Commun. Math. Phys. 176, 447 – 465 (1996)

Communications in
Mathematical
Physics

Quasi-Galois Symmetries of the Modular S-Matrix

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Received: 7 December 1994 / in revised form: 5 April 1995

Abstract: The recently introduced Galois symmetries of rational conformal field theory are generalized, for the case of WZW theories, to "quasi-Galois symmetries." These symmetries can be used to derive a large number of equalities and sum rules for entries of the modular matrix S, including some that previously had been observed empirically. In addition, quasi-Galois symmetries allow us to construct modular invariants and to relate S-matrices as well as modular invariants at different levels. They also lead us to a convenient closed expression for the branching rules of the conformal embeddings $\mathbf{g} \hookrightarrow \widehat{\mathrm{so}}(\dim \bar{\mathbf{g}})$.

1. Introduction

In the study of rational conformal field theories, modular transformations play an essential role. They turn the set of the characters of all primary fields into a unitary module of $SL(2,\mathbb{Z})$, the twofold covering of the modular group of the torus. Via the Verlinde formula, they are also closely related to the fusion rules.

In all cases where the modular matrix S is explicitly known, one observes that it contains surprisingly few different numbers, and that among the distinct numbers there are linear relations. While it has been known for a long time that simple currents lead to relations between individual S-matrix elements [1–3], many other relations, in particular sum rules, have remained so far somewhat mysterious. Recently it has become clear that Galois symmetries [4, 5] are an independent source for relations between individual elements of S [6, 7]. Both simple current and Galois symmetries exist for arbitrary rational conformal field theories, independent of the structure of the chiral algebra.

In this paper we will show that in the special case of WZW theories, Galois symmetries can be generalized to what we will call *quasi-Galois symmetries*. A crucial ingredient of our construction (which is not available for other conformal field theories than WZW theories) is the Kac–Peterson formula for the S-matrix. These new symmetries turn out to be rather powerful and allow us to derive three

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new types of relations between the entries of S: first, a sum rule which relates signed sums of S-matrix elements, see (3.4); second, the equality, modulo signs, of certain specific S-matrix elements, see (4.1); third, a new systematic reason for S-matrix elements to vanish, see the remarks after (2.9).

Just as in the case of Galois symmetries, the relations we find can be employed to construct elements of the commutant of S, and therefore to generate modular invariants. Moreover, they can be used to obtain relations between invariants at different values of the level, i.e. between different WZW theories. Finally, we show that our results allow to determine the branching rules of certain conformal embeddings.

The rest of the paper is organized as follows. In Sect. 2 we recall the basic facts about Galois symmetries of rational conformal field theories, and of WZW theories in particular, and show how in the WZW case they can be generalized to quasi-Galois symmetries. Also, as a first application, we describe how these symmetries force certain S-matrix elements to vanish. In Sect. 3 we construct integral-valued matrices that commute with the S-matrix; as a by-product we obtain an interesting sum rule (3.4) for the entries of S. In Sect. 4 we obtain another symmetry, (4.1), of S as well as relations (see (4.8), (4.9)) between the S-matrices for WZW theories at different heights h_1, h_2, where h_1 is a multiple of h_2. Again, these results lead to a prescription for constructing S-matrix invariants, now both at the smaller and at the larger height (see (4.16) and (4.20), respectively). Finally, in Sect. 5 we consider a special case of the latter invariants, which leads us to a closed formula for the branching rules of the conformal embeddings $\mathbf{g} \hookrightarrow \widehat{\mathrm{so}}(\dim \overline{\mathbf{g}})$, which can easily be evaluated explicitly.

2. Quasi-Galois Scalings

When analyzing the mathematical structure of a WZW theory, we are dealing with integrable highest weight representations of an untwisted affine Lie algebra $\mathbf{g}$ at a fixed integral level $k^\vee$. As the level is fixed, the $\mathbf{g}$-weights are already fully determined by their horizontal part, i.e. by the weight with respect to the horizontal subalgebra $\overline{\mathbf{g}}$ of $\mathbf{g}$. In the following it will be convenient to shift all weights according to $a \mathrel{\hat{=}} \lambda_a + \rho$ by the Weyl vector ρ. Note that if the non-shifted weight λ_a is at level $k^\vee$, the shifted weight a is at level h, where

$$h := k^\vee + g^\vee \,, \tag{2.1}$$

with $g^\vee$ the dual Coxeter number of $\overline{\mathbf{g}}$; we will call h the *height* of the weight a. The set of (shifted) integrable weights of the affine Lie algebra $\mathbf{g}$ at height h is

$$P_h := \{a \in L^{\mathrm{w}} | 0 < a^i \leqq k^\vee + 1 \quad \text{for } i = 0, 1, \ldots, r\} \,. \tag{2.2}$$

Here L^{w} denotes the weight lattice, i.e. the $\mathbb{Z}$-span of the fundamental weights. In other words, the weights (2.2) are precisely the integral weights in the interior of the dominant affine Weyl chamber at level $k^\vee + g^\vee$.

An important tool for studying the modular properties of WZW theories is the Kac–Peterson formula [8],

$$S_{a,b} = \mathcal{N} \sum_{w \in W} \mathrm{sign}(w) \exp\left[-\frac{2\pi \mathrm{i}}{h}(w(a), b)\right] \tag{2.3}$$

for the modular matrix S. Here the summation is over the Weyl group W of the finite-dimensional horizontal subalgebra $\bar{\mathbf{g}}$ of $\mathbf{g}$. Some immediate consequences of this formula are the following. First, the fact that according to (2.3) $S_{a,b}$ depends on a and b only via the inner products $(w(a),b)$ and the identity $(w(la),b) = l(w(a),b) = (w(a),lb)$ imply that

$$S_{la,b} = S_{a,lb} \; ; \tag{2.4}$$

and second, for any element $\hat{w}$ of the affine Weyl group $\hat{W}$ (i.e. the horizontal projection of the Weyl group of the affine algebra $\mathbf{g}$), one has

$$S_{\hat{w}(a),b} = \mathrm{sign}(\hat{w})\, S_{a,b} \, . \tag{2.5}$$

This implies in particular that $S_{a,b} = 0$ whenever a or b lies on the boundary of an affine Weyl chamber. Note that in (2.4) and (2.5) it is implicit that the quantity $S_{a,b}$ given by (2.3) can be considered also for weights which are not integrable. This is possible because we are free to take the formula (2.3) (which for integrable weights yields the entries of the actual S-matrix, i.e. of the matrix which realizes the modular transformation $\tau \mapsto -1/\tau$ on the characters) for arbitrary weights a,b as the definition of $S_{a,b}$. Analogously, these weights need not even be integral, and hence (2.4) is valid for arbitrary numbers l, not just for integers.

To apply Galois theory to conformal field theory, one considers the number field that is obtained as the extension of the rationals $\mathbb{Q}$ by all S-matrix elements. One can show [5] that this extension is a Galois extension and that its Galois group is abelian, implying that the number field is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$. The Galois group of the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is isomorphic to $\mathbb{Z}_n^*$, the multiplicative group of all elements of $\mathbb{Z}_n$ that are coprime with n. The Galois automorphism corresponding to an element $l \in \mathbb{Z}_n^*$ acts as $\zeta_n \mapsto (\zeta_n)^l$.

In the special case of the WZW theory based on the untwisted affine Lie algebra $\mathbf{g}$ at height h, the relevant root of unity is given by ζ_{Mh}, with M the smallest positive integer for which the M-fold of all entries of the metric on the weight space of $\bar{\mathbf{g}}$ is integral.[1] A Galois transformation labeled by $l \in \mathbb{Z}_{Mh}^*$ then induces the permutation $\Lambda \mapsto \hat{w}(l(\Lambda + \rho)) - \rho$ of the highest weights carried by the primary WZW fields, or equivalently, the permutation

$$\dot{\sigma} \equiv \dot{\sigma}_{(l)}: \quad a \mapsto \dot{\sigma}a := \hat{w}_a(la) \tag{2.6}$$

of shifted highest weights. Here $\hat{w}_a$ is an element of the affine Weyl group at level h, i.e.

$$\hat{w}_a(b) = w_a(b) + h t_a \, , \tag{2.7}$$

where w_a is some element of the finite Weyl group W and t_a some weight which belongs to the coroot lattice $L^\vee$ of $\bar{\mathbf{g}}$. They are defined by the condition that $\hat{w}_a(la) \in P_h$, which determines w_a and t_a uniquely. Substituting (2.6) into the formula for WZW conformal dimensions one easily obtains a condition for T-invariance, namely $l^2 = 1 \bmod 2Mh$ (or mod Mh if all integers $M(a,a)$ are even).[2]

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[1] Actually the cyclotomic field $\mathbb{Q}(\zeta_{Mh})$ does not yet always contain the normalization $\mathcal{N}$ appearing in (2.3); rather, sometimes a slightly larger cyclotomic field must be used [5]. However, the permutation $\dot{\sigma}$ can already be determined from the generalized quantum dimensions, which do not depend on $\mathcal{N}$. Accordingly, the correct Galois treatment of $\mathcal{N}$ just amounts to an overall sign factor which is irrelevant for our purposes.

