

# Spectral Analysis and the Haar Functional on the Quantum $SU(2)$ Group

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**Abstract:** The Haar functional on the quantum  $SU(2)$  group is the analogue of invariant integration on the group  $SU(2)$ . If restricted to a subalgebra generated by a self-adjoint element the Haar functional can be expressed as an integral with a continuous measure or with a discrete measure or by a combination of both. These results by Woronowicz and Koornwinder have been proved by using the corepresentation theory of the quantum  $SU(2)$  group and Schur's orthogonality relations for matrix elements of irreducible unitary corepresentations. These results are proved here by using a spectral analysis of the generator of the subalgebra. The spectral measures can be described in terms of the orthogonality measures of orthogonal polynomials by using the theory of Jacobi matrices.

## 1. Introduction

The existence of the Haar measure for locally compact groups is a cornerstone in harmonic analysis. The situation for general quantum groups is not (yet) so nice, but for compact matrix quantum groups Woronowicz [22, Thm. 4.2] has proved that a suitable analogue of the Haar measure exists. This analogue of the Haar measure is a state on a  $C^*$ -algebra. In particular, the analogue of the Haar measure on the deformed  $C^*$ -algebra  $A_q(SU(2))$  of continuous functions on the group  $SU(2)$  is explicitly known. This Haar functional plays an important role in the harmonic analysis on the quantum  $SU(2)$  group. For instance, the corepresentations of the  $C^*$ -algebra are similar to the representations of the Lie group  $SU(2)$ , and the matrix elements of the corepresentations can be expressed in terms of the little  $q$ -Jacobi polynomials, cf. [14, 17, 20], and the orthogonality relations for the little  $q$ -Jacobi polynomials are equivalent to the Schur orthogonality relations on the  $C^*$ -algebra  $A_q(SU(2))$  involving the Haar functional. This was the start of a fruitful connection

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between  $q$ -special functions and the representation theory of quantum groups, see e.g. [12, 15, 18] for more information.

The Haar functional can be restricted to a  $C^*$ -subalgebra of  $A_q(SU(2))$  generated by a self-adjoint element. It turns out that the Haar functional restricted to specific examples of such  $C^*$ -subalgebras can be written as an infinite sum, as an integral with an absolutely continuous measure or as an integral with an absolutely continuous measure on  $[-1, 1]$  and a finite number of discrete mass points off  $[-1, 1]$ . These measures are orthogonality measures for subclasses of the little  $q$ -Jacobi polynomials, the big  $q$ -Jacobi polynomials and the Askey–Wilson polynomials. These formulas for the Haar functional have been proved in several cases by Woronowicz [22, App. A.1] and Koornwinder [16, Thm. 5.3], see also [15, Thm. 8.4], by using the representation theory of the quantum  $SU(2)$  group, i.e. the corepresentations of the  $C^*$ -algebra  $A_q(SU(2))$  equipped with a suitable comultiplication. See also Noumi and Mimachi [19, Thm. 4.1].

The proofs by Woronowicz and Koornwinder use the Schur orthogonality relations for the matrix elements of irreducible unitary corepresentations of  $A_q(SU(2))$ . They determine a combination of such matrix elements of one irreducible unitary corepresentation as an orthogonal polynomial, say  $p_n$ , in a simple self-adjoint element, say  $\rho$ , of the  $C^*$ -algebra  $A_q(SU(2))$ . Here the degree  $n$  of the polynomial  $p_n$  is directly related to the spin  $l$  of the irreducible unitary corepresentation. Hence, they conclude that the Haar functional on the  $C^*$ -subalgebra generated by  $\rho$  is given by an integral with respect to the normalised orthogonality measure for the polynomials  $p_n$ . The result by Noumi and Mimachi is closely related to a limiting case of Koornwinder's result, but the invariant functional lives on a quantum space on which the quantum  $SU(2)$  group acts. Their proof follows by checking that the moments agree. However, the invariant functional takes values in a commutative subalgebra of a non-commutative algebra.

It is the purpose of the present paper to prove these results in an alternative way by only using the  $C^*$ -algebra  $A_q(SU(2))$ . And in particular we study the spectral properties of the generator of the  $C^*$ -subalgebra on which the Haar functional is given as a suitable measure. In order to do so we use the infinite dimensional irreducible representations of the  $C^*$ -algebra. The infinite dimensional irreducible representations of the  $C^*$ -algebra  $A_q(SU(2))$  are parametrised by the unit circle, and the intersection of the kernels of these representations is trivial. So this set of representations of  $A_q(SU(2))$  contains sufficiently many representations. We determine the spectral properties of the operators that correspond to the self-adjoint element which generates the  $C^*$ -subalgebra on which the Haar functional is given by Woronowicz, Koornwinder and Noumi and Mimachi. An important property of the corresponding self-adjoint operators is that they can be given as Jacobi matrices, i.e. as tridiagonal matrices, in a suitable basis, and hence give rise to orthogonal polynomials. These orthogonal polynomials can be determined explicitly and can then be used to derive the explicit form of the Haar functional.

The contents of this paper are as follows. In Sect. 2 we recall Woronowicz's quantum  $SU(2)$  group and the Haar functional on the corresponding  $C^*$ -algebra. The spectral theory of the Jacobi matrices is briefly recalled in Sect. 3. Woronowicz's [22] expression for the Haar functional on the algebra of cocentral elements is then proved in Sect. 4 by use of the continuous  $q$ -Hermite polynomials and its Poisson kernel. In Sect. 5 we prove the statement of Noumi and Mimachi [19] and Koornwinder [16] that the Haar functional on a certain  $C^*$ -subalgebra can be written as a  $q$ -integral. Here we use Al-Salam–Carlitz polynomials and  $q$ -Charlier

polynomials. Finally in Sect. 6 we give a proof of Koornwinder’s [16] result that the Haar functional on certain elements can be written in terms of an Askey–Wilson integral. Here we use the Al-Salam–Chihara polynomials and the corresponding Poisson kernel. It must be noted that the result of Sect. 5 can be obtained by a formal limit transition of the result of Sect. 6, cf. [16, Rem. 6.6], but we think that the proof in Sect. 5 is of independent interest, since it is much simpler. Moreover, the basis of the representation space introduced in Sect. 5 is essential in Sect. 6.

To end this introduction we recall some definitions from the theory of basic hypergeometric series. In this we follow the excellent book [10, Ch. 1] by Gasper and Rahman. We always assume  $0 < q < 1$ . The  $q$ -shifted factorial is defined by

$$(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i), \quad (a_1, \dots, a_r; q)_k = \prod_{i=1}^r (a_i; q)_k$$

for  $k \in \mathbb{Z}_+ \cup \{\infty\}$ . The  $q$ -hypergeometric series is defined by

$$\begin{aligned} {}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} z^k ((-1)^k q^{k(k-1)/2})^{s+1-r}. \end{aligned}$$

Note that for  $a_i = q^{-n}$ ,  $n \in \mathbb{Z}_+$ , the series terminates, and we find a polynomial. We also need the  $q$ -integral;

$$\int_0^b f(x) d_q x = (1 - q)b \sum_{k=0}^{\infty} f(bq^k) q^k, \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

For the very-well-poised  ${}_8\phi_7$ -series we use the abbreviation, cf. [10, Ch. 2],

$${}_8W_7(a; b, c, d, e, f; q, z) = {}_8\phi_7 \left( \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f \end{matrix}; q, z \right). \tag{1.1}$$

## 2. The Quantum $SU(2)$ Group

We recall in this section Woronowicz’s first example of a quantum group, namely the analogue of the Lie group  $SU(2)$ , cf. [21, 22]. In the general theory of compact matrix quantum groups Woronowicz has proved the existence of the analogue of a left and right invariant measure [22].

