i} (A_1|\cdots|A_{i-1}|B|A_i \setminus B|A_{i+1}|\cdots|A_r). \tag{A.2}$$

We shall also need to define the homotopy as used in Lemma 2.3. If $R \subseteq \{1,\dots,q\}$, let $n_i = q - |R|$ if $i \in R$ and $n_i = -|R|$ otherwise. Then

$$T_R(\theta, z_1,\dots,z_q,w_1,\dots,w_q) = (z_1,\dots,z_q,w_1e^{in_1\theta},\dots,w_qe^{in_q\theta})$$

is a partially defined map from $[0,2\pi/q] \times F$ to F. If σ is a r-chain in F, such that T_R is everywhere defined on $[0,2\pi/q] \times |\sigma|$ ($|\sigma|$ being the carrier of σ), then $T_R(\sigma)$ defines in an obvious way an $r+1$-chain whose orientation we shall take to be $O_{T_R(\sigma)} = \frac{\partial}{\partial\theta} \wedge O_\sigma$. The basic property of T_R is

$$\partial T_R + T_R\partial = g^{-|R|} - 1.$$

Now, there is a map

$$P_q \times P_q \times \mathbb{Z}/q\mathbb{Z} \xrightarrow{\alpha} F$$

defined by

$$\alpha(x_1,\dots,x_q,y_1,\dots,y_q,k) = (x_1 + iy_1,\dots,x_q + iy_q, \zeta^k,\dots,\zeta^k).$$

The desired cycle in F will be built up from linear combinations of the chains

$$T_{R_1}\cdots T_{R_{r-1}}\alpha((A_1|\cdots|A_r),(12\cdots q),k).$$

It follows from the above definitions that the above chain is defined when each R_i is a union of a subcollection of the sets $A_1,\dots,A_r$. This is because being in the face $(A_1|\cdots|A_r)$ guarantees us that if $a \in A_i$, $b \in A_j$, $i \neq j$ then x_a and x_b, hence $x_a + iy_a$ and $x_b + iy_b$, are separated, which allows us to move w_a apart from w_b.

Note that we only use the top dimensional face of P_q to restrict the imaginary parts of the z_i. This will result in a cycle relative to the subset F^0 of F defined by

$$F^0 = \{(z_1,\dots,z_q,w_1,\dots,w_q) : (\operatorname{Im}(z_1),\dots,\operatorname{Im}(z_q)) \in \partial P_q\}$$

and on applying β, a cycle relative to the subset X^0 of X_q defined by

$$X^0 = \{\varphi/\psi : |\varphi(z_i)| = 1, (\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_q)) \in \partial P_q\} .$$

(This definition makes sense because P_q is invariant under permutation of coordinates.)

Define

$$[A_1|\cdots|A_r]_p = \sum_{k<q} \zeta^{-pk}\, \alpha((A_1|\cdots|A_r), (12\cdots q), k) ,$$

so $[A_1|\cdots|A_r]_p$ is a "p-type" chain, that is $g[A_1|\cdots|A_r]_p = \zeta^p[A_1|\cdots|A_r]_p$. Let $U_R = (\zeta^{-p|R|} - 1)^{-1} T_R$, so that

$$\partial U_R + U_R \partial = 1 \tag{A.3}$$

on p-type chains. Here it is used that p and q are coprime.

We are now in a position to construct the cycle we are interested in. Let

$$\sigma_p = \beta\left[\frac{1}{q!} \sum_{(A_i)} (-1)^{\binom{r-1}{2}} U_{A_1} U_{A_1\cup A_2} \cdots U_{A_1\cup\cdots\cup A_{r-1}} [A_1|\cdots|A_r]_p\right] , \tag{A.4}$$

the sum being over all ordered partitions of $\{1, \dots, q\}$. Note that $gT_R = T_R g$, so all terms are of p-type. It follows from (A.2) that, modulo chains in F^0,

$$\partial[A_1|\cdots|A_r]_p = \sum_{i=1}^{r} (-1)^i \sum_{\emptyset \neq B \subsetneq A_i} [A_1|\cdots|A_{i-1}|B|A_i \setminus B|A_{i+1}|\cdots|A_r]_p .$$

Some checking using the relation (A.3) then shows that σ_p is a relative cycle in the pair (X_q, X^0).