[2] For more details, see in particular Appendix A of [7].

The key idea in the present paper is to allow in the transformation (2.6) for arbitrary integers l rather than only elements of $\mathbb{Z}^*_{Mh}$. As we will show, these generalized transformations lead to interesting new information. Note that if $l \notin \mathbb{Z}^*_{Mh}$, then in order for the map (2.6) of the integrable weights to be still well-defined, we must slightly extend the prescription for the Weyl group element $\hat{w}_a$. Namely, $\hat{w}_a$ is now determined by the condition that either la lies on the boundary of some affine Weyl chamber (in which case $\hat{w}_a$ can simply be taken to be the identity), or else that $\hat{w}_a(la) \in P_h$. In the latter case, $\hat{w}_a$ is the unique element of $\hat{W}$ with this property, and we write

$$\mathrm{sign}(\hat{w}_a) = \mathrm{sign}(w_a) =: \varepsilon_l(a)\,, \tag{2.8}$$

while in the former case we put $\varepsilon_l(a) = 0$. While the map (2.6) is thus still well-defined for $l \notin \mathbb{Z}^*_{Mh}$, it can no longer be induced by a mapping $\zeta_{Mh} \mapsto (\zeta_{Mh})^l$ of the number field, and hence in particular it no longer corresponds to a Galois transformation. Nevertheless the similarity with Galois transformations is still so close that we call the map $a \mapsto la$, with l not coprime with Mh, a *quasi-Galois scaling* and the associated map $\dot{\sigma}$ (2.6) a *quasi-Galois transformation.*

Note that it is not true that an arbitrary integral weight b can be mapped into P_h by an appropriate affine Weyl transformation. However, if b is of the special form $b = la$ with $a \in P_h$ and l coprime with Lh, this is indeed possible [7]; here L denotes the "lacedness" of $\bar{\mathfrak{g}}$, i.e. $L = 2$ for $\bar{\mathfrak{g}}$ of type B or C or F_4, $L = 3$ for $\bar{\mathfrak{g}} = G_2$, and $L = 1$ else. The condition that l is coprime with Lh is in particular fulfilled whenever the scaling corresponds to an element of the Galois group, and hence in the case of genuine Galois transformations a suitable unique $\hat{w}_a \in \hat{W}$ exists for any $a \in P_h$, implying that the map $\dot{\sigma}$ is indeed a permutation of the weights in P_h. In contrast, for a quasi-Galois scaling there will in general exist some $a \in P_h$ for which la lies on the boundary of an affine Weyl chamber, so that $\dot{\sigma}$ is not even an endomorphism of the set of integrable weights. However, in terms of WZW primary fields the latter situation corresponds to mapping the primary field with highest weight a to zero, so that $\dot{\sigma}$ can still be interpreted as a linear map on the fusion ring that is spanned by the primary fields. Moreover, this can also be translated back to the language of weights by adding to the set P_h a single element $\mathscr{B}$ which stands for the union of all boundaries of affine Weyl chambers. In this setting, the map (2.6) supplemented by $\dot{\sigma}(\mathscr{B}) = \mathscr{B}$ is an endomorphism of the set $P_h \cup \{\mathscr{B}\}$, though it is no longer a permutation.

Consider now an arbitrary scaling $a \mapsto la$, $l \in \mathbb{Z}\backslash\{0\}$, with associated (quasi-) Galois transformation given by (2.6). As follows immediately by applying the identities (2.4) and (2.5) to $S_{\dot{\sigma}a,b}$, we then have the identity

$$\varepsilon_l(a)\, S_{\dot{\sigma}a,b} = \varepsilon_l(b)\, S_{a,\dot{\sigma}b}\,. \tag{2.9}$$

For genuine Galois scalings, this result was already obtained in [5]. In the quasi-Galois case, the two sides of (2.9) are not necessarily non-vanishing, and this provides us with an explanation for the vanishing of certain S-matrix elements. Namely, if for the quasi-Galois scaling l the weights b and $c := \dot{\sigma}a$ are contained in P_h, but $\dot{\sigma}b$ is not (i.e. lb lies on the boundary of an affine Weyl chamber), then (2.9) tells us that $S_{c,b} = 0$. (Another systematic reason for S-matrix elements to be zero is provided by simple current symmetries: $S_{a,b} = 0$ if a is a fixed point of the simple current J and b has non-vanishing monodromy charge [2] with respect to J.)

3. Quasi-Galois Modular Invariants

Consider for a given quasi-Galois scaling l the matrix Π with entries in $\{0, \pm 1\}$ that describes the mapping induced by the scaling on the primary fields, i.e.

$$\Pi_{a,b} \equiv \Pi^{(l)}_{a,b} := \varepsilon_l(a)\, \delta_{b,\dot{\sigma}a} \,. \tag{3.1}$$

Equation (2.9) can then be written as

$$(\Pi S)_{a,b} = \varepsilon_l(a)\, S_{\dot{\sigma}a,b} = \varepsilon_l(b)\, S_{a,\dot{\sigma}b} = (S\Pi^t)_{a,b} \,. \tag{3.2}$$

Multiplying this equation from both the left and the right with S^+, the hermitian conjugate of S, using the unitarity of S and taking the hermitian conjugate of this equation, we see that

$$(\Pi^t S)_{a,b} = (S\Pi)_{a,b} \,. \tag{3.3}$$

This relation describes in fact a rather remarkable sum rule for S-matrix elements: writing the matrix multiplication in (3.3) explicitly, it reads

$$\sum_{c \in P_h} \varepsilon_l(c)\, \delta_{a,\dot{\sigma}c}\, S_{c,b} = \sum_{c \in P_h} \varepsilon_l(c)\, \delta_{b,\dot{\sigma}c} S_{a,c} \,. \tag{3.4}$$

Generically the sums appearing in (3.4) contain more than one non-vanishing term; to our knowledge it is the first time that a relation of this type between S-matrix elements has been established in a general framework.

By introducing the pre-images of a quasi-Galois transformation,

$$\Sigma^{-1}(a) := \{c \in P_h | \dot{\sigma}(c) = a\} \tag{3.5}$$

for any $a \in P_h$, we may rewrite the sum rule (3.4) in the more suggestive manner

$$\sum_{c \in \Sigma^{-1}(a)} \varepsilon_l(c)\, S_{c,b} = \sum_{c \in \Sigma^{-1}(b)} \varepsilon_l(c)\, S_{a,c} \,. \tag{3.6}$$

If the map (2.6) is invertible, then (3.6) reduces to the relation

$$\varepsilon_l(\dot{\sigma}^{-1}a)\, S_{\dot{\sigma}^{-1}a,b} = \varepsilon_l(\dot{\sigma}^{-1}b)\, S_{a,\dot{\sigma}^{-1}b} \,, \tag{3.7}$$

which is equivalent to the identity (2.9) applied to the map $\dot{\sigma}^{-1}$.

Combining the two relations (3.2) and (3.3), it follows that the matrix

$$Z^{(l)} := \Pi + \Pi^t \tag{3.8}$$

commutes with the modular matrix S,

$$[Z^{(l)}, S] = 0 \,. \tag{3.9}$$

Typically the S-matrix invariant $Z^{(l)}$ obtained this way is not positive, nor does it commute with T. This pattern already arises for ordinary Galois scalings. However, just as in the Galois case [6, 7], it is still possible to construct physical modular invariants, because one can get rid of the minus signs and achieve T-invariance by suitably adding up various invariants of the type above and possibly combining with other methods such as simple currents. Note that in the invariant (3.8) typically some of the fields are projected out, and hence when using quasi-Galois transformations it is in fact easier to obtain T-invariance than in the Galois case.

