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Scalar Conservation Laws with Discontinuous Flux Function: II. On the Stability of the Viscous Profiles

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Abstract: The equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(H(x)f(u) + \big(1 - H(x)\big)g(u)\big) = 0$, where H is Heaviside's step function, appears for example in continuous sedimentation of solid particles in a liquid, in two-phase flow, in traffic-flow analysis and in ion etching. The discontinuity of the flux function at $x = 0$ causes a discontinuity of a solution, which is not uniquely determined by the initial data. By a viscous profile of this discontinuity we mean a stationary solution of $u_t + (F^\delta)_x = \varepsilon u_{xx}$, where F^δ is a smooth approximation of the discontinuous flux, i.e., H is smoothed. We present some results on the stability of the viscous profiles, which means that a small disturbance tends to zero uniformly as $t \to \infty$. This is done by weighted energy methods, where the weight (depending on f and g) plays a crucial role.

1. Introduction

The scalar conservation law with discontinuous flux function

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\Big(F^0\big(u(x,t),x\big)\Big) = 0, \text{ where } F^0(u,x) = \begin{cases} f(u), & x > 0 \\ g(u), & x < 0 \end{cases} \quad (1.1)$$

arises in several applications, for example in continuous sedimentation of solid particles in a liquid, see Diehl [4] and Chancelier et al. [1], in two-phase flow in porous media, see Gimse and Risebro [5], in traffic-flow analysis, see Mochon [15], and in ion etching in the fabrication of semiconductor devices, see Ross [17]. The Cauchy problem for a more general equation than (1.1), including a point source $s(t)$ at $x = 0$, has been analysed by Diehl [3]. Generally, solutions of (1.1) contain a discontinuity along the t-axis and curves of discontinuity that go into and emanate from it. This discontinuity along the t-axis is not uniquely determined by the initial data $u(x,0)$ and a uniqueness condition, Condition Γ, was introduced in [3]. Another way to pick out the physically relevant discontinuity is by means of the viscous profile condition. By a viscous profile we mean a stationary solution of $u_t + (F^\delta)_x = \varepsilon u_{xx}$, where F^δ is a smooth approximation of the discontinuous flux, i.e., H is smoothed. In [2] the equivalence between Condition Γ and the viscous profile condition is presented, as well as the non-unique augmentation of Eq. (1.1) to a 2×2 triangular non-strictly hyperbolic system. In terms of such a system, depending on f and g, the discontinuity at $x = 0$ is either a regular Lax, under- or overcompressive, marginal under- or overcompressive or a degenerate shock wave.

To examine the stability of viscous profiles, the elementary energy method was introduced, independently, by Matsumura and Nishihara [13] and by Goodman [6]. Some other important references, where the energy method has been used for systems of viscous conservation laws, strictly hyperbolic and with convexity, are [10, 11, 19]. In [12] Liu and Xin present stability results for a non-strictly hyperbolic 2×2 triangular system, which qualitatively corresponds to letting $f(u) = u^2$ and $g(u) = u^2 + C$ with $C > 0$ in (1.1).

In the scalar case with $f = g$ and under various assumptions, stability results have been obtained by several authors, e.g. [8, 9, 10, 14, 16, 18]. In particular, Matsumura and Nishihara [14] and Jones, Gardner and Kapitula [9] present stability results as well as decay rates for non-convex flux $f = g$.

In this paper we consider the case of general non-convex f and g in (1.1). In Sect. 3 we investigate the stability of the different viscous profiles in the sense that we show that a small disturbance decays to zero uniformly as $t \to \infty$. In terms of the 2×2 system, there is only a disturbance in the first equation. We use weighted energy estimates, where the weight (depending on f and g) plays a crucial role. If the total initial mass of the disturbance is zero, we show stability for any type of wave. For non-zero initial mass, we show stability for undercompressive waves without any further assumption and for overcompressive waves if the absolute integral of the initial disturbance is small. Some cases are not resolved. Matsumura and Nishihara [14] also use the weighted energy method (for the case $f = g$) and it is interesting to note that their weight behaves asymptotically precisely as ours. The weight is constant if the characteristics have different slope than the discontinuity (the shock is compressive) and it increases linearly if the characteristics have the same slope as the discontinuity.

The main results of this paper are contained in Theorems 3.2, 3.3 and 3.4.

2. Definitions and Notation

Equation (1.1) should be interpreted in distribution sense and a solution is defined as a weak solution in the standard way. In the class of piecewise smooth functions $u(x,t)$ the Cauchy problem can be formulated

$$\begin{aligned} &u_t + f(u)_x = 0, && x > 0,\ t > 0\,, \\ &u_t + g(u)_x = 0, && x < 0,\ t > 0\,, \\ &f\big(u^+(t)\big) = g\big(u^-(t)\big), && t > 0\,, \\ &u(x,0) = u_0(x), && x \in \mathbb{R}\,, \end{aligned} \tag{2.1}$$

where $u^{\pm}(t) = u(\pm 0, t)$. We assume that $f, g \in C^\infty$ and that they both satisfy $|f'(u)| = C > 0$ for $|u| \geq R$, for some $R > 0$.

The existence of viscous profiles corresponding to the different types of shock wave is shown in [2]. We now recall the notation. Let h be a scalar smooth function of one variable with $h'(x) \equiv 0$ for $|x| \geq 1$ and $h(x)$ strictly increasing from 0 to 1 in $|x| < 1$. Then $h(x/\delta) \to H(x)$ as $\delta \searrow 0$ in distribution sense and

$$F^\delta(u,x) \equiv h(x/\delta)f(u) + \big(1 - h(x/\delta)\big)g(u) \to F^0(u,x), \quad \delta \searrow 0\,.$$

For physical reasons, we are interested in solutions of (1.1) that are limits of solutions of the parabolic equation

$$u_t + \left(F^\delta\left(u(x,t),x\right)\right)_x = \varepsilon u_{xx} \tag{2.2}$$

as δ and $\varepsilon \searrow 0$ at the same speed. By a *viscous profile* for a discontinuity between u^- and u^+ of a solution of (1.1) we mean a stationary solution $u(x,t) = v(\xi)$, with $\xi = x/\varepsilon$, of (2.2) satisfying $v(\xi) \to u^\pm$ as $\xi \to \pm\infty$. Hence v satisfies

$$v'(\xi) = F^\delta\left(v(\xi), \varepsilon\xi\right) - g(u^-) . \tag{2.3}$$

Note that $g(u^-) = f(u^+)$ by (2.1).

Equation (1.1) can be written as a system

$$\begin{pmatrix} u \\ a \end{pmatrix}_t + \begin{pmatrix} af(u) + (1-a)g(u) \\ k(a) \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where k is defined such that the second equation has the stationary solution $a(x,t) \equiv H(x)$ (i.e. $k(0) = k(1) = 0$, $k(u) > 0$ for $0 < u < 1$). The function k is not uniquely determined; see [2] for a natural definition. The eigenvalues of the Jacobian of the flux matrix are

$$\begin{aligned} \lambda_1(u,a) &= af'(u) + (1-a)g'(u) , \\ \lambda_2(u,a) &= k'(a) , \end{aligned}$$

so generally (2.4) is a non-strictly hyperbolic system. Consider a shock wave between $(u^-, 0)$ and $(u^+, 1)$ of the system (2.4). The first eigenvalue satisfies $\lambda_1(u^-,0) = g'(u^-)$, $\lambda_1(u^+,1) = f'(u^+)$ and the second $\lambda_2(u^-,0) = k'(0) > 0$, $\lambda_2(u^+,1) = k'(1) < 0$. We say that the wave is

undercompressive if	$g'(u^-) < 0, f'(u^+) > 0,$
marginal undercompressive if	$g'(u^-) = 0, f'(u^+) > 0$ or $g'(u^-) < 0, f'(u^+) = 0,$
overcompressive if	$g'(u^-) > 0, f'(u^+) < 0,$
marginal overcompressive if	$g'(u^-) = 0, f'(u^+) < 0$ or $g'(u^-) > 0, f'(u^+) = 0,$
a regular Lax wave if	$g'(u^-) < 0, f'(u^+) < 0$ or $g'(u^-) > 0, f'(u^+) > 0,$
degenerate if	$g'(u^-) = f'(u^+) = 0.$

However, analysis of a general solution of (2.1), see [3], yields that there are three qualitatively different types of discontinuity, depending on f and g. Furthermore, see [2], the viscous profiles of (2.2) are naturally divided into these three cases. The description of these three cases in terms of f and g needs, in the context of this paper, cumbersome notation and we refer to [3, 2]. In this paper we are concerned with the stability of a given viscous profile, so we define the three cases in terms of the qualitatively different viscous profiles obtained. Conditions on the existence of viscous profiles and their behaviour are given in [2]. In some cases the ratio δ/ε must be small enough (depending on f, g, u^- and u^+) in order for a profile to exist. By symmetry we can assume that $u^- \leq u^+$. Some examples of the three different cases are given in the figures below, where orbits of Eq. (2.3) are shown. The figures are obtained by computer simulations, with $\delta/\varepsilon = 1$ and $h(x) = \left(1 + \sin(x\pi/2)\right)/2$ in $|x| < 1$.

Case 1. $u^+ = u^-$ with f and g such that there is a unique viscous profile of (2.3), $v(\xi) \equiv u^+ = u^-$, see Fig. 1. Generically, the wave is *undercompressive*, but it may

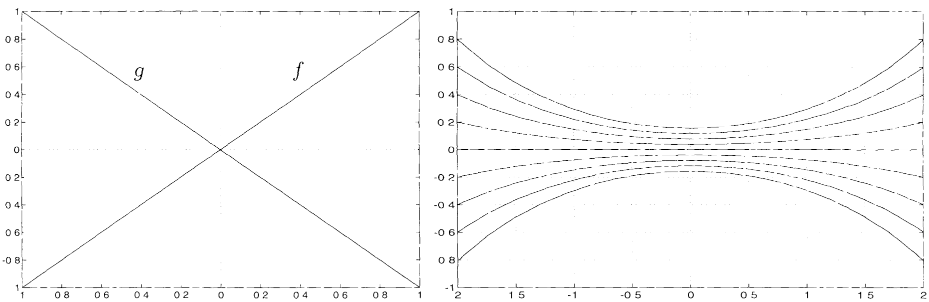

Fig. 1. Case 1. Solutions of (2.3) (right) when $g(u) = -u$, $f(u) = u$ and $u^+ = u^- = 0$.

also be marginal undercompressive or degenerate.

