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Yang–Mills and Dirac Fields in a Bag, Constraints and Reduction

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Abstract: The structure of the constraint set in the Yang–Mills–Dirac theory in a contractible bounded domain is analysed under the bag boundary conditions. The gauge symmetry group is identified, and it is proved that its action on the phase space is proper and admits slices. The reduced phase space is shown to be the union of symplectic manifolds, each of which corresponds to a definite mode of symmetry breaking.

1. Introduction

In a previous paper we have proved the existence and uniqueness theorems for minimally interacting Yang–Mills and Dirac fields in a bounded contractible domain $M \subset \mathbb{R}^3$, [1]. The aim of this paper is to study the structure of the space of solutions.

Our results were obtained for Cauchy data $\mathbf{A} \in H^2(M)$, $\mathbf{E} \in H^1(M)$, and $\Psi \in H^2(M)$, where $H^k(M)$ is the Sobolev space of fields on M which are square integrable together with their derivatives up to the order k, satisfying the boundary conditions

$$n\mathbf{E} = 0, \quad t\mathbf{B} = 0, \quad in_j\gamma^j\Psi|_{\partial M} = \Psi|_{\partial M}\,, \tag{1.1a}$$

$$n\mathbf{A} = 0, \quad in_j\gamma^j\{\gamma^0(\gamma^k\partial_k + im)\Psi\}|_{\partial M} = \gamma^0(\gamma^k\partial_k + im)\Psi|_{\partial M}\,. \tag{1.1b}$$

Here we use the notation established in [1]. In particular, $n\mathbf{E}$ denotes the normal component of the "electric" part, $t\mathbf{B}$ the tangential component of the "magnetic" part of the field strength on the boundary ∂M of M. Thus, the extended phase space of the theory under consideration is

$$\mathbf{P} = \{(\mathbf{A}, \mathbf{E}, \Psi) \in H^2(M) \times H^1(M) \times H^2(M) |\ \text{satisfying (1.1a,b)}\}\,. \tag{1.2}$$

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The variational principle underlying the theory gives rise to a (weak) symplectic structure on $\mathbf{P}$. Let θ be a 1-form on $\mathbf{P}$ such that, for every $\mathbf{p} = (\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{P}$ and

$$\mathbf{a}\frac{\delta}{\delta \mathbf{A}} + \mathbf{e}\frac{\delta}{\delta \mathbf{E}} + \psi\frac{\delta}{\delta \Psi} \in T_{\mathbf{p}}\mathbf{P},$$

$$\left\langle \theta(\mathbf{A}, \mathbf{E}, \Psi) \middle| \mathbf{a}\frac{\delta}{\delta \mathbf{A}} + \mathbf{e}\frac{\delta}{\delta \mathbf{E}} + \psi\frac{\delta}{\delta \Psi} \right\rangle = \int_M (\mathbf{E} \cdot \mathbf{a} + \Psi^\dagger \psi)\, d_3x, \tag{1.3}$$

The symplectic form ω of $\mathbf{P}$ is the exterior differential of θ,

$$\omega = d\theta. \tag{1.4}$$

Let G be the structure group of the theory, presented as a matrix group, and $\mathfrak{g}$ be the Lie algebra of G. We assume that G is compact, and that $\mathfrak{g}$ admits an ad-invariant metric. The group $GS(\mathbf{P})$ of gauge symmetries consists of maps $\phi : M \to G$ such that their action on the variables $(\mathbf{A}, \mathbf{E}, \Psi)$, given by

$$\mathbf{A} \mapsto \phi\mathbf{A}\phi^{-1} + \phi \operatorname{grad} \phi^{-1}, \qquad \mathbf{E} \mapsto \phi\mathbf{E}\phi^{-1}, \qquad \Psi = \phi\Psi, \tag{1.5}$$

leaves the extended phase space $\mathbf{P}$ invariant. The infinitesimal action of an element ξ of the Lie algebra $gs(\mathbf{P})$ of $GS(\mathbf{P})$ is given by

$$\mathbf{A} \mapsto \mathbf{A} - D_{\mathbf{A}}\xi, \qquad \mathbf{E} \mapsto \mathbf{E} - [\mathbf{E}, \xi], \qquad \Psi = \Psi + \xi\Psi, \tag{1.6}$$

where

$$D_{\mathbf{A}}\xi = \operatorname{grad} \xi + [\mathbf{A}, \xi] \tag{1.7}$$

is the covariant derivative of ξ with respect to the connection defined by $\mathbf{A}$. It gives rise to a vector field $\xi_{\mathbf{P}}$ on $\mathbf{P}$ such that

$$\xi_{\mathbf{P}}(\mathbf{A}, \mathbf{E}, \Psi) = -(D_{\mathbf{A}}\xi)\frac{\delta}{\delta \mathbf{A}} - [\mathbf{E}, \xi]\frac{\delta}{\delta \mathbf{E}} + \xi\Psi\frac{\delta}{\delta \Psi}. \tag{1.8}$$

The action of $GS(\mathbf{P})$ preserves the 1-form θ. Hence, it is Hamiltonian with the equivariant momentum map $J : \mathbf{P} \to gs(\mathbf{P})^*$ such that

$$\langle J(\mathbf{A}, \mathbf{E}, \Psi) | \xi \rangle = \langle \theta | \xi_{\mathbf{P}}(\mathbf{A}, \mathbf{E}, \Psi) \rangle = \int_M \{-\mathbf{E} \cdot D_{\mathbf{A}}\xi + \Psi^\dagger \xi \Psi\}\, d_3x. \tag{1.9}$$

Here $gs(\mathbf{P})^*$ denotes the L^2 dual of $gs(\mathbf{P})$, that is the space of square integrable maps from M to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the structure group G. For each $\xi \in gs(\mathbf{P})$, the function $J_\xi : \mathbf{P} \to \mathbb{R}$ given by

$$J_\xi(\mathbf{A}, \mathbf{E}, \Psi) = \langle J(\mathbf{A}, \mathbf{E}, \Psi) | \xi \rangle \tag{1.10}$$

is called the momentum associated to ξ. The vector field $\xi_{\mathbf{P}}$ is the Hamiltonian vector field of J_ξ, i.e.

$$\xi_{\mathbf{P}} \lrcorner\, \omega = dJ_\xi. \tag{1.11}$$

Integrating by parts on the right-hand side of Eq. (1.9), and taking into account the boundary condition $n\mathbf{E} = 0$, we obtain

$$\langle J(\mathbf{A}, \mathbf{E}, \Psi) | \xi \rangle = \int_M \{(\operatorname{div} \mathbf{E} + [\mathbf{A}; \mathbf{E}])\xi + \Psi^\dagger \xi \Psi\}\, d_3x. \tag{1.12}$$

For every $\xi \in gs(\mathbf{P})$,

$$\Psi^\dagger \xi \Psi = -j \cdot \xi, \tag{1.13}$$

where j is the source term in the Yang–Mills–Dirac theory. Hence, the constraint equation of the theory

$$\operatorname{div} \mathbf{E} + [\mathbf{A}; \mathbf{E}] = j\,, \tag{1.14}$$

is equivalent to the vanishing of the momentum map J.

The presentation of the constraint set as the zero level $J^{-1}(0)$ of the momentum map J, enables one to study its structure in terms of the action of the group of gauge symmetries. It was first done by Arms [2], who discussed the structure of the constraint set for pure Yang–Mills fields in compact spaces (no boundary) in general terms, without specifying the topology of the function spaces under consideration. The structure of the zero level of the momentum map, corresponding to a Hamiltonian action of a Hilbert–Lie group on a Hilbert manifold was studied, under additional technical assumptions, by Arms, Marsden and Moncrief, [3]. Special cases were considered by Mitter and Vialet [4], Atiyah and Bott [5], Kondracki and Rogulski [6] and Huebschmann [7, 8].

Functional analytic assumptions made in this paper are consequences of the results of [1]. They fail to satisfy two basic assumptions made in [3]: (i) neither the differential of J nor its adjoint are elliptic, (ii) the extended phase space $\mathbf{P}$ is not invariant under the interchange of $\mathbf{A}$ and $\mathbf{E}$. Hence, we cannot use the results of Arms, Marsden and Moncrief, [3]. Instead, we follow the main idea of their paper, and prove the necessary intermediate steps. In particular, we prove the properness of the action of $GS(\mathbf{P})$ and of the existence of slices for this action. From this we show that the reduced phase space is the union of symplectic manifolds labelled by the conjugacy classes of compact subgroups of $GS(\mathbf{P})$. Each of these symplectic manifolds consists of the fields $(\mathbf{A}, \mathbf{E}, \Psi)$ with a definite mode of symmetry breaking.

In the finite dimensional case the partition of the reduced phase space into symplectic manifolds can be described algebraically in terms of the Poisson algebra, cf. [9, 10]. Similar results for central Yang–Mills connections on surfaces has been obtained in [8]. An adaptation of this approach to our phase space will be studied elsewhere.

The paper is organized as follows. In Sect. 2 we discuss, in a proper functional analytic framework, the gauge symmetry group and its action. The structure of the zero level of the momentum map is analysed in Sect. 3. A stratification of the reduced phase space is studied in Sect. 4. Section 5 contains discussion of symmetry breaking corresponding to each stratum. The almost complex structure in the L^2 completion of $\mathbf{P}$ is discussed in Appendix A. The properness of the action of $GS(\mathbf{P})$ is proved in Appendix B. The slice theorem is proved in Appendix C.

2. Gauge Symmetries and the Momentum Map

The requirement that (1.6) gives an action of $\xi \in gs(\mathbf{P})$ in the space $\mathbf{P}$, defined by (1.2), implies that $\operatorname{grad} \xi \in H^2(M)$. Since M is bounded, it follows that $\xi \in H^3(M)$. Moreover, the action of ξ has to preserve the boundary conditions. The conditions (1.1a) are the usual bag boundary conditions and are gauge invariant. The conditions (1.1b) are satisfied if and only if $n \cdot \operatorname{grad} \xi = 0$. Hence,

$$gs(\mathbf{P}) = \{\xi : M \to \mathrm{g} \,|\, \xi \in H^3(M) \text{ and } n \cdot \operatorname{grad}\, \xi = 0\}\,. \tag{2.1}$$

The L^2 dual $gs(\mathbf{P})^*$ of $gs(\mathbf{P})$, considered here, is the space of square integrable maps from M to the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, that is

$$gs(\mathbf{P})^* = \{v : M \to \mathfrak{g}^* | v \in L^2(M)\} . \tag{2.2}$$

The evaluation of $v \in gs(\mathbf{P})^*$ on $\xi \in gs(\mathbf{P})$ is given by pointwise evaluation and integration

$$\langle v|\xi\rangle = \int_M v \cdot \xi \, d_3x . \tag{2.3}$$

The momentum map J defined in Eq. (1.9) is a continuous map from $\mathbf{P}$ to $gs(\mathbf{P})^*$.