To calculate the intersection number $\sigma_p\sigma_{p'}$ we proceed as follows. A map $m : X_q \to X_q$ is exhibited such that the following properties hold:

(i) m is a diffeomorphism.

(ii) $m(X^0) \cap \beta(F) = m(X^0) \cap X^0 = \emptyset$.

(iii) Suppose $\lambda_1 c_1$ and $\lambda_2 c_2$ are component chains in $m(\sigma_p)$ and $\sigma_{p'}$ respectively. That is, $\lambda_i \in \mathbb{C}$ and, for $i = 1, 2$, c_i is a map from some polyhedron, S_i, in $\mathbb{R}^{2q-2}$ together with an orientation of S_i. Suppose also that $c_1(x_1) = c_2(x_2) = y$. Then $x_i \notin \partial S_i$, c_i is smooth in a neighbourhood of x_i and $c_1 \cap c_2$ is transverse at x_1, x_2 ($c_{1*}T_{x_1}S_1 + c_{2*}T_{x_2}S_2 = T_y X_q$).

It follows from these conditions that we can calculate $\sigma_p\sigma_{p'}$ by summing $\lambda_1\lambda_2$ over pairs (x_1, x_2) as in (iii) with the usual sign. To see this we use the general theory of [de R], Sect. 20 together with the following observation. The proof of Lemma 2.3 implies that $H_{2q-2}(X^0)_p = 0$. Therefore there is a cycle $\sigma_p + \alpha_p$ of X_q with $|\alpha_p| \subseteq X^0$. Condition (ii) above now assures us that α_p does not enter into the intersection calculation.

Exactly similar reasoning allows us to calculate the intersection numbers $\sigma_p\tau_{p'}$ by moving $\tau_{p'}$. This is easier, so let us deal with it first. Move Y_ω from $\{\varphi/z^q : \varphi(0) = \omega\}$ to $Y'_\omega = \{\varphi/\psi : |\varphi(0) - \omega| < \delta\}$ for some small $\delta > 0$ and some fixed choice of $\psi(z)$ whose roots are small and distinct. It is clear that the only term in (A.4) which intersects this is the "top" term, i.e., the one with $r = 1$. Taking the orientations into consideration we have $\sigma_p \tau_{p'} = \beta(\frac{1}{q!}[12\cdots q]_p)\frac{1}{q}\sum_s \zeta^{-p's} Y'_{\zeta s} = S\frac{1}{q \cdot q!}\sum_{r,s}\zeta^{-pr+p's}\delta_{rs} q! = S\delta_{p,-p'}$, where $S = \pm 1$ is a overall sign (depending only on q) which is unimportant since Theorem A.1 is asserting something about $\sigma_p \sigma_{p'}$.

Returning to $\sigma_p \sigma_{p'}$, define m as the composition of two maps, m_1 and m_2.

(i) The map m_1 takes φ/ψ to

$$\frac{\varphi(z)e^{iz} \bmod \psi(z)}{\psi(z)} .$$

(Here $\varphi(z)e^{iz} \bmod \psi(z)$ means the unique polynomial of degree less than q congruent to $\varphi(z)e^{iz}$ modulo $\psi(z)$.)

(ii) To define the map m_2 we fix a real number λ with $0 < \lambda < 1$, and map φ/ψ to

$$\frac{\varphi(z/\lambda)}{\lambda^q \psi(z/\lambda)} .$$

Both maps are analytic diffeomorphisms (since ψ is always monic) which preserve the resultant, so we may define m to be $m_2 \circ m_1$.

If $m(\varphi/\psi) = \varphi_m/\psi_m$, where the roots of ψ_m are z'_i and $\varphi_m(z'_i) = w'_i$, then

$$w'_i = e^{iz_i} w_i \quad \text{and} \quad z'_i = \lambda z_i . \tag{A.5}$$

This establishes property (ii) of m, since the condition $|w'_i| = 1$ implies that $\mathrm{Im}(z_i) = 0$. Furthermore, since $\lambda P_q \cap \partial P_q = \emptyset$, we are reduced to finding intersections of $m(\sigma_p)$ with the top term, $\beta(\frac{1}{q!}[12\cdots q]_{p'})$, of $\sigma_{p'}$. This implies that φ_m is a constant polynomial.