To give an example for a matrix that commutes with the S-matrix and that is obtained by the above prescription, let us consider the scaling $l = 3$ for the A_1 WZW theory at height $h = 6$. In terms of non-shifted highest weights, this scaling maps $\Lambda = 0$ and $\Lambda = 4$ with a positive sign ε_l on $\Lambda = 2$, the weight $\Lambda = 2$ with a negative sign on itself, and the weights $\Lambda = 1, 3$ on the boundary $\mathscr{B}$. Thus the matrix $Z^{(3)}$ defined by (3.8) reads

$$Z^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \tag{3.10}$$

While this matrix has negative entries and is hence unphysical, the combination

$$\hat{Z} = (Z^{(3)})^2 + 2Z^{(3)} \tag{3.11}$$

is a physical invariant, namely the D-type invariant of the height 6 A_1 theory. As the number of primary fields is rapidly increasing with the rank and level, most applications of our prescription which lead to physical invariants involve rather complex expressions; therefore we will not display more complicated examples explicitly.

Actually the invariant (3.11) can also be obtained from genuine Galois transformations [7]. An example for a physical modular invariant which cannot be explained that way, but which is obtainable as a linear combination of quasi-Galois invariants is the exceptional E_7-type invariant of A_1 at level 16. However, the concrete expression is rather lengthy so that we refrain from presenting it here. As we shall see later, also for the E_7-type invariant there exists a close relation to the matrix $Z^{(3)}$ displayed in (3.10) even though they are invariants at different heights.

4. S-Matrix Invariants: Increasing and Lowering the Height

In this section we consider the special case where the scaling factor $l \in \mathbb{Z}_{>0}$ is a divisor of the height; to simplify notation, we will make this explicit by denoting the height of the theory to which the scaling is applied by lh. As we will see, in this situation there exist intimate relations between the WZW theories at height lh and at height h.[3] As we are now dealing with weights at two distinct heights, we find it convenient to denote the elements of P_h by lower case and the elements of P_{lh} by upper case roman letters, respectively. Similarly, we use the capital letter "S" for the S-matrix of the height lh theory and the symbol "s" for the S-matrix of the height h theory.

Before describing the relationship between height h and height lh theories, let us first prove another new symmetry property of the S-matrix: if the height is divisible by l, then for any $B \in P_{lh}$ the signed S-matrix elements

$$\varepsilon_l(C) \cdot \mathrm{S}_{la,C} \tag{4.1}$$

are identical for all $C \in \Sigma^{-1}(B)$. To check this statement, take any fixed $B \in P_{lh}$ and any $C \in \Sigma^{-1}(B)$. Then considering weights of the form $A = la$ with $a \in P_h$,

[3] We are grateful to T. Gannon for remarks that triggered the work presented in this section.

and using the fact that $\dot\sigma C = w_C(lC) + lh\,t_C$ with $w_C \in W$ and $t_C \in L^\vee$, as well as $\varepsilon_l(C) = \mathrm{sign}(w_C)$, we find

$$\begin{aligned} \mathrm{S}_{la,C} &= \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp\left[-\frac{2\pi\mathrm{i}}{lh}(w(la), l^{-1}w_C^{-1}(B) + ht'_C)\right] \\ &= \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp\left[-\frac{2\pi\mathrm{i}}{h}(w_C w(a), l^{-1}B)\right] \\ &= \mathrm{sign}(w_C)\cdot \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp\left[-\frac{2\pi\mathrm{i}}{h}(w(a), l^{-1}B)\right] . \end{aligned} \tag{4.2}$$

The only dependence of the right-hand side on the weight C is thus via the sign $\varepsilon_l(C) \equiv \mathrm{sign}(w_C)$, and hence we have established the symmetry (4.1).

The primary WZW fields φ_a and ϕ_A which are associated to the weights in P_h and in P_{lh}, respectively, can be viewed as the generators of the fusion rings $\mathcal{R}_h$ and $\mathcal{R}_{lh}$ of the height h and height lh WZW theories, respectively. Let us introduce the mappings

$$\begin{gathered} P\colon\ \mathcal{R}_{lh} \to \mathcal{R}_h \\ \phi_A \mapsto P(\phi_A) = \sum_{b\in P_h} P_{A,b}\varphi_b, \qquad P_{A,b} := \varepsilon_l(A)\,\delta_{\dot\sigma A,\,lb}\,, \end{gathered} \tag{4.3}$$

and

$$\begin{gathered} D\colon\ \mathcal{R}_h \to \mathcal{R}_{lh} \\ \varphi_a \mapsto D(\varphi_a) = \sum_{B\in P_{lh}} D_{a,B}\phi_B, \qquad D_{a,B} := \delta_{la,B} \end{gathered} \tag{4.4}$$

between these two fusion rings. Note that because of

$$l^{-1}\dot\sigma A = l^{-1}(w_A(lA) + lh\,t_A) = w_A(A) + ht_A \tag{4.5}$$

with $w_A \in W$ and $t_A \in L^\vee$ for any $A \in P_{lh}$, the weight $l^{-1}\dot\sigma A$ is integral and either an element of P_h or else on the boundary of an affine Weyl chamber at height h. Also, $P_{b,b} = 1$ (here the first label b is to be considered as an element of P_{lh}) which shows that the map P is always non-zero.

The relation (4.5) implies that there is a close connection, which will prove to be useful later on, between the conformal dimensions Δ mod $\mathbb{Z}$ of all those fields which belong to the same pre-image under the map $\dot\sigma$. Namely, from the definition $\Delta_a = [(a,a) - (\rho,\rho)]/2h$ of the conformal dimensions at height h (and the fact that any Weyl group element $w \in W$ is an isometry), it follows that

$$\begin{aligned} l(\Delta_b - \Delta_c) &= (2hl)^{-1}[(a+ht_b, a+ht_b) - (a+ht_c, a+ht_c)] \\ &= l^{-1}(a, t_b - t_c) + \frac{1}{2}hl^{-1}[(t_b,t_b) - (t_c,t_c)] \end{aligned} \tag{4.6}$$

for $b,c \in \Sigma^{-1}(a)$; we will use this equation only modulo $\mathbb{Z}$. Since $t_b, t_c \in L^\vee$, we have $(a,t_b) \in \mathbb{Z}$, $(t_b,t_b) \in 2\mathbb{Z}$, and analogously for t_c, and hence the right-hand side of (4.6) is an integral multiple of l^{-1}. If in addition the height is divisible by l, then according to (4.5) this is also true for the Dynkin components of any a for which $\Sigma^{-1}(a)$ is non-empty, and hence in this case the right-hand side is in fact an integer, so that $\Delta_b - \Delta_c \in l^{-1}\mathbb{Z}$ for $h = lh'$ and $b, c \in \Sigma^{-1}(a)$. In the notation appropriate to the height lh theory we thus have, for all $A \in P_{lh}$,

$$\Delta_B - \Delta_C \in l^{-1}\mathbb{Z} \quad \text{for } B, C \in \Sigma^{-1}(A)\,. \tag{4.7}$$

The relevance of the maps P and D that we introduced in (4.3) and (4.4) comes from the fact that they provide direct relations between the two modular matrices S and s. Namely, we find

$$\mathrm{S}D^t = l^{-r/2}P\mathrm{s}\,, \tag{4.8}$$

$$P^t\mathrm{S} = l^{r/2}\mathrm{s}D\,. \tag{4.9}$$

Equivalently, by taking the transpose, we can write these identities as

$$D\mathrm{S} = l^{-r/2}\mathrm{s}P^t\,, \tag{4.10}$$

$$\mathrm{S}P = l^{r/2}D^t\mathrm{s}\,. \tag{4.11}$$

To prove (4.8), we first separate the height-independent part of the normalization factor $\mathcal{N}$ in the Kac–Peterson formula (2.3) from the rest,

$$\mathcal{N} \equiv \mathcal{N}_{(h)} = \mathrm{i}^{(d-r)/2}|L^{\mathrm{w}}/L^{\vee}|^{-1/2}h^{-r/2} =: h^{-r/2}\,\overline{\mathcal{N}}\,, \tag{4.12}$$

where d is the dimension of $\bar{\mathfrak{g}}$. Then we compute

$$\begin{aligned}(\mathrm{S}D^t)_{A,b} = \mathrm{S}_{A,lb} &= (lh)^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp\left[-\frac{2\pi\mathrm{i}}{lh}(w(A),lb)\right]\\ &= (lh)^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp\left[-\frac{2\pi\mathrm{i}}{h}(w(A),b)\right]\end{aligned} \tag{4.13}$$

and, once again making use of $\dot{\sigma}A = w_A(lA) + lh\,t_A$ with $w_A \in W$ and $t_A \in L^{\vee}$, and of $\varepsilon_l(A) = \mathrm{sign}(w_A)$,

$$\begin{aligned}(P\mathrm{s})_{A,b} &= \varepsilon_l(A)\mathrm{s}_{l^{-1}\dot{\sigma}A,b}\\ &= h^{-r/2}\,\overline{\mathcal{N}}\,\mathrm{sign}(w_A)\sum_{w\in W}\mathrm{sign}(w)\exp\left[-\frac{2\pi\mathrm{i}}{h}(w(w_A(A)+ht_A),b)\right]\\ &= h^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp\left[-\frac{2\pi\mathrm{i}}{h}(w(A),b)\right]\,.\end{aligned} \tag{4.14}$$

Comparing (4.13) and (4.14), we obtain (4.8).