$GS(\mathbf{P})$ has a manifold structure with the tangent bundle space spanned by $gs(\mathbf{P})$. The presentation of the structure group G as a matrix group, and boundedness of M, enable us to present $GS(\mathbf{P})$ as a group of maps ϕ from M to G of Sobolev class $H^3(M)$. Moreover, the boundary conditions (1.1) require that $n \cdot \text{grad } \phi = 0$. Hence,

$$GS(\mathbf{P}) = \{\phi : M \to G | \phi \in H^3(M) \text{ and } n \cdot \text{grad } \phi = 0\} . \tag{2.4}$$

Since M is contractible and G is connected, $GS(\mathbf{P})$ is connected. However, it need not be simply connected.

Proposition 2.1. *The exponential mapping* $\exp : gs(\mathbf{P}) \to GS(\mathbf{P})$ *is a diffeomorphism of a neighbourhood of* $0 \in gs(\mathbf{P})$ *onto a neighbourhood of the identity in* $GS(\mathbf{P})$.

Proof. Let U be a neighbourhood of $0 \in \mathfrak{g}$ and V a neighbourhood of the identity $e \in G$, such that the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism of U onto V, and let $\ln : V \to U$ be the inverse of this diffeomorphism. Since, by the Sobolev embedding theorem, each $\phi \in GS(\mathbf{P})$ is a continuous map from M to G, the sets

$$\mathbf{V} = \{\phi \in GS(\mathbf{P}) | \text{range } \phi \subseteq V\}$$

is open in $GS(\mathbf{P})$. Similarly, the set

$$\mathbf{U} = \{\xi \in gs(\mathbf{P}) | \text{range } \xi \subseteq U\}$$

is open in $gs(\mathbf{P})$. For every $\phi \in \mathbf{V}$, $\ln \circ \phi$ is in $gs(\mathbf{P})$, and its range is in U. Hence, $\ln \circ \phi \in \mathbf{U}$. Let $\exp : gs(\mathbf{P}) \to GS(\mathbf{P})$ denote the exponential for the gauge algebra. For every $\xi \in gs(\mathbf{P}), \exp(\xi) = \exp \circ \xi$. Hence, for every $\phi \in \mathbf{V}$, $\exp(\ln \circ \phi) = \exp \circ \ln \circ \phi = \phi$, which implies that $\exp(\mathbf{U}) = \mathbf{V}$. □

The main property of the action of $GS(\mathbf{P})$ in $\mathbf{P}$ used in this paper is its properness.

Theorem 1. *The action of* $GS(\mathbf{P})$ *in* $\mathbf{P}$ *is proper. That is, for every sequence* $\mathbf{p}_n$ *converging to* $\mathbf{q}$ *in* $\mathbf{P}$ *and every sequence* ϕ_n *in* $GS(\mathbf{P})$ *such that* $\phi_n\mathbf{p}_n$ *converges to* $\mathbf{p}$, *the sequence* ϕ_n *has a convergent subsequence with limit* ϕ, *and* $\phi\mathbf{q} = \mathbf{p}$.

Proof is given in Appendix B.

For each $\mathbf{p} \in \mathbf{P}$, we denote by $O_\mathbf{p}$ the orbit of $GS(\mathbf{P})$ through $\mathbf{p}$,

$$O_\mathbf{p} = \{\phi\mathbf{p} | \phi \in GS(\mathbf{P})\} . \tag{2.5}$$

All orbits $O_{\mathbf{p}}$ of $GS(\mathbf{P})$ are closed since, if $\phi_n\mathbf{p}$ is a convergent sequence of points in $O_{\mathbf{p}}$ with limit $\mathbf{q}$, then the sequence ϕ_n has a convergent subsequence with limit ϕ and $\mathbf{q} = \phi\mathbf{p}$, which implies that $\mathbf{q} \in O_{\mathbf{p}}$.

For every subspace $\mathbf{V}$ of $T_{\mathbf{p}}\mathbf{P}$, we denote by $\mathbf{V}^{\omega}$ the symplectic annihilator of $\mathbf{V}$, that is

$$\mathbf{V}^{\omega} = \{\mathbf{w} \in T_{\mathbf{p}}\mathbf{P} | \omega(\mathbf{v}, \mathbf{w}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}\} . \tag{2.6}$$

Note that $\mathbf{V}^{\omega}$ is closed, and if $\mathbf{V}$ is closed, then $(\mathbf{V}^{\omega})^{\omega} = \mathbf{V}$.

Proposition 2.2. *For each* $\mathbf{p} \in \mathbf{P}$,

$$T_{\mathbf{p}}O_{\mathbf{p}} = (\ker dJ_{\mathbf{p}})^{\omega} . \tag{2.7}$$

Proof. If $\xi_{\mathbf{P}}$ is the Hamiltonian vector field of J_{ξ}, cf. Eq. (1.11), then for every $\mathbf{v} \in T_{\mathbf{p}}\mathbf{P}$,

$$\omega(\xi_{\mathbf{P}}(\mathbf{p}), \mathbf{v}) = \langle dJ_{\mathbf{p}}(\mathbf{v}) | \xi \rangle . \tag{2.8}$$

Since $T_{\mathbf{p}}O_{\mathbf{p}} = \{\xi_{\mathbf{P}}(\mathbf{p}) | \xi \in gs(\mathbf{P})\}$ it follows that $\mathbf{v} \in (T_{\mathbf{p}}O_{\mathbf{p}})^{\omega}$ if and only if $\mathbf{v} \in \ker dJ_{\mathbf{p}}$. Hence, $(T_{\mathbf{p}}O_{\mathbf{p}})^{\omega} = \ker dJ_{\mathbf{p}}$, and therefore $T_{\mathbf{p}}O_{\mathbf{p}} = (\ker dJ_{\mathbf{p}})^{\omega}$, since $\ker dJ_{\mathbf{p}}$ is closed. □

Proposition 2.3. *For every* $\mathbf{p} \in \mathbf{P}$, range $dJ_{\mathbf{p}}$ *is a closed subspace of* $gs(\mathbf{P})^*$ *with finite codimension.*

Proof. For $\mathbf{p} = (\mathbf{A}, \mathbf{E}, \Psi)$ and $(\mathbf{a}, \mathbf{e}, \psi) \in T_{\mathbf{p}}\mathbf{P}$, Eq. (1.12) implies that

$$\langle dJ_{\mathbf{p}}(\mathbf{a}, \mathbf{e}, \psi) | \xi \rangle = \int_M \{-(\operatorname{div}(\mathbf{e}) + [\mathbf{A}, \mathbf{e}] + [\mathbf{E}, \mathbf{a}])\xi + \psi^{\dagger}\xi\Psi + \Psi^{\dagger}\xi\psi\} \, d_3x .$$

Hence, $dJ_{\mathbf{p}} = T + S : T_{\mathbf{p}}\mathbf{P} \to L^2(M, \mathfrak{g})$, where

$$T(\mathbf{a}, \mathbf{e}, \psi) = -\operatorname{div}(\mathbf{e}) \quad \text{and} \quad S(\mathbf{a}, \mathbf{e}, \psi) = -[\mathbf{A}, \mathbf{e}] - [\mathbf{E}, \mathbf{a}] + \psi^{\dagger} \otimes \Psi + \Psi^{\dagger} \otimes \psi .$$

The Hodge decomposition, cf. [11], applied to square integrable zero forms on M, implies that $L^2(M, \mathfrak{g}) = \mathscr{C} \oplus \mathscr{H}$, where $\mathscr{H}$ is the space of constant $\mathfrak{g}$-valued functions and $\mathscr{C} = \{\operatorname{div}(\mathbf{v}) | \mathbf{v} \in H^1(M, \mathfrak{g}),\ n\mathbf{v} = 0\}$. Both $\mathscr{C}$ and $\mathscr{H}$ are closed subspaces of $L^2(M, \mathfrak{g})$. Since range $T = \mathscr{C}$, it follows that the range of T is closed. Moreover, cokernel $T = L^2(M, \mathfrak{g})/\text{range } T \simeq \mathscr{H}$ has finite dimension, since $\dim \mathscr{H} = \dim \mathfrak{g}$. Hence, T is semi-Fredholm.

Further, if $\mathbf{v}_n = (\mathbf{a}_n, \mathbf{e}_n, \psi_n)$ is a bounded sequence in $T_{\mathbf{p}}\mathbf{P}$, then the sequence

$$\{S\mathbf{v}_n\} = \{-[\mathbf{A}, \mathbf{e}_n] - [\mathbf{E}, \mathbf{a}_n] + \psi_n^{\dagger} \otimes \Psi + \Psi^{\dagger} \otimes \psi_n\}$$

is bounded in $H^1(M, \mathfrak{g}) \subset L^2(M, \mathfrak{g})$. Since the embedding of $H^1(M, \mathfrak{g})$ into $L^2(M, \mathfrak{g})$ is compact, it follows that the sequence $\{S\mathbf{v}_n\}$ has a convergent subsequence. That is, the operator S is compact. This implies that $dJ_{\mathbf{p}} = T + S$ is semi-Fredholm, that is it has closed range and finite codimension, cf. [12]. □

For each $\mathbf{p} \in \mathbf{P}$ we denote by $gs_{\mathbf{p}}$ the gauge symmetry (isotropy) algebra of $\mathbf{p}$, that is

$$gs_{\mathbf{p}} = \{\xi \in gs(\mathbf{P}) | \xi_{\mathbf{P}}(\mathbf{p}) = 0\} , \tag{2.9}$$

and by $GS_{\mathbf{p}}$ gauge symmetry (isotropy) group of $\mathbf{p}$,

$$GS_{\mathbf{p}} = \{\phi \in GS(\mathbf{P}) | \phi\mathbf{p} = \mathbf{p}\} . \tag{2.10}$$

By properness of the action of $GS(\mathbf{P})$ in $\mathbf{P}$, each sequence $\{\phi_n\}$ in $GS_{\mathbf{p}}$ has a convergent subsequence, which implies that $GS_{\mathbf{p}}$ is compact. Consequently, the Lie algebra $gs_{\mathbf{p}}$ is finite dimensional. It is isomorphic to a subalgebra of the structure algebra $\mathfrak{g}$; a construction of such an isomorphism is given in Sect. 5.

The annihilator of a subalgebra $\mathfrak{h} \subseteq gs(\mathbf{P})$ is the subspace $\mathfrak{h}^a \subseteq gs(\mathbf{P})^*$ defined by

$$\mathfrak{h}^a = \{\nu \in gs(\mathbf{P})^* | \langle \nu | \xi \rangle = 0 \ \forall \ \xi \in \mathfrak{h}\} . \tag{2.11}$$

Proposition 2.4. *The range of the map $dJ_{\mathbf{p}} : T_{\mathbf{p}}\mathbf{P} \to gs(\mathbf{P})^*$ is given by the annihilator of the symmetry algebra of* $\mathbf{p}$, *that is*

$$\operatorname{range} dJ_{\mathbf{p}} = (gs_{\mathbf{p}})^a . \tag{2.12}$$

Proof. By (1.11), for each $\xi \in gs(\mathbf{P})$, and $\mathbf{p} \in \mathbf{P}$,

$$\langle dJ_{\mathbf{p}}(\cdot) | \xi \rangle = \xi_{\mathbf{P}}(\mathbf{p}) \lrcorner \omega . \tag{2.13}$$

Since ω is non-degenerate, it follows from (2.9) that

$$gs_{\mathbf{p}} = \{\xi \in gs(\mathbf{P}) | \langle dJ_{\mathbf{p}}(\mathbf{v}) | \xi \rangle = 0 \ \forall \ \mathbf{v} \in T_{\mathbf{p}}\mathbf{P}\} = (\operatorname{range} dJ_{\mathbf{p}})^a . \tag{2.14}$$

Since $\operatorname{range} dJ_{\mathbf{p}}$ is closed, taking annihilators of both sides we obtain

$$(gs_{\mathbf{p}})^a = (\operatorname{range} dJ_{\mathbf{p}})^{aa} = \operatorname{range} dJ_{\mathbf{p}} ,$$

provided that $(\operatorname{range} dJ_{\mathbf{p}})^{aa}$ is the closure of $\operatorname{range} dJ_{\mathbf{p}}$.

