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Arithmetic Properties of Mirror Map and Quantum Coupling

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Abstract: We study some arithmetic properties of the mirror maps and the quantum Yukawa couplings for some 1-parameter deformations of Calabi–Yau manifolds. First we use the Schwarzian differential equation, which we derived previously, to characterize the mirror map in each case. For algebraic K3 surfaces, we solve the equation in terms of the J-function. By deriving explicit modular relations we prove that some K3 mirror maps are algebraic over the genus zero function field $\mathbf{Q}(J)$. This leads to a uniform proof that those mirror maps have integral Fourier coefficients. Regarding the maps as Riemann mappings, we prove that they are genus zero functions. By virtue of the Conway–Norton conjecture (proved by Borcherds using Frenkel–Lepowsky–Meurman's Moonshine module), we find that these maps are actually the reciprocals of the Thompson series for certain conjugacy classes in the Griess–Fischer group. This also gives, as an immediate consequence, a second proof that those mirror maps are integral. We thus conjecture a surprising connection between K3 mirror maps and the Thompson series. For threefolds, we construct a formal nonlinear ODE for the quantum coupling reduced mod p. Under the mirror hypothesis and an integrality assumption, we derive mod p congruences for the Fourier coefficients. For the quintics, we deduce, (at least for $5\not| d$) that the degree d instanton numbers n_d are divisible by 5^3 – a fact first conjectured by Clemens.

1. Introduction

For background on Mirror Symmetry, the readers are referred to reference [1, 2] (see especially the articles therein by Greene–Plesser, Candelas-de la Ossa–Green–Parkes. Katz, Morrison, Vafa and Witten).

It is known that the so-called mirror map and the quantum coupling have many interesting number theoretic properties based on numerical experiments – as previously observed by many [1, 3, 4, 5]. For example the Fourier coefficients of the

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mirror map appear to be integral in all known cases. In some cases, the coefficients even appear to be alternating. The instanton numbers n_d in the quantum coupling on the other hand, apparently have some striking divisibility property. Clemens conjectured that for the quintics in $\mathbf{CP}^4$, $5^3|n_d$ for all d [6].[1] This was supported by Katz' proof that $5|n_d$, along with similar divisibility properties for other manifolds [7]. Our main motivation here is to develop some techniques, along with mirror symmetry, to understand some of these remarkable "arithmetic" properties.

The technique for studying the mirror map is based on the following simple idea: fix a known integral series $f(q)$. Study when is $z(q)$ commensurable with $f(q)$ (i.e. when do $z(q), f(q)$ satisfy a polynomial relation)? Hopefully when enough is known about $f(q)$, then give a polynomial $p(X, Y)$ we can understand some of the arithmetic properties of the root $z(q)$ to $p(f(q), Y) = 0$. Note that this is still a difficult Diophantine type problem in general, involving infinitely many variables consisting of the Fourier coefficients of $z(q)$.

However, a lot is known about the j-function both number theoretically and geometrically. Thus it is natural to try to find commensurability relations (also known as modular relations) between $j(q), z(q)$. We will see that this idea works well in the case of elliptic curves and K3 surfaces.

The technique for studying the instanton numbers is based on the fact that there is a canonical polynomial ODE for the quantum coupling, which is defined over $\mathbf{Q}$. This raises the possibility of deriving similar equations, but reduced mod p. The mod p arithmetic properties of the quantum couplings should then be reflected in these reduced equations. In [8, 9], in collaboration with A. Klemm and S.S. Roan we have constructed an ODE over $\mathbf{Q}$. However, its mod p counterpart derived here appears much simpler and more manageable. We now summarize our discussion.

First we discuss, along the lines of [4, 5], the construction of deformation coordinates based on which the mirror map is defined. Then we review the polynomial differential equations for the mirror maps, studied in [8, 9]. First we classify the analytic solutions, on a disk, to the n^{th} ($n = 2$, 3 or 4) Schwarzian equations associated to certain n^{th} order linear ODEs of Fuchsian type. As a consequence, the so-called mirror map z can be given a simple characterization. Corresponding to $n = 2, 3$ or 4, there is a universal family of polynomials whose evaluation at certain integral points recovers the Fourier coefficients of a mirror map.

We consider some examples in which the linear ODEs are the Picard–Fuchs equations of several distinguished families of smooth Calabi–Yau varieties in weighted projective spaces. We revisit the case of elliptic curves ($n = 2$).

In the case of $n = 3$, we consider some distinguished families of K3 surfaces in weighted projective spaces. We prove that the mirror map z for each of those families is algebraic over the function field $\mathbf{Q}(J)$ generated by the Dedekind–Klein J-functions – a rather surprising connection between modular functions and K3 surfaces. Our result on the $n = 3$ Schwarzian equation is an important tool in this connection. We then use our explicit modular relations to give a uniform proof that the Fourier coefficients of z are integral – a fact which has been previously observed experimentally. This is also the first confirmation of the integrality property in the case of K3 surfaces.

We then discuss a mysterious connection between our mirror maps and the Thompson series. We offer some speculation as to why the two might be related.

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[1] We thank S. Katz for pointing out some references to us.

We conjecture that any 1-parameter deformation of algebraic K3, determined by an orbifold construction, gives rise to a Thompson series.

In the last section under an integrality assumption and the Mirror Hypothesis, we study our differential equations reduced mod p. We derived some general mod p congruences and then specialize to $p = 2, 3, 5, 7$. For the quintic hypersurface, we deduce Clemens' conjecture that the degree d instanton numbers n_d are divisible by 5^3 (at least for $5 \not| d$).

2. Deformation Coordinates

The purpose here is to review the orbifold construction of the mirror map and to give an explicit description of the deformation coordinates for complex structures. The orbifold construction in weighted projective spaces has now been superceded by toric geometry construction [10, 4]. But in the former, the description of the deformation coordinates can be made rather explicit. Here we will adopt the multi-indexed convention for monomials $y^I = y_1^{i_1} \dots y_m^{i_m}$ whenever the meaning of the variables y is clear.

Let X be a $l := (n - k - 1)$-dimensional Calabi–Yau variety defined as the zero locus of k homogeneous polynomials $p_1, \dots, p_k$ whose degrees are $d_1, \dots, d_k$, in the weighted projective space $\mathbf{P}^{n-1}[w]$. Here $w = (w_1, \dots, w_n)$ are the weights consisting of coprime positive integers. Suppose that X has a mirror X^* given by an orbifold construction [11, 12, 1]. (Thus X^* is assumed to have the usual mirror Hodge diamond.) The space X^* is a resolution $\widehat{X/G}$ of the singular quotient X/G, where G is a finite abelian group acting on the homogeneous coordinates x_i of the ambient space $\mathbf{P}^{n-1}[w]$ by characters $\chi_i : G \to S^1$. They are assumed to satisfy $\chi_1 \dots \chi_n = 1$, which ensures that the holomorphic top form on X is G-invariant, and hence induces a form on X^*. We now want to describe some complex structure deformations of X^*. We do so by describing a family of G-invariant holomorphic top forms $\Omega(z)$ on X. Thus ultimately, our deformation space M will be parameterizing a family of G-invariant deformations of the p_j.

Fix m_j G-invariants monomials which we denote by x^{I_i}, $i \in N_j := \{(m_1 + \dots + m_{j-1}) + 1, \dots, (m_1 + \dots + m_{j-1}) + m_j\}$. Let $m := m_1 + \dots + m_k$. Let F be the following holomorphic family, fibered over $(\mathbf{C}^\times)^m$, of varieties. Its fiber at $a = (a_1, \dots, a_m)$ is defined as the zero locus of

$$\tilde{p}_j(a : x) = \sum_{i \in N_j} a_i x^{I_i} \,. \tag{2.1}$$

We assume that X is the fiber at some limiting value of a and that the generic fibers are homeomorphic to X. Now two fibers can be biholomorphic simply by coordinate transformation. Specifically for any $\alpha = (\alpha_1, \dots, \alpha_{n+k}) \in (\mathbf{C}^\times)^{n+k}$, the transformation $x_i \mapsto x_i/\alpha_i$, $a_i \mapsto a_i \alpha_1^{i_1} \dots \alpha_n^{i_n} \alpha_{n+j} \equiv a_i \alpha^{J_i}$, where $(i_1, \dots, i_n) = I_i$, transforms $\tilde{p}_j \mapsto \alpha_{n+j} \tilde{p}_j$. Call this transformation group H, and consider the quotient family $F/H \to (\mathbf{C}^\times)^m / H = M$. The base M will be our deformation space. The isotropy group of the H-action on $(\mathbf{C}^\times)^m$ is $H' = \{\alpha | \alpha^{J_i} = 1, \; all \; i\}$. The conditions $\alpha^{J_i} = 1$ mean that the dot products $J_i \cdot (\log \alpha_1, \dots, \log \alpha_{n+k}) = 0$. Or if K is the $m \times (n + k)$ *matrix of exponents* whose rows are the J_i, then $K \cdot (\log \alpha)^t = 0$. Thus $rk\ K = \dim H - \dim H'$, implying that $\dim M = m - rk\ K$.

We wish to construct some coordinates on M. Now H acts on the coordinate ring $\mathbf{C}[a_1^{\pm 1},\ldots,a_m^{\pm 1}]$ of $(\mathbf{C}^\times)^m$ by $a_i \mapsto a_i\alpha^{J_i}$. A simple way to coordinatize $M = (\mathbf{C}^\times)^m/H$ would be to find enough suitable H-invariant functions $f : (\mathbf{C}^\times)^m \to \mathbf{C}^\times$. An H-invariant Laurent monomial a^μ is one with $\mu_1 J_1 + \cdots + \mu_m J_m = 0$. The set of such μ is the lattice $L = (\ker K^t) \cap \mathbf{Z}^m$ which has rank $\dim(\ker K^t) = m - rk\, K^t = \dim M$. The subalgebra of H-invariants in $\mathbf{C}[a_1^{\pm 1},\ldots,a_m^{\pm 1}]$ is the group algebra $\mathbf{C}[L]$ which is canonically generated by $a^{\pm B}$, for a given basis $\{B\}$ of L. Thus for every basis $\{B\}$, we get a canonical set of functions $z = (a^B)$ globally defined on M. The z will be our deformation coordinates.

Since any two bases $\{B\}$, $\{B'\}$ are related by $B = \sum m_{BB'}B'$, where $(m_{BB'})$ is an integral matrix, $z = (a^B) \mapsto z' = (a^{B'})$ is a birational change. Note that if the z take the value of *integral* q-series, then so does the image z' under the above change. Thus when the z_i takes the value of the mirror map which has a q-series expansion, the question of integrality of the Fourier coefficients is independent of the choice of basis of L. In case $\dim M = 1$, which is all we are going to deal with here, the coordinate z above is unique up to $z \mapsto 1/z$. But demanding that $z = 0$ is the point with maximum unipotent monodromy [13] for the Picard–Fuchs equation, fixes the choice completely.

2.1. Some examples. Let's first consider the simplest example: cubics in $\mathbf{P}^2$. Let G be a cyclic groups of order 3 and act on the homogeneous coordinates $x_i \mapsto \xi^i x_i$, where $\xi = e^{2\pi i/3}$. Then the G-invariant cubic monomials are $x_1^3, x_2^3, x_3^3, x_1x_2x_3$. We consider the invariant family F which is the zero locus of the polynomial

$$\tilde{p}(a : x) = a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_1x_2x_3 \,. \tag{2.2}$$

The group $H = (\mathbf{C}^\times)^{3+1}$ acts on F as described above. On the base space $(\mathbf{C}^\times)^4$, it acts by $\alpha : a \mapsto (\alpha_1^3 a_1, \alpha_2^3 a_2, \alpha_3^3 a_3, \alpha_1\alpha_2\alpha_3 a_4)\alpha_4$. We see that the H-invariant functions are generated by $z = a_1a_2a_3a_4^{-3}$. It's easy to check that this function defines an isomorphism $(\mathbf{C}^\times)^4/H \to \mathbf{C}^\times$.