We first introduce the  $C^*$ -algebra  $A_q(SU(2))$ . The  $C^*$ -algebra  $A_q(SU(2))$  is the unital  $C^*$ -algebra generated by two elements  $\alpha$  and  $\gamma$  subject to the relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, & \alpha\gamma^* &= q\gamma^*\alpha, & \gamma\gamma^* &= \gamma^*\gamma, \\ \alpha^*\alpha + \gamma^*\gamma &= 1 = \alpha\alpha^* + q^2\gamma\gamma^*, \end{aligned} \tag{2.1}$$

where  $0 < q < 1$ . Here  $q$  is a deformation parameter, and for  $q = 1$  we can identify  $A_q(SU(2))$  with the  $C^*$ -algebra of continuous function on  $SU(2)$ , where  $\alpha$  and  $\gamma$  are coordinate functions. The group multiplication is reflected in the comultiplication, i.e. a  $C^*$ -homomorphism  $\Delta: A_q(SU(2)) \rightarrow A_q(SU(2)) \otimes A_q(SU(2))$ . (Since

$A_q(SU(2))$  is a type I  $C^*$ -algebra, we can take any  $C^*$ -tensor product on the right-hand side.) The Haar functional is then uniquely determined by the conditions  $h(1) = 1$  and  $(\text{id} \otimes h)\Delta(a) = h(a)1 = (h \otimes \text{id})\Delta(a)$  for all  $a \in A_q(SU(2))$ .

The irreducible representations of the  $C^*$ -algebra  $A_q(SU(2))$  have been completely classified, cf. [20]. Apart from the one-dimensional representations  $\alpha \mapsto e^{i\theta}$ ,  $\gamma \mapsto 0$ , we only have the infinite dimensional representations  $\pi_\phi$ ,  $\phi \in [0, 2\pi)$ , of  $A_q(SU(2))$  acting in  $l^2(\mathbb{Z}_+)$ . Denote by  $\{e_n | n \in \mathbb{Z}_+\}$  the standard orthonormal basis of  $l^2(\mathbb{Z}_+)$ , then  $\pi_\phi$  is given by

$$\pi_\phi(\alpha)e_n = \sqrt{1 - q^{2n}}e_{n-1}, \quad \pi_\phi(\gamma)e_n = e^{i\phi}q^n e_n. \tag{2.2}$$

Here we follow the convention that  $e_{-p} = 0$  for  $p \in \mathbb{N}$ . This is a complete list of the irreducible  $*$ -representations of  $A_q(SU(2))$ . Moreover,  $\cap_\phi \ker \pi_\phi$  is trivial, so that the spectral properties of  $a \in A_q(SU(2))$  are determined by the spectral properties of  $\pi_\phi(a)$ .

Woronowicz [22, App. A.1] has given an explicit formula for the Haar functional (not using corepresentations) in terms of an infinite dimensional faithful representation of  $A_q(SU(2))$ . This can be rewritten in terms of the irreducible representations of  $A_q(SU(2))$  as

$$h(a) = (1 - q^2) \sum_{p=0}^{\infty} \frac{q^{2p}}{2\pi} \int_0^{2\pi} \langle \pi_\phi(a)e_p, e_p \rangle d\phi, \quad a \in A_q(SU(2)).$$

Observe that  $h(p(\gamma^* \gamma)) = \int_0^1 p(x) d_q x$ , for any continuous  $p$  on  $\{q^{2k} | k \in \mathbb{Z}_+\}$ . This can be considered as a limit case of the results of Sect. 5, 6, cf. [16, Rem. 6.6].

Introduce the self-adjoint positive diagonal operator  $D: l^2(\mathbb{Z}_+) \rightarrow l^2(\mathbb{Z}_+)$ ,  $e_p \mapsto q^{2p}e_p$ , then we can rewrite this in a basis independent way, cf. [20, Thm. 5.5];

$$h(a) = \frac{(1 - q^2)}{2\pi} \int_0^{2\pi} \text{tr}(D\pi_\phi(a)) d\phi, \quad a \in A_q(SU(2)). \tag{2.3}$$

The trace operation in (2.3) is well-defined due to the appearance of  $D$ . Since  $D^{1/2}$  is a Hilbert–Schmidt operator, so is  $\pi_\phi(a)D^{1/2}$ . The trace of the product of two Hilbert–Schmidt operators is well-defined. Moreover,  $\text{tr}(D\pi_\phi(a)) = \text{tr}(\pi_\phi(a)D)$  and it is independent of the choice of the basis. The trace can be estimated by the product of the Hilbert–Schmidt norms of  $D^{1/2}$  and  $\pi_\phi(a)D^{1/2}$  and then we get  $|\text{tr}(D\pi_\phi(a))| \leq \|a\|_{A_q(SU(2))} / (1 - q^2)$ , so that the function in (2.3) is integrable. See Dunford and Schwartz [9, Ch. XI, Sect. 6] for more details.

### 3. Spectral Theory of Jacobi Matrices

In this section we recall some of the results on the spectral theory of Jacobi matrices and the relation with orthogonal polynomials. For more information we refer to Berezanskiĭ [5, Ch. VII, Sect. 1] and Dombrowski [8].

The operator  $J$  acting on the standard orthonormal basis  $\{e_n | n \in \mathbb{Z}_+\}$  of  $l^2(\mathbb{Z}_+)$  by

$$Je_n = a_{n+1}e_{n+1} + b_n e_n + a_n e_{n-1}, \quad a_n > 0, \quad b_n \in \mathbb{R}, \tag{3.1}$$

is called a Jacobi matrix. This operator is symmetric, and its deficiency indices are  $(0, 0)$  or  $(1, 1)$ , cf. [5, Ch. VII, Sect. 1, Thm. 1.1]. In particular, if the coefficients

$a_n$  and  $b_n$  are bounded,  $J$  is a bounded operator on  $l^2(\mathbb{Z}_+)$  and thus self-adjoint. Then  $e_0$  is a cyclic vector for  $J$ , i.e. the span of finite linear combinations of the form  $J^p e_0$ ,  $p \in \mathbb{Z}_+$ , is dense in  $l^2(\mathbb{Z}_+)$ . This is the case for all Jacobi matrices considered in this paper.

Using the same coefficients  $a_n, b_n$  we can generate polynomials  $p_n(x)$  of degree  $n$  in  $x$  by the recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad p_{-1}(x) = 0, \quad p_0(x) = 1. \quad (3.2)$$

By Favard's theorem, which states that polynomials satisfying a three-term recurrence relation are orthogonal polynomials, cf. e.g. [7], there exists a positive measure  $m$  on the real line such that the polynomials  $p_n(x)$  are orthonormal;

$$\int_{\mathbb{R}} p_n(x)p_m(x) dm(x) = \delta_{n,m}.$$

If the orthogonality measure  $m$  is uniquely determined, we speak of a determined moment problem. In this case the set of polynomials is dense in the weighted  $L^2(m)$ -space, and the polynomials  $\{p_n | n \in \mathbb{Z}_+\}$  form an orthonormal basis for  $L^2(m)$ . The boundedness of  $a_n, b_n$  implies that the moment problem is determined. More generally, the moment problem is determined if and only if the Jacobi matrix is a self-adjoint operator.

So we now assume that the coefficients  $a_n, b_n$  are bounded, so  $J$  is self-adjoint with cyclic vector  $e_0$  and the corresponding moment problem is determined. We can represent the operator  $J$  as a multiplication operator  $M$  on  $L^2(m)$ , where

$$M: L^2(m) \rightarrow L^2(m), \quad (Mf)(x) = xf(x).$$

For this we define

$$A: l^2(\mathbb{Z}_+) \rightarrow L^2(m), \quad (Ae_n)(x) = p_n(x),$$

then  $A$  is a unitary operator, since it maps an orthonormal basis onto an orthonormal basis. Note that we use here that the polynomials are dense in  $L^2(m)$ . From (3.1) and (3.2) it follows that

$$AJv = MAv, \quad \forall v \in l^2(\mathbb{Z}_+),$$

so that  $A$  intertwines the Jacobi matrix  $J$  on  $l^2(\mathbb{Z}_+)$  with the multiplication operator  $M$  on  $L^2(m)$ . Observe that  $\|J\| = \|M\|$ , so that  $M$  is bounded and hence the support of the orthogonality measure  $m$  is compact.