Case 2. $u^- < u^+$ with f and g such that there is a unique viscous profile of (2.3) which satisfies $v(\xi) \equiv u^-$ for $\xi < -\delta/\varepsilon$, see Fig. 2. If the ratio δ/ε is small enough, the profile is increasing for $\xi > -\delta/\varepsilon$. Generically, the wave is a *regular Lax shock*, but it may also be marginal under- or overcompressive or degenerate.

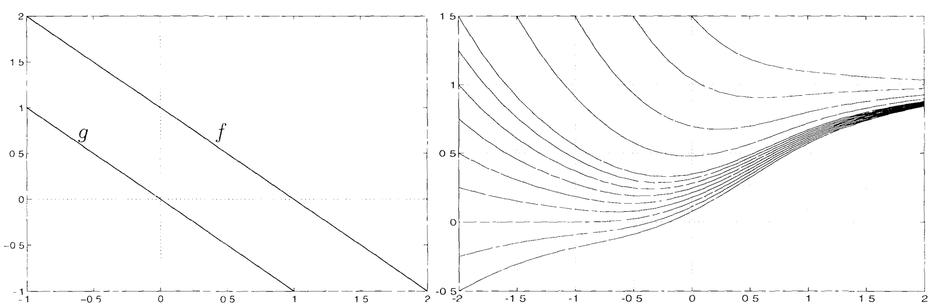

Fig. 2. Case 2. Solutions of (2.3) (right) when $g(u) = -u$, $f(u) = 1 - u$, $u^- = 0$ and $u^+ = 1$. Notice that there is a unique profile.

Case 3. $u^- \leq u^+$ such that there are infinitely many viscous profiles of (2.3), see Figs. 3 and 4. Generically, the wave is *overcompressive*, but it may also be marginal overcompressive or degenerate.

3. Stability of the Viscous Profiles

By simplicity we shall from now on let $\delta = \varepsilon = 1$. Then $\xi = x$ and we write F instead of F^δ. In Subsect. 3.1 we treat a generic situation of Case 1, covering the undercompressive waves. Stability is shown without any restriction on the initial disturbance. In Subsect. 3.2 stability is shown for any viscous profile assuming that the total initial mass of the disturbance is zero and its absolute integral is small. If the initial mass is non-zero we give, in Subsect. 3.3, conditions on when this problem can be brought back to the zero-mass case by jumping to another profile. We shall make use of the following theorem, the proof of which can be found in Appendix A. For $n \geq 0$ and $T \geq 0$ introduce the norm

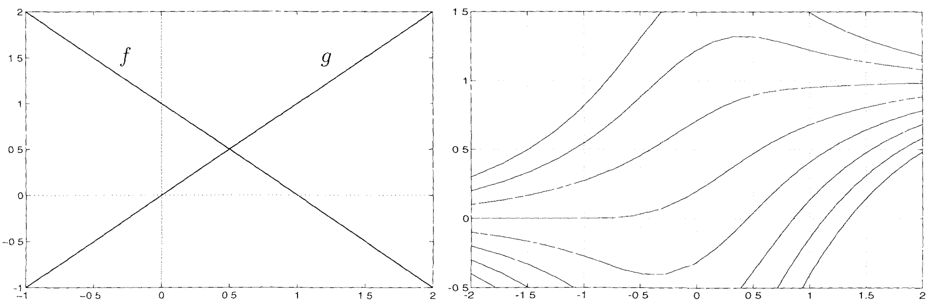

Fig. 3. Case 3. $g(u) = u$, $f(u) = 1 - u$, $u^- = 0$ and $u^+ = 1$. All orbits are viscous profiles. The corresponding shock wave is overcompressive.

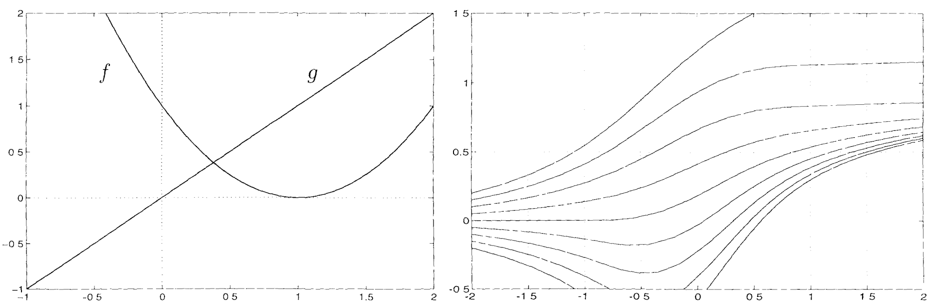

Fig. 4. Case 3. Phase plane diagram (right) when $g(u) = u$, $f(u) = (1-u)^2$, $u^- = 0$ and $u^+ = 1$. There are infinitely many viscous profiles. The corresponding shock wave is marginal overcompressive.

$$\|u\|_{n,T} = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |u(x,t)|(1+x^2)^{n/2} .$$

In particular, the norm of the initial data $u(x,0) = u_0(x)$ is

$$\|u\|_{n,0} = \sup_{x\in\mathbb{R}} |u_0(x)|(1+x^2)^{n/2} .$$

Theorem 3.1. *Consider the Cauchy problem*

$$\begin{aligned} &u_t + \left(\tilde{F}(u,x)\right)_x = u_{xx}, \quad && x \in \mathbb{R},\ t > 0 , \\ &u(x,0) = u_0(x), && x \in \mathbb{R} . \end{aligned} \tag{3.1}$$

Assume that $\tilde{F} \in C^\infty(\mathbb{R}^2)$ with $\tilde{F}(0,x) \equiv 0$ and that $\tilde{F}_u$ and its higher derivatives ($\tilde{F}_{ux}$, $\tilde{F}_{uux}$, $\tilde{F}_{uxx}$, etc.) are bounded for all $u, x \in \mathbb{R}$. If, for some $n \geq 0$, $u_0 \in C^\infty(\mathbb{R})$ satisfies

$$\|\partial_x^\alpha u\|_{n,0} < \infty, \quad \alpha = 0, 1, 2, \ldots , \tag{3.2}$$

then (3.1) *has a unique solution $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$, which for every $T > 0$ satisfies*

$$\|\partial_x^\alpha \partial_t^\beta u\|_{n,T} \leq C_{T,\alpha,\beta} \sum_{k=0}^{\max(\alpha,\beta+1)} \|\partial_x^k u\|_{n,0}, \quad \alpha, \beta = 0, 1, 2, \ldots . \tag{3.3}$$

Furthermore, if $u\tilde{F}_x(u,x) \geq 0$*, for all* $u, x \in \mathbb{R}$*, and* $u_0(x) \to 0$ *as* $|x| \to \infty$*, then*

$$\inf_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq \sup_{x\in\mathbb{R}} u_0(x), \quad (x,t) \in \mathbb{R} \times [0,T] .$$

In the situations below when we shall use this result, the boundedness of the coefficients $\tilde{F}_u$, etc. will be implied by the assumption that f' and g' are identically constant far away. In Case 1 we require that $n > 1/2$, so that $u \in L^2$. In Cases 2 and 3 we require $n > 2$, since we shall integrate functions of the type $U^2\varphi$, where U is a primitive function of u and the weight φ fulfils $0 < \varphi(x) \leq C|x|$.

3.1. Case 1.

The viscous profile in this case is constant, say $v(x) \equiv 0$. A disturbance u will thus satisfy

$$\begin{aligned} &u_t + \Big(h(x)\big(f(u) - g(u)\big) + g(u)\Big)_x = u_{xx}, && x \in \mathbb{R},\ t > 0 , \\ &u(x,0) = u_0(x), && x \in \mathbb{R} , \end{aligned} \tag{3.4}$$

where the flux functions are assumed to satisfy

$$\begin{matrix} g(u) \gtrless 0 \\ f(u) \lessgtr 0 \end{matrix} \qquad u \lessgtr 0 , \tag{3.5}$$

cf. Fig. 1. Concerning f and g this is the generic situation, see [2]. The other cases that do not satisfy (3.5) but still yield a unique constant viscous profile are covered by the more general but weaker Theorem 3.3.

Theorem 3.2. *Let the initial data* $u_0 \in C^\infty(\mathbb{R})$ *for problem* (3.4)*, with the assumption* (3.5)*, satisfy* (3.2) *for some* $n > 1/2$*. Then the solution, guaranteed by Theorem* 3.1*,* $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ *satisfies* $\inf u_0 \leq u \leq \sup u_0$ *and*

$$\sup_x |u(x,t)| \to 0, \quad t \to \infty .$$

Firstly, (3.5) implies $u\tilde{F}_x(u,x) = uh'(x)\big(f(u) - g(u)\big) \geq 0$, so by Theorem 3.1 $\inf u_0 \leq u \leq \sup u_0$ holds. To prove the decay to zero we need the following lemma.