Consider now the sextics in $\mathbf{P}^3[1, 1, 2, 2]$. Let G be a finite abelian group of type $(6,3,3)$, which acts on the homogeneous coordinates by

$$x_1 \mapsto \xi_1\xi_2\xi_3 x_1, \qquad x_2 \mapsto \xi_1^{-1}x_2, \qquad x_3 \mapsto \xi_2^{-1}x_3, \qquad x_4 \mapsto \xi_3^{-1}x^4\,, \tag{2.3}$$

where ξ_1, ξ_2, ξ_3 are respectively arbitrary 6th, 3rd, 3rd roots of 1. The G-invariants monomials are $x_1^6, x_2^6, x_3^3, x_4^3, x_1x_2x_3x_4, x_1^3x_2^3$. As before we can write down a linear sum of them with coefficients $a_1, \ldots, a_6$. The matrix of exponents K defined above is the 6×5 matrix with rows: $(6,0,0,0,1)$, $(0,6,0,0,1)$, $(0,0,3,0,1)$, $(0,0,0,3,1)$, $(1,1,1,1,1)$, $(3,3,0,0,1)$. The lattice L therefore has a base $(1,1,0,0,0,-2)$, $(0,0,1,1,-3,1)$. This gives us the deformation coordinates $z = (z_1, z_2)$ with $z_1 = a_1a_2a_6^{-2}$, $z_2 = a_3a_4a_5^{-3}a_6$.

As a third example, we consider the sextics in $\mathbf{P}^3[1, 1, 1, 3]$. Let G be a finite abelian group of type $(6,6,2)$, which acts on the homogeneous coordinates by

$$x_1 \mapsto \xi_1\xi_2\xi_3 x_1, \qquad x_2 \mapsto \xi_1^{-1}x_2, \qquad x_3 \mapsto \xi_2^{-1}x_3, \qquad x_4 \mapsto \xi_3^{-1}x_4\,, \tag{2.4}$$

where ξ_1, ξ_2, ξ_3 are respectively arbitrary 6th, 6th, 2nd roots of 1. The G-invariants monomials are $x_1^6, x_2^6, x_3^6, x_4^2, x_1x_2x_3x_4, x_1^2x_2^2x_3^2$. As before we can write down a general linear sum of them to define our family of sextics. But note than any sum of the form $ax_4^2, +bx_1x_2x_3x_4 + cx_1^2x_2^2x_3^2$ can be written as $ax_4^2 + b'x_1x_2x_3x_4$, by a suitable redefinition $x_1 \mapsto x_1,\ x_2 \mapsto x_2,\ x_3 \mapsto x_3,\ x_4 \mapsto x_4 + \lambda x_1x_2x_3$. Thus we consider the family:

$$p(a:x) = a_1x_1^6 + a_2x_2^6 + a_3x_3^6 + a_4x_4^2 + a_5x_1x_2x_3x_4\,. \tag{2.5}$$

The matrix of exponents K is the 5×5 matrix with rows: $(6,0,0,0,1)$, $(0,6,0,0,1)$, $(0,0,3,0,1)$, $(0,0,0,3,1)$, $(1,1,1,1,1)$. The lattice L therefore has a base $(1,1,1,3,-6)$. This gives us the deformation coordinate $z = a_1a_2a_3a_4^3a_5^{-6}$.

2.2. Definition of the mirror map. Following [1,13,4], we now define the mirror map in terms of the coordinates z by studying variation of the periods for X^*. By construction, the holomorphic top form $\Omega(z)$ on X induces a form on X^*. Integrating the form against the G-invariant cycles, we obtain periods for X^*. By the Dwork–Griffith–Katz reduction method, we get a system of Picard–Fuchs differential equation(s) for those periods. If $z = 0$ is a maximal unipotent point, then the system admits a unique solution $\omega_0(z)$ which is holomorphic near $z = 0$ with $\omega_0(0) = 1$, and $s := \dim M$ independent solutions of the form $\omega_i(z) = \omega_0(z)\log z_i + g_i(z)$, where the g_i are holomorphic near $z = 0$ with $g_i(0) = 0$. We call the mapping defined by $q_j := e^{2\pi i t_j} = z_j e^{\frac{g_j(z)}{\omega_0(z)}}$ the mirror map. For convenience, we also refer to the inverse $z_j(q)$ as the mirror map. Thus by the construction above, the mirror map for an s-parameter family of Calabi–Yau mirror pairs can be regarded as collection of s q-series $z_j(q)$ which are determined by X, the holomorphic family of G-invariant deformations, and the choice of basis of a lattice.

3. Construction of the Schwarzian Equations

We now discuss the construction of the differential equation which governs the mirror map $z(t)$. We begin with an n^{th} order ODE of Fuchsian type:

$$Lf := \left(\frac{d^n}{dz^n} + \sum_{i=0}^{n-1} q_i(z)\frac{d^i}{dz^i}\right) f = 0 \tag{3.1}$$

(n will specialize to $2, 3$ and 4 later). In particular, the $q_i(z)$ are rational functions of z. Let f_1, f_2 be two linearly independent solutions of this equation and consider the ratio $t := f_2(z)/f_1(z)$. Inverting this relation (at least locally), we obtain z as a function of t. Our goal is to derive a polynomial ODE, in a canonical way, for $z(t)$.

We first perform a change of coordinates $z \to t$ on (3.1) and obtain:

$$\sum_{i=0}^{n} b_i(t)\frac{d^i}{dt^i} f(z(t)) = 0\,, \tag{3.2}$$

where the $b_i(t)$ are rational expressions of the derivatives $z^{(k)}$ (including $z(t)$). For example we have $b_n(t) = a_n(z(t))z'(t)^{-n}$. It is convenient to put the equation in *reduced* form. We do a change of variable $f = Ag$, where $A = \exp\left(-\int \frac{b_{n-1}(t)}{nb_n(t)}\right)$, and multiply (3.2) by $\frac{1}{Ab_n}$ so that it becomes

$$\tilde{L}g := \left(\frac{d^n}{dt^n} + \sum_{i=0}^{n-2} c_i(t)\frac{d^i}{dt^i}\right) g(z(t)) = 0\,, \tag{3.3}$$

where c_i is now a rational expression of $z(t), z'(t), \ldots, z^{(n-i+1)}$ for $i = 0, \ldots, n-2$. Now $g_1 := f_1/A$ and $g_2 := f_2/A = tg_1$ are both solutions to Eq. (3.3). In particular we have

$$P := \tilde{L}g_1 = \left(\frac{d^n}{dt^n} + \sum_{i=0}^{n-2} c_i(t)\frac{d^i}{dt^i}\right) g_1 = 0\,,$$

$$Q := \tilde{L}(tg_1) - t\tilde{L}g_1 = \left(n\frac{d^{n-1}}{dt^{n-1}} + \sum_{i=0}^{n-3} (i+1)c_{i+1}(t)\frac{d^i}{dt^i}\right) g_1 = 0\,. \tag{3.4}$$

Note that since c_i is a rational expression of $z(t)$, $z'(t), \ldots, z^{(n-i+1)}(t)$, it follows that P involves $z(t), \ldots, z^{(n+1)}(t)$ while Q involves only $z(t), \ldots, z^{(n)}(t)$. Equations (3.4) may be viewed as a coupled system of differential equations for $g_1(t), z(t)$. Our goal is to eliminate $g_1(t)$ so that we obtain an equation for just $z(t)$. One way to construct this is as follows. By (3.4), we have

$$\frac{d^i}{dt^i}P = 0, \quad i = 0, 1, \ldots, n-2\,,$$

$$\frac{d^j}{dt^j}Q = 0, \quad j = 0, 1, \ldots, n-1\,. \tag{3.5}$$

We now view (3.5) as a homogeneous *linear* system of equations:

$$\sum_{l=0}^{2n-2} M_{kl}(z(t), \ldots, z^{(2n-1)}(t))\,\frac{d^l}{dt^l}\,g_1 = 0, \quad k = 0, \ldots, 2n-2\,, \tag{3.6}$$

where (M_{kl}) is the following $(2n-1) \times (2n-1)$ matrix:

$$\begin{pmatrix}
c_0 & c_1 & \cdots & c_{n-2} & 0 & 1 & 0 & \cdots & 0 \\
c_0' & c_0 + c_1' & \cdots & c_{n-3} + c_{n-2}' & \cdot & 0 & 1 & 0 & 0 \\
 & \cdots & \cdots & \cdots & & & & \cdots & \\
c_0^{(n-2)} & (n-2)c_0^{(n-3)} + c_1^{(n-2)} & \cdots & \cdots & \cdot & \cdot & \cdot & 0 & 1 \\
c_1 & 2c_2 & \cdots & (n-2)c_{n-2} & n & 0 & 0 & \cdots & 0 \\
c_1' & c_1 + 2c_2' & \cdots & (n-3)c_{n-3} + (n-2)c_{n-2}' & 0 & n & 0 & \cdots & 0 \\
 & \cdots & \cdots & \cdots & & & \cdots & & \\
c_1^{(n-1)} & (n-1)c_1^{(n-2)} + 2c_2^{(n-1)} & \cdots & \cdots & \cdot & & \cdot & 0 & n
\end{pmatrix}. \tag{3.7}$$

More precisely, if we define the 1st and n^{th} (n fixed) row vectors to be $(M_{1l}) = (c_0, c_1, \ldots, c_{n-2}, 0, 1, 0, \ldots, 0)$ and $(M_{nl}) = (c_1, 2c_2, \ldots, (n-2)c_{n-2}, 0, n, 0, \ldots, 0)$ respectively, then the matrix (M_{kl}) is given by the recursion relation:

$$M_{k+1,l} = M_{k,l-1} + M_{k,l}', \quad l = 1, \ldots, 2n-1; \quad k = 1, \ldots, n-2, n, \ldots 2n-2\,. \tag{3.8}$$

Thus the (M_{kl}) depends rationally on $z(t), \ldots, z^{(2n-1)}(t)$. Since g_1 is nonzero, it follows that

$$\det(M_{kl}(z(t), \ldots, z^{(2n-1)}(t))) = 0\,. \tag{3.9}$$

We call this the n^{th} Schwarzian equation associated with (3.1). Note that by suitably clearing denominators, the equation becomes a $(2n-1)^{st}$ order polynomial ODE for $z(t)$ with constant coefficients. It is clear that this equation depends only on the data $q_i(z)$ we began with.

4. The Analytic Solutions to the Schwarzian Equation

It was already known to H.A. Schwarz that all the solutions to his equation ($n = 2$) can be constructed from the original 2[nd] order ODE. In this section, we wish to generalize his result to the other higher Schwarzian equations. We will focus on those which arise only from cases in which the data $q_i(z)$ come from the Picard–Fuchs equation of a smooth Calabi–Yau hypersurface (or complete intersections) in a weighted projective space. We will show that there is exactly 1-parameter family of single-valued analytic solutions, on a disk, to the Schwarzian equation. It is instructive to first go back to the $n = 2$ case (elliptic curves). While Schwarz's treatment in this case focuses on the solution ratios (triangle functions), it is less direct for studying their inverses. Instead we will study the inverses directly. This will have the advantages of a) exhausting all the analytic solutions directly; b) seeing manifestly that the solutions have polynomial dependence on the parameters; and c) generalizing immediately to higher Schwarzian equations.