What we have described in the previous paragraph is essentially the spectral theorem for self-adjoint operators. The theorem states that there exists a projection valued measure  $E$  on  $\mathbb{R}$ , the spectral decomposition, such that

$$J = \int_{\mathbb{R}} x dE(x).$$

The relation between the spectral decomposition  $E$  and the orthogonality measure  $m$  is given by  $m(B) = \|E(B)e_0\|$ , where  $B \subset \mathbb{R}$  is a Borel set and  $e_0$  is the cyclic vector for  $J$ . More generally we have

$$\langle E(B)e_n, e_m \rangle = \int_B p_n(x)p_m(x) dm(x).$$

The point spectrum of  $J$  corresponds to discrete mass points of the measure  $m$ . Let us finally note that Favard’s theorem can be proved from the spectral theorem for the Jacobi matrix  $J$ .

#### 4. The Haar Functional on Cocentral Elements

Using the characters of the irreducible unitary corepresentations of the quantum  $SU(2)$  group Woronowicz [22, App. A.1] has proved an expression for the Haar functional on the  $C^*$ -subalgebra generated by  $\alpha + \alpha^*$ . The set of characters is known as the set of cocentral elements and it is generated by the element  $\alpha + \alpha^*$ . The purpose of this section is to give a proof of this theorem based on the spectral analysis of the generator  $\alpha + \alpha^*$ . The method used in this section is also used in Sect. 6 in somewhat greater computational complexity.

**Theorem 4.1.** *The Haar functional on the  $C^*$ -subalgebra generated by the self-adjoint element  $\alpha + \alpha^*$  is given by the integral*

$$h(p((\alpha + \alpha^*)/2)) = \frac{2}{\pi} \int_{-1}^1 p(x) \sqrt{1 - x^2} dx$$

for any continuous function  $p \in C([-1, 1])$ .

In order to give an alternative proof of this theorem, we start with considering

$$2\pi_\phi((\alpha + \alpha^*)/2)e_n = \sqrt{1 - q^{2n}}e_{n-1} + \sqrt{1 - q^{2n+2}}e_{n+1}. \tag{4.1}$$

So the operator  $\pi_\phi((\alpha + \alpha^*)/2)$  is represented by a Jacobi matrix with respect to the standard basis of  $l^2(\mathbb{Z}_+)$ .

Recall Rogers’s continuous  $q$ -Hermite polynomials  $H_n(x|q)$  defined by

$$2xH_n(x|q) = H_{n+1}(x|q) + (1 - q^n)H_{n-1}(x|q), \quad H_{-1}(x|q) = 0, \quad H_0(x|q) = 1. \tag{4.2}$$

The continuous  $q$ -Hermite polynomials satisfy the orthogonality relations

$$\int_0^\pi H_n(\cos \theta|q)H_m(\cos \theta|q)w(\cos \theta|q) d\theta = \delta_{n,m} \frac{2\pi(q; q)_n}{(q; q)_\infty}, \tag{4.3}$$

with  $w(\cos \theta|q) = (e^{2i\theta}, e^{-2i\theta}; q)_\infty$ , cf. [1]. The Poisson kernel for the continuous  $q$ -Hermite polynomials is given by, cf. [1, 6],

$$\begin{aligned} P_t(\cos \theta, \cos \psi|q) &= \sum_{n=0}^\infty \frac{H_n(\cos \theta|q)H_n(\cos \psi|q)t^n}{(q; q)_n} \\ &= \frac{(t^2; q)_\infty}{(te^{i\theta+i\psi}, te^{i\theta-i\psi}, te^{-i\theta+i\psi}, te^{-i\theta-i\psi}; q)_\infty} \end{aligned} \tag{4.4}$$

for  $|t| < 1$ .

Compare (4.1) with the three-term recurrence relation (4.2) to see that (4.1) is solved by the orthonormal continuous  $q$ -Hermite polynomials  $H_n(x|q^2)/\sqrt{(q^2; q^2)_n}$ . Using the spectral theory of Jacobi matrices as in Sect. 3 we have obtained the

spectral decomposition of the self-adjoint operator  $\pi_\phi((\alpha + \alpha^*)/2)$ , which has spectrum  $[-1, 1]$ . This link between  $\alpha + \alpha^*$  and the continuous  $q$ -Hermite polynomials is already observed in [12, Sect. 11]. So the spectrum of  $(\alpha + \alpha^*)/2 \in A_q(SU(2))$  is  $[-1, 1]$  and by the functional calculus  $p((\alpha + \alpha^*)/2) \in A_q(SU(2))$  for any continuous function  $p$  on  $[-1, 1]$ .

We use the results of Sect. 3 here with  $p_n(x) = H_n(x|q^2)/\sqrt{(q^2; q^2)_n}$  as the orthonormal polynomials and  $dm(x|q^2) = (2\pi)^{-1}(q^2; q^2)_\infty w(x|q^2)(1 - x^2)^{-1/2} dx$  as the (normalised) orthogonality measure, which is absolutely continuous in this case. With the unitary mapping  $A$ , intertwining  $\pi_\phi((\alpha + \alpha^*)/2)$  with the multiplication operator  $M$  on  $L^2([-1, 1], dm(x|q^2))$  as in Sect. 3, we get in this particular case

$$\begin{aligned} \text{tr}(D\pi_\phi(p((\alpha + \alpha^*)/2))) &= \sum_{n=0}^\infty q^{2n} \langle \pi_\phi(p((\alpha + \alpha^*)/2)) e_n, e_n \rangle \\ &= \sum_{n=0}^\infty q^{2n} \langle A\pi_\phi(p((\alpha + \alpha^*)/2)) e_n, Ae_n \rangle \\ &= \sum_{n=0}^\infty q^{2n} \int_{-1}^1 p(x) |(Ae_n)(x)|^2 dm(x|q^2) \\ &= \sum_{n=0}^\infty q^{2n} \int_{-1}^1 p(x) \frac{(H_n(x|q^2))^2}{(q^2; q^2)_n} dm(x|q^2) \\ &= \int_{-1}^1 p(x) P_{q^2}(x, x|q^2) dm(x|q^2), \end{aligned} \tag{4.5}$$

by (4.4). Interchanging summation and integration is justified by (4.3) and (4.4) and estimating  $p(x)$  by its supremum norm on  $[-1, 1]$ .

From (4.4) we see that

$$w(x|q^2)P_{q^2}(x, x|q^2) = \frac{4(1 - x^2)}{(1 - q^2)(q^2; q^2)_\infty},$$

so that

$$\text{tr}(D\pi_\phi(p((\alpha + \alpha^*)/2))) = \frac{2}{\pi(1 - q^2)} \int_{-1}^1 p(x) \sqrt{1 - x^2} dx.$$

Since this is independent of the parameter  $\phi$  of the infinite dimensional representation, we obtain Woronowicz’s Theorem 4.1 from (2.3).

### 5. The Haar Functional on Special Spherical Elements

We start with introducing the self-adjoint element

$$\rho_{\tau, \infty} = iq^\tau(\alpha^*\gamma - \gamma^*\alpha) - (1 - q^{2\tau})\gamma^*\gamma \in A_q(SU(2)).$$

This element is a limiting case of the general spherical element considered in Sect. 6, and from that section the notation for this element is explained. The Haar functional on the  $C^*$ -algebra generated by  $\rho_{\tau, \infty}$  can be written as a  $q$ -integral as proved by Koornwinder [16, Thm. 5.3, Rem. 6.6] and Noumi and Mimachi [19, Thm. 4.1] using the corepresentations of the quantum  $SU(2)$  group.