Lemma 3.1. *Let* M *be a constant satisfying*

$$|f'(u)| \leq M, \quad |g'(u)| \leq M \quad \textit{for} \quad \inf u_0 \leq u \leq \sup u_0 ,$$

$m = \sup h'$ *and introduce*

$$d(u) = f(u) - g(u) \quad \textit{and} \quad D(u) = \int_0^u d(v)\, dv .$$

The solution u *of* (3.4) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx + \int_{-\infty}^{\infty} h'(x)D(u)\,dx + \int_{-\infty}^{\infty} u_x^2\,dx = 0\,, \tag{3.6}$$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,dx - \frac{m}{2}\int_{-\infty}^{\infty} h'(x)d^2(u)\,dx - \frac{M^2}{2}\int_{-\infty}^{\infty} u_x^2\,dx \le 0\,. \tag{3.7}$$

Proof. Multiply the equation of (3.4) with u and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx + \int_{-\infty}^{\infty} \big(h(x)d(u)+g(u)\big)_x u\,dx = \int_{-\infty}^{\infty} u_{xx}u\,dx\,.$$

Partial integration of the second term yields

$$-\int_{-\infty}^{\infty} h(x)d(u)u_x\,dx - \int_{-\infty}^{\infty} g(u)u_x\,dx = \int_{-\infty}^{\infty} h'(x)D(u)\,dx - 0\,.$$

Doing the same with the right-hand side, $\int u_{xx}u\,dx = -\int u_x^2\,dx$, we obtain (3.6). Differentiating (3.4) with respect to x, multiplying by u_x and integrating yields

$$\int_{-\infty}^{\infty} u_{tx}u_x\,dx + \int_{-\infty}^{\infty} \big(h(x)d(u)+g(u)\big)_{xx} u_x\,dx = \int_{-\infty}^{\infty} u_{xxx}u_x\,dx \quad \Longleftrightarrow$$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,dx - \int_{-\infty}^{\infty} h'(x)d(u)u_{xx}\,dx - \int_{-\infty}^{\infty} \big(h(x)d'(u)+g'(u)\big)u_x u_{xx}\,dx =$$

$$= -\int_{-\infty}^{\infty} u_{xx}^2\,dx\,.$$

Here, the second and third term can be estimated by, respectively,

$$\frac{1}{2}\int_{-\infty}^{\infty} h'(x)\left(md^2(u) + \frac{u_{xx}^2}{m}\right)dx \le \frac{m}{2}\int_{-\infty}^{\infty} h'(x)d^2(u)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} u_{xx}^2\,dx$$

and

$$\int_{-\infty}^{\infty} |hf' + (1-h)g'||u_x u_{xx}|\,dx \le \frac{M}{2}\int_{-\infty}^{\infty}\left(Mu_x^2 + \frac{u_{xx}^2}{M}\right)dx =$$

$$= \frac{M^2}{2}\int_{-\infty}^{\infty} u_x^2\,dx + \frac{1}{2}\int_{-\infty}^{\infty} u_{xx}^2\,dx\,.$$

Hence (3.7) follows.

Proof of Theorem 3.2. Equation (3.6) implies $\frac{d}{dt}\int u^2\,dx \le 0$, hence

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} u^2\,dx \quad \text{exists} \ge 0\,. \tag{3.8}$$

Add s times (3.7) to (3.6):

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(u^2+su_x^2)\,dx+\int_{-\infty}^{\infty}h'(x)\left(D(u)-\frac{sm}{2}d^2(u)\right)\,dx+$$
$$+\int_{-\infty}^{\infty}\left(1-\frac{sM^2}{2}\right)u_x^2\,dx\le 0\,. \tag{3.9}$$

Let $0<s\le\frac{1}{2M}\min(\frac{1}{m},\frac{1}{M})$, then the last term is non-negative. The second term is non-negative too, because $d'(u)\le 2M$, $ud(u)\ge 0$ for all u and $d(0)=0$ imply

$$0\le\int_0^u\big(2M-d'(v)\big)d(v)\,dv=2MD(u)-\frac{1}{2}d^2(u)\,,$$

which in turn implies

$$D(u)-\frac{sm}{2}d^2(u)\ge D(u)-\frac{d^2(u)}{4M}\ge 0\,.$$

Thus (3.9) is reduced to $\frac{d}{dt}\int(u^2+su_x^2)\,dx\le 0$, so the limit $\lim_{t\to\infty}\int(u^2+su_x^2)\,dx$ exists. Hence, by (3.8)

$$\lim_{t\to\infty}\int_{-\infty}^{\infty}u_x^2\,dx \quad \text{exists } =A\ge 0\,.$$

Integrating (3.6) from 0 to T implies $(h'D\ge 0)$

$$\int_0^T\int_{-\infty}^{\infty}u_x^2\,dx\,dt\le\frac{1}{2}\int_{-\infty}^{\infty}u_x^2(x,0)\,dx \quad \forall T>0\,,$$

which implies that $A=0$. Finally,

$$u^2=2\int_{-\infty}^{x}uu_x\,dx\le 2\left(\int_{-\infty}^{\infty}u^2\,dx\int_{-\infty}^{\infty}u_x^2\,dx\right)^{1/2}$$

shows that $\sup_x|u(x,t)|\to 0$ as $t\to\infty$.

3.2. Cases 2 and 3 with Zero Initial Mass for the Disturbance.

Let v be any viscous profile of Case 2 or 3 and consider a disturbance w. Then

$$u(x,t) = v(x) + w(x,t)$$

satisfies the viscous equation

$$u_t + \big(F(u,x)\big)_x = u_{xx} \quad \text{where} \quad F(u,x) = h(x)f(u) + \big(1 - h(x)\big)g(u) .$$

Using this equation and that v satisfies $\big(F(v,x)\big)_x = v_{xx}$ we obtain the Cauchy problem for the disturbance

$$\begin{aligned} &w_t + \big(F(v+w,x) - F(v,x)\big)_x = w_{xx} , \\ &w(x,0) = w_0(x) . \end{aligned} \tag{3.10}$$

In order to prove that the disturbance w decays to zero as $t \to \infty$ we shall use weighted energy estimates, where the weight function φ depends on the viscous profile v and on the flux functions f and g. In Subsect. 3.4 we show the following lemma.

Lemma 3.2. *Given v, f and g, there exists a $\varphi \in C^2(\mathbb{R})$ such that for all $x \in \mathbb{R}$*

$$\begin{aligned} &0 < C_1 \le \varphi(x) \le C_2 + C_3|x| , \\ &|\varphi'(x)| \le C'\varphi(x) , \\ &|\varphi''(x)| \le C''\varphi(x) , \\ &\frac{d}{dx}\Big(\Big(h(x)f'\big(v(x)\big) + \big(1 - h(x)\big)g'\big(v(x)\big)\Big)\varphi(x) + \varphi'(x)\Big) \le 0 . \end{aligned} \tag{3.11}$$

The main result is the following and the proof is given at the end of this subsection.

Theorem 3.3. *Let the initial data $w_0 \in C^\infty(\mathbb{R})$ for problem* (3.10) *satisfy*

$$\int_{-\infty}^{\infty} w_0(x)\,dx = 0 , \tag{3.12}$$

$$\int_{-\infty}^{\infty} |w_0(x)|\,dx < \epsilon , \tag{3.13}$$

and (3.2) *for some $n > 2$. If ϵ is sufficiently small, then the solution, guaranteed by Theorem* 3.1, *$w \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ of* (3.10) *satisfies* (3.3), *and there exists a $\varphi \in C^2(\mathbb{R})$ satisfying* (3.11) *such that*

$$\sup_x w^2(x,t)\varphi(x) \to 0, \quad t \to \infty .$$

For the proof we need the following estimates.

Lemma 3.3. *Let* (3.12) *and* (3.13) *hold. Then the solution w of* (3.10) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} W^2\varphi\,dx + K\int_{-\infty}^{\infty} w^2\varphi\,dx \le 0\,, \tag{3.14}$$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (W^2+sw^2)\varphi\,dx + \frac{K}{2}\int_{-\infty}^{\infty} w^2\varphi\,dx + \frac{s}{2}\int_{-\infty}^{\infty} w_x^2\varphi\,dx \le 0\,, \tag{3.15}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} (W^2+sw^2+rw_x^2)\varphi\,dx \le 0\,, \tag{3.16}$$

where $W_x = w$ and K, s and r are non-negative constants and φ satisfies (3.11).

Proof. The conservation law (3.10) implies $\frac{d}{dt}\int_{-\infty}^{\infty} w(x,t)\,dx = 0$. The assumption (3.12) then implies

$$\int_{-\infty}^{\infty} w(x,t)\,dx = 0, \quad \forall t \ge 0\,. \tag{3.17}$$

Let

$$W(x,t) = \int_{-\infty}^{x} w(y,t)\,dy\,.$$

By the decay property (3.3) it follows that $W_t = \int_{-\infty}^{x} w_t(y,t)\,dy \to 0$, $W_x = w \to 0$ and $W_{xx} = w_x \to 0$ as $x \to \infty$. Hence the integrated equation of (3.10) is

$$W_t + F(v+W_x,x) - F(v,x) = W_{xx}\,. \tag{3.18}$$

By (3.17) we can write

$$W(x,t) = \begin{cases} \int_{-\infty}^{x} w(y,t)\,dy, & x \le 0 \\ -\int_{x}^{\infty} w(y,t)\,dy, & x > 0 \end{cases}$$

and conclude that W satisfies (3.3) with $n > 1$. Write

$$F(v+w,x) - F(v,x) = w\psi(x) + w^2\rho_2(w,x)\,,$$

where

$$\psi(x) = F_u\big(v(x),x\big) = h(x)f'\big(v(x)\big) + \big(1-h(x)\big)g'\big(v(x)\big)$$

and

$$\rho_2(w,x) = \int_0^1 (1-\theta)F_{uu}\big(v(x)+\theta w,x\big)\,d\theta\,.$$

Both these expressions are bounded because of the assumption that f', g', f'' and g'' are bounded functions. The integrated Eq. (3.18) can thus be written

$$W_t + \psi(x)W_x + \rho_2(W_x, x)W_x^2 = W_{xx} \,. \tag{3.19}$$

Since this equation has no W-term and the coefficients are bounded, the maximum principle holds (see Appendix A), that is,

$$\sup_{x\in\mathbb{R},t>0} |W(x,t)| \leq \sup_{x\in\mathbb{R}} |W(x,0)| \leq \int_{-\infty}^{\infty} |w_0(x)|\,dx \,. \tag{3.20}$$