4.1. $n = 2$ Schwarzian equation. We begin with a Fuchsian equation of the general form:

$$(\Theta^2 - \delta z(\Theta + a)(\Theta + b))f(z) = 0\,, \tag{4.1}$$

where the δ, a, b are constants with $\delta \neq 0$ (later we will restrict to rational numbers with $0 < a, b < 1$), and $\Theta = z\frac{d}{dz}$. The Picard–Fuchs equations for elliptic curves in a weighted projective space (for the general form of the Picard–Fuchs equation, see [4, 5]) is of this form where z is a suitable coordinate for the complex structure moduli space of the curves. The periods of the curves are given by the solutions to (4.1). This equation is projectively equivalent to a hypergeometric equation:

$$z(1 - z)y'' + (1 - (a + b + 1)z)y' - aby = 0\,, \tag{4.2}$$

i.e. there exists a function A and a Mobius transformation $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$ such that $f = Ay$ transforms (4.1) into (4.2). (Take $z \rightarrow z/\delta$.) The Schwarzian equation associated to (4.1) is

$$2Q(z(t))z'^2 + \{z, t\} = 0\,, \tag{4.3}$$

where $\{z, t\} = \frac{z'''}{z'} - \frac{3}{2}(\frac{z''}{z'})^2$ and

$$Q(z) = \frac{1 + 2(-1 + a + b - 2ab)\delta z + (-1 + a - b)(-1 - a + b)\delta^2 z^2}{4z^2(1 - \delta z)^2}\,. \tag{4.4}$$

The Schwarzian equation associated with (4.2) is (4.3) with $\delta = 1$. For convenience, we let $\delta = 1$ for now. To restore the generality, one simply replaces z by δz below. It was known to H.A. Schwarz that the every solution $z(t)$ to (4.3) is the inverse of a ratio (known as a Schwarz triangle function of type $s(0, b - a, 1 - a - b; z)$) of two hypergeometric functions which solve (4.2). In this note, we will only be interested in *solutions $z(q)$ to (4.3) which are analytic in some disk $|q| < R$, where $q = e^{2\pi i t}$.* It turns out that every analytic solution z must have either $z(q = 0) = 0$ or 1. We will only consider those with $z(0) = 0$. They can be obtained as follows. Equation (4.1) has a unique power series solution $f_1(z)$ with $f_1(0) = 1$, namely $f_1(z) = \sum_{n \geqq 0} c_n z^n$, where $c_n = \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(a)\Gamma(b)\Gamma(n+1)^2}$. There is also a unique solution of the form $f_2(z) = \log(z) f_1(z) + \sum_{n \geqq 1} d_n z^n$. Note that both the c_n, d_n depends polynomially on the parameters a, b. Let $t = \frac{f_2(z)}{2\pi i f_1(z)}$. Then

$$q = z\, \exp\left(\sum d_n z^n / \sum c_n z^n\right) . \tag{4.5}$$

This defines an invertible analytic map from a disk $|z| < S$ to some $|q| < R$ sending 0 to 0. The inverse map $z_f(q)$ therefore gives a particular analytic solution to (4.3), which we will call *the fundamental solution.* Note also the Fourier coefficients of $z_f(q)$ also depend polynomially on a, b. (Thus the above construction makes sense with no restriction at all on the values of a, b.) It is convenient to transform (4.3) by $q = e^{2\pi i t}$ so that the equation now has q as an independent variable and $z(q)$ dependent variable.

We will argue that $\{z_f(kq)\}_{k \in \mathbf{C}^\times}$ exhausts all analytic solutions $z(q)$ with $z(0) = 0$ and $\frac{dz(0)}{dq} \neq 0$, i.e. every such solution can be obtained from the fundamental one by scaling q. First note that scaling q by k corresponds to a translation $t \to t + \beta$ for some β. But we know that (4.3) is invariant under Mobius transformations on t. This shows that each $z_f(kq)$ is an analytic solution to (4.3). It remains to show that if $z(q) = \sum_{n \geqq 1} a_n q^n$ is an analytic solution to (4.3) with $a_1 \neq 0$, then the a_n are determined by a_1. Since (4.3) is 3$^{\text{rd}}$ order and is linear in $\frac{d^3 z}{dq^3}$, a_n for $n = 3, 4, \ldots$ are determined by a_1 and a_2. But a simple computation using (4.3) gives

$$a_2 = (2ab - a - b) a_1 . \tag{4.6}$$

As argued, the Fourier coefficients of $z_f(kq)$ are polynomials in a, b, k. The first few terms of $z_f(q)$ are

$$\begin{aligned} z_f(q) = {} & q + (-a - b + 2ab) q^2 \\ & - (a - 5a^2 + b - 11ab + 19a^2 b - 5b^2 + 19ab^2 - 19a^2 b^2) q^3/4 \\ & - (6a - 45a^2 + 93a^3 + 6b - 94ab + 408a^2 b - 515a^3 b - 45b^2 \\ & + 408ab^2 - 1116a^2 b^2 + 987a^3 b^2 + 93b^3 - 515ab^3 \\ & + 987a^2 b^3 - 658a^3 b^3) q^4/54 + \cdots . \end{aligned} \tag{4.7}$$

This is obtained by either inverting (4.5) or by directly solving (4.3).

4.2. $n = 3$ Schwarzian equation. It is now clear how to generalize the above. Consider the following 3$^{\text{rd}}$ order ODE of Fuchsian type:

$$\left(\Theta^3 - z\left(\sum_{i=0}^{3} r_i \Theta^i\right)\right) f(z) = 0 , \tag{4.8}$$

where the r_i are constants with $r_3 \neq 0$. The Picard–Fuchs equations for (a 1-parameter deformation of) smooth K3 hypersurfaces or complete intersections in weighted projective space is of this form (see [4, 5]) where the r_i are integer valued and satisfy some restriction given as follows. There is a "uniformizing" coordinate τ in which (4.8) becomes $\frac{d^3\tilde{f}}{d\tau^3}=0$. This implies the following three possibilities:

$$\begin{aligned}&\text{a) } r_0=r_1=r_2=0\,;\\&\text{b) } 3r_0=r_1=r_2=3r_3\,;\\&\text{c) } r_0=\frac{2r_1-r_3}{4}, \qquad r_2=\frac{3r_3}{2}\,.\end{aligned} \tag{4.9}$$

Cases a) and b) turn out to be projectively equivalent, i.e. they both result in the same reduced form for (4.8):

$$\left(\frac{d^3}{dz^3}+\frac{1}{4z^2}\frac{d}{dz}-\frac{1}{4z^3}\right)\tilde{f}(z)=0\,. \tag{4.10}$$

This case will not arise in our discussion below. Thus from now on we impose c) and hence (4.8) becomes, after simplification:

$$\left(\Theta^3-r_3z\left(\Theta+\frac{1}{2}\right)\left(\Theta^2+\Theta+\frac{r_1}{r_3}-\frac{1}{2}\right)\right)f(z)=0\,. \tag{4.11}$$

According to our general construction, the Schwarzian equation associated with (4.11) is then (see Eq. (3.9))

$$\det(M_{kl}):=27c_0^2+4c_1^3-18c_1c_0'-3c_1'^2+6c_1c''_1=0\,, \tag{4.12}$$

where

$$\begin{aligned}c_1(t)&:=q_1(z)z'^2+2\{z,t\}\,,\\c_0(t)&:=q_0(z)z'^3+q_1(z)z'z''+\frac{3z''^3}{z'^3}-\frac{4z''z^{(3)}}{z'^2}+\frac{z^{(4)}}{z'}\,,\\q_1(z)&:=\frac{4-4r_1z-2r_3z+4r_1r_3z^2+r_3^2z^2}{4z^2(-1+r_3z)^2}\,,\\q_0(z)&:=\frac{1}{2}\frac{dq_1(z)}{dz}\,.\end{aligned} \tag{4.13}$$

It is easy to check that $c_0(t)=\frac{1}{2}c_1'(t)$. This simplifies (4.12) to

$$15c_1'^2+16c_1^3-12c_1c_1''=0\,. \tag{4.14}$$

Note that this simplification is a direct consequence of imposing c) above. This $n=3$ Schwarzian equation will be useful for understanding the mirror map for K3 surfaces (see below).

We now construct all analytic solutions $z(q)$ to (4.14) with $z(0)=0$ and $\frac{dz(0)}{dq}\neq 0$. The situation here is completely analogous to the $n=2$ case. Equation (4.11) has a unique power series solution $f_1(z)=\sum_{n\geqq 0}c_nz^n$ with $f_1(0)=1$. There is also a unique solution of the form $f_2(z)=\log(z)f_1(z)+\sum_{n\geqq 1}d_nz^n$. The c_n, d_n are polynomials of the parameter r_1, r_3. If we let $t=\frac{f_2(z)}{2\pi i f_1(z)}$, then

$$q:=e^{2\pi it}=z\ \exp\left(\sum d_nz^n/\sum c_nz^n\right) \tag{4.15}$$

defines an invertible analytic map from a disk $|z|<S$ to some $|q|<R$. The inverse map $z_f(q)$ therefore gives a particular analytic solution to (4.14). By an argument

similar to the $n=2$ case, we conclude that $\{z_f(kq)\}_{k\in\mathbf{C}^\times}$ exhausts all such analytic solutions. The Fourier coefficients of the fundamental solution $z_f(q)$ can be computed using (4.14):

$$\begin{aligned} z_f(q) = q &+ (2r_1 - 3r_3)q^2/4 + (76r_1^2 - 212r_1r_3 + 135r_3^2)q^3/256 \\ &+ (2632r_1^3 - 10468r_1^2r_3 + 12862r_1r_3^2 - 5007r_3^3)q^4/13824 \\ &+ (1806544r_1^4 - 9219424r_1^3r_3 + 16526488r_1^2r_3^2 \\ &- 12589560r_1r_3^3 + 3479157r_3^4)q^5/14155776 + \cdots . \end{aligned} \tag{4.16}$$

The above result has an interesting consequence: *every analytic solution to* (4.14) *is a solution to*:

$$c_1 := q_1(z)z'^2 + 2\{z,t\} = 0 . \tag{4.17}$$

To prove this, note that (4.17) is an $n=2$ Schwarzian equation (4.3) with $Q(z) = \frac{1}{4}q_1(z)$. Associated to it are (in general) four 2nd order Fuchsian equations of the type (4.1) with parameters $\delta = r_3$ and a,b satisfying

$$\begin{aligned} 2ab - a - b &= \frac{r_1}{2r_3} - \frac{3}{4} , \\ (a-b)^2 &= -\frac{r_1}{r_3} + \frac{3}{4} . \end{aligned} \tag{4.18}$$

(Note that if (a,b) solve (4.18), so do (b,a) and $(1-a, 1-b)$.) By our previous result on the $n=2$ Schwarzian equation, (4.17) has a family of solutions $\{z_f(kq)\}_{k\in\mathbf{C}^\times}$. Obviously they are solutions to (4.14) as well. Thus by our result on the $n=3$ Schwarzian equation, this family exhausts the solutions to (4.14). Thus we have *effectively reduced the $n=3$ Schwarzian equation to the $n=2$ equation.*

4.3. $n=4$ Schwarzian equation. For completeness, we briefly discuss the $n=4$ case even though we will not be applying the result later. The situation here is quite similar to the $n=3$ case, except the last part above. Consider the following 4th order ODE:

$$(\Theta^4 - z(r_4\Theta^4 + 2r_4\Theta^3 + r_2\Theta^2 + (r_2 - r_4)\Theta + r_0))f(z) = 0 , \tag{4.19}$$

where the r_i are constants with $r_4 \neq 0$. The Picard–Fuchs equations for a certain 1-parameter family of Calabi–Yau threefolds in weighted projective space is of this form where the r_i are integers. According to our general construction, the Schwarzian equation associated with (4.11) is then (see Eq. (3.9))

$$\begin{aligned} \det(M_{kl}) := 16c_2^4c_0 &- 128c_2^2c_0^2 + 256c_0^3 + 4c_2^3c_2'^2 \\ &+ 240c_2c_0c_2'^2 - 15c_2'^4 - 144c_2^2c_2'c_0' - 448c_0c_2'c_0' \\ &+ 256c_2c_0'^2 - 8c_2^4c_2'' + 128c_0^2c_2'' - 48c_2c_2'^2c_2'' \\ &+ 48c_2'c_0'c_2'' + 12c_2^2c_2''^2 - 48c_0c_2''^2 + 32c_2^3c_0'' \\ &- 128c_2c_0c_0'' + 48c_2'^2c_0'' + 32c_2^2c_2'c_2^{(3)} + 64c_0c_2'c_2^{(3)} - 96c_2c_0'c_2^{(3)} \\ &+ 8c_2c_2^{(3)2} - 8c_2^3c_2^{(4)} + 32c_2c_0c_2^{(4)} - 12c_2'^2c_2^{(4)} = 0 . \end{aligned} \tag{4.20}$$

where

$$c_2(t) := q_2(z)z'^2 + 5\{z(t),t\}_2\,,$$

$$c_0(t) := q_0(z)z'^4 + \frac{3}{2}\frac{dq_2(z)}{dz}z'^2z'' - \frac{3}{4}q_2(z)z''^2 - \frac{135z''^4}{16z'^4}$$

$$+\frac{3}{2}q_2(z)z'z^{(3)} + \frac{75z''^2z^{(3)}}{4z'^3} - \frac{15z^{(3)2}}{4z'^2} - \frac{15z''z^{(4)}}{2z'^2} + \frac{3z^{(5)}}{2z'}\,,$$

$$q_2(z) := \frac{5 - 2zr_2 - 4zr_4 + 2z^2r_2r_4 + 2z^2r_4^2}{2z^2(-1+zr_4)^2}\,,$$

$$q_0(z) := (81 - 16zr_0 - 12zr_2 - 280zr_4 + 48z^2r_0r_4 + 44z^2r_2r_4 + 340z^2r_4^2$$
$$- 48z^3r_0r_4^2 - 64z^3r_2r_4^2 - 128z^3r_4^3 + 16z^4r_0r_4^3 + 32z^4r_2r_4^3$$
$$+ 32z^4r_4^4)/(16z^4(-1+zr_4)^4)\,. \tag{4.21}$$