The relation (4.9) can now be proven by multiplying (4.8) from the left with the hermitian conjugate S^+ of S and from the right with s^+. Using the unitarity of S and s and taking the hermitian conjugate yields (4.9).

We can now apply the results just proven to the construction of S-matrix invariants, both at height h and at height lh. Namely, assume first that the matrix Z belongs to the commutant of the S-matrix of the height lh theory, i.e. that

$$[Z,\mathrm{S}] = 0\,. \tag{4.15}$$

Further, define

$$\tilde{z} := P^t Z D^t + D Z P\,. \tag{4.16}$$

Explicitly, we have

$$\tilde{z}_{a,b} = \sum_{A\in\Sigma^{-1}(la)} \varepsilon_l(A)\, Z_{A,lb} + \sum_{B\in\Sigma^{-1}(lb)} \varepsilon_l(B)\, Z_{la,B}\,. \tag{4.17}$$

Using (4.15) as well as the relations (4.8)–(4.11) proven above, we can then derive that

$$\begin{aligned}\tilde{z}\,\mathrm{s} &= P^t\, Z D^t\, \mathrm{s} + D Z P\, \mathrm{s} = l^{-r/2} P^t\, Z \mathrm{S} P + l^{r/2} D Z\, \mathrm{S} D^t \\ &= l^{-r/2} P^t\, \mathrm{S} Z P + l^{r/2} D \mathrm{S} Z D^t = \mathrm{s} D Z P + \mathrm{s} P^t\, Z D^t = \mathrm{s}\,\tilde{z}\,.\end{aligned} \tag{4.18}$$

Similarly, let z be an S-matrix invariant of the height h theory,

$$[z, \mathrm{s}] = 0\,, \tag{4.19}$$

and define

$$\tilde{Z} := D^t z P^t + P z D\,. \tag{4.20}$$

Using the convention that $z_{a,b} = 0$ whenever a or b is not in P_h, the matrix elements of $\tilde{Z}$ read

$$\tilde{Z}_{A,B} = \varepsilon_l(A)\, z_{l^{-1}\dot{\sigma}A,\, l^{-1}B} + \varepsilon_l(B)\, z_{l^{-1}A,\, l^{-1}\dot{\sigma}B}\,. \tag{4.21}$$

By employing (4.19) and again (4.8)–(4.11), we obtain

$$\begin{aligned}\tilde{Z}\,\mathrm{S} &= D^t\, z P^t \mathrm{S} + P z D\, \mathrm{S} = l^{r/2}\, D^t\, z\, \mathrm{s} D + l^{-r/2}\, P z\, \mathrm{s} P^t \\ &= l^{r/2}\, D^t\, \mathrm{s} z D + l^{-r/2}\, P\, \mathrm{s} z P^t = \mathrm{S} P z D + \mathrm{S} D^t\, z P^t = \mathrm{S}\,\tilde{Z}\,.\end{aligned} \tag{4.22}$$

We have thus proven the following remarkable facts: Given an S-matrix invariant Z at height lh, the formula (4.16) provides us with an S-matrix invariant $\tilde{z}$ at height h,

$$[\tilde{z}, \mathrm{s}] = 0\,; \tag{4.23}$$

and conversely, given an S-matrix invariant z at height h, the formula (4.20) defines an S-matrix invariant $\tilde{Z}$ at height lh,

$$[\tilde{Z}, \mathrm{S}] = 0\,. \tag{4.24}$$

Not surprisingly, the prescriptions (4.16) and (4.20) do not respect positivity, i.e. even if Z (respectively z) is a positive invariant, this need not hold for $\tilde{z}\,(\tilde{Z})$.

As an example, let us take for Z the exceptional invariants of A_1 which all occur at heights of a multiple of 6, namely for $h = 12, 18, 30$, and obtain from them by (4.16) invariants of A_1 at height 6. For $h = 12$ and $h = 30$ the prescription (4.16) yields the zero matrix. More interesting is the E_7-type invariant at $h = 18$; in this case $\tilde{z}$ is precisely the quasi-Galois invariant (3.10) obtained in the previous section.

Note that the maps (4.3) and (4.4) are related to the map Π introduced in (3.1) by $\Pi = PD$:

$$\Pi_{A,B} = \varepsilon_l(A)\, \delta_{B,\dot{\sigma}A} \equiv \sum_{c\in P_h} \varepsilon_l(A)\, \delta_{lc,\dot{\sigma}A} \delta_{B,lc} = \sum_{c\in P_h} P_{A,c} D_{c,B}\,. \tag{4.25}$$

The prescription (4.20) actually provides a generalization of the quasi-Galois S-matrix invariant (3.8). Namely, according to (4.25), when considering the diagonal invariant $z = \mathbb{1}$, (4.20) yields

$$\tilde{Z} = PD + D^t P^t = \Pi + \Pi^t , \tag{4.26}$$

i.e. reproduces the invariant (3.8). A still more special case is obtained by performing the scaling by the factor l at height $lg^\vee$. Then the smaller level is in fact zero, so that there is a single primary field with shifted weight $a = \rho$, and hence a single nontrivial invariant $z_{a,b} = \delta_{a,\rho}\delta_{b,\rho}$. In this situation, (4.21) reads

$$\tilde{Z}_{A,B} = \delta_{A,l\rho} \sum_{C\in\Sigma^{-1}(l\rho)} \varepsilon_l(C)\, \delta_{B,C} + \delta_{B,l\rho} \sum_{C\in\Sigma^{-1}(l\rho)} \varepsilon_l(C)\, \delta_{A,C} \,. \tag{4.27}$$

In applications (see in particular Sect. 5 below) it is often not the matrix (4.27) that is directly relevant, but rather the combination

$$\hat{Z} := \tilde{Z}^2 - 2\varepsilon_l(l\rho)\tilde{Z} \tag{4.28}$$

(compare the similar formula (3.11)). The entries of (4.28) read

$$\hat{Z}_{A,B} = |\tilde{\Sigma}^{-1}(l\rho)|\, \delta_{A,l\rho}\delta_{B,l\rho} + \sum_{C,D\in\tilde{\Sigma}^{-1}(l\rho)} \varepsilon_l(C)\varepsilon_l(D)\, \delta_{A,C}\delta_{B,D} \,, \tag{4.29}$$

where

$$\tilde{\Sigma}^{-1}(l\rho) := \Sigma^{-1}(l\rho)\backslash\{l\rho\} \,. \tag{4.30}$$

Note that in the invariant $\hat{Z}$ only fields belonging to $\Sigma^{-1}(l\rho)$ get mixed; by (4.7) this implies that $\hat{Z}$ is not only S-invariant, but also invariant under T^l. It is also easily checked that $\hat{Z}^2 = |\tilde{\Sigma}^{-1}(l\rho)|\hat{Z}$, so that by taking powers of $\hat{Z}$ we cannot produce any new invariants.