Multiply (3.19) by $W\varphi$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} W^2\varphi\,dx + \int_{-\infty}^{\infty} \psi W_x W\varphi\,dx + \int_{-\infty}^{\infty} \rho_2 W_x^2 W\varphi\,dx = \int_{-\infty}^{\infty} W_{xx}W\varphi\,dx \,. \tag{3.21}$$

Partial integration of the second term gives

$$\int_{-\infty}^{\infty} \psi W_x W\varphi\,dx = -\frac{1}{2}\int_{-\infty}^{\infty} (\psi\varphi)' W^2\,dx \,,$$

and the right-hand side of (3.21) can be written

$$\int_{-\infty}^{\infty} W_{xx}W\varphi\,dx = -\int_{-\infty}^{\infty} W_x(W\varphi)_x\,dx = -\int_{-\infty}^{\infty} W_x^2\varphi\,dx - \int_{-\infty}^{\infty} W_x W\varphi'\,dx =$$

$$= -\int_{-\infty}^{\infty} W_x^2\varphi\,dx + \frac{1}{2}\int_{-\infty}^{\infty} W^2\varphi''\,dx \,.$$

Hence (3.21) can be written

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} W^2\varphi\,dx - \frac{1}{2}\int_{-\infty}^{\infty} \big((\psi\varphi)' + \varphi''\big)W^2\,dx +$$

$$+ \int_{-\infty}^{\infty} \rho_2 W_x^2 W\varphi\,dx + \int_{-\infty}^{\infty} W_x^2\varphi\,dx = 0 \,.$$

The second term is positive by (3.11). Because of the maximum principle (3.20) we can choose $\int_{-\infty}^{\infty} |w_0(x)|\,dx$ so small that there exists a constant $K \in (0,1)$ so that the third term can be estimated

$$\left|\int_{-\infty}^{\infty} \rho_2 W_x^2 W\varphi\,dx\right| \leq (1-K)\int_{-\infty}^{\infty} W_x^2\varphi\,dx \,.$$

Hence we get (3.14).

Now consider Eq. (3.10). Let $F(v+w,x) - F(v,x) = w\rho_1(w,x)$, where

$$\rho_1(w,x) = \int_0^1 F_u\big(v(x)+\theta w, x\big)\,d\theta\ .$$

Notice that, since f' and g' are bounded, $|\rho_1| \le M$ for some constant M. Multiply (3.10) by $w\varphi$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2\varphi\,dx + \int_{-\infty}^{\infty}(\rho_1 w)_x w\varphi\,dx = \int_{-\infty}^{\infty} w_{xx}w\varphi\,dx\ . \tag{3.22}$$

The right-hand side can be treated as above and with $\varphi'' \le C''\varphi$ of (3.11) we get

$$-\int_{-\infty}^{\infty} w_x^2\varphi\,dx + \frac{1}{2}\int_{-\infty}^{\infty} w^2\varphi''\,dx \le -\int_{-\infty}^{\infty} w_x^2\varphi\,dx + \frac{C''}{2}\int_{-\infty}^{\infty} w^2\varphi\,dx\ .$$

The second term of (3.22) can be estimated, using $|\varphi'| \le C'\varphi$,

$$\begin{aligned}
\left|\int_{-\infty}^{\infty}(\rho_1 w)_x w\varphi\,dx\right| &= \left|\int_{-\infty}^{\infty}\rho_1 w(w\varphi)_x\,dx\right| \le \int_{-\infty}^{\infty}|\rho_1|\big(|ww_x|\varphi + w^2|\varphi'|\big)\,dx \le \\
&\le \int_{-\infty}^{\infty} M\left(\left(\frac{Mw^2}{2}+\frac{w_x^2}{2M}\right)\varphi + w^2C'\varphi\right)\,dx = \\
&= M\left(\frac{M}{2}+C'\right)\int_{-\infty}^{\infty} w^2\varphi\,dx + \frac{1}{2}\int_{-\infty}^{\infty} w_x^2\varphi\,dx\ .
\end{aligned}$$

Hence (3.22) yields

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2\varphi\,dx - \left(\frac{M^2+C''}{2}+MC'\right)\int_{-\infty}^{\infty} w^2\varphi\,dx + \frac{1}{2}\int_{-\infty}^{\infty} w_x^2\varphi\,dx \le 0\ .$$

Multiplying this inequality by an $s > 0$ small enough and adding to (3.14) yields (3.15).

The third step is to differentiate (3.10) with respect to x, multiply by $w_x\varphi$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w_x^2\varphi\,dx + \int_{-\infty}^{\infty}(\rho_1 w)_{xx} w_x\varphi\,dx = \int_{-\infty}^{\infty} w_{xxx}w_x\varphi\,dx\ . \tag{3.23}$$

As before the right-hand side is

$$-\int_{-\infty}^{\infty} w_{xx}^2\varphi\,dx + \frac{1}{2}\int_{-\infty}^{\infty} w_x^2\varphi''\,dx \le -\int_{-\infty}^{\infty} w_{xx}^2\varphi\,dx + \frac{C''}{2}\int_{-\infty}^{\infty} w_x^2\varphi\,dx\ .$$

The second term of (3.23) is $-\int (\rho_1 w)_x (w_x\varphi)_x \, dx$ and the two factors of this integrand can be estimated as follows, with positive constants $C_1, \dots, C_5$. Boundedness of f', g', f'' and g'' implies

$$|(\rho_1 w)_x| = \left| \frac{d}{dx} \int_0^w F_u\big(v(x)+y,x\big)\,dy \right| = \left| v'(x) \int_0^w F_{uu}\big(v(x)+y,x\big)\,dy + \right.$$

$$\left. + \int_0^w F_{ux}\big(v(x)+y,x\big)\,dy + F_u\big(v(x)+w,x\big) w_x \right| \le C_1\big(|w| + |w_x|\big) ,$$

and, since $|\varphi'| \le C'\varphi$,

$$|(w_x\varphi)_x| \le \big(|w_{xx}| + C'|w_x|\big)\varphi .$$

The second term of (3.23) can thus be estimated by

$$C_2 \int_{-\infty}^{\infty} w^2\varphi\,dx + C_3 \int_{-\infty}^{\infty} w_x^2\varphi\,dx + C_4 \int_{-\infty}^{\infty} w_{xx}^2\varphi\,dx ,$$

where $C_4 < 1$ can be obtained by using the arithmetic-geometric inequality as we have done several times above. Hence (3.23) becomes

$$\frac{1}{2}\frac{d}{dt} \int_{-\infty}^{\infty} w_x^2\varphi\,dx - C_5 \int_{-\infty}^{\infty} (w^2 + w_x^2)\varphi\,dx \le 0 .$$

Multiplying this by an $r > 0$ small enough and adding to (3.15) gives (3.16).

Proof of Theorem 3.3. The three inequalities (3.14), (3.15) and (3.16) imply in turn that the limits

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} W^2\varphi\,dx, \quad \lim_{t\to\infty} \int_{-\infty}^{\infty} w^2\varphi\,dx, \quad \lim_{t\to\infty} \int_{-\infty}^{\infty} w_x^2\varphi\,dx \tag{3.24}$$

exist. Integrating (3.15) from 0 to T gives

$$\int_{-\infty}^{\infty} \big(W^2(x,T) + sw^2(x,T)\big)\varphi\,dx + K \int_0^T \int_{-\infty}^{\infty} w^2\varphi\,dx\,dt + s \int_0^T \int_{-\infty}^{\infty} w_x^2\varphi\,dx\,dt \le$$

$$\le \int_{-\infty}^{\infty} \big(W^2(x,0) + sw^2(x,0)\big)\varphi\,dx .$$

Particularly,

$$K \int_0^{\infty} \int_{-\infty}^{\infty} w^2\varphi\,dx\,dt + s \int_0^{\infty} \int_{-\infty}^{\infty} w_x^2\varphi\,dx\,dt < \infty ,$$

which implies that the second and third limit of (3.24) are zero. Finally,

$$w^2(x,t)\varphi(x) = 2\int_{-\infty}^{x} ww_x\varphi\,dx + \int_{-\infty}^{x} w^2\varphi'\,dx \le$$

$$\le \int_{-\infty}^{\infty}(w^2+w_x^2)\varphi\,dx + C'\int_{-\infty}^{\infty} w^2\varphi\,dx$$

implies that $\sup_x w^2(x,t)\varphi(x) \to 0$ as $t\to\infty$.

3.3. Case 3 with Non-Zero Initial Mass for the Disturbance.

Given a profile $\tilde{v}$ and a disturbance $\tilde{w}$ with non-zero initial mass, i.e.,

$$m_0 = \int_{-\infty}^{\infty} \tilde{w}(x,0)\,dx \neq 0\ .$$

We shall investigate when this case can be brought back to the zero-mass case so that Theorem 3.3 can be applied. Put $u = \tilde{v} + \tilde{w}$. If we can find a profile v with

$$\int_{-\infty}^{\infty} \big(v(x) - \tilde{v}(x)\big)\,dx = m_0\ ,$$

then $w \equiv u - v$ will satisfy $\tilde{v} + \tilde{w} = v + w$ and

$$\int_{-\infty}^{\infty} w(x,0)\,dx = 0\ ,$$

i.e., (3.12) will be satisfied. Assuming that $\tilde{w}(x,0)$ satisfies the other conditions in Theorem 3.3, we shall investigate when $w(x,0)$ also does. Parametrize the solutions of

$$v' = h(x)f(v) + \big(1-h(x)\big)g(v) - g(u^-) \tag{3.25}$$

with their intersection with the v-axis, that is, $v_\alpha(0) = \alpha$. Basic results on ordinary differential equations imply

$$\alpha_1 < \alpha_2 \quad\Longrightarrow\quad v_{\alpha_1} < v_{\alpha_2}\ , \tag{3.26}$$

$$v_\alpha(x) \text{ is continuous with respect to } \alpha\ , \tag{3.27}$$

so that

$$A = \{\alpha\in\mathbb{R} : v_\alpha \text{ satisfies (3.25)}; v_\alpha(0)=\alpha; v_\alpha(x)\to u^\pm,\ x\to\pm\infty\}$$

is an interval. (In Cases 1 and 2 the set A is a single point, since the profiles are unique.) Furthermore, the properties (3.26) and (3.27) together with Lebesgue's majorant theorem imply that given $\tilde{v} = v_{\alpha_0}$ with $\alpha_0 \in A$ and

$$\int_{-\infty}^{\infty} |v_\alpha(x) - v_{\alpha_0}(x)|\, dx < \infty, \quad \forall \alpha \in A\,, \tag{3.28}$$

then

$$I(\alpha) \equiv \int_{-\infty}^{\infty} \big(v_\alpha(x) - v_{\alpha_0}(x)\big)\, dx$$

is continuous and increasing.