Once again, the set of analytic solutions to (4.20) with $z(q=0)=0$ and $\frac{dz(0)}{dq} \neq 0$ is of the form $\{z_f(kq)\}_{k\in\mathbf{C}^\times}$, where $z_f(q)$ is the fundamental solution whose Fourier coefficients are polynomials in the data (r_4, r_2, r_0). The construction is almost identical to that for $n = 2, 3$. The first few coefficients $z_f(q)$ are given by:

$$z_f(q) = q + q^2(4r_0 - r_2 + r_4)$$
$$+ q^3(163r_0^2 - 84r_0r_2 + 11r_2^2 + 90r_0r_4 - 24r_2r_4 + 13r_4^2)/8$$
$$+ q^4(112798r_0^3 - 88840r_0^2r_2 + 23610r_0r_2^2 + 2115r_2^3$$
$$+ 98969r_0^2r_4 - 53280r_0r_2r_4 + 7245r_2^2r_4 + 29958r_0r_4^2$$
$$- 8253r_2r_4^2 + 3123r_4^3)/972$$
$$+ q^5(702344153r_0^4 - 747165436r_0^3r_2 + 300965204r_0{}^2r_2^2 - 54365280r_0r_2^3$$
$$+ 3713328r_2^4 + 853805796r_0^3r_4 - 694587576r_0^2r_2r_4 + 189938400r_0r_2^2r_4$$
$$- 17448192r_2^3r_4 + 400090708r_0^2r_4^2 - 220959840r_0r_2r_4^2 + 30726432r_2^2r_4^2$$
$$+ 85542240r_0r_4^3 - 24023808r_2r_4^3 + 7032240r_4^4)/995328$$
$$+ q^6(2322744173252r_0^5 - 3118439852795r_0^4r_2 + 1688073976516r_0^3r_2^2$$
$$- 460304004892r_0^2r_2^3 + 63195785760r_0r_2^4 - 3493177200r_2^5$$
$$+ 3628517760463r_0^4r_4 - 3959225703216r_0^3r_2r_4 + 1631456931956r_0^2r_2^2r_4$$
$$- 300755456640r_0r_2^3r_4 + 20919798000r_2^4r_4 + 2319777533708r_0^3r_4^2$$
$$- 1926736953572r_0^2r_2r_4^2 + 536717533440r_0r_2^2r_4^2 - 50124477600r_2^3r_4^2$$
$$+ 757958047180r_0^2r_4^3 - 425557676160r_0r_2r_4^3 + 60050628000r_2^2r_4^3$$
$$+ 126449580000r_0r_4^4 - 35961865200r_2r_4^4$$
$$+ 8609094000r_4^5)/518400000 + \cdots\,. \tag{4.22}$$

It is interesting to note that this series appears to be integral when evaluated at certain integral points. For example, if the data $(r_4, r_2, r_0) = (3125, 4375, 120)$ corresponding to the quintic threefold in $\mathbf{P}^4$, then (4.22) becomes:

$$z(q) = q - 770q^2 + 171525q^3 - 81623000q^4 - 35423171250q^5 - 54572818340154q^6 - \cdots . \tag{4.23}$$

This integrality phenomenon apparently continues to hold in other examples as has been previously observed.

At this point, one might wonder if z_f would satisfy an $n = 2$ Schwarzian equation (4.3) as in the previous case, i.e. is there a rational function $Q(z)$ similar to (4.4) such that (4.22) solves the equation (4.3)? The answer is no in general. In fact, we have checked that in many cases of a 1-parameter family of Calabi–Yau threefolds, such a $Q(z)$ doesn't exist (cf. [1], Sect. 2). However the mirror map z_f of such a threefold does satisfy an equation similar to (4.3), but with some "quantum" corrections, namely:

$$2Q(z)z'^2 + \{z, t\} = \frac{2}{5}y'' - \frac{1}{10}y'^2 , \tag{4.24}$$

where $Q(z) = \frac{1}{10}q_2(z)$ (Eq. (4.21)) and $y = \log K(t)$, K being the quantum Yukawa coupling (also written as $\partial^3 F$). Thus the right-hand side of (4.24) should be thought of as a "quantum" correction to the classical equation (4.3).

4.4. *Remarks*

1. In the last section, we have seen that the Schwarzian equations can essentially be solved in terms of certain hypergeometric functions. Moreover the analytic solution is essentially unique. In fact, it follows easily from the previous discussion that in each case $n = 2, 3$ or 4, the fundamental solution to the n^{th} Schwarzian equation is the unique analytic solution (single-valued) $z_f(q)$ with $z_f(0) = 0, \frac{dz_f(0)}{dq} = 1$. This is exactly the so-called mirror map. Thus *the mirror map can be characterized by means of an ODE and analyticity*. We also find that there is a similar characterization for the quantum coupling $K(t)$ in a number of examples.

2. Even though we have obtained only the singled-valued analytic solutions to the Schwarzian equations in the disk $|q| < R$, the results actually allow us to classify certain multi-valued solutions as well. For simplicity, let's focus on the $n = 2$ case. What we say here will hold true for $n = 3, 4$ as well. For example, given any positive constant α, one can easily classify the analytic solutions $z(q)$ to (4.3) such that $z(q) \sim kq^\alpha$ as $q \to 0$ for some constant k. First note that there is a 1-parameter family of such solutions, namely $z_f(kq^\alpha)$, where $z_f(q)$ is our fundamental solution. This follows immediately from Mobius invariance of (4.3). This family also exhausts all such solutions.

3. We studied the Schwarzian equations associated with Fuchsian equations of special type which arises as the Picard–Fuchs equations for certain Calabi–Yau manifolds. It is clear that one can study these Schwarzian equations without reference to special geometry, in which case, some of our assumptions can be weakened. For example, we have checked that the uniqueness of analytic solutions to these equations continues to hold true under much weaker assumptions on the form of the associated Fuchsian equations. (For instance, take any Fuchsian equation whose

indicial equation is maximally degenerate, i.e. whenever the differential operator takes the form $\Theta^n - zp(z, \Theta)$, where p is a polynomial of degree at most n in Θ.

4. We have shown that the mirror map in the case of K3 surfaces satisfies an $n = 2$ Schwarzian equation, as in the case of elliptic curves. This is the first hint that there might be some relationship between K3 surfaces and elliptic curves. It also says that the mirror map is nothing but the inverse of a Schwarz triangle function. As even more interesting hint comes from the following numerical experiment. If we consider a 1-parameter family of sextic hypersurfaces in the weighted projective space $\mathbf{P}^3[1, 1, 1, 3]$ and compute the Fourier series of the mirror map, we get

$$z(g) = q - 744q^2 + 356652q^3 - 140361152q^4 + 49336682190q^5 \\ - 16114625669088q^6 + \cdots . \tag{4.25}$$

The observant reader would have realized that the coefficient 744 is the constant term in the J-function times 1728. In fact if we compute $1/z(q)$, we get exactly the first 6 coefficients of the $1728J$. We will prove that $1728J = 1/z(q)$. This will be another confirmation that the mirror map for a Calabi–Yau variety is integral.

5. Transcendence of the Mirror Map over $Q(J)$

In this section analytic functions are assumed to be defined on some disk $|q| < R$. Let $\mathbf{Q}(f)$ be the field generated by the analytic functions $f(q)$ over the rationals $\mathbf{Q}$. Two analytic functions $f(q)$, $g(q)$ are called commensurable if there exists a nontrivial polynomial relation (with coefficients in $\mathbf{Q}$) $P(f(q), g(kq^\alpha)) = 0$ for some rational numbers k, α. The statement that $f(q)$ and $g(q)$ are commensurable is equivalent to the statement that $g(kq^\alpha)$ is algebraic over the field $\mathbf{Q}(f)$. Note also that commensurability is an equivalence relation.

We will be mainly concerned with the question of commensurability of an analytic function $f(q)$ with the J-function. Perhaps the simplest example of a function commensurable to J is the so-called elliptic modular function (see below) λ. It bears the following well-known modular relation to J:

$$J = \frac{4}{27} \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2} . \tag{5.1}$$

The function λ will show up again in the next section.

In this section, we will prove that the mirror map $z(q)$ for K3 surfaces modelled in various weighted projective spaces is commensurable with J. Before discussing K3 surfaces, it is useful to recall some known examples of mirror maps which are commensurable with J [14, 15] (see also [8] Sect. 3.1). For both the elliptic curves and the K3 surfaces, we will use the commensurability relation to prove that the mirror maps in those cases are integral.

5.1. Elliptic curves. In the following table, we have four realizations of a 1-parameter family of elliptic curves as hypersurfaces (or complete intersections) in weighted projective spaces $\mathbf{P}^2(1,1,1)$, $\mathbf{P}^2(1,1,2)$, $\mathbf{P}^2(1,2,3)$ and $\mathbf{P}^3(1,1,1,1)$.

	deformations	diff. operator	$1728J(z)$
1.	$x_1^3+x_2^3+x_3^3+z^{-1/3}x_1x_2x_3=0$	$\theta^2-3z(3\theta+2)(3\theta+1)$	$\frac{(1+216z)^3}{z(1-27z)^3}$
2.	$x_1^4+x_2^4+x_3^2+z^{-1/4}x_1x_2x_3=0$	$\theta^2-4z(4\theta+3)(4\theta+1)$	$\frac{(1+192z)^3}{z(1-64z)^2}$
3.	$x_1^6+x_2^3+x_3^2+z^{-1/6}x_1x_2x_3=0$	$\theta^2-12z(6\theta+5)(6\theta+1)$	$\frac{1}{z(1-432z)}$
4.	$x_1^2+x_2^2+z^{-1/4}x_3x_4=0$ $x_3^2+x_4^2+z^{-1/4}x_1x_2=0$	$\theta^2-4z(2\theta+1)^2$	$\frac{(1+224z+256z^2)^3}{z(-1+16z)^4}$

Thus in those four examples, the mirror maps z are algebraic over the field $\mathbf{Q}(J)$. The relations between z and J in some of these examples are derived in [8] by using the Weierstrass model for the elliptic curves. We will instead illustrate the proof using a slightly different approach which will prove useful in the case of K3 surfaces. Namely, we will effectively use the results on the uniqueness of analytic solutions to the associated $n=2$ Schwarzian equations. We will consider Example 4 because it has some interesting connection with elliptic functions and it hasn't been treated in [8]. Our goal is to prove the relation between z and J in Example 4 above. (We also have similar proofs for all the other cases above.)

Recall that $J(t):=\frac{g_2^3(t)}{\Delta(t)}$, where $\Delta(t)$ is the discriminant of the Weierstrass elliptic curve W with period ratio t:

$$W: y^2=4x^3-g_2(t)x-g_3(t)\,. \tag{5.2}$$

The periods Ω of W are solutions to the Picard–Fuchs equation:

$$\frac{d^2\Omega}{dJ^2}+\frac{1}{J}\frac{d\Omega}{dJ}+\frac{31J-4}{144J^2(1-J)^2}\Omega=0\,. \tag{5.3}$$

Thus J is a solution to the associated Schwarzian equation (4.3) with:

$$Q(z):=\frac{32-41z+36z^2}{144z^2(-1+z)^2}\,. \tag{5.4}$$

The function $J(q)$ $(q:=e^{2\pi it})$ can be characterized as the unique meromorphic solution to the Schwarzian equation on a disk $|q|<R$ with the leading behavior $J(q)\sim\frac{1}{1728q}$ near 0. It is more convenient to work with $z_W(q):=\frac{1}{1728J(q)}$ which satisfies (4.3) with:

$$Q(z)=Q_W(z):=\frac{1-1968z+2654208z^2}{4z^2(-1+1728z)^2} \tag{5.5}$$

which is in the standard form (4.4). Since z_W has the leading behavior $z_W(q)\sim q$, it follows that (Sect. 4.1) z_W is the fundamental solution to (4.3) with (5.5).