We can also apply the constructions (4.20) and (4.16) consecutively to a height h S-matrix invariant, or in the opposite order to a height lh invariant. The computation then involves the identities $PD = \Pi$, $DD^t = \mathbb{1}$, $P^tP = l^r\mathbb{1}$, as well as $DP = \pi$ and $D^tD = Q$ with

$$\pi_{a,b} := \varepsilon_l(la)\, \delta_{lb,\dot{\sigma}(la)} \tag{4.31}$$

and

$$Q_{A,B} := \delta_{A,B} \cdot \sum_{b\in P_h} \delta_{A,lb} \,. \tag{4.32}$$

We find

$$\tilde{\tilde{z}} = 2l^r z + \pi z \pi + \pi^t z \pi^t , \tag{4.33}$$

and a similar formula for $\tilde{\tilde{Z}}$. The result (4.33) means that whenever z commutes with s, then so does the matrix $\pi z \pi + \pi^t z \pi^t$. Also note that in (4.31) the map $\dot{\sigma}$ is the quasi-Galois transformation with scale factor l at height lh. This implies that $\dot{\sigma}(la) = l(w_{la}(la) + ht_{la})$, and hence the δ-symbol in (4.31) imposes the constraint that the weight b is related to a by a quasi-Galois transformation with the same scale factor l, but now at height h. In other words, as already anticipated in the notation, the map $\pi = DP$ implements the same quasi-Galois scaling for the height h theory as the map $\Pi = PD$ (4.25) implements for the height lh theory.

5. Conformal Embeddings

Conformal embeddings are embeddings $\mathbf{g} \hookrightarrow \mathbf{h}$ of untwisted affine Lie algebras for which the irreducible highest weight modules possess finite branching rules. The explicit form of these branching rules has been determined for various cases (see e.g. [9–15]), but a general formula is not known, and there are still many conformal embeddings for which all known methods are inapplicable.

The list of conformal embeddings [16, 17] contains several infinite series. Here we are interested in a particular infinite series, namely the embedding $\mathbf{g}_{g^\vee} \hookrightarrow \widehat{\mathrm{so}}(d)_1$, i.e. of $\mathbf{g}$ at level $g^\vee$ (with $\mathbf{g}$ an arbitrary untwisted affine Lie algebra) into $\widehat{\mathrm{so}}(d)$, with $d \equiv \dim \bar{\mathbf{g}}$, at level one. In terms of the horizontal algebras, the embedding is the one for which the vector representation of $\mathrm{so}(d)$ branches to the adjoint representation of the smaller algebra $\bar{\mathbf{g}}$. Such embeddings are of particular interest because they are connected with the "fermionization" of WZW models with level $g^\vee$, which is due to the fact that $\widehat{\mathrm{so}}(d)$ can be written in terms of free fermions. This will play a rôle in the following.

The diagonal level one $\widehat{\mathrm{so}}(d)$ partition function is

$$\mathscr{Z}_{\mathrm{so}(d)}(\tau, \bar{\tau}) = |\mathscr{X}_\mathrm{o}|^2 + |\mathscr{X}_\mathrm{v}|^2 + |\mathscr{X}_\mathrm{s}|^2 + |\mathscr{X}_\mathrm{c}|^2 \quad \text{for } d \text{ even}, \tag{5.1}$$

and

$$\mathscr{Z}_{\mathrm{so}(d)}(\tau, \bar{\tau}) = |\mathscr{X}_\mathrm{o}|^2 + |\mathscr{X}_\mathrm{v}|^2 + |\mathscr{X}_\mathrm{s}|^2 \quad \text{for } d \text{ odd}, \tag{5.2}$$

where o, v, s and c refer to the singlet, vector, spinor, and conjugate spinor representation of $\mathrm{so}(d)$, respectively. Our objective is to write each of these characters in terms of the characters χ_Λ of $\mathbf{g}$ at level $g^\vee$.

The branching rule for the $\widehat{\mathrm{so}}(d)$ spinor(s) is already known explicitly ([18], see also [19, 10, 20]). Up to a multiplicity, they branch to a single irreducible representation, namely the one whose (unshifted) highest weight is the Weyl vector ρ. We will denote this irreducible representation by L_ρ. The dimension of the analogous irreducible representation of the horizontal algebra $\bar{\mathbf{g}}$ is 2^{N_+}, where $N_+ = (d-r)/2$ is the number of positive roots (and r is the rank of $\bar{\mathbf{g}}$); hence the multiplicity with which L_ρ is contained in the $\widehat{\mathrm{so}}(d)$ spinors is $2^{r/2-1}$ if d is even, and $2^{(r-1)/2}$ if d is odd. A closed formula for the branching rules of the $\widehat{\mathrm{so}}(d)$ singlet and vector is also known [10], but (see (5.20) below) it involves the image $\hat{W}(\rho)$ of the Weyl vector under the affine Weyl group and hence is not convenient for explicit calculations. (As a matter of fact, only in very few cases, such as for $\bar{\mathbf{g}} = G_2$ [12], the branching has already been determined explicitly). Accordingly, we will not employ this formula, but rather prove an equivalent formula which allows for an immediate evaluation on a computer. To start, we make the following general ansatz for the relation between level one $\widehat{\mathrm{so}}(d)$ and $\mathbf{g}_{g^\vee}$ characters:

$$\mathscr{X}_\mathrm{o} = \sum_{\Lambda \in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda \chi_\Lambda, \qquad \mathscr{X}_\mathrm{v} = \sum_{\Lambda \in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda \chi_\Lambda, \qquad \mathscr{X}_\mathrm{s} = \mathscr{X}_\mathrm{c} = 2^{r/2-1} \chi_\rho \tag{5.3}$$

for d even, and

$$\mathscr{X}_\mathrm{o} = \sum_{\Lambda \in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda \chi_\Lambda, \qquad \mathscr{X}_\mathrm{v} = \sum_{\Lambda \in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda \chi_\Lambda, \qquad \mathscr{X}_\mathrm{s} = 2^{(r-1)/2} \chi_\rho \tag{5.4}$$

for d odd. Here and below we label the integrable $\mathbf{g}_{g^\vee}$ representations by their *unshifted* highest weights (in particular we will use $\Lambda = \rho$ in place of $a = 2\rho$);

accordingly, the summations in (5.3) and (5.4) are over the unshifted fundamental chamber $P_{g^\vee}(\mathbf{g})$; also, m_o and m_v are non-negative integral vectors in the space of all characters. The equality of the decomposition of the two $\widehat{\mathrm{so}}(d)$ spinor characters for even d implies that these representations will appear as a fixed point of order 2 in the $\mathbf{g}_{g^\vee}$ modular invariant. Hence the invariant will have the form

$$\mathscr{Z}_{\text{c.e.}} = \Big|\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda \chi_\Lambda\Big|^2 + \Big|\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda \chi_\Lambda\Big|^2 + 2\cdot\left|2^{r/2-1}\chi_\rho\right|^2 \tag{5.5}$$

for d even, and

$$\mathscr{Z}_{\text{c.e.}} = \Big|\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda \chi_\Lambda\Big|^2 + \Big|\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda \chi_\Lambda\Big|^2 + \left|2^{(r-1)/2}\chi_\rho\right|^2 \tag{5.6}$$

for d odd.

The identity and vector characters of $\widehat{\mathrm{so}}(d)$ branch to distinct $\mathbf{g}_{g^\vee}$ characters, since the difference of conformal dimensions of identity and vector is non-integral. Thus the vectors m_o and m_v are orthogonal. We will focus first on the cases where also the spinor(s) have different conformal weights modulo integers than identity and vector, which holds if $d \neq 0 \bmod 8$. Then by the same argument the spinor(s) branch to different $\mathbf{g}_{g^\vee}$ characters than identity and vector characters, and hence we have $\mathrm{m}_\mathrm{o}^\rho = \mathrm{m}_\mathrm{v}^\rho = 0$. This situation is covered by the following simple theorem. Consider any S-invariant (such as (5.5), (5.6)) that is a sum of squares, i.e. of the form

$$\mathscr{M} = \sum_p N_p \Big|\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_p^\Lambda \chi_\Lambda\Big|^2 . \tag{5.7}$$

This can be written as $\sum_{\Lambda,\Lambda'\in P_{g^\vee}} \chi_\Lambda M_{\Lambda,\Lambda'} \chi^*_{\Lambda'}$, where M is the matrix with entries

$$M_{\Lambda,\Lambda'} = \sum_p N_p \mathrm{m}_p^\Lambda \mathrm{m}_p^{\Lambda'} . \tag{5.8}$$

Further, suppose that the vectors m_p are orthogonal,

$$\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_p^\Lambda \mathrm{m}_{p'}^\Lambda = R_p \delta_{pp'} . \tag{5.9}$$