In order to prove the last assertion, denote by $R_{\mathbf{p}}$ the closure of $\operatorname{range} dJ_{\mathbf{p}}$ in the topological dual $gs(\mathbf{P})'$ of $gs(\mathbf{P})$. The polar of $R_{\mathbf{p}}$ is

$$(R_{\mathbf{p}})^0 = \{\xi \in gs(\mathbf{P}) | \langle \nu | \xi \rangle = 0 \ \forall \ \nu \in R_{\mathbf{p}}\} ,$$

and the bi-polar

$$(R_{\mathbf{p}})^{00} = \{\nu \in gs(\mathbf{P})' | \langle \nu | \xi \rangle = 0 \ \forall \ \xi \in (R_{\mathbf{p}})^0\}$$

is the closure of $R_{\mathbf{p}}$ in $gs(\mathbf{P})'$, cf. [13]. By definition $R_{\mathbf{p}}$ is closed so that $R_{\mathbf{p}} = (R_{\mathbf{p}})^{00}$. Since $\operatorname{range} dJ_{\mathbf{p}}$ is dense in $R_{\mathbf{p}}$, it follows that

$$(R_{\mathbf{p}})^0 = (\operatorname{range} dJ_{\mathbf{p}})^a .$$

Hence,

$$(\operatorname{range} dJ_{\mathbf{p}})^{aa} = (R_{\mathbf{p}})^{00} \cap gs(\mathbf{P})^* = R_{\mathbf{p}} \cap gs(\mathbf{P})^* ,$$

which implies that $(\operatorname{range} dJ_{\mathbf{p}})^{aa}$ is the closure of $\operatorname{range} dJ_{\mathbf{p}}$ in $gs(\mathbf{P})^*$. □

We conclude from Proposition 2.4 that $\mathbf{p}$ is a regular point of the momentum map J if and only if $\mathbf{p}$ has no infinitesimal symmetries, i.e. $gs_{\mathbf{p}} = \{0\}$. In this case $J^{-1}(J(\mathbf{p}))$ is a manifold in a neighbourhood of $\mathbf{p}$ with the tangent space

$$T_{\mathbf{p}}J^{-1}(J(\mathbf{p})) = \ker dJ_{\mathbf{p}} .$$

Singular points of the momentum map have non-trivial algebras of infinitesimal symmetries.

The next essential property of the action of $GS(\mathbf{P})$ in $\mathbf{P}$ needed here is the existence of slices. A slice through a point $\mathbf{p} \in \mathbf{P}$ for the action of $GS(\mathbf{P})$ is a submanifold $\mathbf{S_p}$ of $\mathbf{P}$ containing $\mathbf{p}$, and such that

(1) $\mathbf{S_p}$ is transverse and complementary to the orbit $O_\mathbf{p}$ at $\mathbf{p}$, that is

$$T_\mathbf{p}\mathbf{S_p} \oplus T_\mathbf{p}O_\mathbf{p} = T_\mathbf{p}\mathbf{P} . \tag{2.15}$$

(2) $\mathbf{S_p}$ is transverse to all $GS(\mathbf{P})$ orbits, that is, for each $\mathbf{q} \in \mathbf{S_p}$,

$$T_\mathbf{q}\mathbf{S_p} + T_\mathbf{q}O_\mathbf{q} = T_\mathbf{q}\mathbf{P} . \tag{2.16}$$

(3) $\mathbf{S_p}$ is invariant under the action of the gauge symmetry group $GS_\mathbf{p}$ of $\mathbf{p}$.
(4) For $\mathbf{q} \in \mathbf{S_p}$ and $\phi \in GS(\mathbf{P})$, if $\phi\mathbf{q} \in \mathbf{S_p}$ then $\phi \in GS_\mathbf{p}$.
The last condition implies that

$$GS_\mathbf{q} \subseteq GS_\mathbf{p} \; \forall \; \mathbf{q} \in \mathbf{S_p} . \tag{2.17}$$

A slice $\mathbf{S_p}$ through $\mathbf{p}$ gives rise to an open neighbourhood $\mathbf{U_p}$ of $\mathbf{p} \in \mathbf{P}$ of the form

$$\mathbf{U_p} = \mathbf{S_p} \times \mathbf{V_p} , \tag{2.18}$$

where $\mathbf{V_p}$ is an open neighbourhood of $\mathbf{p}$ in the orbit $O_\mathbf{p}$. It will be referred to as a slice neighbourhood of $\mathbf{p}$. A slice $\mathbf{S_p}$ will be called affine if it is an open subset of a closed affine subspace of $\mathbf{P}$.

Theorem 2 (Slice Theorem). *For each* $\mathbf{p} \in \mathbf{P}$, *there exists an affine slice* $\mathbf{S_p}$ *through* $\mathbf{p}$ *for the action of* $GS(\mathbf{P})$, *which is* L^2*-orthogonal to* $T_\mathbf{p}O_\mathbf{p}$.

Proof is given in Appendix C.

Let H be a compact subgroup of H of $GS(\mathbf{P})$. We denote by $\mathbf{P}_H$, $\mathbf{P}_{[H]}$, and $\mathbf{P}_{(H)}$ the sets of points $\mathbf{p}$ in $\mathbf{P}$ such that $GS_\mathbf{p} = H$, $GS_\mathbf{p} \supseteq H$, and $GS_\mathbf{p}$ is conjugate to H, respectively,

$$\mathbf{P}_H = \{\mathbf{p} \in \mathbf{P} | GS_\mathbf{p} = H\} , \tag{2.19}$$

$$\mathbf{P}_{[H]} = \{\mathbf{p} \in \mathbf{P} | GS_\mathbf{p} \supseteq H\} , \tag{2.20}$$

$$\mathbf{P}_{(H)} = \{\mathbf{p} \in \mathbf{P} | \; \exists \; \phi \in GS(\mathbf{P}) \text{ such that } GS_\mathbf{p} = \phi H \phi^{-1}\} . \tag{2.21}$$

Note that $\mathbf{P}_{(H)}$ is the union of the $GS(\mathbf{P})$ orbits through the points of $\mathbf{P}_H$,

$$\mathbf{P}_{(H)} = \{\phi\mathbf{p} | \phi \in GS(\mathbf{P}), \; \mathbf{p} \in \mathbf{P}_H\} . \tag{2.22}$$

Proposition 2.5.

(1) $\mathbf{P}_{[H]}$ *is a closed affine subspace of* $\mathbf{P}$.
(2) *For every* $\mathbf{p} \in \mathbf{P}_H$,

$$\mathbf{P}_H \cap \mathbf{S_p} = \mathbf{P}_{[H]} \cap \mathbf{S_p} , \tag{2.23}$$

where $\mathbf{S_p}$ *is an affine slice through* $\mathbf{p}$, *is an open subset of a closed affine subspace of* $\mathbf{P}$.

(3) $\mathbf{P}_H$ *is locally a submanifold of* $\mathbf{P}$, *that is connected components of* $\mathbf{P}_{(H)}$ *are submanifolds of* $\mathbf{P}$.

Proof.
(1) Follows from the fact that the action of $GS(\mathbf{P})$ is continuous and affine.

(2) Clearly, $\mathbf{P}_H \cap \mathbf{S_p} \subseteq \mathbf{P}_{[H]} \cap \mathbf{S_p}$. Suppose $\mathbf{q} \in \mathbf{P}_{[H]} \cap \mathbf{S_p}$. By definition, $GS_\mathbf{q} \supseteq H$. However, (2.17) implies that $GS_\mathbf{q} \subseteq H$. Hence, $GS_\mathbf{q} = H$ and $\mathbf{q} \in \mathbf{P}_H$. Therefore, $\mathbf{P}_H \cap \mathbf{S_p} = \mathbf{P}_{[H]} \cap \mathbf{S_p}$. Since $\mathbf{P}_{[H]}$ is a closed affine subspace and $\mathbf{S_p}$ is an open subset of a closed affine subspace, it follows that $\mathbf{P}_H \cap \mathbf{S_p}$ is an open subset of a closed affine subspace of $\mathbf{P}$.

(3) Each $\mathbf{q} \in \mathbf{P}_{(H)}$ has a neighbourhood in $\mathbf{P}_{(H)}$ of the form $\mathbf{V_q} \times (\mathbf{S_q} \cap \mathbf{P}_{\tilde{H}})$, where $\tilde{H} = GS_\mathbf{q}$ and $\mathbf{V_q}$ is a neighbourhood of $GS(\mathbf{P})$ orbit through $\mathbf{q}$, cf. (2.18). Since both factors are submanifolds of $\mathbf{P}$, it follows that $\mathbf{P}_{(H)}$ is locally a submanifold of $\mathbf{P}$. □

3. Constraints

The constraint set is the zero level of the momentum map J. It follows from Proposition 2.4 that $J^{-1}(0)$ need not be a manifold in neighbourhoods of points admitting infinitesimal symmetries. We shall show that it is partitioned by presymplectic submanifolds labelled by conjugacy classes (H) of compact subgroups of $GS(\mathbf{P})$.

For each compact subgroup H of $GS(\mathbf{P})$, we denote by $\mathbf{M}_{(H)}$ the intersection of $J^{-1}(0)$ with the submanifold $\mathbf{P}_{(H)}$,

$$\mathbf{M}_{(H)} = J^{-1}(0) \cap \mathbf{P}_{(H)} \,. \tag{3.1}$$

If $(H_1) \neq (H_2)$, then $\mathbf{P}_{(H_1)} \cap \mathbf{P}_{(H_2)} = \emptyset$. Hence, the constraint set is the union of disjoint sets $\mathbf{M}_{(H)}$,

$$J^{-1}(0) = \bigcup_{(H)} \mathbf{M}_{(H)} \,, \tag{3.2}$$

where the union is taken over the conjugacy classes of compact subgroups of $GS(\mathbf{P})$.

Theorem 3. *For every compact subgroup H of $GS(\mathbf{P})$, $\mathbf{M}_{(H)}$ is locally a submanifold of $(\mathbf{P}, \omega)$. The null distribution of ω restricted to $\mathbf{M}_{(H)}$ consisting of the vectors tangent to the $GS(\mathbf{P})$ orbits in $\mathbf{M}_{(H)}$.*

The proof of this theorem will be given in a series of propositions.