Now consider the mirror map z_X for Example 4 above. This is the fundamental solution to the Schwarzian equation (4.3) with:

$$Q(z)=Q_X(z):=\frac{1-16z+256z^2}{4z^2(-1+16z)^2}\,. \tag{5.6}$$

We want to prove that

$$z_W = \frac{z_X(-1+16z_X)^4}{(1+224z_X+256z_X^2)^3}\,. \tag{5.7}$$

Note that the right-hand side, which we will call ζ, has the correct leading behavior. By substituting ζ into (4.3) with (5.5) and use the fact that z_X satisfies (4.3) with (5.6), we find that indeed (4.3) holds. By uniqueness of analytic solution to the Schwarzian equation we conclude that $\zeta = z_W$. We remark that one can also prove (5.7), simply by applying the variable change $z \to \frac{z(-1+16z)^4}{(1+224z+256z^2)^3}$ to the (reduced) Fuchsian equation $f''(z) + Q_W(z)f(z) = 0$ and see that you get the new reduced Fuchsian equation $\tilde{f}''(z) + Q_X(z)\tilde{f}(z) = 0$. It follows that their respective associated Schwarzian equations must transform from one to another under the same variable change.

5.2. Integrality of z_X and the elliptic modular function λ. There is an amusing connection which we should point out between z_X and λ. The latter is defined to be the inverse function of the Schwarz triangle function [16]:

$$t = S(0,0,0;z) := \frac{\sqrt{-1}F(\frac{1}{2},\frac{1}{2},1;1-z)}{F(\frac{1}{2},\frac{1}{2},1;z)}\,, \tag{5.8}$$

where the F are hypergeometric functions with $(a,b,c) = (\frac{1}{2},\frac{1}{2},1)$. As a result we can also characterize λ as the unique analytic solution, with the leading behavior $\lambda(q) \sim 16q^{\frac{1}{2}}$, to the Schwarzian equation (4.3) (associated to (4.2)) with:

$$Q(z) = Q_E(z) := \frac{1-z+z^2}{4z^2(-1+z)^2}\,. \tag{5.9}$$

The function λ has a q-series expansion:

$$\lambda(q) = 16q^{1/2} - 128q + 704q^{3/2} - 3072q^2 + 11488q^{5/2} - 38400q^3 + 117632q^{7/2} - \cdots\,. \tag{5.10}$$

From its expression in terms of theta functions, it is known that λ has integral Fourier coefficients divisible by 16. It is also well-known that λ defines a modular function of level 2 on the upper half-plane and it is related to J by:

$$J = \frac{4}{27}\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}\,. \tag{5.11}$$

(This relation can also be proved using the approach we used to prove (5.7)). Since commensurability is an equivalence relation, it follows from (5.7) and (5.11) that $z_X(q), \lambda(q^2)$ are commensurable also. In fact, using the method above, one can easily prove that

$$\lambda(q^2) = 16z_X(q)\,. \tag{5.12}$$

Since the Fourier coefficients on the left-hand side are integers divisible by 16, it follows that z_X also has integral Fourier coefficients. We should also point out that combining the relations (5.7), (5.11) and (5.12), we get a rather peculiar identity for λ:

$$\frac{16\lambda(q^2)(-1+\lambda(q^2))^4}{(1+14\lambda(q^2)+\lambda(q^2)^2)^3} = \frac{\lambda(q)^2(1-\lambda(q))^2}{(1-\lambda(q)+\lambda(q)^2)^3}\,. \tag{5.13}$$

It also says that you can write J in terms of λ in two different ways.

5.3. Remarks

1. The above approach for studying the commensurability of mirror maps also applies easily to Examples 1–3 in the previous section.

2. In the last example we studied, it is less clear how to deduce, from the relation (5.7), the integrality of z_X from the known integrality property of z_W. But upon relating z_X to λ via (5.12), this property becomes immediately clear. The upshot of this is that using the arithmetic properties of any particular one mirror map, such as z_W, alone may only give partial information about mirror maps commensurable to it. One should instead use other series, such as λ, in the same commensurability class to help obtain further information about other members in the same class. The lesson is that the larger the commensurability class, the more arithmetical information we can get about its members because every pair of members are related by some modular relations.

3. The key steps we used repeatedly above to prove commensurability of two fundamental solutions are:

a) identify the appropriate modular relation;

b) relate their corresponding Schwarzian equations (or equivalently their reduced Fuchsian equations) by a change of variables using the modular relation;

c) use analyticity (asymptotic behavior) and uniqueness of solutions to the Schwarzian equation.

This idea will continue to work well, as we will see, in the case of fundamental solutions arising from the Picard–Fuchs equations for K3 surfaces.

4. Finally, we note that in e.g. 3 of elliptic curves above, we have the relation, for $\operatorname{Im} t \gg 0$,

$$w(t) = \frac{j(t) + (j(t)(j(t)-1728))^{1/2}}{2}\,, \tag{5.14}$$

where $w = 1/z$, $j = 1728J$. We claim that $w(t)$ admits a double-valued analytic continuation in the upper half plane $\operatorname{Im} t > 0$. This implies in particular that the Fourier series of the mirror map $z = 1/w$ has radius of convergence strictly less than 1 – a fact that is not obvious from the construction of z. If we denote by $z(t)$, $\tilde{z}(t)$ the two branches of z, we see that $z(t) + \tilde{z}(t) = \frac{1}{432}$. Thus the two branches differ by an affine transformation $x \to -x + \frac{1}{432}$.

To prove our claim, let's recall some properties of the modular function $j(t)$. We know that $j(t)$ is a single-valued function in $\operatorname{Im} t > 0$. Thus (5.14) implies that $w(t)$ admits an analytic continuation which is *at most* double-valued. Also for $\rho := \frac{1}{2} + \frac{i\sqrt{3}}{2}$ we have $j(\rho) = 0$. In a small punctured disk centered at ρ, $(j(t) - 1728)^{1/2}$ is single-valued. If we move around a small loop enclosing ρ in that disk, $j(t)$ will move around a small loop (3 times) enclosing 0 in the j-plane. It follows that $(j(t)(j(t)-1728))^{1/2}$ is necessarily double-valued in that disk, implying that $w(t)$ is *at least* double-valued.

5.4. K3 surfaces: $X_z : x_1^6 + x_2^6 + x_3^6 + x_4^2 + z^{-1/6} x_1 x_2 x_3 x_4 = 0$. We now apply what we learned in Sect. 4.2 to this family of K3 surfaces. The Picard–Fuchs equation for the above 1-parameter family of K3 hypersurfaces in $\mathbf{P}^3[1,1,1,3]$ is given by:

$$(\Theta^3 - 8z(6\Theta + 1)(6\Theta + 3)(6\Theta + 5))f(z) = 0, \tag{5.15}$$

which is of the form (4.11) with $(r_3, r_1) = (1728, 1104)$. By the results in Sect. 4.2, the analytic solutions of the associated $n = 3$ Schwarzian equation (4.14) can be obtained from the $n = 2$ Schwarzian equation (4.3) with:

$$Q(z) = Q_X(z) := \frac{1 - 1968z + 2654208z^2}{4z^2(-1 + 1728z)^2}. \tag{5.16}$$

In particular the mirror map z_X for our K3 surfaces X which is the fundamental solution to (4.14), is now the fundamental solution to (4.3) with (5.16). But observe that $Q_X(z)$ is identical to $Q_W(z)$ given in (5.5). It follows that our mirror map is given by:

$$z_X = z_W := \frac{1}{1728J}. \tag{5.17}$$

This also proves, in particular, that z_X also has an integral Fourier coefficients, and that the mirror map z_X is commensurable to J.

5.5. Other deformations of K3 surfaces. We now consider three other families of K3 surfaces: quartic hypersurfaces $X_{(4)}$ in $\mathbf{P}^3$, the complete intersections of quadrics and cubics $X_{(2,3)}$ in $\mathbf{P}^4$ and the complete intersections of 3 quadrics $X_{(2,2,2)}$ in $\mathbf{P}^5$. The following table lists the types of abelian group G for the orbifold constructions, the 1-parameter deformations, the data (4.4) for the corresponding $n = 2$ Schwarzian equation (4.3), (4.17). We also include the previous example in $\mathbf{P}^3$ $[1, 1, 1, 3]$, which we denote $X_{(6)}$.

X, G	deformations	$Q(z)$ in (4.3)
$X_{(6)}, (6, 6, 2)$	$x_1^6 + x_2^6 + x_3^6 + x_4^2 + z^{-1/6} x_1 x_2 x_3 x_4 = 0$	$\frac{1 - 1968z + 2654208z^2}{4z^2(-1 + 1728z)^2}$
$X_{(4)}, (4, 4)$	$x_1^4 + x_2^4 + x_3^4 + x_4^4 + z^{-1/4} x_1 x_2 x_3 x_4 = 0$	$\frac{1 - 304z + 61440z^2}{4z^2(-1 + 256z)^2}$
$X_{(2,3)}, (2, 2, 3)$	$x_1^2 + x_2^2 + x_3^2 + z^{-1/5} x_4 x_5 = 0$	$\frac{1 - 132z + 11340z^2}{4z^2(-1 + 108z)^2}$
	$x_4^3 + x_5^3 + z^{-1/5} x_1 x_2 x_3 = 0$	
$X_{(2,2,2)}, (2, 2, 2)$	$x_1^2 + x_2^2 + z^{-1/6} x_3 x_4 = 0$	$\frac{1 - 80z + 4096z^2}{4z^2(-1 + 64z)^2}$
	$x_3^2 + x_4^2 + z^{-1/6} x_5 x_6 = 0$	
	$x_5^2 + x_6^2 + z^{-1/6} x_1 x_2 = 0$	

The respective values of (r_3, r_1) in Eq. (4.11) are (1728, 1104), (256, 176), (108, 78), (64, 48). The corresponding fundamental solutions in the 4 cases have Fourier series:

$$\begin{aligned} z_{X_{(6)}}(q) &= q - 744q^2 + 356652q^3 - 140361152q^4 + 49336682190q^5 \\ &\quad - 16114625669088q^6 + \cdots , \\ z_{X_{(4)}}(q) &= q - 104q^2 + 6444q^3 - 311744q^4 + 13018830q^5 - 493025760q^6 + \cdots , \\ z_{X_{(2,3)}}(q) &= q - 42q^2 + 981q^3 - 16988q^4 + 244230q^5 - 3089394q^6 + \cdots , \\ z_{X_{(2,2,2)}}(q) &= q - 24q^2 + 300q^3 - 2624q^4 + 18126q^5 - 105504q^6 + \cdots . \end{aligned} \tag{5.18}$$

We claim that, as for $z_{X_{(6)}}$, all the other mirror maps are commensurable with the J-function (or equivalently with z_W). By numerical experiment, we identified the following modular relations:

$$\begin{aligned} P_{X_{(4)}}(z_W, z_{X_{(4)}}) &:= - z_W^2 + z_W z_{X_{(4)}} - 432 z_W^2 z_{X_{(4)}} - 207 z_W z_{X_{(4)}}^2 - 62208 z_W^2 z_{X_{(4)}}^2 , \\ &\quad - z_{X_{(4)}}^3 + 3456 z_W z_{X_{(4)}}^3 - 2985984 z_W^2 z_{X_{(4)}}^3 = 0 , \\ P_{X_{(2,3)}}(z_W, z_{X_{(2,3)}}) &:= z_W^2 - z_W z_{X_{(2,3)}} + 576 z_W^2 z_{X_{(2,3)}} + 126 z_W z_{X_{(2,3)}}^2 \\ &\quad + 110592 z_W^2 z_{X_{(2,3)}}^2 - 2944 z_W z_{X_{(2,3)}}^3 \\ &\quad + 7077888 z_W^2 z_{X_{(2,3)}}^3 + z_{X_{(2,3)}}^4 = 0 , \\ P_{X_{(2,2,2)}}(z_W, z_{X_{(2,2,2)}}) &:= z_W^2 - z_W z_{X_{(2,2,2)}} + 624 z_W^2 z_{X_{(2,2,2)}} + 96 z_W z_{X_{(2,2,2)}}^2 \\ &\quad + 129840 z_W^2 z_{X_{(2,2,2)}}^2 - 2352 z_W z_{X_{(2,2,2)}}^3 + 9018880 z_W^2 z_{X_{(2,2,2)}}^3 \\ &\quad + 10495 z_W z_{X_{(2,2,2)}}^4 + 2077440 z_W^2 z_{X_{(2,2,2)}}^4 + z_{X_{(2,2,2)}}^5 \\ &\quad - 1488 z_W z_{X_{(2,2,2)}}^5 + 159744 z_W^2 z_{X_{(2,2,2)}}^5 + 4096 z_W^2 z_{X_{(2,2,2)}}^6 = 0 . \end{aligned} \tag{5.19}$$