**Theorem 5.1.** *The Haar functional on the  $C^*$ -subalgebra generated by the self-adjoint element  $\rho_{\tau,\infty}$  is given by the  $q$ -integral*

$$h(p(\rho_{\tau,\infty})) = \frac{1}{1+q^{2\tau}} \int_{-1}^{q^{2\tau}} p(x) d_{q^2} x$$

for any continuous function  $p$  on  $\{-q^{2k} | k \in \mathbb{Z}_+\} \cup \{q^{2\tau+2k} | k \in \mathbb{Z}_+\} \cup \{0\}$ .

To prove this theorem from a spectral analysis of  $\pi_\phi(\rho_{\tau,\infty})$  we have to recall the following result. It is shown in [11, Prop. 4.1, Cor. 4.2] that  $l^2(\mathbb{Z}_+)$  has an orthogonal basis of eigenvectors of  $\pi_0(\rho_{\tau,\infty})$ . The proof of that proposition only needs a minor adaptation to handle the general case  $\pi_\phi(\rho_{\tau,\infty})$ . In general we have the following proposition.

**Proposition 5.2.**  *$l^2(\mathbb{Z}_+)$  has an orthogonal basis of eigenvectors  $v_\lambda^\phi$ , where  $\lambda = -q^{2k}$ ,  $k \in \mathbb{Z}_+$ , and  $\lambda = q^{2\tau+2k}$ ,  $k \in \mathbb{Z}_+$ , for the eigenvalue  $\lambda$  of the self-adjoint operator  $\pi_\phi(\rho_{\tau,\infty})$ . The squared norm is given by*

$$\begin{aligned} \langle v_\lambda^\phi, v_\lambda^\phi \rangle &= q^{-2k} (q^2; q^2)_k (-q^{2-2\tau}; q^2)_k (-q^{2\tau}; q^2)_\infty, \quad \lambda = -q^{2k}, \\ \langle v_\lambda^\phi, v_\lambda^\phi \rangle &= q^{-2k} (q^2; q^2)_k (-q^{2+2\tau}; q^2)_k (-q^{-2\tau}; q^2)_\infty, \quad \lambda = q^{2\tau+2k}. \end{aligned}$$

Moreover,  $v_\lambda^\phi = \sum_{n=0}^\infty i^n e^{in\phi} p_n(\lambda) e_n$  with the polynomial  $p_n(\lambda)$  defined by

$$\begin{aligned} p_n(\lambda) &= \frac{q^{-n\tau} q^{\frac{1}{2}n(n-1)}}{\sqrt{(q^2; q^2)_n}} {}_2\phi_1(q^{-2n}, q^{2\tau}/\lambda; 0; q^2; -q^2 \lambda) \\ &= \frac{(-1)^n q^{n\tau} q^{\frac{1}{2}n(n-1)}}{\sqrt{(q^2; q^2)_n}} {}_2\phi_1(q^{-2n}, -1/\lambda; 0; q^2; q^{2-2\tau} \lambda). \end{aligned} \tag{5.1}$$

*Remark. 5.3.* The polynomials in  $\lambda$  in (5.1) are Al-Salam–Carlitz polynomials  $U_n^{(a)}$ . The orthogonality relations obtained from  $\langle v_\lambda^\phi, v_\mu^\phi \rangle = 0$  for  $\lambda \neq \mu$  are the orthogonality relations for the  $q$ -Charlier polynomials, cf. [11, Cor. 4.2].

*Remark. 5.4.* The basis described in Proposition 5.2 induces an orthogonal decomposition of the representation space  $l^2(\mathbb{Z}_+) = V_1^\phi \oplus V_2^\phi$ , where  $V_1^\phi$  is the subspace with basis  $v_{-q^{2k}}^\phi$ ,  $k \in \mathbb{Z}_+$ , and  $V_2^\phi$  is the subspace with basis  $v_{q^{2\tau+2k}}^\phi$ ,  $k \in \mathbb{Z}_+$ .

In this section we use the basis described in Proposition 5.2 to calculate the trace. The spectrum of  $\pi_\phi(\rho_{\tau,\infty})$  is independent of  $\phi$ , so the spectrum of  $\rho_{\tau,\infty} \in A_q(SU(2))$  equals  $\{-q^{2k} | k \in \mathbb{Z}_+\} \cup \{q^{2\tau+2k} | k \in \mathbb{Z}_+\} \cup \{0\}$ . Hence, for any function  $p$  continuous on the spectrum  $\{-q^{2k} | k \in \mathbb{Z}_+\} \cup \{q^{2\tau+2k} | k \in \mathbb{Z}_+\} \cup \{0\}$  we have  $p(\rho_{\tau,\infty}) \in A_q(SU(2))$ . This time we do not need the mapping  $A$  of Sect. 3, since we have a basis of eigenvectors. So we can calculate the trace with respect to the orthogonal basis of eigenvectors described in Proposition 5.2;

$$\begin{aligned} \text{tr}(D\pi_\phi(p(\rho_{\tau,\infty}))) &= \sum_{k=0}^\infty p(-q^{2k}) \frac{\langle Dv_{-q^{2k}}^\phi, v_{-q^{2k}}^\phi \rangle}{\langle v_{-q^{2k}}^\phi, v_{-q^{2k}}^\phi \rangle} \\ &\quad + \sum_{k=0}^\infty p(q^{2\tau+2k}) \frac{\langle Dv_{q^{2\tau+2k}}^\phi, v_{q^{2\tau+2k}}^\phi \rangle}{\langle v_{q^{2\tau+2k}}^\phi, v_{q^{2\tau+2k}}^\phi \rangle}. \end{aligned} \tag{5.2}$$



So it remains to calculate the matrix coefficients on the diagonal of the operator  $D$  with respect to this basis. We give all the matrix coefficients in the following lemma.

**Lemma 5.5.** *The matrix coefficients of the operator  $D$  with respect to the orthogonal basis  $v_{\lambda}^{\phi}$  are given by*

$$\begin{aligned} \langle Dv_{-q^{2k}}^{\phi}, v_{-q^{2l}}^{\phi} \rangle &= (-q^{2\tau+2}; q^2)_{\infty} (q^2; q^2)_k (-q^{2-2\tau}; q^2)_l, \quad k \geq l, \\ \langle Dv_{q^{2\tau+2k}}^{\phi}, v_{q^{2\tau+2l}}^{\phi} \rangle &= (-q^{2-2\tau}; q^2)_{\infty} (q^2; q^2)_k (-q^{2+2\tau}; q^2)_l, \quad k \geq l, \\ \langle Dv_{-q^{2k}}^{\phi}, v_{q^{2\tau+2l}}^{\phi} \rangle &= (q^2; q^2)_{\infty} (-q^{2-2\tau}; q^2)_k (-q^{2+2\tau}; q^2)_l, \end{aligned}$$

and all other cases follow from  $D$  being self-adjoint.