The momentum map J restricted to $\mathbf{P}_H$ has values in the annihilator $\mathfrak{h}^a$ of the Lie algebra $\mathfrak{h}$ of H because $\langle J(\mathbf{q}), \xi \rangle = \langle \theta, \xi_\mathbf{P}(\mathbf{q}) \rangle$, and $\xi_\mathbf{P}(\mathbf{q}) = 0$ for all $\xi \in \mathfrak{h}$ and $\mathbf{q} \in \mathbf{P}_H$. The Ad^* action of $GS(\mathbf{P})$ in $gs(\mathbf{P})^*$ is given by,

$$\langle \mathrm{Ad}^*_\phi \mu, \xi \rangle = \langle \mu, \mathrm{Ad}_\phi \xi \rangle = \langle \mu, \phi^{-1} \xi \phi \rangle \,, \tag{3.3}$$

for all $\phi \in GS(\mathbf{P})$, $\mu \in gs(\mathbf{P})^*$, and $\xi \in gs(\mathbf{P})$. Since J is Ad^* equivariant,

$$\mathrm{Ad}^*_\phi J(\mathbf{q}) = J(\phi \mathbf{q}) \ \forall \ \phi \in GS(\mathbf{P}) \,, \tag{3.4}$$

it follows that,

$$\mathrm{Ad}^*_\phi J(\mathbf{q}) = J(\mathbf{q}) \ \forall \ \phi \in H \,. \tag{3.5}$$

Let $\mathfrak{h}^a_H$ denote the subspace of $gs(\mathbf{P})^*$ consisting of the $\mu \in \mathfrak{h}^a$ satisfying

$$\mathrm{Ad}^*_\phi \mu = \mu \ \forall \ \phi \in H\,. \tag{3.6}$$

It is a closed subspace of $gs(\mathbf{P})^*$, and hence,

$$gs(\mathbf{P})^* = \mathfrak{h}^a_H \oplus (\mathfrak{h}^a_H)^\perp\,, \tag{3.7}$$

where $(\mathfrak{h}^a_H)^\perp$ denotes the L^2 orthogonal complement of $\mathfrak{h}^a_H$. We denote by $\pi_H : \mathbf{P} \to \mathfrak{h}^a_H$ the projection on the first component, and by K_H the composition of J with π_H,

$$K_H = \pi_H \circ J : \mathbf{P} \to \mathfrak{h}^a_H\,. \tag{3.8}$$

Proposition 3.1.

$$J^{-1}(0) \cap \mathbf{P}_H = K_H^{-1}(0) \cap \mathbf{P}_H\,. \tag{3.9}$$

Proof. Clearly, $J(\mathbf{p}) = 0$ implies $K_H(\mathbf{p}) = 0$. Hence $J^{-1}(0) \cap \mathbf{P}_H$ is contained in $K_H^{-1}(0) \cap \mathbf{P}_H$. Conversely, let $\mathbf{p} \in K_H^{-1}(0) \cap \mathbf{P}_H$. Equations (3.5) and (3.6) imply that $J(\mathbf{p}) \in \mathfrak{h}^a_H$. Since the projection $K_H(\mathbf{p})$ of $J(\mathbf{p})$ to $\mathfrak{h}^a_H$ vanishes, it follows that $J(\mathbf{p}) = 0$. Hence, $J^{-1}(0) \cap \mathbf{P}_H \supseteq K_H^{-1}(0) \cap \mathbf{P}_H$, and $J^{-1}(0) \cap \mathbf{P}_H = K_H^{-1}(0) \cap \mathbf{P}_H$. □

For each $\mathbf{p} \in \mathbf{P}_H$, we denote by $\mathbf{S}_\mathbf{p}$ an affine slice through $\mathbf{p}$ which is L^2 orthogonal to the tangent space $T_\mathbf{p}O_\mathbf{p}$ of the $GS(\mathbf{P})$ orbit through $\mathbf{p}$.

Proposition 3.2.

$K_H^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p}$ *is a submanifold of* $\mathbf{P}$ *in a neighbourhood of* $\mathbf{p}$.

Proof. By Proposition 2.4 range $dJ_\mathbf{p} = \mathfrak{h}^a$. Hence, for every $\mu \in \mathfrak{h}^a_H$, there exists a unique vector $\mathbf{u}_\mu \in T_\mathbf{p}\mathbf{P}$, L^2-orthogonal to $\ker dJ_\mathbf{p}$, such that

$$dJ_\mathbf{p}(\mathbf{u}_\mu) = \mu\,. \tag{3.10}$$

By definition of K_H, Eq. (3.10) is equivalent to

$$dK_\mathbf{p}(\mathbf{u}_\mu) = \mu\,. \tag{3.11}$$

Since the action of $GS(\mathbf{P})$ in $\mathbf{P}$ preserves the Riemannian structure given by the L^2 scalar product, and $\ker dJ_\mathbf{p}$ is invariant under the action of H it follows that the L^2 orthogonal complement of $\ker dJ_\mathbf{p}$ is H invariant. Hence, Eqs. (3.4) and (3.6) imply that $\mathbf{u}_\mu$ is H invariant. This implies that the action of H fixes every point of the affine line $\mathbf{q}(t) = \mathbf{p} + t\mathbf{u}_\mu$ in $\mathbf{P}$. Therefore, for every $t \in \mathbb{R}$, $GS_{\mathbf{q}(t)} \supseteq H$ so that $\mathbf{q}(t) \in \mathbf{P}_{[H]}$. Differentiating with respect to t we get $\mathbf{u}_\mu \in T_\mathbf{p}\mathbf{P}_{[H]}$.

For every ξ and $\zeta \in gs(\mathbf{P})$, $\langle dJ_\mathbf{p}(\xi_\mathbf{P}) | \zeta \rangle = \langle J(\mathbf{p}) | [\xi, \zeta] \rangle$. Hence, $J(\mathbf{p}) = 0$ implies that $\xi_\mathbf{P}(\mathbf{p}) \in \ker dJ_\mathbf{p}$ for all $\xi \in gs(\mathbf{P})$. Since $\{\xi_\mathbf{P}(\mathbf{p}) | \xi \in gs(\mathbf{P})\}$ spans $T_\mathbf{p}O_\mathbf{p}$, it follows that

$$T_\mathbf{p}O_\mathbf{p} \subseteq \ker dJ_\mathbf{p}\,. \tag{3.12}$$

By assumption $T_\mathbf{p}\mathbf{S}_\mathbf{p}$ is the L^2 orthogonal complement of $T_\mathbf{p}O_\mathbf{p}$ and $\mathbf{u}_\mu$ is L^2 orthogonal to $\ker dJ_\mathbf{p}$. Hence, it follows that $\mathbf{u}_\mu \in T_\mathbf{p}\mathbf{S}_\mathbf{p}$.

The above results imply that $\mathbf{u}_\mu \in T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$. Taking into account (3.11), we see this means that $\mathbf{p}$ is a regular point of the restriction $K_H | (\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$ of K_H

to $\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$. Hence, $K_H^{-1}(0) \cap \mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p} = (K_H|(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}))^{-1}(0)$ is a submanifold of $\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$ in a neighbourhood of $\mathbf{p}$. By Proposition 2.5 $K_H^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p} = K_H^{-1}(0) \cap \mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$, and it is a submanifold of $\mathbf{P}$ in a neighbourhood of $\mathbf{p}$. □

Corollary 3.3. $\mathbf{M}_{(H)}$ *is locally a submanifold of* $\mathbf{P}$.

Proof. For each $\mathbf{q} \in \mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p}$, $GS_\mathbf{q}$ is conjugate to H and contained in H. Hence $GS_\mathbf{q} = H$, and $\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p} \subseteq \mathbf{P}_H$. Hence,

$$\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p} = J^{-1}(0) \cap \mathbf{P}_{(H)} \cap \mathbf{S}_\mathbf{p} = J^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p} = K_H^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p}\,,$$

which is a submanifold of $\mathbf{P}$ in a neighbourhood of $\mathbf{p}$ by Proposition 3.2.

As a consequence of the Slice Theorem, each point $\mathbf{p} \in \mathbf{M}_{(H)}$ has an open neighbourhood in $\mathbf{M}_{(H)}$ obtained by the intersection of $\mathbf{M}_{(H)}$ with slice neighbourhood of $\mathbf{p} \in \mathbf{P}$. By (2.18) it is of the form $(\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p}) \times \mathbf{V}_\mathbf{p}$, where $\mathbf{V}_\mathbf{p}$ is an open neighbourhood of $\mathbf{p}$ in the orbit $O_\mathbf{p}$. Since $(\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p}) \times \mathbf{V}_\mathbf{p}$ is a submanifold of $\mathbf{P}$, it follows that $\mathbf{M}_{(H)}$ is locally a submanifold of $\mathbf{P}$. □

Proposition 3.4.

$$T_\mathbf{p}(K_H^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p}) = \ker dJ_\mathbf{p} \cap T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})\,. \tag{3.13}$$

Proof. In the proof of Proposition 3.2 we have shown that $\mathbf{p}$ is a regular point of the restriction $K_H(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$ of K_H to $\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$. By Proposition 2.5, $\mathbf{P}_H \cap \mathbf{S}_\mathbf{p} = \mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$ so that

$$T_\mathbf{p}(K_H^{-1}(0) \cap \mathbf{P}_H \cap \mathbf{S}_\mathbf{p}) = (\ker dK_H) \cap T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})\,. \tag{3.14}$$

Since $\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$ is an open subset a closed affine subspace of $\mathbf{P}$, it follows that, for each $\mathbf{u} \in T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$, a neighbourhood of $\mathbf{p}$ in the affine line $\mathbf{q}(t) = \mathbf{p} + t\mathbf{u}$ is contained in $\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}$. Equation (3.5) implies that $\mathrm{Ad}^*_\phi J(\mathbf{q}(t)) = J(\mathbf{q}(t))$ for all $\phi \in H$. Differentiating with respect to t, we obtain $\mathrm{Ad}^*_\phi dJ_\mathbf{p}(\mathbf{u}) = dJ_\mathbf{p}(\mathbf{u})$ for all ϕ in H. Taking into account Proposition 2.4 we obtain $dJ_\mathbf{p}(\mathbf{u}) \in \mathfrak{h}^a_H$. Hence, $dK_H(\mathbf{u}) = \pi_H \circ dJ_\mathbf{p}(\mathbf{u}) = dJ_\mathbf{p}(\mathbf{u})$ for every $\mathbf{u} \in T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$. Hence,

$$dJ|T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}) = dK_H|T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})\,. \tag{3.15}$$

In particular, $\ker dJ \cap T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p}) = \ker dK_H \cap T_\mathbf{p}(\mathbf{P}_{[H]} \cap \mathbf{S}_\mathbf{p})$ which, together with (3.14), implies (3.13). □

Proposition 3.5. *For each* $\mathbf{p} \in \mathbf{P}_H$, *the restriction of* ω *to* $T_\mathbf{p}(\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p})$ *is non-degenerate.*

Proof. Propositions 3.1 and 3.4 imply that

$$T_\mathbf{p}(\mathbf{M}_{(H)} \cap \mathbf{S}_\mathbf{p}) = (\ker dJ_\mathbf{p}) \cap T_\mathbf{p}\mathbf{P}_{[H]} \cap T_\mathbf{p}\mathbf{S}_\mathbf{p}\,. \tag{3.16}$$

In order to show that it is symplectic we need the almost complex structure $\mathcal{J}$ discussed in Appendix A.