Let us also impose the physical requirement that there is a unique vacuum, i.e. that M satisfies $M_{00} = 1$; then among the vectors m_p there must be precisely one, conventionally labeled by $p = 0$, which contains the identity character, i.e. we must have $N_0 = 1$ and $\mathrm{m}_0^0 = 1$. Next consider the matrix M^2; it has entries $(M^2)_{\Lambda,\Lambda'} = \sum_p N_p^2 R_p \mathrm{m}_p^\Lambda \mathrm{m}_p^{\Lambda'}$; in particular, $(M^2)_{00} = R_0$. Thus the matrix $M^2 - R_0 M$ has entries $(M^2 - R_0 M)_{\Lambda,\Lambda'} = \sum_p (N_p^2 R_p - N_p R_0) \mathrm{m}_p^\Lambda \mathrm{m}_p^{\Lambda'}$. Finally, the square Z of the latter matrix has entries

$$Z_{\Lambda,\Lambda'} \equiv ([M^2 - R_0 M]^2)_{\Lambda,\Lambda'} = \sum_p (N_p R_p - R_0)^2 N_p R_p \mathrm{m}_p^\Lambda \mathrm{m}_p^{\Lambda'} . \tag{5.10}$$

This is a manifestly non-negative matrix, it obeys $Z_{00} = 0$, and because it is a polynomial in M it commutes with S. Thus $0 = Z_{00} = \sum_{\Lambda,\Lambda'\in P_{g^\vee}} S_{0\Lambda} Z_{\Lambda,\Lambda'} S_{0\Lambda'} \geqq 0$, with equality only if $Z_{\Lambda,\Lambda'}=0$ for all $\Lambda,\Lambda'\in P_{g^\vee}$; i.e., any such matrix must vanish. By (5.10), the vanishing of Z implies that for any p the sum rule

$$N_p \sum_{\Lambda\in P_{g^\vee}} (\mathrm{m}_p^\Lambda)^2 \equiv N_p R_p = R_0 \tag{5.11}$$

holds. This is equivalent to the property $M^2=R_0 M$, so that M is idempotent up to a normalization.

In the situation of our interest, these sum rules give useful information because we know N_p and m_p for the spinor characters. For even d, the spinors have $N=2$, and hence (5.11) tells us that

$$R_{\mathrm{o}} = N_{\mathrm{v}} R_{\mathrm{v}} = 2\cdot(2^{r/2-1})^2 = 2^{r-1}\,, \tag{5.12}$$

and for d odd we get

$$R_{\mathrm{o}} = N_{\mathrm{v}} R_{\mathrm{v}} = (2^{(r-1)/2})^2 = 2^{r-1}\,. \tag{5.13}$$

Since for $d \not\equiv 8 \bmod 16$ the vector representation of level one $\widehat{\mathrm{so}}(d)$ has different conformal dimension modulo integers than the other representations, we have $N_{\mathrm{v}}=1$. As we will see below, the matrix M has all entries except the spinor entries equal to 0 or 1, and in that case the sum rule (5.11) tells us that the identity and the vector of $\widehat{\mathrm{so}}(d)$ each branch to 2^{r-1} different irreducible representations of the conformal subalgebra $\mathbf{g}$.

For the following argument it is convenient to summarize the spinor branching rules in (5.3) and (5.4) as $\tilde{\mathscr{X}}_{\mathrm{s}}=2^{[r/2]}\chi_\rho$, where $[n]$ stands for the integer part of n, and where $\tilde{\mathscr{X}}_{\mathrm{s}}=\mathscr{X}_{\mathrm{s}}$ for odd d and $\tilde{\mathscr{X}}_{\mathrm{s}}=(\mathscr{X}_{\mathrm{s}}+\mathscr{X}_{\mathrm{c}})/2$ for even d. Then by performing the modular transformation $\tau\mapsto -1/\tau$ and using the explicit form of the S-matrix of the $\widehat{\mathrm{so}}(d)$ theory, we have

$$2^{[r/2]-r/2}(\mathscr{X}_{\mathrm{o}}-\mathscr{X}_{\mathrm{v}})(\tau) = \tilde{\mathscr{X}}_{\mathrm{s}}\left(-\frac{1}{\tau}\right) = 2^{[r/2]}\chi_\rho\left(-\frac{1}{\tau}\right) = 2^{[r/2]}\sum_{\Lambda\in P_{g^\vee}} (S_{\mathbf{g}})_{\rho,\Lambda}\chi_\Lambda(\tau)\,. \tag{5.14}$$

This formula holds in fact for the full characters, not just for the Virasoro specialized ones. Since the full characters form a basis of the relevant module of $SL(2,\mathbb{Z})$, and since in the expansions of $\mathscr{X}_{\mathrm{o}}$ and $\mathscr{X}_{\mathrm{v}}$ into powers of $q=\exp(2\pi i\tau)$ the fractional powers of q are different, it follows that (5.14) already determines the branching rules of the singlet and vector characters uniquely. In particular the knowledge that χ_0 must appear with multiplicity one in the branching rule for $\mathscr{X}_{\mathrm{o}}$ implies that $(S_{\mathbf{g}})_{\rho,0}=2^{-r/2}$, and that for any $\Lambda\in P_{g^\vee}$, $(S_{\mathbf{g}})_{\rho,\Lambda}$ must be an integral multiple of this number.

All the properties of the conformal embedding invariants that were obtained above follow by rather general arguments. We will now discuss how one can obtain these invariants (i.e. the form of the vectors m_{o} and m_{v}) in a much more explicit manner by employing a quasi-Galois scaling by a factor 2. Thus consider $\mathbf{g}$ at height $h=2g^\vee$, and the quasi-Galois scaling $l=2$. Applying the prescription (4.20), we obtain the special case $l=2$ of the S-matrix invariant (4.29). In terms of unshifted weights, (4.29) reads

$$\hat{Z}_{\Lambda,\Lambda'} = |\tilde{\Sigma}^{-1}(\rho)|\delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\mu,\mu'\in\tilde{\Sigma}^{-1}(\rho)} \varepsilon(\mu)\varepsilon(\mu')\delta_{\Lambda,\mu}\delta_{\Lambda',\mu'}\,. \tag{5.15}$$

As it turns out, the sign ε is not constant on $\Sigma^{-1}(\rho)$, so that (unlike in the, otherwise similar, situation of (3.10)) the invariant $\hat{Z}$ (5.15) is not positive. By the remark after (4.30) it follows, however, that it does commute with T^2.

Furthermore, according to (2.9) we have

$$\varepsilon(0)\,(S_{\mathbf{g}})_{\rho,\Lambda} \equiv \varepsilon(0)\,(S_{\mathbf{g}})_{\dot{\sigma}0,\Lambda} = \varepsilon(\Lambda)\,(S_{\mathbf{g}})_{0,\dot{\sigma}\Lambda} \tag{5.16}$$

for any $\Lambda \in P_{g^\vee}$, and hence the observation after (5.14) implies that $\varepsilon(0) = 1$ and

$$(S_{\mathbf{g}})_{\rho,\Lambda} = \varepsilon(\Lambda)\cdot 2^{-r/2} \tag{5.17}$$

for all $\Lambda \in P_{g^\vee}$. Combining this information with (5.14) and the fact that the full characters form a basis, we learn that

$$\mathscr{X}_{\mathrm{o}} = \sum_{\substack{\Lambda\in\Sigma^{-1}(\rho)\\ \varepsilon(\Lambda)=1}} \chi_\Lambda, \qquad \mathscr{X}_{\mathrm{v}} = \sum_{\substack{\Lambda\in\Sigma^{-1}(\rho)\\ \varepsilon(\Lambda)=-1}} \chi_\Lambda\,. \tag{5.18}$$

This is the announced closed formula for the branching rules of the embedding $\mathbf{g} \hookrightarrow \widehat{\mathrm{so}}(\dim \mathbf{g})$. Note that in terms of unshifted weights the explicit form of the quasi-Galois transformation reads $2\rho = \rho + \rho = \dot{\sigma}\Lambda + \rho \equiv \hat{w}_\Lambda(2(\Lambda+\rho)) = 2w_\Lambda(\Lambda+\rho) + 2g^\vee\beta_\Lambda$ with $w_\Lambda \in W$ and $\beta_\Lambda \in L^\vee$, which can be rewritten as

$$\Lambda = w_\Lambda^{-1}(\rho) - \rho - g^\vee w_\Lambda^{-1}(\beta_\Lambda) = \hat{u}(\rho) - \rho\,, \tag{5.19}$$

where the last equality defines a unique element $\hat{u}$ of the affine Weyl group $\hat{W}$ at level $g^\vee$. Thus our result (5.18) can be rewritten as

$$\mathscr{X}_{\mathrm{o}} = \sum_{\Lambda\in P_{g^\vee}\cap \mathrm{R}_+} \chi_\Lambda, \qquad \mathscr{X}_{\mathrm{v}} = \sum_{\Lambda\in P_{g^\vee}\cap \mathrm{R}_-} \chi_\Lambda \tag{5.20}$$

with

$$\mathrm{R}_\pm := \{\hat{w}(\rho) - \rho \mid \hat{w} \in \hat{W},\ \mathrm{sign}(w) = \pm 1\}\,. \tag{5.21}$$

The formula (5.20) has already been obtained in [10]. It is equivalent to (5.18), but for explicit calculations has the disadvantage that it involves the sets $\mathrm{R}_\pm$; these sets are infinite due to the fact that all elements of the affine Weyl group must be taken into account.