*Proof.* The proof is based on calculations involving the  $q$ -Charlier polynomials  $c_n(x; a; q) = {}_2\phi_1(q^{-n}, x; 0; q, -q^{n+1}/a)$ . Define a moment functional  $\mathcal{L}$  by

$$\mathcal{L}(p) = \sum_{n=0}^{\infty} \frac{q^{2n\tau} q^{n(n-1)}}{(q^2; q^2)_n} p(q^{-2n}), \tag{5.3}$$

for any polynomial  $p$ . Note that all moments, i.e.  $\mathcal{L}(x^n)$ ,  $n \in \mathbb{Z}_+$ , exist. The orthogonality relations  $\langle v_{-q^{2k}}^{\phi}, v_{-q^{2l}}^{\phi} \rangle = 0$  are rewritten as the orthogonality relations for the  $q$ -Charlier polynomials;

$$\mathcal{L}(c_k(x; q^{2\tau}; q^2) c_l(x; q^{2\tau}; q^2)) = \delta_{k,l} q^{-2k} (q^2, -q^{2-2\tau}; q^2)_k (-q^{2\tau}; q^2)_{\infty},$$

cf. [10, Ex. 7.13; 11, Cor. 4.2].

Using Proposition 5.2 and the definition of the self-adjoint operator  $D$  we see that

$$\langle Dv_{-q^{2k}}^{\phi}, v_{-q^{2l}}^{\phi} \rangle = \mathcal{L}\left(\frac{1}{x} c_k(x; q^{2\tau}; q^2) c_l(x; q^{2\tau}; q^2)\right).$$

Note that this expression is well-defined, since 0 is not an element of the support of the measure representing  $\mathcal{L}$ . Now use the orthogonality of the  $q$ -Charlier polynomials to see that for  $k \geq l$  this equals  $c_l(0; q^{2\tau}; q^2) \mathcal{L}(c_k(x; q^{2\tau}; q^2)/x)$ . Use the  ${}_2\phi_1$ -series representation for the  $q$ -Charlier polynomials and (5.3) to evaluate the moment functional on this particular function. We get

$$\mathcal{L}\left(\frac{1}{x} c_k(x; q^{2\tau}; q^2)\right) = \sum_{i=0}^k \frac{(q^{-2k}; q^2)_i}{(q^2; q^2)_i} (-q^{2k+2+2\tau})^i \sum_{n=1}^{\infty} \frac{(q^{-2n}; q^2)_l}{(q^2; q^2)_n} q^{2n(\tau+1)} q^{n(n-1)},$$

where interchanging the summations is allowed as all sums are absolutely convergent. Replace the summation parameter  $n = m + l$  and use that  $(q^{-2(m+l)}; q^2)_l / (q^2; q^2)_{m+l} = (-1)^l q^{-l(l+1+2m)} / (q^2; q^2)_m$ . Now the inner sum can be summed using  ${}_0\phi_0(-; -; q^2, z) = (z; q^2)_{\infty}$ , cf. [10, (1.3.16)]. The remaining finite sum can be summed using the terminating  $q$ -binomial formula  ${}_1\phi_0(q^{-2n}; -; q^2, q^{2n}x) = (x; q^2)_n$ , cf. [10, (1.3.14)], which can also be used to evaluate the  $q$ -Charlier polynomial at  $x = 0$ . This proves the first part of the lemma. The second part is proved in the same way, but with  $\tau$  replaced by  $-\tau$ .

For the last part of the lemma we use the same strategy, but now we have to use the moment functional  $\mathcal{M}$  defined by

$$\mathcal{M}(p) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(q^2; q^2)_n} p(q^{-2n}),$$

for which we have  $\mathcal{M}(c_k(x; q^{2\tau}; q^2)c_l(x; q^{-2\tau}; q^2)) = 0$ , cf. [11, Cor. 4.2].  $\square$

Now use Lemma 5.5 and Proposition 5.2 in (5.2) to find

$$\text{tr}(D\pi_\phi(p(\rho_{\tau, \infty}))) = \frac{1}{1 + q^{2\tau}} \left( \sum_{k=0}^{\infty} p(-q^{2k})q^{2k} + q^{2\tau} \sum_{k=0}^{\infty} p(q^{2\tau+2k})q^{2k} \right).$$

This expression is independent of  $\phi$ , so that we obtain from (2.3),

$$h(p(\rho_{\tau, \infty})) = \frac{1 - q^2}{1 + q^{2\tau}} \left( \sum_{k=0}^{\infty} p(-q^{2k})q^{2k} + q^{2\tau} \sum_{k=0}^{\infty} p(q^{2\tau+2k})q^{2k} \right),$$

which is precisely the statement of Theorem 5.1 using the definition of the  $q$ -integral given in Sect. 1.

### 6. The Haar Functional on Spherical Elements

In this section we give a proof of Koornwinder’s theorem expressing the Haar functional on a  $C^*$ -subalgebra of  $A_q(SU(2))$  as an Askey–Wilson integral from the spectral analysis of the generator of the  $C^*$ -subalgebra.

We first introduce the self-adjoint element

$$\begin{aligned} \rho_{\tau, \sigma} = & \frac{1}{2}(\alpha^2 + (\alpha^*)^2 + q\gamma^2 + q(\gamma^*)^2 + iq(q^{-\sigma} - q^\sigma)(\alpha^*\gamma - \gamma^*\alpha) \\ & - iq(q^{-\tau} - q^\tau)(\gamma\alpha - \alpha^*\gamma^*) - q(q^{-\sigma} - q^\sigma)(q^{-\tau} - q^\tau)\gamma^*\gamma) \in A_q(SU(2)). \end{aligned}$$

Note that the element  $\rho_{\tau, \infty}$ , as introduced in Sect. 5, is a limiting case of this general spherical element, namely

$$\rho_{\tau, \infty} = \lim_{\sigma \rightarrow \infty} 2q^{\sigma+\tau-1} \rho_{\tau, \sigma}.$$

The Haar functional on the  $C^*$ -algebra generated by  $\rho_{\tau, \sigma}$  can be written as an integral with respect to the orthogonality measure for Askey–Wilson polynomials, as proved by Koornwinder [16, Thm. 5.3].

**Theorem 6.1.** *The Haar functional on the  $C^*$ -subalgebra generated by the self-adjoint element  $\rho_{\tau, \sigma}$  is given by*

$$h(p(\rho_{\tau, \sigma})) = \int_{\mathbb{R}} p(x) dm(x; a, b, c, d|q^2) \tag{6.1}$$

for any continuous function  $p$  on the spectrum of  $\rho_{\tau, \sigma}$ , which coincides with the support of the orthogonality measure in (6.1). Here  $a = -q^{\sigma+\tau+1}$ ,  $b = -q^{-\sigma-\tau+1}$ ,  $c = q^{\sigma-\tau+1}$ ,  $d = q^{-\sigma+\tau+1}$  and  $dm(x; a, b, c, d|q^2)$  denotes the normalised Askey–Wilson measure.

We recall that the normalised Askey–Wilson measure is given by, cf. Askey and Wilson [4, Thm. 2.1, 2.5],

$$\int_{\mathbb{R}} p(x) dm(x; a, b, c, d|q) = \frac{1}{h_0 2\pi} \int_0^\pi p(\cos \theta) w(\cos \theta) d\theta + \frac{1}{h_0} \sum_k p(x_k) w_k. \quad (6.2)$$

Here we use the notation  $w(\cos \theta) = w(\cos \theta; a, b, c, d|q)$ ,  $h_0 = h_0(a, b, c, d|q)$  and

$$h_0(a, b, c, d|q) = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},$$

$$w(\cos \theta; a, b, c, d|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}, \quad (6.3)$$

and we suppose  $a, b, c$  and  $d$  real and such that all pairwise products are less than 1. The sum in (6.2) is over the points  $x_k$  of the form  $(eq^k + e^{-1}q^{-k})/2$  with  $e$  any of the parameters  $a, b, c$  or  $d$  whose absolute value is larger than one and such that  $|eq^k| > 1, k \in \mathbb{Z}_+$ . The corresponding mass  $w_k$  is the residue of  $z \mapsto w(\frac{1}{2}(z + z^{-1}))$  at  $z = eq^k$  minus the residue at  $z = e^{-1}q^{-k}$ . The value of  $w_k$  in case  $e = a$  is given in [4, (2.10)], but  $(1 - aq^{2k})/(1 - a)$  has to be replaced by  $(1 - a^2q^{2k})/(1 - a^2)$ . Explicitly,

$$w_k(a; b, c, d|q) = \frac{(a^{-2}; q)_\infty}{(q, ab, b/a, ac, c/a, ad, d/a; q)_\infty} \times \frac{(1 - a^2q^{2k})}{(1 - a^2)} \frac{(a^2, ab, ac, ad; q)_k}{(q, aq/b, aq/c, aq/d; q)_k} \left(\frac{q}{abcd}\right)^k, \quad (6.4)$$

see [10, (6.6.12)].