We now proceed to prove the first of the relations (5.19). The other cases are similar. By solving $P_{X_{(4)}}(w, z_{X_{(4)}}(q)) = 0$ for w, we see that there is a branch of w which admits a q-series expansion near $q = 0$ and which has the leading behavior $w(q) \sim q$. Now it is enough to show that every branch of w solves the Schwarzian equation (4.3) with (5.5). For then $w(q)$ above must coincide with the fundamental solution, by uniqueness. Now applying the relation $P_{X_{(4)}}(w(q), z_{X_{(4)}}(q)) = 0$, we can compute $w'(q), w''(q), w'''(q)$ in terms of $w(q), z_{X_{(4)}}(q)$ and its derivatives. Substituting these expressions into (4.3) and using the fact that $z_{X_{(4)}}(q)$ satisfies its Schwarzian equation, we find that (4.3) with (5.5) holds identically.

6. Uniform Proof of Integrality

Applying the modular relations derived above between the mirror maps and the j-function, we will now prove that the Fourier coefficients of the mirror maps are integral. We begin with the following lemma. Let $z_0(q)$ be an integral q-series and

$$P(x,y) = \sum a_{i,j} x^i y^j \tag{6.1}$$

be a nonzero polynomial with integral coefficients $a_{i,j}$. Let $z(q) = \sum_{n \geqq 1} c_n q^n$ with $c_1 = 1$. Let $l := \min\{i + j | a_{i,j} \neq o\}$. Now suppose that

$$m := \sum_{i \geqq 0} a_{l-i,i} i \neq 0\,. \tag{6.2}$$

We claim that the relation $P(z_0(q), z(q)) = 0$ determines $z(q)$ uniquely and that the coefficients c_n are in $\mathbf{Z}[\frac{1}{m}]$.

Consider the coefficient K_N of q^N, for $N > l$, in the series $\mathbf{P}(z_0(q), z(q))$. Since $z_0(q)^j = q^j + O(q^{j+1})$ for $j \geqq 0$, the contribution to K_N from the term $a_{p-i,i} z_0(q)^{p-i} z(q)^i (p \geqq 1)$ must be an integral linear combination of monomials of the form $c_{n_1} \ldots c_{n_i}$, with $n_1 + \ldots + n_i \leqq N - (p-i)$. From this inequality, we find that $n_k \leqq N - p + i - (n_1 + \cdots n_i - n_k) \leqq N - p + 1$ for $k = 1, \ldots, i$. Thus the contribution to K_N from the term $a_{p-i,i} z_0(q)^{p-i} z(q)^i$ lives in $\mathbf{Z}[c_1, \ldots, c_{N-p+1}]$. In particular, for $p > l$ this contribution is in $\mathbf{Z}[c_1, \ldots, c_{N-l}]$.

Consider the cast $p = l$. It is easy to see that the order q^N term in $a_{l-i,i}$ $z_0(q)^{l-i} z(q)^i$ takes the form

$$a_{l-i,i} q^{l-i} i q^{i-1} c_{N-l+1} q^{N-l+1} + g q^N\,, \tag{6.3}$$

where $g \in \mathbf{Z}[c_1, \ldots, c_{N-l}]$. Setting $c_1 = 1$, we see that

$$K_N = m c_{N-l+1} + h \tag{6.4}$$

for some $h \in \mathbf{Z}[c_1, \ldots, c_{N-l}]$. By induction, the relations $K_N = 0$ imply that $c_{N-l+1} \in \mathbf{Z}[\frac{1}{m}]$.

It follows immediately from the above lemma that if $m = \pm 1$, then $z(q)$ is an integral q-series.

Now consider the polynomial relation between $z_W(q)$ and the mirror map $z(q)$ for each case of elliptic curves (see the table in Sect. 5.1). For each of the polynomials, we have $l = 1$, $m = 1$. It follows that $z(q)$ is integral in each case. Now for the four cases of K3 surfaces, we found the relations (5.19). For these polynomials, we have $l = 2$, $m = 1$. It follows again that $z(q)$ is integral in each case.

We should remark that the above lemma gives us an effective way to construct many integral q-series $f(q)$ which are commensurable with $1/1728J(q)$. For example let $r(x,y)$ be any integral polynomial with no constant or linear terms. Then our lemma implies that the equation

$$\frac{1}{1728J(q)} - z(q) + r\left(\frac{1}{1728J(q)}, z(q)\right) = 0 \tag{6.5}$$

determines a unique integral q-series $z(q)$. This also implies that $z(q)$ admits an analytic continuation (with at most finitely many sheets) to the upper half plane $\operatorname{Im} t > 0$. Moreover, since $J(q)$ satisfy the Schwarzian equation (4.3) with rational $Q(z), z(q)$ also satisfies a similar equation with algebraic $Q(z)$.

6.1. Relations with genus zero functions and the Thompson series. In this section, we point out some tantalizing evidence that the mirror maps for K3 surfaces are possibly related to the theory of the Griess–Fischer group. We will in fact give an alternative proof of integrality for the mirror maps. This proof, though more technical than the previous one, suggests some deeper connection with the theory of modular functions and the Thompson series.

Let's recall a few facts about genus zero functions and the Thompson series (see [17]). Let H be the upper half plane, G be a discrete subgroup of $PSL(2,\mathbf{R})$ which acts on H by linear fractional transformations. We consider meromorphic functions f on H which has a Fourier expansion $\sum_{n>m} a_n q^n$ for $\mathrm{Im}\, t \gg 0$ $(q = e^{2\pi i t})$.

If f is invariant under G, then f defines a function on the quotient H/G. If H/G is a genus zero Riemann surface with finitely many punctures, we call G a genus zero group and f a genus zero function. In this case, the field of genus zero functions for G has a canonical generator h known as the normalized hauptmodul for G. It has a Fourier expansion of the form $h(t) = \frac{1}{q} + O(q)$. For example $\Gamma_0(1) := PSL(2,\mathbf{Z})$ is a genus zero group and the Dedekind–Klein j-function $j(t) - 744$ is the hauptmodul for this group. More generally, a list of 174 hauptmoduls corresponding to certain congruent subgroups is known [18]. Let $\Gamma_0(N)$ be the group consisting of integral transformations $t \to (at+b)/(ct+d)$ with $N|c$. Then the 174 genus zero groups are subgroups of $PSL(2,\mathbf{R})$ generated by $\Gamma_0(N)$ together with a set of Atkin–Lehner involutions (see [19]).

It turns out that there is a deep connection between the theory of modular functions and the Griess–Fischer group M also known as the Monster. It is the largest sporadic simple finite group, whose order is:

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71\,. \tag{6.6}$$

For a long and glorious history of this group, see the paper of Conway–Norton [17]. Based on some remarkable observations of MacKay and Thompson, Conway–Norton conjectured that there exists a natural $\mathbf{Z}$-graded representation V of M with the following property: for every $g \in M$, the q-series, called the Thompson series of g,

$$T_g(q) := \sum \mathrm{tr}(g|V_n)q^{n-1} \tag{6.7}$$

is a normalized hauptmodul in the list of 174 mentioned above. It turns out that all but 3 of those 174 hauptmoduls correspond to Thompson series. At the time the conjecture was made, neither M nor V had been known to exist, though the evidence for them was overwhelming. On some hypotheses, the first few coefficients of the Thompson series were computed in [17] and were seen to coincide with the coefficients of the appropriate hauptmoduls (see Table 4 in [17]). Later, Griess proved that M exists by constructing a 196883-dimensional algebra B in which M acts by automorphisms. Subsequently, Frenkel–Lepowsky–Meurman [20] constructed a $\mathbf{Z}$-graded representation $V^\natural$ of M with the property that $V_2^\natural$ is the direct sum of B with the 1-dimensional representation of M and that $\sum \mathrm{tr}(1|_{V_n^\natural})q^{n-1} = j(q) - 744$. Borcherds [21] finally completed the proof of the Conway–Norton conjecture by showing that $V^\natural$ does indeed have the property that the corresponding Thompson series coincide with the hauptmoduls suggested in [17].

Our brief account of the Monster by no means does justice to the many remarkable developments surrounding the theory of the Monster. The above paragraph is meant to provide just enough background to enter a discussion on the relation, which we are about to show, between the Thompson series and our mirror maps. Based on numerical experiments, we have

Observation. *The reciprocal of the four mirror maps* (5.18) *agree respectively, up to an additive constant, with the Thompson series* T_{1A}, T_{2A}, T_{3A}, T_{4A} *(see Table* 4 *of* [17]).

Before we give a proof of this assertion, we argue that together with the Conway–Norton conjecture, the above assertion implies the integrality of mirror maps (5.18). The character values of a finite group are algebraic integers. But since $V^\natural$ is defined over the rationals, the coefficients of the Thompson series which are character values of M, must be rational integers. Note also that this property can be deduced from the explicit representation of the hauptmoduls in terms of the Dedekind eta function.

To prove our assertion above, we will use Schwarz' theorem on the Riemann mapping and the fact that the Thompson series above are hauptmoduls for the following genus zero groups $\Gamma_0(1)$, $\Gamma_0(2)+$, $\Gamma_0(3)+$, $\Gamma_0(4)+$ (see [17] on notations). If G is a genus zero group and $h(t)$ its normalized hauptmodul, then it is easy to check that the expression $\{h(t),t\}/(2h'(t)^2)$ is a meromorphic function which is invariant under G. Since $h(t)$ is a generator of the function field for G, it follows that $\{h(t),t\}/(2h'(t)^2)$ is a rational expression Q in $h(t)$. This shows that a hauptmodul satisfies a Schwarzian equation of the type (4.3). Our problem is to construct Q in each of the cases of interest. Once we determine the Schwarzian equations governing the hauptmoduls T_{1A}, T_{2A}, T_{3A}, T_{4A} in each case, it is enough to see that they coincide with the corresponding Schwarzian equation whcih we already know governing $\frac{1}{z}+c$, where z is one of the $z_{X_{(6)}}, z_{X_{(4)}}, z_{X_{(2,3)}}, z_{X_{(2,2,2)}}$. For if two q-series satisfy the same Schwarzian equation and has the same asymptotic behavior $1/q$, then they must coincide by uniqueness. The constant c above will turn out to be the coefficient of q^2 in each case (so that $\frac{1}{z}+c$ has no constant term).