Let us describe some aspects of the formula (5.18) in more detail. First, for all simple $\bar{\mathbf{g}}$ except $\bar{\mathbf{g}} = A_r$ with r even, we observe the following. A certain number K of representations with integer conformal weight is mapped via the quasi-Galois transformation to L_ρ with a positive sign; an equal number of representations with half-integer conformal weight flows to L_ρ with a negative sign; all other representations as well as L_ρ itself flow to the boundary. (This has been checked explicitly for rank less than 9; the continuation of this specific result to higher rank is only a conjecture.) For A_r with r even, there are two differences with respect to the foregoing. First of all the numbers K and K' of fields with integral and half-integral conformal weight, respectively, that flow to L_ρ are different, and secondly L_ρ does not flow to the boundary, but to itself. In this case $d = r(r+2)$, which is a multiple of 8, implying that the $\widehat{\mathrm{so}}(d)$ spinor has integral or half-integral conformal weight. The sign associated with the flow of L_ρ to itself is plus or minus for these two cases respectively.

In matrix notation, we thus have $\tilde{Z} = \Pi + \Pi^t$, with

$$\Pi = \begin{pmatrix} 0 & 0 & \vec{e} & 0 \\ 0 & 0 & -\vec{e} & 0 \\ 0 & 0 & \varepsilon(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.22}$$

for the matrix (4.20) that underlies (4.28), and hence

$$\hat{Z} = \begin{pmatrix} \mathrm{E} & -\mathrm{E} & 0 & 0 \\ -\mathrm{E} & \mathrm{E} & 0 & 0 \\ 0 & 0 & K+K' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.23}$$

Here the third column/row corresponds to L_ρ, the first one to all K fields with integral conformal weight which flow to L_ρ under the quasi-Galois transformation, the second to the K' fields with half-integral weight flowing to L_ρ, and the fourth to all remaining fields. The symbol $\vec{e}$ stands for a K, respectively K', component vector with all entries equal to 1, and $\mathrm{E} \equiv \vec{e} \otimes \vec{e}^{\,t}$ denotes the matrix of appropriate size (i.e., $K \times K, K \times K', K' \times K$, and $K' \times K'$, respectively) each of whose entries is equal to 1; the 0's indicate matrices of zeroes of the proper size. Thus in particular for all cases except A_r with even rank, (5.23) can also be written as

$$\hat{Z} = \begin{pmatrix} \mathrm{E} & -\mathrm{E} & 0 & 0 \\ -\mathrm{E} & \mathrm{E} & 0 & 0 \\ 0 & 0 & 2K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.24}$$

with all matrices E of size $K \times K$. Also recall that if L_ρ flows to the boundary, then $\varepsilon(\rho) = 0$ so that the entry $\Pi_{\rho,\rho}$ of the matrix (5.22) vanishes. Further, if d is a multiple of 8, then not only the matrix (5.23), but also

$$\hat{Z}' := \hat{Z} + \varepsilon(\rho)\tilde{Z} = \begin{pmatrix} \mathrm{E} & -\mathrm{E} & \varepsilon(\rho)\vec{e} & 0 \\ -\mathrm{E} & \mathrm{E} & -\varepsilon(\rho)\vec{e} & 0 \\ \varepsilon(\rho)\vec{e}^{\,t} & -\varepsilon(\rho)\vec{e}^{\,t} & K+K'+2\varepsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.25}$$

commutes with both S and T^2.

These results can be related to the conformal embedding invariant in the following way. Consider first the case of even d. The diagonal $\widehat{\mathrm{so}}(d)$ invariant can be written in terms of Jacobi theta functions and the Dedekind eta function, using

$$\mathcal{X}_\mathrm{o} = \frac{1}{2}\eta^{-d/2}(\theta_3^{d/2} + \theta_4^{d/2}), \qquad \mathcal{X}_\mathrm{v} = \frac{1}{2}\eta^{-d/2}(\theta_3^{d/2} - \theta_4^{d/2}),$$

$$\mathcal{X}_\mathrm{s} = \frac{1}{2}\eta^{-d/2}(\theta_2^{d/2} + \mathrm{i}^{d/2}\theta_1^{d/2}), \qquad \mathcal{X}_\mathrm{c} = \frac{1}{2}\eta^{-d/2}(\theta_2^{d/2} - \mathrm{i}^{d/2}\theta_1^{d/2}), \tag{5.26}$$

where the arguments τ and z are suppressed ((5.26) reflects the possible description of the $\widehat{\mathrm{so}}(d)$ theory by free fermions). We are only considering Virasoro specialized characters here, i.e. these functions are in fact $\theta_i(\mathrm{z} = 0, \tau)$. Since $\theta_1(\mathrm{z} = 0, \tau) = 0$, in this setting the partition function (5.1) reads $\mathscr{Z}_{\mathrm{so}(d)} = \frac{1}{2}|\eta|^{-d}[|\theta_3|^d + |\theta_4|^d + |\theta_2|^d]$. This is modular invariant because S interchanges θ_4 and θ_3, while T interchanges θ_4 and θ_2, and all overall factors cancel.

This diagonal partition function is however not the one we obtain from quasi-Galois transformations. Using the modular transformation properties of the θ-functions one can write down another partition function that is only invariant under S and T^2, namely (fixing the normalization such as to make the square of the identity character appear exactly once) $\hat{\mathscr{Z}}_{\mathrm{so}(d)} = |\eta|^{-d/2}[|\theta_4|^d + |\theta_2|^d]$, or, re-expressed in terms of the $\widehat{\mathrm{so}}(d)$ characters (5.26),

$$\hat{\mathscr{Z}}_{\mathrm{so}(d)} = |\mathscr{X}_{\mathrm{o}} - \mathscr{X}_{\mathrm{v}}|^2 + |\mathscr{X}_{\mathrm{s}} + \mathscr{X}_{\mathrm{c}}|^2 \,. \tag{5.27}$$

Both the diagonal modular invariant (5.1) and the partition function (5.27) contain more information than one strictly gets from specialized characters; one may check explicitly that both are S-invariant if the spinor characters are distributed symmetrically, as indicated.

If we write the matrix M corresponding to (5.27) in terms of **g**-representations we get

$$\begin{pmatrix} \mathrm{E}_{\mathrm{oo}} & -\mathrm{E}_{\mathrm{ov}} & 0 & 0 \\ -\mathrm{E}_{\mathrm{vo}} & \mathrm{E}_{\mathrm{vv}} & 0 & 0 \\ 0 & 0 & 2^r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.28$'$}$$

where $(\mathrm{E}_{pp'})_{\Lambda,\Lambda'} = \mathrm{m}_p^{\Lambda}\,\mathrm{m}_{p'}^{\Lambda'}$. The result (5.18) implies that $\mathrm{E}_{\mathrm{oo}} = \mathrm{E}_{\mathrm{ov}} = \mathrm{E}_{\mathrm{vo}} = \mathrm{E}_{\mathrm{vv}} = \mathrm{E}$, or in other words, that $\vec{\mathrm{m}}_{\mathrm{o}} = \vec{\mathrm{m}}_{\mathrm{v}} = \vec{\mathrm{e}}$. Thus (5.28) can be identified with (5.24). There is also an independent consistency check of this identification. Namely, we find that $K = 2^{r-1}$, so that both m_{o} and m_{v} have 2^{r-1} components, each equal to 1. Hence they do satisfy the sum rule (5.12), so this rather nontrivial requirement for the matrix

$$Z_{\mathrm{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & 0 & 0 \\ 0 & \mathrm{E} & 0 & 0 \\ 0 & 0 & 2^{r-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.29}$$

to commute with S is fulfilled. The matrix (5.29) is the modular invariant that corresponds to the branching rules (5.18). Note that the quasi-Galois symmetries imply that (5.24) commutes with S and T^2, while the step from (5.24) to (5.29) does not follow from any symmetry we know.