We prove Theorem 6.1 from a spectral analysis of the self-adjoint operators  $\pi_\phi(\rho_{\tau,\sigma})$ . We realise  $\pi_\phi(\rho_{\tau,\sigma})$  as a Jacobi matrix in the basis of  $l^2(\mathbb{Z}_+)$  introduced in Sect. 5.

**Proposition 6.2.** *Let  $v_\lambda^\phi$  be the orthogonal basis of  $l^2(\mathbb{Z}_+)$  as in Proposition 5.2, then*

$$2\pi_\phi(\rho_{\tau,\sigma})v_\lambda^\phi = qe^{-2i\phi}v_{\lambda q^2}^\phi + q^{-1}e^{2i\phi}(1 - q^{-2\tau}\lambda)(1 + \lambda)v_{\lambda/q^2}^\phi + \lambda q^{1-\tau}(q^{-\sigma} - q^\sigma)v_\lambda^\phi, \quad (6.5)$$

where  $\lambda = -q^{2k}, k \in \mathbb{Z}_+$ , and  $\lambda = q^{2\tau+2k}, k \in \mathbb{Z}_+$ .

*Proof.* We use a factorisation of  $\rho_{\tau,\sigma}$  in elements, which are linear combinations in the generators of the  $C^*$ -algebra  $A_q(SU(2))$ . Explicitly,

$$2q^{\tau+\sigma}\rho_{\tau,\sigma} - q^{2\sigma-1} - q^{2\tau+1} = (\beta_{\tau+1,\infty} - q^{\sigma-1}\alpha_{\tau+1,\infty})(\gamma_{\tau,\infty} + q^\sigma\delta_{\tau,\infty}), \quad (6.6)$$

where

$$\begin{aligned} \alpha_{\tau,\infty} &= q^{1/2}\alpha + iq^{\tau+1/2}\gamma, & \beta_{\tau,\infty} &= iq^{1/2}\gamma^* + q^{\tau-1/2}\alpha^*, \\ \gamma_{\tau,\infty} &= -q^{\tau+1/2}\alpha + iq^{1/2}\gamma, & \delta_{\tau,\infty} &= -iq^{\tau+1/2}\gamma^* + q^{-1/2}\alpha^*, \end{aligned} \quad (6.7)$$

cf. [13, Prop. 3.3, (2.14), (2.2)]. Of course, (6.6) can also be checked directly from the commutation relations (2.1) in  $A_q(SU(2))$ . As proved in [13, Prop. 3.8], the operators corresponding to the elements in (6.7) under the representation  $\pi_0$  act nicely in the basis  $v_\lambda^0$ . The proof immediately generalises to the basis  $v_\lambda^\phi$ . Using the

notation  $v_\lambda^\phi(q^\tau) = v_\lambda^\phi$  to stress the dependence on  $q^\tau$ , we get

$$\begin{aligned} \pi_\phi(\alpha_{\tau,\infty})v_\lambda^\phi(q^\tau) &= e^{i\phi}iq^{\frac{1}{2}-\tau}(1+\lambda)v_{\lambda/q^2}^\phi(q^{\tau-1}), \\ \pi_\phi(\beta_{\tau,\infty})v_\lambda^\phi(q^\tau) &= e^{-i\phi}iq^{\frac{1}{2}}v_\lambda^\phi(q^{\tau-1}), \\ \pi_\phi(\gamma_{\tau,\infty})v_\lambda^\phi(q^\tau) &= e^{i\phi}iq^{\frac{1}{2}}(q^{2\tau}-\lambda)v_\lambda^\phi(q^{\tau+1}), \\ \pi_\phi(\delta_{\tau,\infty})v_\lambda^\phi(q^\tau) &= -e^{-i\phi}iq^{\frac{1}{2}+\tau}v_{\lambda q^2}^\phi(q^{\tau+1}). \end{aligned}$$

From this result and (6.6) the proposition follows.  $\square$

Proposition 6.2 implies that  $\pi_\phi(\rho_{\tau,\sigma})$  respects the orthogonal decomposition  $l^2(\mathbb{Z}_+) = V_1^\phi \oplus V_2^\phi$ , cf. Remark 5.4. We denote by  $w_m^\phi$ ,  $m \in \mathbb{Z}_+$ , the orthonormal basis of  $V_1^\phi$  obtained by normalising  $v_{-q^{2m}}^\phi$ ,  $m \in \mathbb{Z}_+$ , and by  $u_m^\phi$ ,  $m \in \mathbb{Z}_+$ , the orthonormal basis of  $V_2^\phi$  obtained by normalising  $v_{q^{2\tau+2m}}^\phi$ ,  $m \in \mathbb{Z}_+$ . Then we get, using Proposition 5.2,

$$\begin{aligned} 2\pi_\phi(\rho_{\tau,\sigma})w_m^\phi &= e^{-2i\phi}a_mw_{m+1}^\phi + b_mw_m^\phi + e^{2i\phi}a_{m-1}w_{m-1}^\phi, \\ a_m &= \sqrt{(1-q^{2m+2})(1+q^{2m+2-2\tau})}, \quad b_m = q^{2m+1-\tau}(q^\sigma - q^{-\sigma}), \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} 2\pi_\phi(\rho_{\tau,\sigma})u_m^\phi &= e^{-2i\phi}a_mu_{m+1}^\phi + b_mu_m^\phi + e^{2i\phi}a_{m-1}u_{m-1}^\phi, \\ a_m &= \sqrt{(1-q^{2m+2})(1+q^{2m+2+2\tau})}, \quad b_m = q^{2m+1+\tau}(q^{-\sigma} - q^\sigma). \end{aligned} \tag{6.9}$$

In order to solve the corresponding three-term recurrence relations for the orthonormal polynomials, we recall the Al-Salam–Chihara polynomials  $p_n(x) = p_n(x; a, b|q)$ . These polynomials are Askey–Wilson polynomials [4] with two parameters set to zero, so the orthogonality measure for these polynomials follows from (6.2). The Al-Salam–Chihara polynomials are defined by

$$p_n(\cos \theta; a, b|q) = a^{-n}(ab; q)_n {}_3\phi_2 \left( \begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}; q, q \right). \tag{6.10}$$

The Al-Salam–Chihara polynomials are symmetric in  $a$  and  $b$  and they satisfy the three-term recurrence relation

$$2xp_n(x) = p_{n+1}(x) + (a+b)q^n p_n(x) + (1-abq^{n-1})(1-q^n)p_{n-1}(x).$$

The Al-Salam–Chihara polynomials are orthogonal with respect to a positive measure for  $ab < 1$ . Then the orthogonality measure is given by  $dm(x; a, b, 0, 0|q)$ , cf. (6.2). We use the following orthonormal Al-Salam–Chihara polynomials;

$$h_n(x; s, t|q) = \frac{1}{\sqrt{(q, -qs^{-2}; q)_n}} p_n \left( x; q^{\frac{1}{2}} \frac{t}{s}, -q^{\frac{1}{2}} \frac{1}{st} \middle| q \right).$$

Now the corresponding three-term recurrence relations (6.8), respectively (6.9), are solved by  $e^{2im\phi}h_m(x; q^\tau, q^\sigma|q^2)$ , respectively  $e^{2im\phi}h_m(x; q^{-\tau}, q^{-\sigma}|q^2)$ . So we see that

$\pi_\phi(\rho_{\tau,\sigma})$  preserves the orthogonal decomposition  $l^2(\mathbb{Z}_+) = V_1^\phi \oplus V_2^\phi$  of the representation space and that this operator is represented by a Jacobi matrix on each of the components.