We now proceed to determine Q for each of the four hauptmoduls $h(t)$ above. Since $h(t)$ is an isomorphism from H/G onto a punctured Riemann sphere, it defines a univalent mapping from a simply connected region R (half a fundamental region of G) onto the upper half h-plane. If R is a polygon with circular edges, then Schwarz' Theorem tells us that $h(t)$ satisfies Eq. (4.3) with Q having the form

$$Q(z)=\sum_{i=1}^{n}\left(\frac{A_i}{(z-a_i)^2}+\frac{B_i}{z-a_i}\right), \tag{6.8}$$

where $a_1,\dots,a_n$ are images of the vertices of R and the A_i, B_i are constants. To determine Q completely it is enough to know the number n of vertices (actually a reasonable bound would suffice). For then we can use the first few Fourier coefficients of $h(t)$ to determine the A_i, B_i. To find the number of vertices, we need to determine a domain R and see that it is a circular polygon in each case. But since we know G explicitly as a congruent subgroup in each case, constructing R is easy. We find that the R corresponding to the hauptmodul $T_{kA}(k=1,2,3,4)$ is $R=\{t\in H\,|\,0<\operatorname{Im} t<\frac{1}{2},|t|>\frac{1}{\sqrt{k}}\}$. Each of these domains is a circular polygon with 3 vertices. Using the first 6 Fourier coefficients (actually 3 is enough because the A_i, B_i satisfies 3 relations) of the T_{kA}, we find that the respective Q for their Schwarzian equations are:

$$T_{1A} : Q(z) = \frac{19896824 + 53527z + 36z^2}{144(743 + z)^2(744 + z)^2} ,$$

$$T_{2A} : Q(z) = \frac{40640 - 96z + z^2}{4(-152 + z)^2(104 + z)^2} ,$$

$$T_{3A} : A(z) = \frac{7560 - 48z + z^2}{4(-66 + z)^2(42 + z)^2} ,$$

$$T_{4A} : Q(z) = \frac{2752 - 32z + z^2}{4(-40 + z)^2(24 + z)^2} . \tag{6.9}$$

Now consider the four Fourier series $\frac{1}{z} + c$ mentioned above. We know that each one satisfies a Schwarzian equation which can be easily obtained from that of z. We find that the Q indeed agrees with (6.9) in each case. This completes the proof of our assertion.

Our observation above raises various tantalizing questions. Why do the mirror maps of our K3 surfaces correspond to the Thompson series of the Monster group? What happens to the other realizations of K3 surfaces in weighted projective spaces? Do they realize other Thompson series? On the last two questions our joint work with A. Klemm in this direction is now underway. Early indications have shown that the correspondence continues to hold. On the first question, we offer the following speculative remarks. It is known that there are K3 surfaces on which a subgroup of the Mathieu group M_{24} acts by automorphisms [22]. It happens that M_{24} is also a subgroup of the Monster. Thus it makes sense to speak of the Thompson series for those elements in M_{24}. It is possible that the appearance of Thompson series as mirror maps is related to some appropriate action of a subgroup of M_{24}. Precisely how this happens is unclear at this point. However, enough evidence has convinced us the following:

Conjecture. *If $z(q)$ is the mirror map for a 1-parameter deformation of an algebraic K3 surface from an orbifold construction, then for some $c \in \mathbf{Z}$, $\frac{1}{z(q)} + c$ is a Thompson series $T_g(q)$ for some $g \in M$.*

It seems that the above connection between the mirror map and Thompson series is peculiar to K3 surfaces. In the case of elliptic curves for instance, unlike the K3 case, the correspondence there is only partial. Applying the same technique as above, we find that all but the 3rd family of elliptic curves we studied in Sect. 5.1, correspond to Thompson series. They are T_{2B}, T_{3B}, and T_{4C} respectively. The mirror map for the 3rd family is in fact a double-valued function on the upper half plane, hence cannot be a modular function of any type.

7. Congruences of the Quantum Coupling

We now move on to Calabi–Yau threefolds. We consider a 1-parameter family of hypersurfaces or complete intersections X_z in $\mathbf{P}^n[w]$, where z is the complex structure deformation coordinate as defined in Sect. 2 in terms of a finite orbifold group G. In this case, the period vector $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ of the holomorphic

3-form for the mirror manifold X^* satisfies a 4^{th} order Picard–Fuchs equation which has the form (4.19). The prepotential is defined as

$$F := \frac{1}{\omega_0^2}(\omega_3\omega_0 + \omega_1\omega_2) \tag{7.1}$$

which is a holomorphic section of a line bundle over the moduli space M. It is also related to the period vector by [1, 23, 24] $\omega = \omega_0(1, t, \partial_t F, 2F - t\partial_t F)$, where t is a special coordinate defined by $t = \omega_1(z)/\omega_0(z)$. Let

$$K = \partial_t^3 F\,. \tag{7.2}$$

The mirror hypothesis [1] identifies t with the flat coordinate on the Kähler cone of X and asserts that K is the quantum coupling for X, which has the form [1, 25, 26]

$$K = \int_X J \wedge J \wedge J + \sum_{d \geqq 1} \frac{d^3 n_d e^{2\pi i t d}}{1 - e^{2\pi i t d}}\,. \tag{7.3}$$

Here n_d is the "number" of rational curves of degree d in a generic deformation of X, and J is the Kähler class on X.

In this section, we would like to study some arithmetic properties of K as a q-series ($q = e^{2\pi i t}$) under the *assumption of the mirror hypothesis with the n_d integers, and that the mirror map $z(q)$ has integral Fourier coefficients.* Specifically, we will study congruences mod p of the instanton numbers under the above hypothesis, using a formal nonlinear ODE. We should point out that the integrality of $z(q)$ has been checked numerically in many examples of threeholds up to at least order q^{30}, and has been proved for K3 surfaces and elliptic curves in numerous cases.

7.1. Differential equations mod p. In [8, 9], we derived certain polynomial differential equations for $z(q), K(q)$. An important feature of these equaitons is that they are defined over the rationals. This is so because the Picard–Fuchs equation, from which our ODE are constructed, are defined over the rationals. The main idea here is to derive the mod p version of our nonlinear ODE from the Picard–Fuchs equation. We then use some data – basically an integer N_e – which is extracted from the Picard–Fuchs equation in each case to derive congruences modulo certain prime powers. We will illustrate this in several well-known examples.

We begin with the following form of the Picard–Fuchs equation (4.19):

$$\left(\frac{d^4}{dz^4} + q_2(z)\frac{d^2}{dz^2} + q_2'(z)\frac{d}{dz} + q_0(z)\right) f(z) = 0\,, \tag{7.4}$$

where $q_2(z)$, $q_0(z)$ are defined in (4.21). By a change of coordinate $z \mapsto t$ (see [9] for details) and using the list of solutions $f_0(z)(1, t, \partial_t F, 2F - t\partial_t F)$, we show that z, K as functions of t, satisfy a pair of coupled nonlinear ODEs: The simpler of the two is given by

$$\lambda_z K^4 = \rho_K \quad (*)\,, \tag{7.5}$$

where

$$\rho_K := 5^2 \cdot 7K'^4 - 2^3 \cdot 5 \cdot 7KK'^2K'' + 7^2K^2K''^2$$
$$+ 2 \cdot 5 \cdot 7K^2K'K^{(3)} - 2 \cdot 5K^3K^{(4)},$$
$$\lambda_z := \frac{e(z)z'^4}{\Delta(z)^4},$$
$$e(z) := 10(-10r_2 + 3r_2 - 4r_4)z + (-9r_2^2 + 300r_0r_4 - 46r_2r_4 + 64r_4^2)z^2$$
$$+ 2r_4(9r_2^2 - 150r_0r_4 + 7r_2r_4 - 22r_4^2)z^3$$
$$+ r_4^2(-9r_2^2 + 100r_0r_4 + 2r_2r_4 + 11r_4^2)z^4 . \tag{7.6}$$

Here $\Delta(z) = z(1 - r_4z)$ is the discriminant of Eq. (4.19). (Actually, the following discussion requires only that $e(z), \Delta(z) \in \mathbf{Z}[z]$, with $\Delta(z)$ having leading coefficient 1. No specific forms need to be assumed). We *define N_e to be the g.c.d. of the coefficients in $e(z)$*. Then it is clear that $n|N_e$ iff n divides all Fourier coefficients of $e(z(q))$. *All the arithmetic properties we derive of $K(q)$ will be entirely controlled by this one integer N_e and the classical intersection number $K_{cl} := \int_X J \wedge J \wedge J$.*

Notice that (a) Eq. (*) is defined over $\mathbf{Z}$; (b) Eq. (*) is invariant under an overall constant scaling of K; (c) the primes 2, 5, 7 feature prominently. It turns out that, though less obvious, the equation also simplifies considerably modulo 3. The main purpose of having Eq. (*) is as follows. Suppose a 1-parameter family of mirror Calabi–Yau varieties is given such that $p|N_e$ for some prime p. Then the left-hand side of (*) becomes zero over the field of q-series $\mathbf{Z}/p\mathbf{Z}((q))$. We can then study the q-series solution K (7.3) of this simplified equation modulo p or its powers. We should point out that in [8, 9], we derived a similar equation but over $\mathbf{Q}$. The result there was much more complicated.

Notations. In the following discussion, K will be the quantum coupling (7.3). The $'$ in the equation (*) means $q\frac{d}{dq}$. Thus $(q^n)' \equiv 0 \bmod p$ whenever $p|n$. More generally when p is prime and $h = \sum a_nq^n$ is any integral q-series, the p^{th} derivative $h^{(p)} \equiv h'$ because $n^p \equiv n$ for all n. In what follows, we will use these few tricks repeatedly, sometimes without mention, to simplify our computations. We write $n|h$ if $n|a_n$ for all n. We also write ρ_h to denote the right-hand side of (7.5) but with K replaced with h:

$$\rho_h := 5^2 \cdot 7h'^4 - 2^3 \cdot 5 \cdot 7hh'^2h'' + 7^2h^2h''^2 + 2 \cdot 5 \cdot 7h^2h'h^{(3)} - 2 \cdot 5h^3h^{(4)} . \tag{7.7}$$

For each prime p, we introduce the notation:

$$h_p := \sum_{p \not| n} a_nq^n . \tag{7.8}$$

Thus we have $h' \equiv h'_p \bmod p$. Finally, given K (7.3), we denote the largest integer divisor of K by $m(K)$.

7.2. mod p

Claim 1. Let $h := \frac{K}{m(K)}$. For any integer n, $n|h'$ implies that $n|\rho_h$ and $n|\rho_h$ implies that $n|N_e$. Conversely, if $n|N_e$ then $n|\rho_h$.

Proof. If $n|h'$ then from (7.7), it follows that $n|\rho_h$. Assuming $n|\rho_h$, Eq. (*) becomes $e(z)z'^4h^4 \equiv 0 \bmod n$. Since neither h nor z' has any overall factor, we have $e(z) \equiv 0$. This implies that $N_e \equiv 0$. The converse follows immediately from Eq. (*).

This says that the integer N_e tells us much above congruences of $\frac{K}{m(K)}$.

Claim 2. Fix a prime p and suppose $p^r|K_p$. Then $p^r|n_d$ for all $d \not\equiv 0 \bmod p$, and $p^{\min(3,r)}|(K - K_{cl})$.

Proof. From (7.3), the m^{th} $(m > 0)$ Fourier coefficient of K is

$$k_m = \sum_{d|m} d^3 n_d \,. \tag{7.9}$$

By supposition, $p^r|k_m$ for all $m \not\equiv 0 \bmod p$. Let $d \not\equiv 0 \bmod p$ be the smallest such integer for which $p^r \not| n_d$. Now $p^r|k_d$ means that $p^r|(d^3 n_d + \sum_{d'|d,d'<d} d'^3 n_{d'})$. Since $p \not| d$, that $d'|d$ implies that $p \not| d'$, hence $p^r|n_{d'}$ by the minimality of d. So $p^r|(d^3 n_d + \sum_{d'|d,d'<d} d'^3 n_{d'})$ implies that $p^r|n_d$, which is a contradiction.

In particular, we have $p^r|d^3 n_d$ for $p \not| d$. Trivially $p^3|d^3 n_d$ for $p|d$, and so we have $p^{\min(3,r)}|d^3 n_d$ for all d. Now it follows immediately from (7.3) that $p^{\min(3,r)}|(K - K_{cl})$.

Now let h be an integral q-series. As pointed out above, $h' \equiv h_p' \bmod p$. It follows that $p|h_p$ implies $p|\rho_h$. On the other hand, we have the following:

Claim 3. Assume prime $p \neq 2, 5$. Suppose for any integral q-series h with $p \not| h$, $p|\rho_h$ implies $p|h_p$. Then for all $r > 0$, $p^r|\rho_h$ implies that $p^r|h_p$ and $p^r|\rho_{h-h_p}$.

Proof. If $p|\rho_h$ then $p|h_p$, and so we can write $h_p = pf$ for some integral series f. Write also $g = h - h_p$, which has only q-powers q^n with $p|n$. Thus $p|g'$, hence $p|\rho_g$. Thus our statement holds for $r = 1$.