By assumption $T_{\mathbf{p}}\mathbf{S}_{\mathbf{p}}$ is L^2 orthogonal to $T_{\mathbf{p}}O_{\mathbf{p}}$. Hence, Eqs. (2.7) and (A.7) imply, that

$$\mathcal{J}(\ker dJ_{\mathbf{p}}) = ((\ker dJ_{\mathbf{p}})^{\perp})^{\omega} = ((\ker dJ_{\mathbf{p}})^{\omega})^{\perp} = (T_{\mathbf{p}}O_{\mathbf{p}})^{\perp} \subseteq T_{\mathbf{p}}S_{\mathbf{p}}$$

and

$$\mathcal{J}(T_{\mathbf{p}}\mathbf{S}_{\mathbf{p}}) = ((T_{\mathbf{p}}\mathbf{S}_{\mathbf{p}})^{\perp})^{\omega} = (T_{\mathbf{p}}O_{\mathbf{p}})^{\omega} = \ker dJ_{\mathbf{p}} \,,$$

so that

$$\mathcal{J}((\ker dJ_{\mathbf{p}}) \cap T_{\mathbf{p}}\mathbf{S}_{\mathbf{p}}) = (\ker dJ_{\mathbf{p}}) \cap T_{\mathbf{p}}\mathbf{S}_{\mathbf{p}} \,. \tag{3.17}$$

Moreover, the action of $GS(\mathbf{P})$ preserves $\mathcal{J}$, which implies that

$$\mathcal{J}(T_{\mathbf{p}}\mathbf{P}_{[H]}) = T_{\mathbf{p}}\mathbf{P}_{[H]} \,. \tag{3.18}$$

Let $\mathbf{v} \in T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ be such that $\omega(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$. Then $\tilde{\omega}(\mathbf{v}, \tilde{\mathbf{w}}) = 0$ for all $\tilde{\mathbf{w}}$ in the L^2 closure $\tilde{T}_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ of $T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$. By (3.17) and (3.18), $\mathcal{J}$ maps $\tilde{T}_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ to itself. Hence, taking $\tilde{\mathbf{w}} = \mathcal{J}\mathbf{v}$, we get

$$\|\mathbf{v}\|_{L^2}^2 = \langle \mathbf{v}|\mathbf{v}\rangle_{L^2} = \tilde{\omega}(\mathcal{J}\mathbf{v}, \mathbf{v}) = 0 \,.$$

Therefore $\mathbf{v} = 0$, which implies that the restriction of ω to $T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ is non-degenerate. □

Corollary 3.6. *$\mathbf{M}_{(H)}$ is locally a submanifold of $(\mathbf{P}, \omega)$. The null distribution of ω restricted to $\mathbf{M}_{(H)}$ consisting of the vectors tangent to the $GS(\mathbf{P})$ orbits in $\mathbf{M}_{(H)}$.*

Proof. It follows from Proposition 2.2 and (3.12) that $T_{\mathbf{p}}O_{\mathbf{p}} = (\ker dJ_{\mathbf{p}})^{\omega} \subseteq (\ker dJ_{\mathbf{p}})$. Hence, $\omega(\mathbf{v}, \mathbf{w}) = 0$ for every $\mathbf{v} \in TO_{\mathbf{p}}$ and $\mathbf{w} \in T_{\mathbf{p}}\mathbf{M}_{(H)}$. Since $T_{\mathbf{p}}\mathbf{M}_{(H)} = T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}}) + T_{\mathbf{p}}O_{\mathbf{p}}$, and $T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ symplectic by Proposition 3.5, it follows that $T_{\mathbf{p}}O_{\mathbf{p}}$ is the null space of $T_{\mathbf{p}}\mathbf{M}_{(H)}$. □

4. Reduction

The reduced phase space $\check{\mathbf{P}}$ of the system is defined as the space of $GS(\mathbf{P})$ orbits in the constraint set $J^{-1}(0)$,

$$\check{\mathbf{P}} = J^{-1}(0)/GS(\mathbf{P}) \,. \tag{4.1}$$

We denote by $\rho : J^{-1}(0) \to \check{\mathbf{P}}$ the natural projection, assigning to each $\mathbf{p} \in J^{-1}(0)$ the orbit $O_{\mathbf{p}} \in \check{\mathbf{P}}$,

$$\rho(\mathbf{p}) = O_{\mathbf{p}} \,. \tag{4.2}$$

Since the action of $GS(\mathbf{P})$ in $\mathbf{P}$ is proper, the quotient topology in $\check{\mathbf{P}}$ is Hausdorff. This can be seen as follows. If $\mathbf{p}, \mathbf{q} \in J^{-1}(0)$ are such that $\rho(\mathbf{p})$ and $\rho(\mathbf{q})$ cannot be separated by open sets, then there exists a sequence $\mathbf{p}_n$ in $J^{-1}(0)$ such that $\rho(\mathbf{p}_n)$ converges both to $\rho(\mathbf{p})$ and $\rho(\mathbf{q})$. Let $\mathbf{S}_{\mathbf{p}}$ and $\mathbf{S}_{\mathbf{q}}$ be slices through $\mathbf{p}$ and $\mathbf{q}$, respectively. For sufficiently large n, there exist $\phi_n, \psi_n \in GS(\mathbf{P})$ such that $\phi_n \mathbf{p}_n \in \mathbf{S}_{\mathbf{p}}$

and $\psi_n \mathbf{p}_n \in \mathbf{S_q}$. Hence, $\phi_n \mathbf{p}_n \to \mathbf{p}$ and $\psi_n \mathbf{p}_n \to \mathbf{q}$ as $n \to \infty$. Thus, $\phi_n \psi_n^{-1}(\psi_n \mathbf{p}_n) \to \mathbf{p}$, while $\psi_n \mathbf{p}_n \to \mathbf{q}$, which implies that $\phi_n \psi_n^{-1}$ has a convergent subsequence with limit χ and $\chi \mathbf{q} = \mathbf{p}$. Hence, $\mathbf{p} \in O_\mathbf{q}$ and $\rho(\mathbf{p}) = \rho(\mathbf{q})$.

For every compact subgroup H of $GS(\mathbf{P})$, we denote by $\check{\mathbf{P}}_{(H)}$ the projection of $\mathbf{M}_{(H)} = J^{-1}(0) \cap \mathbf{P}_{(H)}$ to $\check{\mathbf{P}}$,

$$\check{\mathbf{P}}_{(H)} = \rho(\mathbf{M}_{(H)}) , \tag{4.3}$$

and by $\rho_{(H)} : \mathbf{M}_{(H)} \to \check{\mathbf{P}}_{(H)}$ the restriction of ρ to $\mathbf{M}_{(H)}$, considered as a map to $\check{\mathbf{P}}_{(\mathfrak{h})}$. Thus,

$$\check{\mathbf{P}} = \bigcup_{(H)} \check{\mathbf{P}}_{(H)} , \tag{4.4}$$

where the union is taken over the conjugacy classes of compact subgroups of $GS(\mathbf{P})$.

Theorem 4. *For each conjugacy class (H), $\check{\mathbf{P}}_{(H)}$ is a quotient manifold of $\mathbf{M}_{(H)}$ endowed with a weakly symplectic form*

$$\check{\omega}_{(H)} = d\check{\theta}_{(H)} \tag{4.5}$$

such that

$$\rho^*_{(H)}\check{\theta}_{(H)} = \imath^*_{(H)}\theta \quad \text{and} \quad \rho^*_{(H)}\check{\omega}_{(H)} = \imath^*_{(H)}\omega , \tag{4.6}$$

where $\imath_{(H)} : \mathbf{M}_{(H)} \to \mathbf{P}$ is the inclusion map.

Proof. For each $\mathbf{p} \in \mathbf{M}_{(H)}$, let $\mathbf{S_p}$ be an affine slice through $\mathbf{p}$ normal to $T_\mathbf{p} O_\mathbf{p}$. By Proposition 3.1, $\mathbf{M}_{(H)} \cap \mathbf{S_p}$ is a submanifold of $\mathbf{P}$ in a neighbourhood $\mathbf{U}$ of $\mathbf{p}$ and it is contained in $\mathbf{P}_H$, where $H = GS_\mathbf{p}$. The image of $\mathbf{U}$ under $\rho_{(H)}$ is the space of the equivalence classes under the equivalence relation $\sim$ in $\mathbf{U}$, given by $\mathbf{p}_1 \sim \mathbf{p}_2$ if and only if $\mathbf{p}_1 = \phi \mathbf{p}_2$ for some $\phi \in GS(\mathbf{P})$. The Slice Theorem implies that $\phi \in H$. Since $\mathbf{U} \subseteq \mathbf{P}_H$, it follows that $\phi \mathbf{p}_2 = \mathbf{p}_2$. Hence, $\rho_{(H)}|\mathbf{U}$ is a bijection of $\mathbf{U}$ onto its image, and it is a local homeomorphism in the quotient topology of $\check{\mathbf{P}}_{(H)}$.

Since $\mathbf{U}$ is a submanifold of $\mathbf{P}$, the collection of maps $\{(\rho_{(H)}|\mathbf{U})^{-1}\}$ induces an atlas in $\check{\mathbf{P}}$. Suppose that $\check{\mathbf{p}} \in \rho_{(H)}(\mathbf{U}_1) \cap \rho_{(H)}(\mathbf{U}_2)$. Then, there exists an open neighbourhood $\check{\mathbf{V}}$ of $\check{\mathbf{p}}$ in $\rho_{(H)}(\mathbf{U}_1) \cap \rho_{(H)}(\mathbf{U}_2)$. Let $\mathbf{V}_i = \rho_{(H)}^{-1}(\check{\mathbf{V}}) \cap \mathbf{U}_i$, and $\mathbf{p}_i$ be in the intersection of $\mathbf{V}_i$ with the fibre $\rho_{(H)}^{-1}(\check{\mathbf{p}})$, $i = 1, 2$. Then, there exists $\phi \in GS(\mathbf{P})$ such that $\mathbf{p}_2 = \phi \mathbf{p}_1$. Consider the affine slice $\mathbf{S}_{\mathbf{p}_1}$ through $\mathbf{p}_1$ orthogonal to $T_{\mathbf{p}_1} O_{\mathbf{p}_1}$ such that $\rho_{(H)}(\mathbf{S}_{\mathbf{p}_1} \cap \mathbf{M}_{(H)}) = \mathbf{V}$ (such a slice always exists for a sufficiently small $\mathbf{V}$). Since $\mathbf{V}_1$ and $\mathbf{S}_{\mathbf{p}_1} \cap \mathbf{M}_{(H)}$ are smooth submanifolds of $\mathbf{P}$, projecting onto $\mathbf{V}$ and $\mathbf{p}_1 \in \mathbf{V}_1 \cap (\mathbf{S}_{\mathbf{p}_1} \cap \mathbf{M}_{(H)})$, it follows that there exists a smooth map $\Phi_1 : \mathbf{S}_{\mathbf{p}_1} \cap \mathbf{M}_{(H)} \to GS(\mathbf{P})$ such that the map

$$\hat{\Phi}_1 : \mathbf{S}_{\mathbf{p}_1} \cap \mathbf{M}_{(H)} \to \mathbf{V}_1 : \mathbf{q} \mapsto \hat{\Phi}_1(\mathbf{q})\mathbf{q}$$

is a diffeomorphism. In a similar way we can construct a diffeomorphism $\hat{\Phi}_2$: $\mathbf{S}_{\mathbf{p}_2} \cap \mathbf{M}_{(H)} \to \mathbf{V}_2$, where $\mathbf{S}_{\mathbf{p}_2} = \phi \mathbf{S}_{\mathbf{p}_1}$ is an affine slice through $\mathbf{p}_2$ orthogonal to $T_{\mathbf{p}_2} O_{\mathbf{p}_2}$ and such that $\rho_{(H)}(\mathbf{S}_{\mathbf{p}_2} \cap \mathbf{M}_{(H)}) = \mathbf{V}$. The map $\mathbf{q} \mapsto \hat{\Phi}_2(\phi \hat{\Phi}_1^{-1}(\mathbf{q}))\phi \hat{\Phi}_1^{-1}(\mathbf{q})$ is a diffeomorphism of $\mathbf{V}_1$ onto $\mathbf{V}_2$. This guarantees that the atlas induced by the maps $\{(\rho_{(H)}|\mathbf{U})^{-1}\}$ defines a differentiable structure in $\check{\mathbf{P}}$ of class C^∞.