The operator  $D: l^2(\mathbb{Z}_+) \rightarrow l^2(\mathbb{Z}_+)$  introduced in Sect. 2 does not preserve the orthogonal decomposition  $l^2(\mathbb{Z}_+) = V_1^\phi \oplus V_2^\phi$  as follows from Lemma 5.5. Let

$$D = \begin{pmatrix} D_{1,1}^\phi & D_{1,2}^\phi \\ D_{2,1}^\phi & D_{2,2}^\phi \end{pmatrix}$$

be the corresponding decomposition of  $D$ , then

$$\text{tr}(D\pi_\phi(p(\rho_{\tau,\sigma}))) = \text{tr}_{V_1^\phi}(D_{11}^\phi\pi_\phi(p(\rho_{\tau,\sigma}))) + \text{tr}_{V_2^\phi}(D_{22}^\phi\pi_\phi(p(\rho_{\tau,\sigma}))), \quad (6.11)$$

since  $\pi_\phi(p(\rho_{\tau,\sigma}))$  does preserve the orthogonal decomposition. Moreover, as in Sect. 3, we define

$$\begin{aligned} A_1: V_1^\phi &\rightarrow L^2(dm_1), & w_m^\phi &\mapsto e^{2im\phi} h_m(x; q^\tau, q^\sigma | q^2), \\ A_2: V_2^\phi &\rightarrow L^2(dm_2), & u_m^\phi &\mapsto e^{2im\phi} h_m(x; q^{-\tau}, q^{-\sigma} | q^2), \end{aligned}$$

where

$$\begin{aligned} dm_1(x) &= dm(x; q^{1+\sigma-\tau}, -q^{1-\sigma-\tau}, 0, 0 | q^2), \\ dm_2(x) &= dm(x; q^{1-\sigma+\tau}, -q^{1+\sigma+\tau}, 0, 0 | q^2) \end{aligned}$$

are the corresponding normalised orthogonality measures. Then, by using the spectral theory of Jacobi matrices as described in Sect. 3 in a similar way as in the derivation of (4.5), we get

$$\begin{aligned} &\text{tr}_{V_1^\phi}(D_{11}^\phi\pi_\phi(p(\rho_{\tau,\sigma}))) \\ &= \int_{\mathbb{R}} p(x) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle Dw_n^\phi, w_m^\phi \rangle h_n(x; q^\tau, q^\sigma | q^2) h_m(x; q^\tau, q^\sigma | q^2) e^{2i(m-n)\phi} dm_1(x), \end{aligned} \quad (6.12)$$

and similarly

$$\begin{aligned} &\text{tr}_{V_2^\phi}(D_{22}^\phi\pi_\phi(p(\rho_{\tau,\sigma}))) \\ &= \int_{\mathbb{R}} p(x) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle Du_n^\phi, u_m^\phi \rangle h_n(x; q^{-\tau}, q^{-\sigma} | q^2) h_m(x; q^{-\tau}, q^{-\sigma} | q^2) e^{2i(m-n)\phi} dm_2(x). \end{aligned} \quad (6.13)$$

The double sum in both (6.12) and (6.13) is absolutely convergent, uniformly in  $\phi$  and uniformly in  $x$  on the support of the corresponding orthogonality measure. To see this we observe that Proposition 5.2 and Lemma 5.5 imply

$$|\langle Dw_n^\phi, w_m^\phi \rangle| \leq q^{n+m} (-q^2, -q^{2-2\tau}; q^2)_\infty / (q^2; q^2)_\infty,$$

so that for some constant  $C$

$$\left| \sum_{n,m=0}^{\infty} \langle Dw_n^\phi, w_m^\phi \rangle (h_n h_m)(x; q^\tau, q^\sigma | q^2) e^{2i(m-n)\phi} \right| \leq C \left( \sum_{n=0}^{\infty} q^n |h_n(x; q^\tau, q^\sigma | q^2)| \right)^2.$$

Now we use the asymptotic behaviour of the Al-Salam–Chihara polynomials on  $[-1, 1]$ , cf. Askey and Ismail [2, Sect. 3.1], and, if  $(aq^k + a^{-1}q^{-k})/2$  is a discrete mass point of the orthogonality measure of the Al-Salam–Chihara polynomials we have  $p_n((aq^k + a^{-1}q^{-k})/2; a, b|q) \sim a^{-n}q^{-nk}$  as  $n \rightarrow \infty$ , cf. (6.10), to find our assertion. The other double sum is treated analogously.

Now that we have established (6.11), (6.12) and (6.13), we have an explicit expression for the Haar functional on the  $C^*$ -subalgebra of  $A_q(SU(2))$  generated by  $\rho_{\tau, \sigma}$ . Integrate (6.12) and (6.13) with respect to  $\phi$  over  $[0, 2\pi]$ , cf. (2.3), and interchange summations, which is justified by the previous remark. Since the inner products in (6.12) and (6.13) are independent of  $\phi$  by Lemma 5.5, the integration over  $\phi$  reduces the double sum to a single sum. Now use (2.3), Lemma 5.5 and Proposition 5.2 to prove the following proposition.

**Proposition 6.3.** *For any continuous function  $p$  on the spectrum of  $\rho_{\tau, \sigma} \in A_q(SU(2))$ , i.e. the union of the supports of the measures  $dm_1$  and  $dm_2$ , we have the following expression for the Haar functional:*

$$\begin{aligned}
 h(p(\rho_{\tau, \sigma})) &= \frac{1 - q^2}{1 + q^{2\tau}} \int_{\mathbb{R}} p(x) P_{q^2}(x, x; q^{1+\sigma-\tau}, -q^{1-\sigma-\tau}|q^2) dm_1(x) \\
 &+ \frac{1 - q^2}{1 + q^{-2\tau}} \int_{\mathbb{R}} p(x) P_{q^2}(x, x; q^{1-\sigma+\tau}, -q^{1+\sigma+\tau}|q^2) dm_2(x), \quad (6.14)
 \end{aligned}$$

where

$$P_t(x, y; a, b|q) = \sum_{k=0}^{\infty} t^k \frac{p_k(x; a, b|q) p_k(y; a, b|q)}{(q, ab; q)_k} \quad (6.15)$$

is the Poisson kernel for the Al-Salam–Chihara polynomials (6.10).

*Remark 6.4.* For all values needed the Poisson kernel in (6.14) is absolutely convergent for  $t = q^2$  by the asymptotic behaviour of the Al-Salam–Chihara polynomials, cf. the remarks following (6.12) and (6.13).