Suppose it holds up to some $r > 0$. Assume $p^{r+1}|\rho_h$. By inductive hypothesis we have at least $p^r|h_p$, and so we can write $h_p = p^r f$, $g = h - h_p$ as before. Because $h = p^r f + g$, and $p \not| h$ by assumption, we have $p \not| g$. The equation $\rho_h \equiv 0 \bmod p^{r+1}$ becomes, after some simplifications using $p^i|g^{(i)}$.

$$\rho_g - 10p^r g^3 f'''' \equiv 0 \bmod p^{r+1} \,. \tag{7.10}$$

But g only has q-powers q^n with $p|n$, and hence so does ρ_g. On the other hand f (hence f'''') has only q^n with $p \not| n$. Thus (7.10) implies that $p^{r+1}|\rho_g$ and $p^{r+1}|(10p^r g^3 f'''')$. From the fact that $p \neq 2, 5$ and that $p \not| g$, it follows that $p|f''''$ and hence $p|f$. Thus $h_p = p^r f$ implies that $p^{r+1}|h_p$.

The cases $p = 2, 5$ are dealt with separately. We will also discuss the cases $p = 3, 7$ in more detail.

7.3. mod 2

Claim 4. Let h be integral q-series with $2 \not| h$. Then $2|\rho_h$ implies that $2|h_2$ or $2|(h - h_2)$. If moreover $2|h_2$, then (a) $2^r|\rho_h$ implies that $2^r|\rho_h - h_2$, and (b) $2^{r+1}|\rho_h$ implies that $2^r|h_2$. for all $r > 0$.

Proof. Suppose $r|\rho_h$. Then we get

$$h'^4 + h^2h'^2 \equiv 0 \mod 2\,. \tag{7.11}$$

This says that either $2|h'$ or $2|(h+h')$. But $h' \equiv h_2' \equiv h_2 \mod 2$, implying that either $2|h_2$ or $2|(h-h_2)$. This proves the first half.

For part (a), the proof of *Claim 3* applies to $p=2$ without change.

For part (b), we modify the inductive argument there as follows. Suppose $2^3|\rho_h$ and write $h_2 = 2f,\ g = h - h_2$ as before. The equation $\rho_h \equiv 0 \mod 2^3$ becomes

$$\rho_g + 49.2^2 g^2 f''^2 - 10.2 g^3 f'''' \equiv 0 \mod 2^3\,. \tag{7.12}$$

Part (a) implies that $2^3|\rho_g$. Also $g^2f''^2$ only has q-powers q^n with $2|n$ while g^3f'''' has only those with $2\not|n$. Thus (7.12) implies that all three terms vanish separately. It follows that $2|f''''$ and hence $2|f$. Thus $h_2 = 2f$ implies that $2^2|h_2$. Suppose (b) holds up to some $r > 1$. Assume $2^{r+2}|\rho_h$. By inductive hypothesis we have at least $2^r|h_2$, and so we can write $h_2 = 2^r f,\ g = h - h_2$ as before. The equation $\rho_h \equiv 0 \mod 2^{r+2}$ becomes

$$\rho_g - 10.2^r g^3 f'''' \equiv 0 \mod 2^{r+2}\,. \tag{7.13}$$

Part (a) implies that $2^{r+2}|\rho_g$. By assumptions, $2\not|h$ and $2|h_2$ hence $2\not|g$. It follows that $2|f''''$, hence $2|f$. Thus $h_2 = 2^r f$ implies that $2^{r+1}|h_2$. This completes our proof of part (b).

Claim 5. If $2|(K+K')$ then $2|n_d$ for all odd, d and $2|K$.

Proof. Let l be the smallest odd integer such that $2\not|n_l$. Now $2|(K+K')$ means that $2|k_m$ for all even m, where k_m is the m^{th} Fourier coefficients of K . In particular $2|k_{2l}$. But

$$k_{2l} = \sum_{d|2l} d^3 n_d = \sum_{d|2l,\, d\ odd} d^3 n_d + \sum_{d|2l,\, d\ even} d^3 n_d\,. \tag{7.14}$$

The even part obviously divides 2, while the odd part is

$$\sum_{d|2l,\, d\ odd} d^3 n_d = l^3 n_l + \sum_{d|l,\, d\ odd,\, l>d} d^3 n_d\,. \tag{7.15}$$

By the minimality of l, the second sum divides 2, implying that $2|l^3 n_l$. But l is odd, hence $2|n_l$ which is a contradiction. Thus $2|n_d$ for odd d.

Now consider (7.3). If $2|n_d$ for odd d, then obviously $2|d^3 n_d$ for all d. Also $2|(K+K')$ implies that $2|K_{cl}$. So we have $2|K$.

Remarks. Let $h := \frac{K}{m(K)}$ and suppose $2^r|N_e, r > 0$. By Claim 1, we have $2^r|\rho_h$. By Claim 4, we have either $2|h_2$ or $2|(h-h_2)$. Suppose first $2|h_2$. The $2^{r-1}|h_2$ by Claim 4b, hence $(m(K)2^{r-1})|K_2$. Thus by Claim 2, we see that $2^{s+r-1}|n_d$ for all d odd, where s is the largest integer such that $2^s|m(K)$. Moreover, $2^{\min(3,s+r-1)}|(K - K_{cl})$. To summarize: *if* $2^r|N_e$ *and* $2|\frac{K_2}{m(K)}$, *then* $(a)\, 2^{s+r-1}|n_d$ *for all* d *odd, where* s *is the largest integer such that* $2^s|m(K)$, *and* $(b) 2^{\min(3,s+r-1)}|(K - K_{cl})$.

Suppose now $2|(h-h_2)$ instead. This means in particular that $2|(K+K')$, and so by Claim 5 we have $2|n_d$ for all odd d, and $2|K$. Thus $m(K) \geqq 2$. So *if* $2|N_e$ *and* $2|\left(\frac{K-K_2}{m(K)}\right)$, *then* $2|n_d$ *for all odd* d *and* $2|K$.

7.4. mod 3. Suppose h is any integral series with $3|\rho_h$ and $3 \nmid h$. We get by differentiating $\rho_h \equiv 0 \bmod 3$ once,

$$hh'(2h^2 + h'^2 + hh'') \equiv 0 \bmod 3 . \tag{7.16}$$

If $3 \nmid h'$ then $3|(2h^2 + h'^2 + hh'')$. Differentiating $2h^2 + h'^2 + hh''$ and simplifying it by $h''' \equiv h' \bmod 3$, we get $3|(2hh')$ implying that $3|h'$, hence $3|h_3$. Thus we have shown that $3|\rho_h$ implies $3|h'$ and $3|h_3$.

Remark. Let $h := \frac{K}{m(K)}$ and suppose $3^r|N_e, r > 0$. Then by Claim 1, we have $3^r|\rho_h$. We have just shown that $3|\rho_h$ implies $3|h_3$. Then $3^r|h_3$ by Claim 3, hence $(m(K)3^r)|K_3$. Thus by Claim 2, we see that $3^{s+r}|n_d$ for all $d \not\equiv 0 \bmod 3$, where s is the largest integer such that $2^s|m(K)$. Moreover, $3^{\min(3,s+r)}|(K - K_{cl})$. To summarize: *if* $3^r|N_e$, *then* (a) $3^{s+r}|n_d$ *for all* $d \not\equiv 0 \bmod 3$, *where* s *is the largest integer such that* $3^s|m(K)$, *and* (*b*) $3^{\min(3,s+r)}|(K - K_{cl})$.

7.5. mod 5

Claim 6. Let h be an integral q-series with $5 \nmid h$. Then $5|\rho_h$ implies that $5|h_5$. Moreover we have (*a*) $5^r|\rho_h$ implies that $5^r|\rho_{h-h_5}$, and (b) $5^{r+1}|\rho_h$ implies that $5^r|h_5$, for all $r > 0$.

Proof. Suppose $5|\rho_h$. Then the equation $\rho_h \equiv 0 \bmod 5$ reads $4hh'' \equiv 0 \bmod 5$. Hence $5|h''$ which implies $5|h_5$. This proves the first statement.

For part (a), the proof of Claim 3 applies to $p = 5$ without change.

For part (b), the proof of Claim 4b applies here with only minor changes.

Remarks. Let $h := \frac{K}{m(K)}$ and suppose $5^r|N_e$, $r > 0$. By Claim 1, we have $5^r|\rho_h$. Then $5^{r-1}|h_5$ by Claim 6b, hence $(m(K)5^{r-1})|K_5$. Thus by Claim 2, we see that $5^{s+r-1}|n_d$ for all $d \not\equiv 0 \bmod 5$, where s is the largest integer such that $5^s|m(K)$. Moreover, $5^{\min(3,s+r-1)}|(K - K_{cl})$. To summarize: *if* $5^r|N_e$, *then* (a) $5^{s+r-1}|n_d$ *for all* $d \not\equiv 0 \bmod 5$, *where* s *is the largest integer such that* $5^s|m(K)$, and (b) $5^{\min(3,s+r-1)})|(K - K_{cl})$.

7.6. mod 7. We summarize the result here: *if* $7^r|N_e$, *then* (a) $7^{s+r}|n_d$ *for all* $d \not\equiv 0 \bmod 7$, *where* s *is the largest integer such that* $7^s|m(K)$, and (b) $7^{\min(3,s+r)}|(K - K_{cl})$. The argument here is quite similar to the case of mod 3.

7.7. Examples. Under the assumption of the integrality of the mirror map $z(q)$ and the mirror hypothesis assertion that quantum coupling $K(q)$ is given by (7.3) with integers n_d, we consider the following examples applying the above results:

1. Let X be the Fermat quintics in $\mathbf{P}^4$. The orbifold group here is an abelian group of type (5,5,5) [12, 1]. In this case, we have $N_e = -5^3$ and $K_{cl} = 5$. It follows from our mod 5 analysis that at least $5^2|(K - K_{cl})$. But since $K_{cl} = 5$, we have $5|K$ and $m(K) = 5$. Thus $5^3|n_d$ for all $d \not\equiv 0 \bmod 5$. Because $n_1 = 23.5^3, 5^3$ is the upper bound on the prime power dividing all n_d. Experimentally however, it seems that 5^3 divides *all* n_d, as previously observed by others (see Table 4 of [1]). Note that since N_e has no prime divisor other than 5, there is no other prime p dividing all n_d, by Claim 1.

2. Consider X the complete intersection of a quartic $x_1^4+x_2^4+x_3^4=0$ and a sextic $x_4^3+x_5^3+x_6^2=0$ in $\mathbf{P}^5[1,1,1,2,2,3]$. The orbifold group is of type $(4,4,3)$. In this case, we have $N_e=-2^6,3^5$ and $K_{cl}=2$. It follows from our mod 3 analysis that $3^5|n_d$ for all $d\not\equiv 0 \bmod 3$. Once again, this lower bound on the powers of 3 is sharp because already $n_1=2^6,3^5$. For mod 2, our analysis shows that either $2|h_2$ our $2|(h-h_2)$, where $h=\frac{K}{m(K)}$. Suppose $2|(h-h_2)$. Then our analysis above implies that $2|K$ and $m(K)\geqq 2$. But $K_{cl}=2\geqq m(K)$, implying that $m(K)=2$. Thus $(h-h_2)$ has constant term $\frac{K_{cl}}{m(K)}=1$, contradicting $2|(h-h_2)$. It follows that we must have $2|h_2$. By the mod 2 analysis above, we conclude that at least $2^6|n_d$ for all d odd. Again this lower bound on the power of 2 is also sharp because already $n_1=2^6,3^5$. Note that since N_e has no prime divisor other than 2.3, there is no other prime p dividing all n_d, by t Claim 1.

3. Finally consider X the degree 10 hypersurface of Fermat type in $\mathbf{P}^4[1,1,1,2,5]$. The orbifold group is of type $(10,5,2)$. In this case, we have $N_e=-2^6,5^3$ and $K_{cl}=1$. Thus $m(K)=1$. It follows from our mod 2 and mod 5 analyses that at least $2^5|n_d$ for d odd: and $5^2|n_d$ for $d\not\equiv 0 \bmod 5$. These bounds on the prime power are also sharp because already $n_1=2^5,5^2.17$. Note that since N_e has no prime divisor other than $2,5$, there is no other prime p dividing all n_d, by Claim 1.

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