For each $\mathbf{p} \in \mathbf{M}_{(H)}$ and $\xi \in GS(\mathbf{P})$, $\langle\theta|\xi_{\mathbf{P}}(\mathbf{p})\rangle = J_{\xi}(\mathbf{p}) = 0$. Hence, $\theta_{\mathbf{p}}$ is annihilated by the vectors in $T_{\mathbf{p}}O_{\mathbf{p}}$. Moreover, θ is $GS(\mathbf{P})$ invariant. Hence, the pullback $\iota_{(H)}^{*}\theta$ of θ to $\mathbf{M}_{(H)}$ pushes forward to a 1-form $\check{\theta}$ on $\check{\mathbf{P}}$ satisfying (4.6).

By Proposition 3.5, the restriction of ω to $T_{\mathbf{p}}(\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}})$ is non-degenerate. Since $\rho_{(H)}$ is a diffeomorphism of a neighbourhood of $\mathbf{p}$ in $\mathbf{M}_{(H)} \cap \mathbf{S}_{\mathbf{p}}$ onto its image, and $T_{\mathbf{p}}O_{\mathbf{p}}$ is the null space of $\iota_{(H)}^{*}\omega_{\mathbf{p}}$, it follows that $\iota_{(H)}^{*}\omega$ pushes forward to a non-degenerate form $\check{\omega}$ on $\check{\mathbf{P}}$ which satisfies (4.6). Equation (4.5) follows from $\iota_{(H)}^{*}\omega = d(\iota_{(H)}^{*}\theta)$. □

5. Symmetry Breaking

Yang–Mills potentials represent connections in a right principal bundle Q over M with structure group G. Since M is contractible, the bundle Q is trivial,

$$Q = M \times G \tag{5.1}$$

and the action of G in Q is given by

$$Q \times G \to Q : ((x,g),h) \mapsto ((x,g) \cdot h) = (x, gh)\,. \tag{5.2}$$

The associated bundle $Q[G]$ of Q with typical fibre G and the adjoint action of G on itself is called the group bundle of Q. Sections of $Q[G]$ correspond to automorphisms of Q covering the identity transformation in M. In this context, the group $GS(\mathbf{P})$ of gauge symmetries of $\mathbf{P}$ can be identified with the group of sections of $Q[G]$, of class $H^3(M)$, which satisfy the boundary condition (2.4).

Sections of associated bundles correspond to equivariant maps from the principal bundle to the typical fibre. Thus, each element $\phi \in GS(\mathbf{P})$ corresponds to a map $\phi^{\#} : Q \to G$ such that, for every $(x,g) \in Q$,

$$\phi^{\#}((x,g)) = g^{-1}\phi(x)g\,. \tag{5.3}$$

The adjoint bundle of Q is the associated bundle $Q[\mathfrak{g}]$ with typical fibre $\mathfrak{g}$ and the adjoint action of G on $\mathfrak{g}$. The space of sections of $Q[\mathfrak{g}]$ is the Lie algebra of the group of sections of the group bundle $Q[G]$. The Lie algebra $gs(\mathbf{P})$ consists of sections of the adjoint bundle, which are of Sobolev class $H^3(M)$ and satisfy the boundary condition (2.1). Each $\xi : M \to \mathfrak{g}$ in $gs(\mathbf{P})$ corresponds to an equivariant map $\xi^{\#} : P \to \mathfrak{g}$ such that

$$\xi^{\#}(x,e) = \xi(x)\,. \tag{5.4}$$

The aim of this section is to describe the symmetry breaking by the fields $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{P}_H$, that is the fields with gauge symmetry group H, where H is a compact subgroup of $GS(\mathbf{P})$.

Let x_0 be a fixed point in M, then

$$H_0 = \{\phi(x_0) | \phi \in H\} \tag{5.5}$$

is a closed subgroup of G isomorphic to H. We denote by $Z[H_0]$ the centralizer of H_0, defined by

$$Z[H_0] = \{g \in G | hg = gh \;\forall\; h \in H_0\}\,. \tag{5.6}$$

It is a closed subgroup of G with the Lie algebra

$$\mathfrak{z}[H_0] = \{\xi \in \mathfrak{g} | h\xi h^{-1} = \xi \;\forall\; h \in H_0\} . \tag{5.7}$$

The subset Q_0 of Q, given by

$$Q_0 = \{(x, g) \in P | \phi^{\#}(x, g) = \phi(x_0) \;\forall\; \phi \in H\} , \tag{5.8}$$

is a right principal bundle over M with structure group $Z[H_0]$. Since M is contractible, Q_0 is trivial, that is it is diffeomorphic to the product of M and $Z[H_0]$. Actually, we could have chosen the product structure (5.1) in Q in such a way that

$$Q_0 = M \times Z[H_0] . \tag{5.9}$$

For the sake of simplicity of presentation, we assume that (5.9) holds. With this choice of the trivialization elements $\phi \in H$ are constant maps from M to G with values in H_0. By assumption $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{P}_H$, and (1.5) implies that

$$\phi \mathbf{A} \phi^{-1} = \mathbf{A} \;\forall\; \phi \in H . \tag{5.10}$$

Comparing with (5.7) we see that the Yang–Mills potential $\mathbf{A}$ takes the values in $\mathfrak{z}[H_0]$. This means that the connection in Q described by $\mathbf{A}$ reduces to a connection in Q_0. Similarly, the electric component $\mathbf{E}$ of the field strength is a $\mathfrak{g}$-valued l-form on M. The transformation law (1.5) implies that

$$\phi \mathbf{E} \phi^{-1} = \mathbf{E} \;\forall\; \phi \in H . \tag{5.11}$$

Hence, $\mathbf{E}$ has values in $\mathfrak{z}\,[H_0]$.

The matter field Ψ is a section of the associated bundle of Q, with typical fibre $\mathbb{R}^n \otimes \mathbb{C}^4$, where $\mathbb{R}^n$ is the space of the fundamental representation of (the matrix group) G, and the factor $\mathbb{C}^4$ describes the spin degrees of freedom. It follows from (1.5) that

$$\phi \Psi = \Psi \;\forall\; \phi \in H . \tag{5.12}$$

Hence, Ψ has values in the space

$$V_0 = \{z \in \mathbb{R}^n \otimes \mathbb{C}^4 | hz = z \;\forall\; h \in H_0\} , \tag{5.12}$$

and it corresponds to a section of the associated bundle $Q_0[V_0]$ of Q_0 with typical fibre V_0. Thus, we have proved

Theorem 5. *For every* $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{P}_H$, *the Cauchy data* $(\mathbf{A}, \mathbf{E})$ *for the Yang–Mills theory with the structure (internal symmetry) group* G *reduce to Cauchy data for a Yang–Mills theory with the structure (internal symmetry) group*

$$Z[H_0] = \{g \in G | g\phi(x_0) = \phi(x_0) g \;\forall\; \phi \in H\} .$$

The matter field Ψ *reduces to a section of an associated bundle with typical fibre* $V_0 = \{z \in \mathbb{R}^n \otimes \mathbb{C}^4 | hz = z \;\forall\; h \in H_0\}$.

It should be noted that the change of the point $x_0 \in X$, used in the definition of H_0, Eq. (5.5), corresponds to passing from Q_0 to another principal sub-bundle of Q with conjugate structure group.

Symmetry breaking can now be described in terms of the centre $C[H_0]$ of H_0 given by

$$C[H_0] = Z[H_0] \cap H_0 . \tag{5.13}$$

It is an abelian Lie group with Lie algebra $\mathfrak{c}[H_0]$. Elements of the center $C[H]$ of H, analyzed in terms of the principal bundle Q_0, correspond to sections γ of the associated bundle $Q_0[C[H_0]]$. As before, we denote by $\gamma^{\#}$ the $Z[H_0]$-equivariant map from Q_0 to $C[H_0]$ corresponding to a section γ of $Q_0[C[H_0]]$. In analogy to (5.8) we define

$$Q_1 = \{(x,g) \in Q_0 | \gamma^{\#}(x,g) = \gamma(x_0) \ \forall \ \gamma \in C[H]\} . \tag{5.14}$$

It is a principal sub-bundle of Q_0 with structure group $C[H_0]$. The assumption (5.9) about the product structure of Q_0 gives the product structure

$$Q_1 = M \times C[H_0] , \tag{5.15}$$

and we shall continue our discussion in terms of this product structure. If $\mathfrak{c}[H_0] \neq 0$, we can decompose $\mathfrak{z}[H_0]$ into $\mathfrak{c}[H_0]$ and its orthogonal complement $\mathfrak{b}_0$,

$$\mathfrak{z}[H_0] = \mathfrak{c}[H_0] \oplus \mathfrak{b}_0 . \tag{5.16}$$

Similarly, we can decompose the Yang–Mills potential

$$\mathbf{A} = \mathbf{A}_{\mathfrak{c}} + \mathbf{A}_{\mathfrak{b}} , \tag{5.17}$$

where $\mathbf{A}_{\mathfrak{c}}$ has values in $\mathfrak{c}[H_0]$, and $\mathbf{A}_{\mathfrak{b}}$ in $\mathfrak{b}_0$. The component $\mathbf{A}_{\mathfrak{c}}$ describes a connection in the $C[H_0]$ principal bundle Q_1, while $\mathbf{A}_{\mathfrak{b}}$ gives rise to a tensorial form on Q_1. In terms of the terminology used in the Higgs mechanism for symmetry breaking, they correspond to the residual Yang–Mills potential and the vector boson field, respectively, [14].

It should be noted that the symmetry breaking described here is purely intrinsic. There is no need for the Higgs field. However, the vector boson fields corresponding to $\mathbf{A}_{\mathfrak{b}}$ are massless. In the Higgs mechanism the mass of vector bosons is derived from the kinetic energy term for the Higgs boson, [15], which is absent here. On the other hand, the mass of the vector bosons might appear in quantization as an anomaly, [16].