If (3.28) holds and $m_0 \in I(A)$ (which is an interval), then we may bring the problem back to the zero-mass case (Theorem 3.3). Firstly, (3.13) holds if $\int |\tilde{w}(x,0)|\, dx$ is sufficiently small, because then $|m_0|$ is small and

$$\int_{-\infty}^{\infty} |w(x,0)|\, dx \le \int_{-\infty}^{\infty} |\tilde{v}(x) - v(x)|\, dx + \int_{-\infty}^{\infty} |\tilde{w}(x,0)|\, dx =$$

$$= |m_0| + \int_{-\infty}^{\infty} |\tilde{w}(x,0)|\, dx\,.$$

Secondly, assume that $\tilde{w}$ satisfies (3.2) for some $n > 2$. Since $w = \tilde{v} - v + \tilde{w}$, then (3.3) holds for w and for some $n > 2$, if and only if $\tilde{v}(x) - v(x)$ and its derivatives decay as $o(|x|^{-2})$, $|x| \to \infty$.

In the following lemma, which we also need in Subsect. 3.4, we focus on the asymptotic behaviour of the profiles as $x \to +\infty$. Similar results are easily obtained for large negative x. Recall that a profile is either $v(x) \neq u^+$ or $v(x) \equiv u^+$ for all $x > 1$.

Lemma 3.4. *Let v be a viscous profile with $v(x) \to u^+$ as $x \to \infty$ and $v(x) \neq u^+$, $x > 1$. If $f'(u^+) < 0$, then there exists an $a > 0$ such that*

$$|v(x) - u^+| = O\left(e^{-ax}\right), \quad \textit{for large } x. \tag{3.29}$$

If $f^{(i)}(u^+) = 0$, $i = 1, \ldots, p-1$, and $f^{(p)}(u^+) \neq 0$ for some integer $p \ge 2$, then there exist a constant l and a function L such that

$$\big(v(x) - u^+\big)^{p-1} = \frac{1}{ax + l + L(x)}, \quad |L(x)| = \begin{cases} O(\log x), & p = 2 \\ O\left(x^{\frac{p-2}{p-1}}\right), & p \ge 3 \end{cases} \quad \textit{for large } x, \tag{3.30}$$

where $a = (1-p) f^{(p)}(u^+)/p!$.

Proof. Let $u^+ = 0$ in the proof. Put $k = f^{(p)}(0)/p!$. Then for $x > 1$, Eq. (3.25) for the viscous profile becomes

$$v' = f(v) - f(0) = kv^p + r(x,v)v^{p+1} \quad \Longleftrightarrow \quad \frac{v'}{v^p} = k + r(x,v)v\,, \tag{3.31}$$

where $\big|r\big(x, v(x)\big)v(x)\big| \le |k|/2$ for x large enough. Thus (3.31) implies

$$\frac{k}{2} \le \frac{v'(x)}{v^p(x)}, \quad k > 0\,, \tag{3.32}$$

$$\frac{v'(x)}{v^p(x)} \le \frac{k}{2}, \quad k < 0\,. \tag{3.33}$$

First let $p = 1$, i.e., $k = f'(0) < 0$. Then integrating (3.33) from $x_0 > 1$ to $x > x_0$ implies (3.29) with $a = -k/2 = -f'(0)/2 > 0$.

Now let $p \ge 2$. Integrating (3.32) from x_0 to x with $x > x_0 > 1$ implies, in the case $k > 0$,

$$(p-1)\frac{k}{2}(x - x_0) \le -\frac{1}{v^{p-1}(x)} + \frac{1}{v^{p-1}(x_0)}\,.$$

Since the left-hand side is positive and $v(x) \to 0$, $x \to \infty$, we conclude that $v^{p-1}(x) < 0$ for $x > x_0$. Hence p is even and $v(x) < 0$ for $x > x_0$. Rearranging gives

$$|v(x)^{p-1}| \le \frac{C_+}{x}, \quad \text{for large } x.$$

Similarly, this can be obtained from (3.33) in the case $k < 0$. Now (3.31) implies

$$\frac{v'}{v^p} = k + r(x,v)v, \quad \text{with } r(x,v)v = O\left(x^{\frac{1}{1-p}}\right). \tag{3.34}$$

Integrating this from $x_0 > 1$ to $x > x_0$ gives

$$v^{p-1}(x) = \frac{1}{ax + l + L(x)}\,,$$

with $a = (1-p)k$, $l = 1/v^{p-1}(x_0) - (1-p)kx_0$ and

$$L(x) = (1-p)\int_{x_0}^{x} r\big(x, v(x)\big)v(x)\,dx = \begin{cases} O(\log x), & p = 2 \\ O\left(x^{\frac{p-2}{p-1}}\right), & p \ge 3\,. \end{cases} \tag{3.35}$$

The lemma implies that if $f'(u^+) < 0$ ($p = 1$) the difference $\tilde{v}(x) - v_\alpha(x)$ decays exponentially as $x \to \infty$ for all $\alpha \in A$. This together with the arguments preceding the lemma gives the following result.

Theorem 3.4. *Given a profile $\tilde{v}$ of Case* 3 *with $f'(u^+) < 0$, $g'(u^-) > 0$ (overcompressive wave) and a disturbance $\tilde{w}$ with mass $m_0 = \int \tilde{w}(x,0)\,dx$. If $m_0 \in I(A)$, $\int |\tilde{w}(x,0)|\,dx$ is sufficiently small and $\tilde{w}(x,0)$ satisfies* (3.2) *for some $n > 2$, then there exist a profile v and a weight φ such that $w(x,t) \equiv \tilde{v}(x) + \tilde{w}(x,t) - v(x)$ satisfies* (3.3) *and*

$$\sup_x w^2(x,t)\varphi(x) \to 0, \quad t \to \infty\,.$$

As an example, consider Fig. 3, where $g'(u^-) > 0$, $f'(u^+) < 0$ ($p = 1$) and all profiles are solutions. Thus $A = \mathbb{R}$, hence $I(A) = \mathbb{R}$. We have thus stability for any disturbance, provided $\int |\tilde{w}(x,0)|\,dx$ is small enough. For other examples where $A \neq \mathbb{R}$, see [2].

In the following we write $h_1(x) \sim h_2(x)$, $x \to \infty$, when there exist positive constants C_1, C_2 and x_0 such that $C_1 < h_1(x)/h_2(x) < C_2$ for all $x \ge x_0$.

Now let $p = 2$ and $u^+ = 0$, and use the notation of Lemma 3.4. The proof of that lemma gives that $v(x) < 0$ for $x > x_0$ and $f''(0) = 2k > 0$. Then (3.30) gives

$$v(x) = \frac{1}{ax + l + L(x)} = \frac{1}{ax} - \frac{l + L(x)}{ax\big(ax + l + L(x)\big)}, \quad L(x) = O(\log x) .$$

Firstly, this implies that given $\tilde{v}(x) \equiv 0$, $x > 1$, the difference $|\tilde{v}(x) - v_\alpha(x)| \sim x^{-1}$ as $x \to \infty$ for all $v_\alpha \neq \tilde{v}$, $\alpha \in A$. Hence any disturbance of $\tilde{v}$ can not result in an asymptotic state $v_\alpha \neq \tilde{v}$. Secondly, assume that both $\tilde{v}(x) < 0$ and $v(x) < 0$ for $x > 1$, i.e., the two profiles are increasing. Then the difference

$$\tilde{v}(x) - v(x) = \frac{L - \tilde{L} + l - \tilde{l}}{a^2x^2 + O(x\log x)} = O\left(\frac{\log x}{x^2}\right) \tag{3.36}$$

used in (3.35) gives $|L(x) - \tilde{L}(x)| \leq C_+$. Since $\tilde{l} \neq l$ (see the proof of Lemma 3.4), (3.36) gives

$$|\tilde{v}(x) - v(x)| \sim x^{-2} \quad \text{as } x \to \infty .$$

For $p \geq 3$ the difference decays even slower, for example $p = 3$ gives $\tilde{v}(x) - v(x) = O\left(\log x/x^{3/2}\right)$. As will be seen in Subsect. 3.4 the weight functions, for $p \geq 2$, grow as $\varphi(x) \sim x$, so that $W^2\varphi \sim x^{-1}$ is not integrable, where $W_x = w = \tilde{v} - v + \tilde{w}$. Consequently, for $p \geq 2$, stability can not be shown with the present method.