If d is a multiple of 8, then the above argument has to be slightly extended. Since in this case both (5.23) and (5.25) are S-T^2-invariants, we have in addition to (5.29) another matrix $Z'_{\mathrm{c.e.}}$, and hence any physical linear combination $Z(u,v) := u\,Z_{\mathrm{c.e.}} + v\,Z'_{\mathrm{c.e.}}$, as candidates for the conformal embedding invariant. Explicitly, the matrix $Z'_{\mathrm{c.e.}}$ reads

$$Z'_{\mathrm{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & \vec{\mathrm{e}} & 0 \\ 0 & \mathrm{E} & 0 & 0 \\ \vec{\mathrm{e}}^{\,t} & 0 & 2^{r-1} + \varepsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.30}$$

for $d = 0 \bmod 16$ and

$$Z'_{\text{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & 0 & 0 \\ 0 & \mathrm{E} & \vec{e} & 0 \\ 0 & \vec{e}^{\,t} & 2^{r-1} + \varepsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.31}$$

for $d = 8 \bmod 16$, respectively. Fortunately, it is easy to eliminate all but one of the candidates, namely by imposing the "quantum dimension" sum rule

$$\frac{1}{2} = (S_{\mathrm{so}(d)})_{0,0} = \sum_{\Lambda \in P_{g^\vee}} (S_{\mathbf{g}})_{0,\Lambda} \tag{5.32}$$

(here the summation is over all fields that are combined with the identity field). Inserting the ansatz $Z(u, v)$, we find that for the case of A_r with even r, this yields the unique solution $u = 0,\ v = 1$, so that (5.30), respectively (5.31), is the correct solution (and we also have $\varepsilon^2(\rho) = 1$). In contrast, for all other cases where d is a multiple of 8 (such as $\bar{\mathbf{g}} = E_8$), the unique solution is given by $u = 1,\ v = 0$, i.e. only (5.29) survives the constraint (5.32). Thus in all cases except A_r with r even, the situation is the same as in the general case where d is not divisible by 8.

For odd d the use of theta functions is somewhat awkward, but it suffices to observe that the matrix

$$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{5.33}$$

commutes with the S-matrix

$$S_{\mathrm{so}(d)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} . \tag{5.34}$$

Written in terms of $\mathbf{g}$-characters, (5.33) becomes identical to (5.28), and the rest of the argument is as before.

In the notation of (5.15), the conformal embedding invariant (5.29) reads

$$(Z_{\text{c.e.}})_{\Lambda,\Lambda'} = 2^{r-1}\delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \varepsilon(\mu)=\varepsilon(\mu')=1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \varepsilon(\mu)=\varepsilon(\mu')=-1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} , \tag{5.35}$$

while (5.30) and (5.31) with $\varepsilon(\rho) = \pm 1$ can be summarized as

$$(Z'_{\text{c.e.}})_{\Lambda,\Lambda'} = (2^{r-1}+1)\delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \varepsilon(\mu)=\varepsilon(\mu')=1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \varepsilon(\mu)=\varepsilon(\mu')=-1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} . \tag{5.36}$$

(By inspection one easily verifies that these matrices commute with T, that the correct number $\dim(\mathrm{so}(d)) - \dim(\bar{\mathbf{g}}) = d(d-3)/2$ of spin one currents are combined with the identity field, and that the "quantum dimension" sum rule (5.32) is satisfied also for d not a multiple of 8.) Note that in the summations in (5.35) and (5.36) (and also in those for the branching rules (5.18) of $\mathcal{X}_o$ and $\mathcal{X}_v$) the weight $\mu = \rho$ does not contribute, except for A_r with even r, in which case it contributes to $\mathcal{X}_o$ (if $d \equiv r(r+2) = 0 \mod 16$) and to $\mathcal{X}_v$ (if $d = 8 \mod 16$), respectively.

Let us finally present some examples for the explicit form of the conformal embedding invariants. The most interesting cases are those with exceptional $\bar{\mathbf{g}}$. We will display the result for the algebras $\bar{\mathbf{g}} = F_4$ and $\bar{\mathbf{g}} = E_6$ (in the E_7 and E_8 cases the invariants require too much space, therefore they will be presented elsewhere [21]). The primary fields are again labeled by their unshifted highest weights. We find

$$\begin{aligned}\mathcal{Z}_{\text{c.e.}}(F_{4,9}) = &\,|\,(0,0,0,0)+(0,0,1,6)+(0,0,2,1)+(0,1,0,0)\\ &+(0,1,1,2)+(0,3,0,0)+(1,0,0,5)+(1,1,0,4)\,|^2\\ &+|\,(0,0,0,7)+(0,0,2,0)+(0,0,3,0)+(0,1,0,3)\\ &+(0,1,0,6)+(0,2,0,2)+(1,0,0,0)+(1,0,1,4)\,|^2\\ &+2\cdot\ |\,2(1,1,1,1)\,|^2\,,\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}&\mathcal{Z}_{\text{c.e.}}(E_{6,12})\\ &\quad=|\,(0,0,0,0,0,0)+(0,0,0,0,12,0)+(0,0,1,0,0,0)+(0,0,1,0,9,0)\\ &\quad+(0,0,2,0,3,0)+(0,1,0,0,5,2)+(0,1,0,2,1,0)+(0,2,0,0,1,0)\\ &\quad+(0,2,0,0,7,0)+(1,0,0,0,7,2)+(1,0,0,2,0,0)+(1,0,3,0,1,0)\\ &\quad+(1,1,1,0,3,1)+(1,1,1,1,1,0)+(1,2,0,0,5,1)+(1,2,0,1,0,0)\\ &\quad+(2,0,0,1,3,1)+(2,0,1,0,2,0)+(2,0,1,0,5,0)+(3,0,2,0,0,0)\\ &\quad+(3,0,2,0,3,0)+(3,0,1,1,1,1)+(3,1,0,0,2,1)+(3,1,0,1,3,0)\\ &\quad+(4,0,0,0,4,0)+(5,0,0,2,1,1)+(5,0,0,1,0,2)+(5,0,1,0,2,0)\\ &\quad+(7,0,0,2,0,0)+(7,0,0,0,1,2)+(9,0,1,0,0,0)+(12,0,0,0,0,0)|^2\\ &\quad+|\,(0,0,0,0,0,1)+(0,0,0,0,6,3)+(0,0,0,1,10,0)+(0,0,0,3,0,0)\\ &\quad+(0,0,4,0,0,0)+(0,1,0,0,8,1)+(0,1,0,1,0,0)+(0,1,2,0,2,0)\\ &\quad+(0,2,0,0,4,2)+(0,2,0,2,0,0)+(0,3,0,0,0,0)+(0,3,0,0,6,0)\\ &\quad+(1,0,1,0,4,1)+(1,0,1,1,2,0)+(1,1,0,0,6,1)+(1,1,0,1,1,0)\\ &\quad+(2,0,2,0,2,1)+(2,0,2,1,0,0)+(2,1,0,1,2,1)+(2,1,1,0,1,0)\\ &\quad+(2,1,1,0,4,0)+(3,0,0,0,3,1)+(3,0,0,1,4,0)+(4,0,0,2,0,2)\\ &\quad+(4,0,1,0,1,1)+(4,0,1,1,2,0)+(4,1,0,0,3,0)+(6,0,0,0,0,3)\\ &\quad+(6,0,0,1,1,1)+(6,0,0,3,0,0)+(8,0,0,1,0,1)+(10,1,0,0,0,0)|^2\\ &\quad+2\cdot\ |\,4(1,1,1,1,1,1)\,|^2\ .\end{aligned} \tag{5.38}$$

These results demonstrate the power of quasi-Galois symmetries quite convincingly.

Acknowledgement. It is a pleasure to thank T. Gannon and V.G. Kac for stimulating discussions. We thank one of the referees for suggesting a simplification of one of the proofs.

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Communicated by R.H. Dijkgraaf