Consider now the term in (A.4) corresponding to the partition $A_1, \ldots, A_r$, and suppose that φ/ψ is a point of $|\sigma_p|$ such that $m(\varphi/\psi) = \varphi_m/\psi_m$ is in $|\sigma_{p'}|$. We have that $w'_1 = w'_2 = \cdots = \zeta^s$ for some s. If $a, b \in A_i$ then $w_a = w_b$, so $e^{iz_a} = e^{iz_b}$, and since the r_i were chosen sufficiently small, we have $z_a = z_b$. (This means that intersections can only occur at points arising from the barycentres of faces of P_q.) There is one more restriction on the intersection points we need to determine, namely that $r = q$, in other words $|A_i| = 1$ for all i. To see this, let $v_i = z_{a_i}$ for some (all) $a_i \in A_i$, and let $\psi_i(z) = (z - v_i)^{|A_i|}$. Decompose ζ^s/ψ into partial fractions:

$$\frac{\zeta^s}{\psi} = \sum_{i=1}^{r} \frac{\varphi_i}{\psi_i} .$$

Then it follows from (A.5) and the nature of the map β that

$$\frac{\varphi}{\psi} = \sum_{i=1}^{r} \frac{e^{-iv_i}\varphi_i}{\psi_i} ,$$

and so

$$m_1\left(\frac{\varphi}{\psi}\right) = \sum_{i=1}^{r} \frac{e^{i(z-v_i)}\varphi_i \bmod \psi_i}{\psi_i} .$$

But this must have constant numerator when put over the common denominator, so it is a constant multiple of $\sum \varphi_i/\psi_i$. Consequently $e^{i(z-v_i)}\varphi_i \equiv C\varphi_i \bmod \psi_i$. Multiplying by $\prod_{j \neq i}\psi_j$ tells us that $e^{i(z-v_i)} \equiv C \bmod (z - v_i)^{|A_i|}$ which is impossible unless $|A_i| = 1$.

This completes the identification of the intersection points $m(|\sigma_p|) \cap |\sigma_{p'}|$ which, to summarize, arise from the $r = q$ terms in (A.4) with $w_i = \zeta^s e^{-iz_i}$ for some s. At such points of $|\sigma_p|$, the z_i are all distinct (being equal to a permutation of $r_1, \ldots, r_q$), and the w_i are not equal to q^{th} roots of unity. This shows that property (iii) of m holds, with the possible exception of the transversality condition.

We now have to work out the intersection of

$$m\beta(U_{B_1} U_{B_2} \cdots U_{B_{q-1}} \alpha((a_1| \cdots |a_q), (12 \cdots q), r))$$

with

$$\beta\alpha((12 \cdots q), (12 \cdots q), s) , \tag{A.6}$$

where (a_i) is a permutation of $1, 2, \ldots, q$ and $B_i = \{a_1, \ldots, a_i\}$. This amounts to counting solutions of the equations

$$\sum_{j=1}^{q-1} [(qI_{i \leqq j} - j)\theta_j] + r_i \equiv \frac{2\pi}{q}(s - r) \bmod 2\pi ,$$

where i ranges from 1 to q (one of these equations being redundant), $0 \leqq \theta_j \leqq 2\pi/q$ and $I_{i \leqq j} = 1$ if $i \leqq j$ and $= 0$ otherwise. These equations have a solution only when $s = r$, namely $\theta_j = (r_{j+1} - r_j)/q$.

It remains to calculate the sign associated to this intersection (and in doing so check that it is transverse). We omit the linear algebra, but the outcome is that the intersection number from (A.6) is $\delta_{rs}(-1)^{\binom{q}{2}} q!^2/q$ (making use of the fact that $\prod_{r=1}^{q-1}(1 - \zeta^{-pr}) = q$). This means that

$$\begin{aligned} \sigma_p \sigma_{p'} &= \sum_{r,s<q} \frac{1}{q!} \zeta^{-pr} (-1)^{\binom{q-1}{2}} \zeta^{-p's} \frac{1}{q!} \delta_{rs} (-1)^{\binom{q}{2}} q!^2/q \\ &= (-1)^{q-1} \delta_{p,-p'} . \end{aligned}$$

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