3.4. The Weight Functions.

Given a viscous profile v in Case 2 or 3, we need a weight function φ that satisfies (3.11) to be able to prove the stability results above. We shall thus prove the statements in Lemma 3.2 during the course of this subsection. In the following, let C_+ denote any suitable positive constant and C any real constant. Put

$$\psi(x) = h(x)f'\big(v(x)\big) + \big(1 - h(x)\big)g'\big(v(x)\big) ,$$

$$\Psi(x) = \int_0^x \psi(y)\, dy .$$

Our weight functions should satisfy the differential inequality of (3.11), that is,

$$(\psi\varphi)' + \varphi'' \leq 0 \quad \Longleftrightarrow$$

$$\varphi(x) = e^{-\Psi(x)}\left(C - \int_0^x V(y)e^{\Psi(y)}\, dy\right) , \tag{3.37}$$

where V is a non-decreasing function. We shall let $|V| \leq 1$ and $V \in C^1$ (with bounded derivative), so that $\varphi \in C^2$. Besides the dependence on v, φ depends on the flux functions f and g. Because of this and since we want the weight functions to be as close to a constant function as possible, we need five different weights to establish the inequality $0 < C_1 \leq \varphi(x) \leq C_2 + C_3|x|$ of (3.11). It is then straightforward to check that the other statements of (3.11) are satisfied.

Assume that $\int_0^\infty e^{\Psi(y)}\,dy < \infty$. Let the constant in (3.37) be $C = \int_0^\infty V(y)e^{\Psi(y)}\,dy$. We get the first weight by letting $V \equiv 1$:

$$\varphi_1^+(x) \equiv e^{-\Psi(x)} \int_x^\infty e^{\Psi(y)}\,dy\ ,$$

and the second by letting $V = V_2^+$ be non-decreasing with $V_2^+(x) = 0$, $x < -1$, and $V_2^+(x) = 1$, $x > 1$:

$$\varphi_2^+(x) \equiv e^{-\Psi(x)} \int_x^\infty V_2^+(y)e^{\Psi(y)}\,dy = \begin{cases} C_+e^{-\Psi(x)}, & x < -1 \\ e^{-\Psi(x)} \int_x^\infty e^{\Psi(y)}\,dy, & x > 1\ . \end{cases}$$

If $\int_{-\infty}^0 e^{\Psi(y)}\,dy < \infty$, let $C = -\int_{-\infty}^0 V(y)e^{\Psi(y)}\,dy$. With $V \equiv -1$ (3.37) implies

$$\varphi_1^-(x) \equiv e^{-\Psi(x)} \int_{-\infty}^x e^{\Psi(y)}\,dy\ ,$$

and with $V = V_2^- \equiv V_2^+ - 1 \leq 0$ we get

$$\varphi_2^-(x) \equiv -e^{-\Psi(x)} \int_{-\infty}^x V_2^-(y)e^{\Psi(y)}\,dy = \begin{cases} e^{-\Psi(x)} \int_{-\infty}^x e^{\Psi(y)}\,dy, & x < -1 \\ C_+e^{-\Psi(x)}, & x > 1\ . \end{cases}$$

The four weight functions above are easily seen to be strictly positive. The last weight we need is defined in the following lemma.

Lemma 3.5. *If $\int_{-\infty}^\infty e^{\Psi(y)}\,dy < \infty$, then there exist positive numbers A and B and a non-decreasing function $V_3 \in C^1(\mathbb{R})$ with bounded derivative, satisfying*

$$V_3(x) = \begin{cases} -1, & x \leq -A \\ 1, & x \geq B. \end{cases} \quad \text{and} \quad \int_{-\infty}^\infty V_3(y)e^{\Psi(y)}\,dy = 0\ .$$

Then we can choose $C = -\int_{-\infty}^0 V_3e^\Psi\,dy = \int_0^\infty V_3e^\Psi\,dy$ in (3.37) *to obtain*

$$0 < \varphi_3(x) \equiv e^{-\Psi(x)} \int_x^\infty V_3e^\Psi\,dy = -e^{-\Psi(x)} \int_{-\infty}^x V_3e^\Psi\,dy =$$

$$= \begin{cases} e^{-\Psi(x)} \int_{-\infty}^x e^{\Psi(y)}\,dy, & x < -A \\ e^{-\Psi(x)} \int_x^\infty e^{\Psi(y)}\,dy, & x > B\ . \end{cases}$$

Proof. Since $\int_{-\infty}^{\infty} e^{\Psi(y)}\,dy < \infty$ we can choose positive numbers A and B such that

$$\int_{-\infty}^{-A} e^{\Psi(y)}\,dy = \int_{B}^{\infty} e^{\Psi(y)}\,dy\,. \tag{3.38}$$

Define the coordinate transformation $z = z(y)$ by

$$z = \int_{-A}^{y} e^{\Psi(\theta)}\,d\theta, \quad -A \le y \le B\,, \tag{3.39}$$

and let $\nu(z)$ be an increasing C^1 function in the interval $\big[0, z(B)\big]$, satisfying $\nu(0) = -1$, $\nu\big(z(B)\big) = 1$, $\nu'(0) = \nu'\big(z(B)\big) = 0$ and $\int_0^{z(B)} \nu(z)\,dz = 0$. It follows that

$$V_3(y) \equiv \begin{cases} -1, & y \le -A \\ \nu\big(z(y)\big), & -A < y \le B \\ 1, & y \ge B \end{cases}$$

is C^1. Then (3.38) and (3.39) imply

$$\int_{-\infty}^{\infty} V_3 e^{\Psi}\,dy = -\int_{-\infty}^{-A} e^{\Psi}\,dy + \int_{-A}^{B} \nu\big(z(y)\big)e^{\Psi(y)}\,dy + \int_{B}^{\infty} e^{\Psi}\,dy = \int_{0}^{z(B)} \nu(z)\,dz = 0\,.$$

It remains to show that $\varphi_3 > 0$. Put $I(x) = \int_x^{\infty} V_3(y)e^{\Psi(y)}\,dy$. Then $I(\pm\infty) = 0$, $I'(x) = -V_3(x)e^{\Psi(x)}$ and the properties of V_3 imply $I > 0$, hence $\varphi = e^{-\Psi} I > 0$.

In the following lemma we focus on the asymptotic behaviour of the weights as $x \to \infty$. Similar results are easily obtained for large negative x. Recall that a profile is either $v(x) \not\equiv u^+$ or $v(x) \equiv u^+$ for all $x > 1$. Let $h_1(x) \sim h_2(x)$, $x \to \infty$, mean that there exist positive constants C_1, C_2 and x_0 such that $C_1 < h_1(x)/h_2(x) < C_2$ for all $x \ge x_0$.

Lemma 3.6. *Let v be a viscous profile with $v(x) \to u^+$ as $x \to \infty$. If $f'(u^+) < 0$, then*

$$e^{\Psi(x)} \sim e^{f'(u^+)x}, \quad x \to \infty\,. \tag{3.40}$$

If for some integer $p \ge 2$, $f^{(i)}(u^+) = 0$, $i = 1, \dots, p-1$, and $f^{(p)}(u^+) \ne 0$, then

$$e^{\Psi(x)} \sim \begin{cases} C_+, & \text{if } v(x) \equiv u^+,\ \forall x > 1 \\ 1/x^{\frac{p}{p-1}}, & \text{if } v(x) \not\equiv u^+,\ \forall x > 1 \end{cases} \quad x \to \infty\,. \tag{3.41}$$

Proof. Let $u^+ = 0$ in the proof and put $k = f^{(p)}(0)/p!$. Use the results of Lemma 3.4. Let $p = 1$. Then, for some $0 \le \theta(x) \le v(x)$,

$$f'\big(v(x)\big) = f'(0) + f''\big(\theta(x)\big)v = f'(0) + O\left(e^{-ax}\right)\,,$$

and, with $x_0 > 1$,

$$\Psi(x) = \int_0^x \psi(y)\,dy = C + \int_{x_0}^x f'\big(v(y)\big)\,dy = C + f'(0)x + O\left(e^{-ax}\right) ,$$

hence (3.40) follows. Let $p \geq 2$. Then (3.30) implies

$$\begin{aligned} f'(v) &= \frac{f^{(p)}(0)}{(p-1)!}v^{p-1} + \frac{f^{(p+1)}\big(\theta(x)\big)}{p!}v^p = \\ &= \frac{pk}{(1-p)kx + l + L(x)} + O\left(x^{-\frac{p}{p-1}}\right) = -\frac{p}{(p-1)x} + R(x) , \end{aligned}$$

where $R(x) = O\left(\frac{\log x}{x^2}\right)$ if $p = 2$ and $R(x) = O\left(x^{-\frac{p}{p-1}}\right)$ if $p \geq 3$. Hence

$$\Psi(x) = \int_0^x \psi(y)\,dy = C + \int_{x_0}^x f'\big(v(y)\big)\,dy = C - \frac{p}{(p-1)}\log x + B(x) ,$$

where $B(x)$ is a bounded function, and (3.41) follows.

Let us now deal with the different cases that may appear regarding v, f and g.
Case 2. In Case 2 the viscous profile is unique and it satisfies $v(x) \equiv u^-$, $x < -1$ (assuming $u^- < u^+$). Furthermore, $g'(u^-) \leq 0$ and $f'(u^+) \leq 0$ hold.