Appendix A. Completion and Almost Complex Structure

One of the technical assumptions in [3] is the existence of an appropriate almost complex structure, which in Yang–Mills theory acts by interchanging $\mathbf{A}$ and $\mathbf{E}$. However, in our phase space $\mathbf{P}$ the variables $\mathbf{A}$ and $\mathbf{E}$ appear asymmetrically, and we do not have existence and uniqueness theorems in spaces symmetric under the interchange of $\mathbf{A}$ and $\mathbf{E}$.

Let $\tilde{\mathbf{P}}$ denote the completion of $\mathbf{P}$ in the L^2 norm. The weak symplectic form ω in $\mathbf{P}$ induces a strong symplectic form $\tilde{\omega}$ in $\tilde{\mathbf{P}}$. The L^2 scalar product $\langle \cdot | \cdot \rangle_{L^2}$ defines a Riemannian metric in $\tilde{\mathbf{P}}$. Let $\mathscr{J} : T\tilde{\mathbf{P}} \to T\tilde{\mathbf{P}}$ be defined by

$$\mathscr{J}(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi) = (-\delta\mathbf{E}, \delta\mathbf{A}, i\delta\Psi) \tag{A.1}$$

for every $(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi) \in T\tilde{\mathbf{P}}$. Then, $\mathscr{J}^2 = -1$, and

$$\tilde{\omega}(\mathscr{J}u, \mathscr{J}v) = \tilde{\omega}(u,v) = \langle \mathscr{J}u | v \rangle_{L^2} = -\langle u | \mathscr{J}v \rangle_{L^2} \tag{A.2}$$

for all $u, v \in T\tilde{\mathbf{P}}$. Thus, $\mathcal{J}$ is an almost complex structure on $\tilde{\mathbf{P}}$. The action of $GS(\mathbf{P})$ in $\mathbf{P}$ extends to an action in $\tilde{\mathbf{P}}$ preserving its symplectic form, the Riemannian metric and the almost complex structure.

Let $\mathbf{V}$ be a closed subspace of $T_\mathbf{p}\mathbf{P}$ and let $\tilde{\mathbf{V}}$ be its closure in $T_\mathbf{p}\tilde{\mathbf{P}}$. The symplectic annihilator $\mathbf{V}^\omega$ of $\mathbf{V}$ is defined by

$$\mathbf{V}^\omega = \{\mathbf{u} \in T_\mathbf{p}\mathbf{P} | \omega(\mathbf{u}, \mathbf{v}) = 0 \ \forall \ \mathbf{v} \in \mathbf{V}\} . \tag{A.3}$$

Similarly, the symplectic annihilator of $\tilde{\mathbf{V}}$ in $T_\mathbf{p}\tilde{\mathbf{P}}$ is

$$\tilde{\mathbf{V}}^{\tilde{\omega}} = \{\mathbf{u} \in T_\mathbf{p}\tilde{\mathbf{P}} | \tilde{\omega}(\mathbf{u}, \mathbf{v}) = 0 \ \forall \ \mathbf{v} \in \tilde{\mathbf{V}}\} . \tag{A.4}$$

Since $\mathbf{V}$ is closed, we have

$$(\mathbf{V}^\omega)^\omega = \mathbf{V} . \tag{A.5}$$

We denote by $\mathbf{V}^\perp$ the L^2-orthogonal complement of $\mathbf{V}$ in $T_\mathbf{p}\mathbf{P}$, and $\tilde{\mathbf{V}}^\perp$ the L^2 orthogonal complement of its closure $\tilde{\mathbf{V}}$ in $T_\mathbf{p}\tilde{\mathbf{P}}$. We have

$$(\mathbf{V}^\perp)^\omega = (\tilde{\mathbf{V}}^\perp)^{\tilde{\omega}} \cap T_\mathbf{p}\mathbf{P} . \tag{A.6}$$

Moreover, by Eq. (A.2),

$$(\tilde{\mathbf{V}}^\perp)^{\tilde{\omega}} = \{\mathbf{u} \in T_\mathbf{p}\tilde{\mathbf{P}} | \tilde{\omega}(\mathbf{u}, \mathbf{v}) = 0 \ \forall \ \mathbf{v} \in \tilde{\mathbf{V}}^\perp\} = \{\mathbf{u} \in T_\mathbf{p}\tilde{\mathbf{P}} | \mathcal{J}\mathbf{u} \in (\tilde{\mathbf{V}}^\perp)^\perp\} = \mathcal{J}\tilde{\mathbf{V}} .$$

Hence,

$$(\mathbf{V}^\perp)^\omega = \mathcal{J}\tilde{\mathbf{V}} \cap T_\mathbf{p}\mathbf{P} . \tag{A.7}$$

In the following we shall use the notation

$$\mathcal{J}\mathbf{V} = \mathcal{J}\tilde{\mathbf{V}} \cap T_\mathbf{p}\mathbf{P} . \tag{A.8}$$

Appendix B. Properness of the Action of the Gauge Symmetry Group

The gauge symmetry group $GS(\mathbf{P})$ consists of map $\phi : M \to G$ in the Sobolev class $H^3(M)$ such that $n \cdot \operatorname{grad} \phi = 0$, (2.4). Its action in $\mathbf{P}$ is given by (1.5.). In order to prove that this action is proper, we need to show that, for every sequence $\mathbf{p}_n = (\mathbf{A}_n, \mathbf{E}_n, \Psi_n)$ converging to $\mathbf{p}_\infty = (\mathbf{A}_\infty, \mathbf{E}_\infty, \Psi_\infty) \in \mathbf{P}$, and every sequence ϕ_n in $GS(\mathbf{P})$ such that $\phi_n\mathbf{p}_n$ converges to $\mathbf{p} = (\mathbf{A}, \mathbf{E}, \Psi)$, the sequence ϕ_n has a convergent subsequence with limit ϕ and $\phi\mathbf{p}_\infty = \mathbf{p}$.

The gauge transformations act on $\mathbf{A}$, $\mathbf{E}$, and Ψ independently. Hence, we may consider first the action of $GS(\mathbf{P})$ on the connections. For a sequence $\mathbf{A}_n$ converging to $\mathbf{A}_\infty$, and a sequence ϕ_n in $GS(\mathbf{P})$, let

$$\mathbf{C}_n = \phi_n\mathbf{A}_n\phi_n^{-1} + \phi_n d\phi_n^{-1} \tag{B.1}$$

denote $\mathbf{A}_n$ transformed by ϕ_n. This implies

$$d\phi_n = \phi_n\mathbf{A}_n - \mathbf{C}_n\phi_n . \tag{B.2}$$

By hypothesis, the sequences $\mathbf{A}_n$ and $\mathbf{C}_n$ converge in $H^2(M)$ to $\mathbf{A}_\infty$ and $\mathbf{A}$, respectively. In particular, their $H^2(M)$ norms $\|\mathbf{A}_n\|_{H^2}$ and $\|\mathbf{C}_n\|_{H^2}$ are bounded. Furthermore, the $L^2(M)$ norms $\|\phi_n\|_{L^2}$ of ϕ_n are bounded since M and G are compact. Equation (B.2) implies that also the $L^2(M)$ norms $\|d\phi_n\|_{L^2}$ of $d\phi_n$ are bounded. Hence, the $H^1(M)$ norms $\|\phi_n\|_{H^1}$ of ϕ_n are bounded. Repeating this argument twice, we conclude that the $H^3(M)$ norms of ϕ_n are bounded. By Rellich's Lemma the sequence ϕ_n has a subsequence convergent to ϕ in $H^2(M)$. Without loss of generality, we can restrict our argument to this subsequence, and assume that ϕ_n converges to ϕ in $H^2(M)$. Hence, the sequence $\mathbf{C}_n = \phi_n \mathbf{A}_n \phi_n^{-1} + \phi_n d\phi_n^{-1}$ converges to $\phi \mathbf{A}_\infty \phi^{-1} + \phi d\phi^{-1}$ in $H^1(M)$,

$$\|\phi \mathbf{A}_\infty \phi^{-1} + \phi d\phi^{-1} - \mathbf{C}_n\|_{H^1} \to 0 \quad \text{as} \quad n \to \infty . \tag{B.3}$$

By hypothesis, $\mathbf{C}_n$ converges to $\mathbf{A}$ in $H^2(M)$. Therefore,

$$\begin{aligned}\|\phi \mathbf{A}_\infty \phi^{-1} + \phi d\phi^{-1} - \mathbf{A}\|_{H^1} &\leqq \|\phi \mathbf{A}_\infty \phi^{-1} + \phi d\phi^{-1} - \mathbf{C}_n\|_{H^1} \\ &\quad + \|\mathbf{C}_n - \mathbf{A}\|_{H^1} \to 0 \quad \text{as} \quad n \to \infty .\end{aligned}$$

This implies that

$$\mathbf{A} = \phi \mathbf{A}_\infty \phi^{-1} + \phi d\phi^{-1} , \tag{B.4}$$

and hence,

$$d\phi = \phi \mathbf{A}_\infty - \mathbf{A}\phi . \tag{B.5}$$

Since the right-hand side of (B.5) belongs to $H^2(M)$, it follows that $d\phi \in H^2(M)$, so that $\phi \in H^3(M)$.

Using (B.2) and (B.5), we observe that

$$\begin{aligned}\|d\phi_n - d\phi\|_{H^2} &= \|\phi_n \mathbf{A}_n - \mathbf{C}_n \phi_n - (\phi \mathbf{A}_\infty - \mathbf{A}\phi)\|_{H^2} \\ &\leqq \|\phi_n \mathbf{A}_n - \phi \mathbf{A}_\infty\|_{H^2} + \|\mathbf{C}_n \phi_n - \mathbf{A}\phi)\|_{H^2} .\end{aligned}$$

As $n \to \infty$ the right-hand side tends to zero, because $\phi_n \to \phi, \mathbf{A}_n \to \mathbf{A}_\infty$, and $\mathbf{C}_n \to \mathbf{A}$ in $H^2(M)$. Hence, $\|d\phi_n - d\phi\|_{H^2} \to 0$, which implies that $\phi_n \to \phi$ in $H^3(M)$. This proves the properness of the action of $GS(\mathbf{P})$ on the space of $H^2(M)$ connections satisfying the boundary conditions (1.1).

In remains to show that ϕ takes $\mathbf{E}_\infty$ to $\mathbf{E}$ and Ψ_∞ to Ψ. By hypothesis $\mathbf{E}_n \to \mathbf{E}_\infty$ and $\phi_n \mathbf{E}_n \phi_n^{-1} \to \mathbf{E}$ in $H^1(M)$. Since $\phi_n \to \phi$ in $H^3(M)$, and a pointwise multiplication of functions in $H^1(M)$ by functions in $H^3(M)$ is a continuous map from $H^1(M) \times H^3(M)$ to $H^1(M)$, we obtain

$$\mathbf{E} = \lim_{H^1(M)} (\phi_n \mathbf{E}_n \phi_n^{-1}) = \left(\lim_{H^3(M)} \phi_n\right)\left(\lim_{H^1(M)} \mathbf{E}_n\right)\left(\lim_{H^3(M)} \phi_n^{-1}\right) = \phi \mathbf{E}_\infty \phi^{-1} .$$

In a similar manner we obtain

$$\Psi = \lim_{H^2(M)} (\phi_n \Psi_n) = \lim_{H^3(M)} (\phi_n) \lim_{H^2(M)} (\Psi_n) = \phi \Psi_\infty .$$

This completes the proof of properness of the action of $GS(\mathbf{P})$ in $\mathbf{P}$.