In order to tie (6.14) to Theorem 6.1 we have to use the explicit expression for the Poisson kernel of the Al-Salam–Chihara polynomials given by Askey, Rahman and Suslov [3, (14.8)]. It is given in terms of a very-well-poised  ${}_8\phi_7$ -series, cf. the notation (1.1),

$$\begin{aligned}
 P_t(\cos \theta, \cos \psi; a, b|q) &= \frac{(ate^{i\theta}, ate^{-i\theta}, bte^{i\psi}, bte^{-i\psi}, t; q)_{\infty}}{(te^{i\theta+i\psi}, te^{i\theta-i\psi}, te^{i\psi-i\theta}, te^{-i\psi-i\theta}, abt; q)_{\infty}} \\
 &\times {}_8W_7 \left( \frac{abt}{q}; t, be^{i\theta}, be^{-i\theta}, ae^{i\psi}, ae^{-i\psi}; q, t \right), \quad (6.16)
 \end{aligned}$$

where we also used the transformation formula [10, (2.10.1)]. Askey, Rahman and Suslov [3] are not very specific about the validity of (6.16), but from [3, Sect. 1] we may deduce that it is valid for  $|a| < 1$ ,  $|b| < 1$  and  $|t| < 1$ . We first observe that (6.16) also holds for  $ab < 1$  and  $|t| < 1$ , for which (6.15) is absolutely convergent. To see this we show that the simple poles of the infinite product on the right-hand side (6.16) at  $1/t = abq^l$ ,  $l \in \mathbb{Z}_+$ , are cancelled by a zero of the very-well-poised  ${}_8\phi_7$ -series at  $1/t = abq^l$ , i.e.

$${}_8W_7 \left( q^{-l-1}; \frac{q^{-l}}{ab}, be^{i\theta}, be^{-i\theta}, ae^{i\psi}, ae^{-i\psi}; q, \frac{q^{-l}}{ab} \right) = 0, \quad l \in \mathbb{Z}_+.$$

This follows by writing the very-well-poised  ${}_8\phi_7$ -series as a sum of two balanced  ${}_4\phi_3$ -series by [10, (2.10.10)], in which case both  ${}_4\phi_3$ -series have a factor  $(q^{-l}; q)_\infty = 0$  in front.

We also need the Poisson kernel evaluated in possible discrete mass points of the corresponding orthogonality measure. For  $x$  in a bounded set, containing the support of the orthogonality measure, we see that for  $|t|$  small enough  $P_t(x, x; a, b|q)$  defined by (6.15) is absolutely convergent and uniformly in  $x$ , and by analytic continuation in  $x$  we see that (6.16) is valid for  $|t|$  small enough. Now assume that  $|a| > 1$  and that  $x_k = (aq^k + a^{-1}q^{-k})/2$  is a discrete mass point of the orthogonality measure of the Al-Salam–Chihara polynomials, then the radius of convergence of  $P_t(x_k, x_k; a, b|q)$  in (6.15) is  $a^2q^{2k}$ , so the radius of convergence is greater than 1, cf. (6.2). For this choice of the arguments the right-hand side of (6.16) can be expressed in terms of a terminating very-well-poised  ${}_8\phi_7$ -series;

$$(a^2q^k t, tq^{-k}; q)_k \frac{(abtq^k, btq^{-k}/a; q)_\infty}{(abt, ta^{-2}q^{-2k}; q)_\infty} {}_8W_7 \left( \frac{abt}{q}; t, abq^k, bq^{-k}/a, q^{-k}, a^2q^k; q, t \right). \tag{6.17}$$

This expression has no poles in the disc  $|t| < a^2q^{2k}$ , and coincides with the Poisson kernel (6.15). So we have shown that (6.16) for  $t = q$  with  $x = \cos \theta = \cos \psi$  is valid for  $ab < 1$  and  $x$  in the support of the corresponding orthogonality measure.

After these considerations on the Poisson kernel for the Al-Salam–Chihara polynomials, we can use Bailey’s summation formula, cf. [10, (2.11.7)], in the following form;

$$\begin{aligned} & \frac{1}{(b/a; q)_\infty} {}_8W_7(a; b, c, d, e, f; q, q) + \frac{1}{(a/b; q)_\infty} \\ & \times \frac{(aq, c, d, e, f, bq/c, bq/d, bq/e, bq/f; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a; q)_\infty} \\ & \times {}_8W_7 \left( \frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}; q, q \right) \\ & = \frac{(aq, aq/(cd), aq/(ce), aq/(cf), aq/(de), aq/(df), aq/(ef); q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a; q)_\infty}. \end{aligned}$$

Use Bailey’s formula with  $q$  replaced by  $q^2$  and parameters  $a = -q^{2-2\tau}$ ,  $b = q^2$ ,  $c = -q^{1-\sigma-\tau}e^{i\theta}$ ,  $d = -q^{1-\sigma-\tau}e^{-i\theta}$ ,  $e = q^{1+\sigma-\tau}e^{i\theta}$  and  $f = q^{1+\sigma-\tau}e^{-i\theta}$ , and multiply the resulting identity by

$$\frac{(1 - q^2)(1 - e^{2i\theta})(1 - e^{-2i\theta})(q^2, -q^{2+2\tau}, -q^{2-2\tau}; q^2)_\infty}{(1 - q^{1+\sigma-\tau}e^{i\theta})(1 - q^{1+\sigma-\tau}e^{-i\theta})(1 + q^{1-\sigma-\tau}e^{i\theta})(1 + q^{1-\sigma-\tau}e^{-i\theta})}$$

to find, using the notation of (6.3),

$$\begin{aligned} & \frac{(1 - q^2)P_{q^2}(x, x; q^{\sigma-\tau+1}, -q^{-\sigma-\tau+1}|q^2)w(x; q^{\sigma-\tau+1}, -q^{1-\sigma-\tau}, 0, 0|q^2)}{(1 + q^{2\tau})h_0(q^{\sigma-\tau+1}, -q^{1-\sigma-\tau}, 0, 0|q^2)} \\ & + \frac{(1 - q^2)P_{q^2}(x, x; q^{-\sigma+\tau+1}, -q^{\sigma+\tau+1}|q^2)w(x; q^{\tau-\sigma+1}, -q^{1+\sigma+\tau}, 0, 0|q^2)}{(1 - q^{-2\tau})h_0(q^{\tau-\sigma+1}, -q^{1+\sigma+\tau}, 0, 0|q^2)} \\ & = \frac{w(x; q^{\tau-\sigma+1}, -q^{1+\sigma+\tau}, q^{\sigma-\tau+1}, -q^{1-\sigma-\tau}|q^2)}{h_0(q^{\tau-\sigma+1}, -q^{1+\sigma+\tau}, q^{\sigma-\tau+1}, -q^{1-\sigma-\tau}|q^2)} \tag{6.18} \end{aligned}$$

for  $x \in [-1, 1]$ , the absolutely continuous part of  $dm_1$  and  $dm_2$ . This proves the absolutely continuous part of the Askey–Wilson integral in Theorem 6.1.

In order to deal with the possible discrete mass points in the orthogonality measures on the right-hand side of (6.14), we first observe that the discrete mass points of  $dm_1$  do not occur as discrete mass points of  $dm_2$  and vice versa. Furthermore, note that for  $t = q$  the expression in (6.17) reduces to

$$P_k(a; b|q) = \frac{(abq^k, bq/a; q)_\infty}{(ab, a^{-2}q^{1-2k}; q)_\infty} (q^{-k}, bq^{-k}/a, a^2q^k; q)_k q^k.$$

This can be seen directly from (6.17), since only the last term in the  ${}_8\phi_7$ -series survives, or by applying Jackson's summation formula [10, (2.6.2)]. Now a straightforward calculation using this formula and the explicit values for the weights given in (6.4) proves

$$\frac{1-q}{1-q/(ab)} \frac{P_k(a; b|q)w_k(a; b, 0, 0|q)}{h_0(a, b, 0, 0|q)} = \frac{w_k(a; b, q/a, q/b|q)}{h_0(a, b, q/a, q/b|q)}.$$

This proves that the discrete mass points in (6.14) lead to the discrete mass points in Theorem 6.1. This proves Theorem 6.1 from the spectral analysis of  $\pi_\phi(\rho_{\tau, \sigma})$ .

*Remark. 6.5.* It is not allowed to take residues in (6.18) to prove the statement concerning the discrete mass points. For this we have to know that (6.18) also holds in a neighbourhood of the discrete mass point  $x_k$ , but the explicit expression for the Poisson kernel in (6.16) leading to (6.18) may fail to hold.

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