I. $g'(u^-) < 0,\ f'(u^+) < 0$ *(regular Lax wave).*
For $x < -1$, $\Psi(x) = C + \int_{-1}^x g'(u^-)\,dy = C + g'(u^-)x$ holds, and at $+\infty$ we use Lemma 3.6 to conclude that the weight φ_1^+ is suitable. It satisfies $\varphi_1^+(x) \sim C_+$, $|x| \to \infty$, for

$$\varphi_1^+(x) \sim \begin{cases} e^{-g'(u^-)x}\int_x^\infty e^{g'(u^-)y}\,dy = -\frac{1}{g'(u^-)}, & x \to -\infty \\ e^{-f'(u^+)x}\int_x^\infty e^{f'(u^+)y}\,dy = -\frac{1}{f'(u^+)}, & x \to \infty . \end{cases}$$

II. $g'(u^-) = 0,\ f'(u^+) < 0$ *(marginal overcompressive wave).*
Then $e^{\Psi(x)} \sim C_+$ as $x \to -\infty$, which implies that we choose the weight φ_2^+, which satisfies

$$\varphi_2^+(x) \sim C_+, \quad |x| \to \infty .$$

III. $g'(u^-) < 0,\ f^{(i)}(u^+) = 0,\ i = 1, \ldots, p-1,\ f^{(p)}(u^+) \neq 0$ *for some* $p \geq 2$ *(marginal undercompressive wave).*
Combining I above at $-\infty$ and Lemma 3.6 at $+\infty$ yields

$$\varphi_1^+(x) \sim \begin{cases} -\frac{1}{g'(u^-)}, & x \to -\infty \\ x^{\frac{p}{p-1}}\int_x^\infty \frac{dy}{y^{\frac{p}{p-1}}} = (p-1)x, & x \to \infty . \end{cases}$$

IV. $g'(u^-) = 0,\ f^{(i)}(u^+) = 0,\ i = 1, \ldots, p-1,\ f^{(p)}(u^+) \neq 0$ *for some* $p \geq 2$ *(degenerate wave).*
Combining II at $-\infty$ and III at $+\infty$ yields

$$\varphi_2^+(x) \sim \begin{cases} C_+, & x \to -\infty \\ (p-1)x, & x \to \infty\,. \end{cases}$$

The weight functions φ_1^- and φ_2^- will be suitable for Case 2 with $u^- > u^+$.
Case 3. Recall the symmetry of the behaviour of the viscous profiles between $+\infty$ and $-\infty$. We can for example use the analogue of Lemma 3.6 at $-\infty$.

V. $g'(u^-) > 0,\ f'(u^+) < 0$ *(overcompressive wave).*
The weight φ_3 will be suitable and satisfy $\varphi_3(x) \sim C_+,\ |x| \to \infty$, for Lemma 3.6 gives

$$e^{\Psi(x)} \sim \begin{cases} e^{g'(u^-)x}, & x \to -\infty \\ e^{f'(u^+)x}, & x \to \infty\,, \end{cases}$$

and Lemma 3.5

$$\varphi_3(x) \sim \begin{cases} e^{-g'(u^-)x} \int\limits_{-\infty}^{x} e^{g'(u^-)y}\,dy = \frac{1}{g'(u^-)}, & x \to -\infty \\ e^{-f'(u^+)x} \int\limits_{x}^{\infty} e^{f'(u^+)y}\,dy = -\frac{1}{f'(u^+)}, & x \to \infty\,. \end{cases}$$

VI. $g'(u^-) > 0,\ f^{(i)}(u^+) = 0,\ i = 1, \ldots, p-1,\ f^{(p)}(u^+) \neq 0$ *for some* $p \geq 2$ *(marginal overcompressive wave).* There are two subcases:
A. $v(x) \neq u^+$ *for* $x > 1$. Lemma 3.6 gives

$$\varphi_3(x) \sim \begin{cases} \frac{1}{g'(u^-)}, & x \to -\infty \\ x^{\frac{p}{p-1}} \int\limits_{x}^{\infty} \frac{dy}{y^{\frac{p}{p-1}}} = (p-1)x, & x \to \infty\,. \end{cases}$$

B. $v(x) \equiv u^+$ *for* $x > 1$. Since $e^{\Psi(x)} \sim C_+$ as $x \to \infty$, we choose φ_2^-, which satisfies

$$\varphi_2^-(x) \sim \begin{cases} \frac{1}{g'(u^-)}, & x \to -\infty \\ C_+, & x \to \infty\,. \end{cases}$$

The information in V and VI can now be combined to give the rest of the combinations, having the mentioned symmetry in mind. For example, if $v(x) \neq u^\pm\ \forall |x| > 1$, p is defined as in VI and if, analogously, $q \geq 2$ is the first integer with $g^{(q)}(u^-) \neq 0$, then the weight φ_3 satisfies

$$\varphi_3(x) \sim \begin{cases} -(q-1)x, & x \to -\infty \\ (p-1)x, & x \to \infty\,. \end{cases}$$

3.5. The Special Case $f = g$.

Let us see what the present method gives in the case of a single conservation law $u_t + f(u)_x = 0$, where f is generally non-convex. Consider a discontinuity with $u^- < u^+$. There are infinitely many viscous profiles $v(\cdot + x_0)$, $x_0 \in \mathbb{R}$, which are all increasing and satisfy $u^- < v < u^+$. This is a special case of Case 3 with $f = g$, so the results of Subsects. 3.2 and 3.3 hold. This means that stability is proved for a small disturbance with zero mass. Furthermore, if $f'(u^{\pm}) \neq 0$ (compressive wave), the asymptotic profile is determined by the excessive mass of the disturbance, see Theorem 3.4. According to Subsect. 3.4 (Case 3V) the weight function then behaves as $\varphi_3 \sim C_+$ as $|x| \to \infty$. However, if for example $f'(u^+) = 0$ we can not show stability with the present method, with the arguments at the end of Subsect. 3.3. The weight satisfies in this case $\varphi_3 \sim x$ as $x \to \infty$ (Subsect. 3.4, Case 3VIA).

A. A Quasi-Linear Parabolic Equation

The assumption $\tilde{F}(0,x) \equiv 0$ in Theorem 3.1 implies $\big(\tilde{F}(u,x)\big)_x = \tilde{F}_u(u,x)u_x + \tilde{F}_x(u,x) = \tilde{F}_u(u,x)u_x + \tilde{F}_{ux}(\theta,x)u$ for some θ between 0 and u. Theorem 3.1 will thus follow from the following more general one. Let $C_{T,\alpha,\beta}$ denote any suitable positive constant. Recall that we use the norm

$$\|u\|_{n,T} = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |u(x,t)|(1+x^2)^{n/2} .$$

Theorem A.1. *Consider the Cauchy problem*

$$\begin{aligned} &u_t + F(u,x,t)u_x + G(u,x,t)u = u_{xx}, \quad && x\in\mathbb{R},\ t>0 ,\\ &u(x,0) = u_0(x), && x\in\mathbb{R} , \end{aligned} \tag{A.1}$$

where $F, G \in C^\infty$ and its derivatives are bounded functions. If, for some $n \geq 0$, $u_0 \in C^\infty(\mathbb{R})$ satisfies

$$\|\partial_x^\alpha u\|_{n,0} < \infty, \quad \alpha = 0,1,2,\dots ,$$

then (A.1) *has a unique solution $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$, which for every $T>0$ satisfies*

$$\|\partial_x^\alpha \partial_t^\beta u\|_{n,T} \leq C_{T,\alpha,\beta} \sum_{k=0}^{\max(\alpha,\beta+1)} \|\partial_x^k u\|_{n,0}, \quad \alpha,\beta = 0,1,2,\dots . \tag{A.2}$$

Furthermore, if $G \geq 0$ and $u_0(x) \to 0$ as $|x| \to \infty$, then

$$\inf_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq \sup_{x\in\mathbb{R}} u_0(x), \quad (x,t) \in \mathbb{R}\times[0,T] . \tag{A.3}$$

The construction and the smoothness of a solution of (A.1) in a strip $\mathbb{R}\times[0,T]$ can be shown by standard techniques, for example by using the ideas of Hörmander [7]. The bound (A.2) and the uniqueness can be proved using the following maximum principle.

Lemma A.1. *Assume that $u = u(x,t)$ satisfies*

$$\begin{aligned} &u_t + F(u,x,t)u_x + G(u,x,t)u + H(u,x,t) = u_{xx}, \quad && t>0 ,\\ &u(x,0) = u_0(x), && x\in\mathbb{R} , \end{aligned} \tag{A.4}$$

with u, u_t, u_x and u_{xx} temperate in the strip $\mathbb{R}\times[0,T]$, that is, each satisfies an inequality of the type

$$\sup_{0\leq t\leq T} |u(x,t)| \leq C_T(1+x^2)^N \quad \text{for some } N.$$

Assume that F is bounded and that G is bounded below. If $\|u\|_{n,0} < \infty$, then

$$\|u\|_{n,T} \leq C_T\big(\|u\|_{n,0} + \|H_1\|_{n,T}\big), \quad \text{where } H_1(x,t) = H\big(u(x,t),x,t\big). \tag{A.5}$$

Proof. We can assume that $\|H_1\|_{n,T} < \infty$, otherwise (A.5) is trivial. First we show (A.5) for $n = 0$. Let m be a positive integer and $0 < \phi \in C^\infty(\mathbb{R})$ satisfy $\phi(x) \sim e^{-|x|}$ as $|x| \to \infty$. Multiply the first row of (A.4) by $u^{2m+1}\phi$ and integrate to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{u^{2m+2}}{2m+2}\phi\,dx + \int_{-\infty}^{\infty} F u_x u^{2m+1}\phi\,dx + \int_{-\infty}^{\infty} G u^{2m+2}\phi\,dx + \\ + \int_{-\infty}^{\infty} H u^{2m+1}\phi\,dx = \int_{-\infty}^{\infty} u_{xx}u^{2m+1}\phi\,dx\ . \qquad \text{(A.6)}$$

With partial integrations we can write the right-hand side of (A.6) as

$$-(2m+1)\int_{-\infty}^{\infty} u_x^2 u^{2m}\phi\,dx + \frac{1}{2m+2}\int_{-\infty}^{\infty} u^{2m+2}\phi''\,dx\ .$$

Using the inequality $(a, b > 0)$ $2ab \le ra^2 + r^{-1}b^2$ with $r^{-1} = 2(2m+1)$, the second term of (A.6) can be estimated by

$$\int_{-\infty}^{\infty} |Fu^{m+1}||u^m u_x|\phi\,dx \le \\ \le \frac{1}{4(2m+1)}\int_{-\infty}^{\infty} F^2 u^{2m+2}\phi\,dx + (2m+1)\int_{-\infty}^{\infty} u^{2m}u_x^2\phi\,dx\ .$$