Appendix C. Proof of the Slice Theorem

We establish here the slice theorem for infinite dimensional groups, cf. [17]. Since the assumptions made here are more general than in the body of the paper, we use an independent notation following that of Appendix 2 of [18].

Let M be a Hilbert manifold, and G a Hilbert Lie group, with a continuous proper smooth left action $\Phi : G \times M \to M$. In the following we use the notation $\Phi_g(m) = \Phi(g, m)$. Let $\mathfrak{g}$ be the Lie algebra of G. For each $m \in M$, we denote by G_m the isotropy group of m, by $\mathfrak{g}_m$ the Lie algebra of G_m, and by $O_m = G \cdot m$ the orbit of G through m. Since the action is proper G_m is compact and the orbit O_m is closed. The tangent space $T_m O_m$ can be presented as $\mathfrak{g} \cdot m = T\Phi(\mathfrak{g}, 0)(e, m)$, and $\mathfrak{g}_m \cdot m = 0$.

Hypotheses

(a) *The group G is a Lie group in the sense that the exponential map gives a diffeomorphism of a neighbourhood of $0 \in \mathfrak{g}$ onto a neighbourhood of $e \in G$.*

(b) *The action Φ is proper.*

(c) *Bochner Linearization Lemma*, [19]. *There is a G_m invariant neighbourhood U of $m \in M$ and a diffeomorphism $\psi : U \to T_m M$ such that:*

$$\psi(m) = 0 \quad and \quad T_m\psi = identity \tag{C.1}$$

and, for every $g \in G_m$ and $\mathbf{p} \in U$,

$$\psi(\Phi_g(\mathbf{p})) = T_m\Phi_g(\psi(\mathbf{p})) \, . \tag{C.2}$$

These assumptions are stronger than needed to get slices, but they allow us to control the topology of the space of orbits of the group action. They are satisfied by the gauge symmetry group $GS(\mathbf{P})$ considered in this paper. Proposition 2.1 guarantees assumption (a). Properness of the action of $GS(\mathbf{P})$ is proved in Appendix B. The Bochner Linearization Lemma follows from the fact that the action of $GS(\mathbf{P})$ is affine.

First we need a lemma.

Lemma C.1. *Given $m \in M$, let L be a submanifold of G through e such that*

$$\mathfrak{g} = \mathfrak{g}_m \oplus T_e L \, , \tag{C.3}$$

and let $\mathbf{S}$ be a submanifold of M through m such that

$$T_m M = T_m O_m \oplus T_m \mathbf{S} \, . \tag{C.4}$$

Then there is an open set $U \times V \subseteq L \times \mathbf{S}$ such that $\Phi|(U \times V)$ is a diffeomorphism onto an open neighbourhood W of $m \in M$.

Proof. Let $D\Phi : TG \times TM \to TM$ denote the derivative of Φ, and $D_i\Phi$ be the restriction of $D\Phi$ to the i^{th} factor. Since $\Phi(e, m) = m$ for all $m \in M$, we have that $D_2\Phi_{(e,m)} = identity$, and so $D\Phi_{(e,m)}$ is surjective. Now $\ker D_1\Phi_{(e,m)} = \mathfrak{g}_m$ by definition, and also, by definition image $D_1\Phi_{(e,m)} = T_m O_m$.

Choosing $L \subset \mathfrak{g}$ and $\mathbf{S} \subset T_m M$ so that we can make the identifications

$$T_e L \cong \mathfrak{g}/\mathfrak{g}_m \, , \tag{C.5}$$

$$T_m \mathbf{S} \cong T_m M / T_m O_m \, , \tag{C.6}$$

we have that $D\Phi|(T_e L \times T_m \mathbf{S})$ is an isomorphism. Since M is a Hilbert manifold the lemma now follows by the inverse function theorem. □

Corollary C.2. *If $\Phi_g V \cap V \neq \emptyset$ for some $g \in U \subseteq L \subset G$, and $V \subseteq \mathbf{S}$, then $g = e$.*

Proof. Let $m \in V$ be such that $\Phi(g, m) = \Phi(e, m')$ with $m' \in V$. Since Φ is a local diffeomorphism on $U \times V$ it follows that $(g, m) = (e, m')$, so that $g = e$. □

Lemma C.3. *For every neighbourhood $\tilde{U}$ on M containing m, there is a G_m invariant open set U containing m with $U \subseteq \tilde{U}$.*

Proof. Since M is a Hilbert manifold, it is first countable. Hence, there exists a sequence $\{U_n\}$ of neighbourhoods of m in M such that $U_n \subseteq U_{n-1}, \bigcap_{n=1}^{\infty} U_n = \{m\}$. Suppose now that the statement of the lemma is false. Then $G_m \cdot U_n$ is not contained in $\tilde{U}$ for all n. Hence, there exist sequences $m_n \in U_n$ and $g_n \in G_m$ such that $g_n m_n \notin \tilde{U}$. Since the action of G is proper, the isotropy group G_m is compact and the sequence g_n has a convergent subsequence. Without loss of generality we may assume that g_n converges to $g \in G_m$. The sequence m_n converges to m by construction. The continuity of the action of G in M implies that $g_n m_n$ converges to $g \cdot m = m$, which contradicts the statement that $g_n m_n \notin \tilde{U}$ for all n. □

Slice Theorem. *For each $m \in M$, there exists a smooth submanifold $\mathbf{S}$ of M through m such that*

(1)
$$T_m M = T_m O_m \oplus T_m \mathbf{S} \,. \tag{C.7}$$

(2)
$$T_{\mathbf{p}} M = T_{\mathbf{p}} O_{\mathbf{p}} + T_{\mathbf{p}} \mathbf{S} \ \forall \ \mathbf{p} \in \mathbf{S} \,, \tag{C.8}$$

(3)
$$G_m \cdot \mathbf{S} \subseteq \mathbf{S} \,, \tag{C.9}$$

(4)
$$\text{For } \mathbf{p} \in \mathbf{S}, \ \textit{and } g \in G, \ \text{if } \Phi_g(\mathbf{p}) \in \mathbf{S} \ \textit{then } g \in G_m \,. \tag{C.10}$$

Proof. We prove the existence of a slice by constructing a candidate $\mathbf{S}_\varepsilon$ and showing that properties (1) through (4) hold.

Observe that if $k \in G_m$, $kg \cdot m = kgk^{-1} \cdot m$, or

$$\Phi_k \circ \Phi_g(m) = \Phi_{kgk^{-1}}(m) \,. \tag{C.11}$$

If $g = \exp(t\xi), \xi \in \mathfrak{g}$, then the 1-parameter groups $t \mapsto k[\exp(t\xi)]k^{-1}$ and $t \mapsto \exp(t\mathrm{Ad}_k \xi)$ have the same tangent vector $\mathrm{Ad}_k \xi$ at $t = 0$. Hence, differentiating (C.11) with respect to t at $t = 0$ we get

$$T_m \Phi_k T_e \Phi_m(\xi) = T_e \Phi_m(\mathrm{Ad}_k \xi) \tag{C.12}$$

which tells us that $T_m \Phi_k$ leaves $T_m O_m$ invariant.

Since G_m is compact, there is a G_m invariant inner product on $T_m M$. So $(T_m O_m)^{\perp}$ is a G_m invariant subspace. Using the local linearizing diffeomorphism ψ (from the Bochner Lemma) the submanifold

$$S_\varepsilon = \psi^{-1}((T_m O_m)^{\perp} \cap B_\varepsilon) \,, \tag{C.13}$$

where B_ε is a ball of radius ε in T_mM (with respect to the G_m invariant inner product) is G_m invariant. So $\mathbf{S}_\varepsilon$ has property (3). Moreover, $T_m\mathbf{S}_\varepsilon = (T_mO_m)^\perp$, since $T_m\psi = identity$. Hence, property (1) holds as well.

We argue that Property (2) is an open condition in $\mathbf{S}_\varepsilon$ as follows. Observe that $\Phi|(G \times \mathbf{S}_\varepsilon) : G \times \mathbf{S}_\varepsilon \to M$ is a submersion at (e, m). Hence it is a submersion at (e, p), for all p in a neighbourhood of m in $\mathbf{S}_\varepsilon$.

Now it remains to show that we can find $\varepsilon > 0$ so that (4) holds. Suppose that it does not hold for any $\varepsilon > 0$. This would imply that there is a sequence of points $\{m_n\}$ with $m_n \in \mathbf{S}_{1/n}$, and a sequence $g_n \in G$, such that $g_n \notin G_m$, and $g_nm_n \in \mathbf{S}_{1/n}$. Hence, $m_n \to m$ and $g_nm_n \to m$. Since the action of G in M is proper, it follows that there exists a convergent subsequence of g_n. Without loss of generality, we may assume that $g_n \to g$. Moreover, $g_nm_n \to gm = m$, which implies that $g \in G_m$. Hence, $g^{-1}g_n \to e$, $g \in G_m$ and $g_n \notin G_m$.

G_m acts in G be multiplication on the left, and the orbit of this action through the identity in G coincides with G_m. Applying Lemma C.1 to the action of G_m in G, we conclude that there is a submanifold L of G transverse to G_m at e, and an open set $U \times V \subseteq G_m \times L$ such that the multiplication $(k, l) \mapsto kl$ is a diffeomorphism onto some open neighbourhood W of e in G. Thus, we may assume that $g^{-1}g_n = k_nl_n$, with $k_n \in G_m$ and $l_n \in L$. Since, g and k_n are in G_m and $g_n \notin G_m$, it follows that $l_n = k_n^{-1}g^{-1}g_n \notin G_m$ for all n.

We now apply Lemma C.1 to $U \times V \subseteq L \times \mathbf{S}_\varepsilon$. For sufficiently large n, $g_nm_n = gk_nl_nm_n$ is in $V \subseteq \mathbf{S}_\varepsilon$. It follows from Corollary C.2 that $gk_nl_n = e$ for n large enough. Hence, $l_n = k_n^{-1}g^{-1} \in G_m$, which contradicts the result above. This contradiction establishes (4). □

We should remark that for the case under consideration in this paper, that is for $G = GS(\mathbf{P})$, there is a natural $GS(\mathbf{P})$ invariant weak inner product on the manifold $M = \mathbf{P}$ given by the L^2 scalar product. In this case, we can take $(T_mO_m)^\perp$ to be the L^2 orthogonal complement of T_mO_m. As long as the ball B_ε is defined with respect to the strong G_m invariant inner product on M, the manifold $\mathbf{S}_\varepsilon$ defined by (C.13) will satisfy properties (1) through (4). Hence, for the gauge symmetry group $GS(\mathbf{P})$ one can always choose a slice $\mathbf{S}$ through m satisfying the condition (3.8), requiring that $T_m\mathbf{S}$ is the L^2 orthogonal complement of T_mO_m.

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