The fourth term of (A.6) can be estimated similarly using the inequality $(a, b, \alpha, \beta > 0$, $\alpha + \beta = 1)$ $a^\alpha b^\beta \le \alpha a + \beta b$ with $a = H^{2m+2}$, $b = u^{2m+2}$, $\alpha = \frac{1}{2m+2}$ and $\beta = \frac{2m+1}{2m+2}$, so that

$$|H||u|^{2m+1} \le \frac{1}{2m+2}H^{2m+2} + \frac{2m+1}{2m+2}u^{2m+2}\ .$$

Now (A.6) becomes

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{u^{2m+2}}{2m+2}\phi\,dx + \\ + \int_{-\infty}^{\infty} \left(\left(G - \frac{F^2}{4(2m+1)} - \frac{2m+1}{2m+2}\right)\phi - \frac{\phi''}{2m+2}\right)u^{2m+2}\,dx \le \\ \le \frac{1}{2m+2}\int_{-\infty}^{\infty} H^{2m+2}\phi\,dx\ . \qquad \text{(A.7)}$$

Assume first that $G \ge 2$. Since $\phi(x) \sim e^{-|x|}$, $|\phi''| \le C''\phi$ for some constant $C'' > 0$. Then it follows that the second term of (A.7) is positive for large m (F is bounded) and we get

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^{2m+2}\phi\,dx \leq \int_{-\infty}^{\infty} H^{2m+2}\phi\,dx \leq \sup_{x\in\mathbb{R},0\leq s\leq T} H_1(x,s)^{2m+2}\int_{-\infty}^{\infty}\phi\,dx\ .$$

Integrating from 0 to t implies

$$\int_{-\infty}^{\infty} u^{2m+2}(x,t)\phi(x)\,dx \leq$$

$$\leq \sup_{x\in\mathbb{R}} u(x,0)^{2m+2}\int_{-\infty}^{\infty}\phi\,dx + t\sup_{x\in\mathbb{R},0\leq s\leq T} H_1(x,s)^{2m+2}\int_{-\infty}^{\infty}\phi\,dx\ . \quad \text{(A.8)}$$

If I is an interval and $x_0 \in I$, then

$$\int_{-\infty}^{\infty} u^{2m+2}\phi\,dx \geq \int_I u^{2m+2}\phi\,dx \geq \min_I u^{2m+2}\int_I \phi\,dx\ . \quad \text{(A.9)}$$

Combining (A.8) and (A.9) with the inequality $(a+b)^{1/m} \leq a^{1/m} + b^{1/m}$ we get

$$\min_I |u| \left(\int_I \phi\,dx\right)^{\frac{1}{2m+2}} \leq$$

$$\leq \sup_{x\in\mathbb{R}} |u(x,0)| \left(\int \phi\,dx\right)^{\frac{1}{2m+2}} + t^{\frac{1}{2m+2}} \sup_{x\in\mathbb{R},0\leq s\leq T} |H_1| \left(\int \phi\,dx\right)^{\frac{1}{2m+2}}\ .$$

First let $m \to \infty$ and then let I shrink to the point x_0. We get

$$|u(x_0,t)| \leq \sup_{x\in\mathbb{R}} |u(x,0)| + \sup_{x\in\mathbb{R},0\leq s\leq T} |H_1|\ . \quad \text{(A.10)}$$

If G does not satisfy $G \geq 2$, but only $G \geq -K$ ($K > 0$), then put $v = ue^{-(K+2)t}$. Substituting this into (A.4) gives

$$\begin{aligned} &v_t + Fv_x + (K+2+G)v + He^{-(K+2)t} = v_{xx}\ ,\\ &v(x,0) = u_0(x)\ . \end{aligned}$$

Since $K+2+G \geq 2$ we can use (A.10) to obtain

$$|v(x,t)| \leq \sup_{x\in\mathbb{R}} |v(x,0)| + \sup_{x\in\mathbb{R},0\leq s\leq t} |H_1|e^{-(K+2)s} \leq \sup_{x\in\mathbb{R}} |u(x,0)| + \sup_{x\in\mathbb{R},0\leq s\leq t} |H_1|\ ,$$

that is,

$$|u(x,t)| \leq e^{(K+2)t}\left(\sup_{x\in\mathbb{R}} |u(x,0)| + \sup_{x\in\mathbb{R},0\leq s\leq T} |H_1|\right)\ .$$

Especially,

$$\|u\|_{0,T} \leq C_T\big(\|u\|_{0,0} + \|H_1\|_{0,T}\big)\ . \quad \text{(A.11)}$$

For $n > 0$ let $\rho(x) = (1 + x^2)^{-n}$ and $v(x,t) = u(x,t)/\rho(x)$. Then v, v_t, v_x and v_{xx} are temperate functions. Substituting $u = v\rho$ in (A.4) gives

$$\begin{aligned} & v_t + \left(F - \frac{2\rho'}{\rho}\right) v_x + \left(F\frac{\rho'}{\rho} + G - \frac{\rho''}{\rho}\right) v + \frac{H}{\rho} = v_{xx} \,, \\ & v(x,0) = u_0(x)/\rho(x) \,. \end{aligned}$$

Since the coefficients for v_x and v are bounded, (A.11) implies

$$\|v\|_{0,T} \leq C_T \big(\|v\|_{0,0} + \|H_1/\rho\|_{0,T}\big) \,,$$

which is equivalent to (A.5).

Proof of (A.2). Let u be the constructed solution of (A.1), which is bounded together with all its derivatives. Since $H \equiv 0$, (A.5) gives directly

$$\|u\|_{n,T} \leq C_T \|u\|_{n,0} \,. \tag{A.12}$$

The composite functions $F_1(x,t) \equiv F\big(u(x,t),x,t\big)$ and $G_1(x,t) \equiv G\big(u(x,t),x,t\big)$ and its derivatives are bounded. We can thus consider Eq. (A.1) as being linear:

$$\begin{aligned} & u_t + F_1(x,t)u_x + G_1(x,t)u = u_{xx} \,, \\ & u(x,0) = u_0(x) \,. \end{aligned}$$

Differentiation with respect to x gives

$$\begin{aligned} & \frac{\partial u_x}{\partial t} + F_1 \frac{\partial u_x}{\partial x} + \left(\frac{\partial F_1}{\partial x} + G_1\right) u_x + \frac{\partial G_1}{\partial x} u = \frac{\partial^2 u_x}{\partial x^2} \\ & u_x(x,0) = u_0'(x). \end{aligned} \tag{A.13}$$

Apply (A.5) with $H = u\partial_x G_1$ and combine with (A.12), then

$$\|u_x\|_{n,T} \leq C_T \big(\|u_x\|_{n,0} + \|u\|_{n,T}\big) \leq C_T \big(\|u_x\|_{n,0} + \|u\|_{n,0}\big).$$

Differentiation of (A.13) yields

$$\begin{aligned} & \frac{\partial u_{xx}}{\partial t} + F_1 \frac{\partial u_{xx}}{\partial x} + F_2 u_{xx} + G_2 u_x + G_3 u = \frac{\partial^2 u_{xx}}{\partial x^2} \,, \\ & u_{xx}(x,0) = u_0''(x) \,, \end{aligned}$$

for some bounded functions $F_i(x,t)$ and $G_i(x,t)$. Again (A.5) together with the previously obtained bounds for u_x and u gives

$$\|u_{xx}\|_{n,T} \leq C_T \big(\|u_{xx}\|_{n,0} + \|u_x\|_{n,T} + \|u\|_{n,T}\big) \leq C_T \sum_{k=0}^{2} \|\partial_x^k u\|_{n,0} \,.$$

Now (A.1) implies that also u_t satisfies such a bound. We can repeat this procedure to obtain such bounds for all derivatives, hence (A.2) holds.

Proof of (A.3). Assume that $u(x_0,t_0) > \sup u_0$ with $(x_0,t_0) \in \mathbb{R} \times [0,T]$. Let $\epsilon > 0$ be so small that $u(x_0,t_0) - \epsilon t_0 - \epsilon^3 x_0^2 > \sup u_0$. Then the function

$$u(x,t) - \epsilon t - \epsilon^3 x^2$$

has a maximum at (x_1, t_1) with $0 < t_1 \leq T$. Both u_0 and u are bounded in the strip and decay to zero as $|x| \to \infty$. Thus there exists a constant C with $\epsilon^3 x_1^2 \leq C$ and $u(x_1, t_1) \geq \sup u_0 \geq 0$ holds. Furthermore, at (x_1, t_1) the following hold:

$$\begin{aligned} u_t - \epsilon &\geq 0 \,, \\ u_x - 2\epsilon^3 x_1 &= 0 \,, \\ u_{xx} - 2\epsilon^3 &\leq 0 \,. \end{aligned}$$

With $|F| \leq M$ we get

$$u_t + Fu_x + Gu - u_{xx} \geq \epsilon - M2\epsilon^3 \frac{C^{1/2}}{\epsilon^{3/2}} + 0 - 2\epsilon^3 > 0$$

for $\epsilon > 0$ small enough, which is a contradiction. Hence $u \leq \sup u_0$. Similarly (substitute ϵ by $-\epsilon$ above), $u \geq \inf u_0$.

Proof of the uniqueness in Theorem A.1. With $H(u, x, t, w) = F(u, x, t)w + G(u, x, t)u$, Eq. (A.1) can be written $u_t + H(u, x, t, u_x) = u_{xx}$. Assume that u^1 and u^2 are two solutions with the same initial data. Then

$$\begin{aligned} &(u^1 - u^2)_t + H(u^1, x, t, u_x^1) - H(u^2, x, t, u_x^2) = (u^1 - u^2)_{xx} \,, \\ &(u^1 - u^2)(x, 0) \equiv 0 \,, \end{aligned}$$

where $H(u^1, x, t, u_x^1) - H(u^2, x, t, u_x^2) = H_u(\theta_1, x, t, \xi_1)(u^1 - u^2) +$ $H_w(\theta_2, x, t, \xi_2)(u^1 - u^2)_x$, which implies that we have an equation of the form (A.1) where the coefficients and its derivatives are bounded. Hence (3.3) with $\alpha = \beta = 0$ yields $u^1 = u^2$.

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